Chapter 3
Punctured Convolutional Codes

After describing the fundamental of PCC, this chapter presents its generator polynomial matrices and upper bound on the constraint length in Theorem 1 and Theorem 2, respectively. By virtue of them, the puncturing realizations of the known good nonsystematic and systematic high rate CCs are put forward.

3.1 Introduction

CCs are ones of the most powerful FEC codes, which are widely used in communication systems. In a CC encoder, the parity bits of every block are not only related to the information bits of the corresponding block but also related to other blocks which are located in ahead. Then, CCs usually have better correction ability than the BCs. Unfortunately, the use of CCs is primarily restricted to the low rate CCs with the rate $R = \frac{1}{n}$ or the high rate but short constraint length CCs. However, in many applications, such as wireless channel, their transmission rates must be high while each bandwidth is strictly limited. For compromising power and bandwidth efficiency, the high rate CCs with long constraint length are needed, but their decoding is complex.

The PCCs were suggested in 1979 [36] to make high rate codes from low rate ones simply. Punctured high rate CCs ($R = \frac{l}{nl-m}$) are produced by being periodically nl bits punctured m bits from $R = \frac{1}{n}$ low rate CC (called as the original code). Some PCCs were shown to be almost as good as the known good regular codes. For example, puncturing an original code reduces its free distance, however, this distance of $R = \frac{l}{nl-m}$ PCCs is as large as that achieved with the ones of any $R = \frac{l}{nl-m}$ code. Thus, in this case no loss in minimum distance is caused by puncturing. Besides, PCCs have two other advantages:

- Simplifying the Viterbi decoder for high rate CCs. In the meantime, PCCs can be advantageously decoded by the sequential decoding, too [37][38].
- Being able to implement a multirate (or rate-compatible) CC encoder/decoder[39], which is very useful in multimedia communication systems.

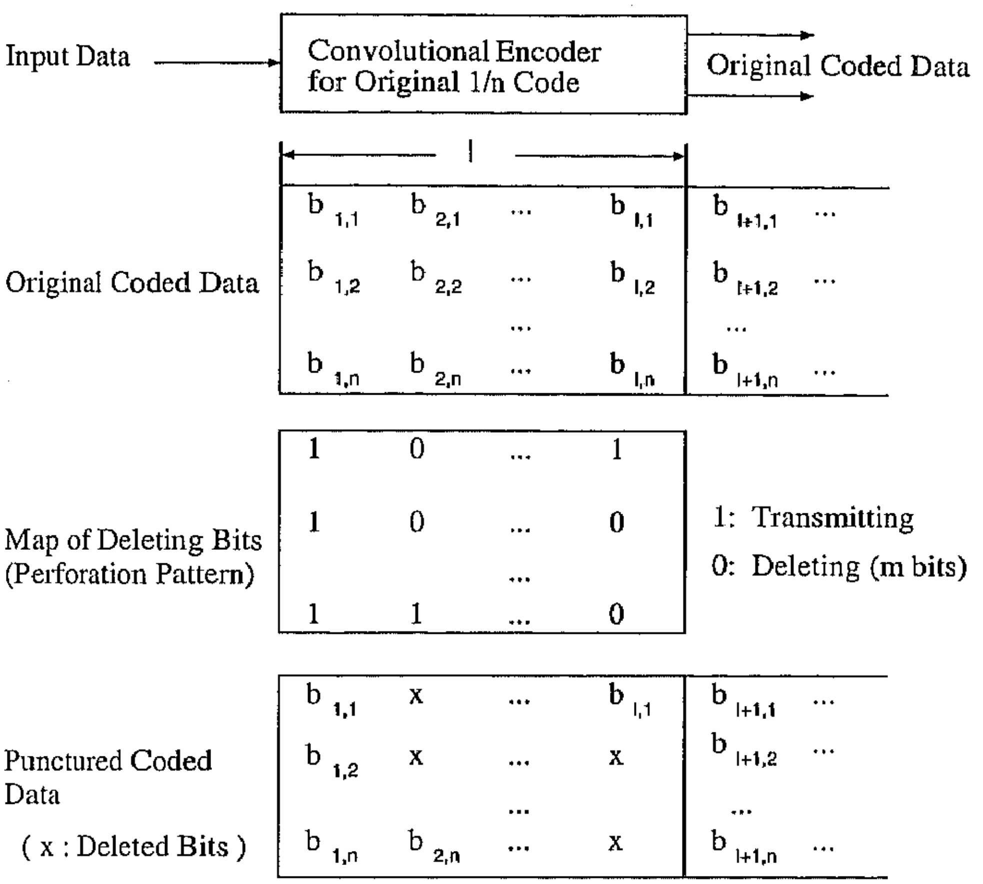

Figure 3.1: The basic steps being punctured from $R = \frac{1}{n}$ CC

Therefore, PCCs are used widely in multirate CDMA systems [40] and wireless cellular systems [41].

The basic steps being punctured from $R = \frac{1}{n}$ CC is shown in Figure 3.1. Firstly, by a good $R = \frac{1}{n}$ CC encoder, an input data sequence is changed into an original coded data sequence. Then, the original coded data sequence is periodically nl bits punctured m bits to produce punctured coded data sequence, according to the map of deleting bits which indicates deleting bit positions.

From the viewpoint of minimum bit error probability, Yasuda et al.[42] have shown a set of good PCCs with different rates from a $R = \frac{1}{2}$ original encoder. Lee[43] found new $R = \frac{l}{l+1}$ PCCs that minimize the required signal-to-noise ratio (SNR) for a target BER of 10^{-9}. Kim[44] derived a group of good high rate systematic PCCs by analyzing their weight spectra and by BER simulation. New rational rate PCCs were proposed for soft decision Viterbi decoding [45][46]. However, though some algebraic properties have been found [37][47][48], no systematic construction method for good

PCCs is known yet. This limit needs the exhaustive search to find the required good PCCs. To give indications for guiding the search for good PCCs, this chapter shows the generator polynomial matrix and the constraint length K of PCCs. By virtue of these properties, the puncturing realizations of the known good nonsystematic high rate $\frac{l}{n}$ CCs from nonsystematic $\frac{1}{n}$ CCs and the known good systematic high rate $\frac{l}{l+1}$ CCs from $\frac{1}{2}$ systematic CCs are given.

3.2 Generator Polynomial Matrix of the PCCs

The generator polynomial matrix is very important to construct CCs. In order to obtain it, this section puts forward Theorem 1 about the generator polynomial matrix $J(D)$ of $R = \frac{l}{nl}$ CCs, firstly.

Theorem 1 *Suppose the generator polynomial matrix of $R = \frac{1}{n}$ CC be:*

$$G(D) = [G_1(D), G_2(D), \ldots, G_n(D)], \tag{3.1}$$

where D is a delay operator in the shift register. Then, the generator polynomial matrix $J(D)$ of $R = \frac{l}{nl}$ CC can be expressed by:

$$J(D) = \begin{pmatrix} J_{1,1}(D) & J_{1,2}(D) & \ldots & J_{1,n}(D) & \ldots & J_{1,(i-1)n+1}(D) & J_{1,(i-1)n+2}(D) & \ldots \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots \\ J_{j,1}(D) & J_{j,2}(D) & \ldots & J_{j,n}(D) & \ldots & J_{j,(i-1)n+1}(D) & J_{j,(i-1)n+2}(D) & \ldots \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots \\ J_{l,1}(D) & J_{l,2}(D) & \ldots & J_{l,n}(D) & \ldots & J_{l,(i-1)n+1}(D) & J_{l,(i-1)n+2}(D) & \ldots \end{pmatrix}$$

$$\begin{pmatrix} J_{1,in}(D) & \ldots & J_{1,(l-1)n+1}(D) & J_{1,(l-1)n+2}(D) & \ldots & J_{1,ln}(D) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ J_{j,in}(D) & \ldots & J_{j,(l-1)n+1}(D) & J_{j,(l-1)n+2}(D) & \ldots & J_{j,ln}(D) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ J_{l,in}(D) & \ldots & J_{l,(l-1)n+1}(D) & J_{l,(l-1)n+2}(D) & \ldots & J_{l,ln}(D) \end{pmatrix} \tag{3.2}$$

where, $i, j = 1, 2, \ldots, l$;

$$J_{j,(i-1)n+s}(D) = D^{\frac{i-i}{l}} G_{s,h}(D^{\frac{1}{l}}), \tag{3.3}$$

for $h = (l + i - j) \bmod l$, $s = 1, 2, \ldots, n$. $G_{s,h}(D)$ is the construction part of $G_s(D)$, expressed by:

$$G_{s,0}(D) + G_{s,1}(D) + \cdots + G_{s,l-1}(D) = \sum_{t=0}^{\infty} (g_{s,0} + g_{s,1}D + \cdots + g_{s,l-1}D^{l-1})D^{tl}. \tag{3.4}$$

Proof: Refer to Appendix A. □

The generator polynomial matrix $Q(D)$ of $R = \frac{l}{nl-m}$ PCCs is derived by being punctured m columns from $J(D)$ in terms of the perforation matrix.

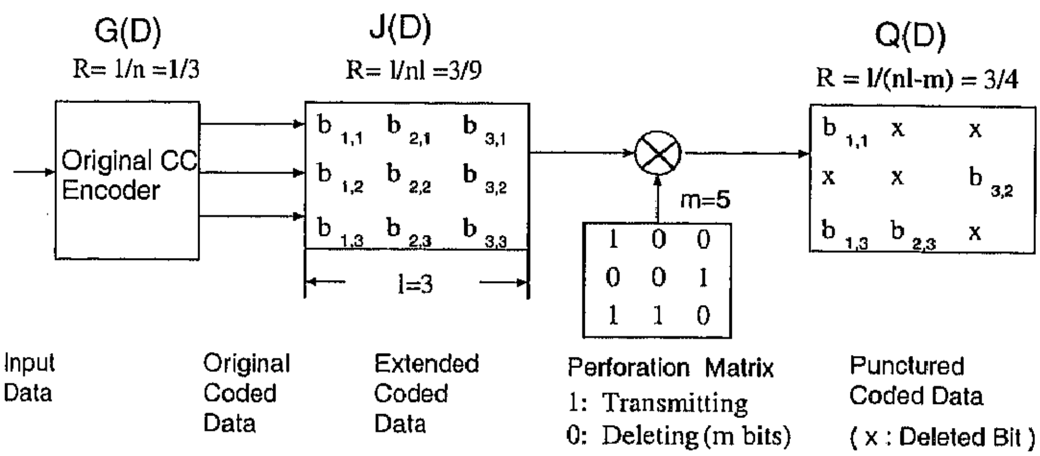

Figure 3.2: **The constructing method of $R = \frac{3}{4}$ PCC from $R = \frac{1}{3}$ CC**

For example, let an original code be the CC with $R = \frac{1}{3}$ and $K = 7$, which has a generator polynomial matrix: $G(D) = [1 + D + D^2 + D^3 + D^5, 1 + D^2 + D^3 + D^4 + D^5 + D^6, 1 + D + D^3 + D^5]$.

From Theorem 1, the generator polynomial matrix $J(D)$ of CC with $R = \frac{3}{9}$ is:

$$\left(\begin{array}{ccccc} 1+D & 1+D+D^2 & 1+D & 1 & D \\ D+D^2 & D+D^2 & D^2 & 1+D & 1+D+D^2 \\ D & D^2 & D & D+D^2 & D+D^2 \end{array}\right.$$

$$\left.\begin{array}{cccc} 1 & 1+D & 1+D & D \\ 1+D & 1 & D & 1 \\ D^2 & 1+D & 1+D+D^2 & 1+D \end{array}\right).$$

Let the corresponding perforation matrix be:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

then, the generator polynomial matrix $Q(D)$ of PCCs with $R = \frac{3}{4}$ is:

$$\begin{pmatrix} 1+D & 1+D & 1 & 1+D \\ D+D^2 & D^2 & 1+D & D \\ D & D & D^2 & 1+D+D^2 \end{pmatrix}. \tag{3.5}$$

The constructing method of the above PCC is shown as Figure 3.2.

In the most practical cases, the original CCs with $R = \frac{1}{2}$ are selected. Suppose its generator polynomial matrix is:

$$G(D) = [G_1(D), G_2(D)]. \tag{3.6}$$

Table 3.1: **The relationship between $R = \frac{1}{2}$ good original CCs and $R = \frac{2}{3}$ PCCs**

$R = \frac{1}{2}$ good original CCs		$R = \frac{2}{3}$ PCCs			P
K_1	$G_1(D)$ $G_2(D)$	$Q_{11}(D)$ $Q_{21}(D)$	$Q_{12}(D)$ $Q_{22}(D)$	$Q_{13}(D)$ $Q_{23}(D)$	
3	$1+D^2$ $1+D+D^2$	$1+D$ 0	$1+D$ D	1 $1+D$	10 11
4	$1+D+D^3$ $1+D+D^2+D^3$	1 $D+D^2$	$1+D$ $D+D^2$	$1+D$ 1	11 10
5	$1+D^3+D^4$ $1+D+D^2+D^4$	$1+D^2$ D^2	$1+D+D^2$ D	D $1+D^2$	11 10
6	$1+D^2+D^4+D^5$ $1+D+D^2+D^3+D^5$	$1+D+D^2$ D^3	$1+D$ $D+D^2+D^3$	$1+D+D^2$ $1+D$	10 11
7	$1+D^2+D^3+D^5+D^6$ $1+D+D^2+D^3+D^6$	$1+D+D^3$ D^2+D^3	$1+D+D^3$ $D+D^2$	$D+D^2$ $1+D+D^3$	11 10
8	$1+D^2+D^5+D^6+D^7$ $1+D+D^2+D^3+D^4$ $+D^7$	$1+D+D^3$ D^3+D^4	$1+D+D^2$ $D+D^2+D^4$	$1+D+D^3$ $1+D+D^2$	10 11
9	$1+D^2+D^3+D^4+D^8$ $1+D+D^2+D^3+D^5$ $+D^7+D^8$	$1+D+D^2+D^4$ D^2	$1+D+D^4$ $D+D^2+D^3+D^4$	D $1+D+D^2+D^4$	11 10

By virtue of Theorem 1, the generator polynomial matrix $J(D)$ of $R = \frac{l}{2l}$ CCs can be expressed by:

$$J(D) = \begin{pmatrix} J_{1,1}(D) & J_{1,2}(D) & \cdots & J_{1,2i-1}(D) & J_{1,2i}(D) & \cdots & J_{1,2l-1}(D) & J_{1,2l}(D) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ J_{j,1}(D) & J_{j,2}(D) & \cdots & J_{j,2i-1}(D) & J_{j,2i}(D) & \cdots & J_{j,2l-1}(D) & J_{j,2l}(D) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ J_{l,1}(D) & J_{l,2}(D) & \cdots & J_{l,2i-1}(D) & J_{l,2i}(D) & \cdots & J_{l,2l-1}(D) & J_{l,2l}(D) \end{pmatrix}, \tag{3.7}$$

where, $i, j = 1, \ldots, l$;

$$\begin{aligned} J_{j,2i-1}(D) &= D^{\frac{j-i}{l}} G_{1,h}(D^{\frac{1}{l}}), \\ J_{j,2i}(D) &= D^{\frac{j-i}{l}} G_{2,h}(D^{\frac{1}{l}}), \end{aligned}$$

for $h = (l + i - j) \bmod l$.

Similarly, the corresponding generator polynomial matrix $Q(D)$ of $R = \frac{l}{2l-m}$ PCCs is derived by being punctured m columns from $J(D)$ in terms of the perforation matrix. For example, Tables 3.1 and 3.2 give the generator polynomial matrices $Q(D)$ of rate $\frac{2}{3}$ and $\frac{3}{4}$ PCCs, being punctured by perforation matrices P from good CCs with $R = \frac{1}{2}$ and constraint length K_1 from 3 to 9. Since the puncturing of a catastrophic original code will always result in catastrophic PCCs[46], the non-catastrophic original $R = \frac{1}{2}$ codes should be selected to produce $R = \frac{2}{3}$ and $R = \frac{3}{4}$ PCCs. Besides, the choosing of perforation matrix refers to the principle as follows [49]:

Table 3.2: **The relationship between $R = \frac{1}{2}$ good original CCs and $R = \frac{3}{4}$ PCCs**

$R=\frac{1}{2}$ good original CCs		$R=\frac{3}{4}$ PCCs				P
K_1	$G_1(D)$ $G_2(D)$	$Q_{11}(D)$ $Q_{21}(D)$ $Q_{31}(D)$	$Q_{12}(D)$ $Q_{22}(D)$ $Q_{32}(D)$	$Q_{13}(D)$ $Q_{23}(D)$ $Q_{33}(D)$	$Q_{14}(D)$ $Q_{24}(D)$ $Q_{34}(D)$	
3	$1+D^2$ $1+D+D^2$	1 D 0	1 D D	1 1 D	1 0 1	101 110
4	$1+D+D^3$ $1+D+D^2+D^3$	$1+D$ 0 D	$1+D$ D D	1 $1+D$ 0	1 1 $1+D$	110 101
5	$1+D^3+D^4$ $1+D+D^2+D^4$	$1+D$ 0 D^2	1 D $D+D^2$	$1+D$ 1 D	0 D $1+D$	101 110
6	$1+D^2+D^4+D^5$ $1+D+D^2+D^3+D^5$	1 $D+D^2$ D^2	$1+D$ $D+D^2$ D	1 $1+D$ $D+D^2$	$1+D$ 1 $1+D$	100 111
7	$1+D^2+D^3+D^5+D^6$ $1+D+D^2+D^3+D^6$	$1+D+D^2$ $D+D^2$ 0	$1+D+D^2$ D D	0 $1+D+D^2$ $D+D^2$	1 1 $1+D+D^2$	110 101
8	$1+D^2+D^5+D^6+D^7$ $1+D+D^2+D^3+D^4$ $+D^7$	$1+D^2$ $D+D^2$ D^3	$1+D$ D $D+D^2+D^3$	D^2 $1+D^2$ $D+D^2$	1 $1+D+D^2$ $1+D$	110 101
9	$1+D^2+D^3+D^4+D^8$ $1+D+D^2+D^3+D^5$ $+D^7+D^8$	$1+D$ $D+D^3$ D^2	$1+D$ $D+D^2+D^3$ $D+D^3$	D $1+D$ $D+D^3$	$1+D^2$ D $1+D$	111 100

1. For the purpose of sequential decoding, do not delete the first and second columns (i.e., the first codeword), since they tend to yield codes with a rapidly increasing column distance function. Besides, it can make the encoder proceed quickly.
2. Do not delete the all columns in one codeword, which makes all the potential (i.e. different generators) of the original code be used fully.

3.3 The Constraint Length of PCCs

The constraint length is a very important parameter of CCs. At encoder, the constraint length determines how many bits of past information are used to determine the parities. At decoder, it determines when to decide about the decoded information. For a CC, the greater its constraint length is, the better it performs. Therefore, this section induces Theorem 2:

Theorem 2 *The constraint length K_l of a high rate $\frac{l}{nl-m}$ CC punctured from a low rate $\frac{1}{n}$ CC is :*

$$K_l \leq \lceil (K_1 - 1)/l \rceil + 1 \tag{3.8}$$

where K_1 is the constraint length of $R = \frac{1}{n}$ CC, and $\lceil x \rceil$ indicates the minimal integer which is larger than or equal to x.

Proof: Let M_1 and M_l be the highest dimension of $G_s(D)$ $(s = 1, 2, \cdots, n)$ and $J_{ji}(D)$ $(j = 1, \ldots, l, i = 1, \ldots, nl)$, respectively, so $M_1 = K_1 - 1$ and $M_l = K_l - 1$.

If $M_1 = tl$, then, $t = \frac{M_1}{l}$.

If $M_1 = tl + l_1$ $(1 \le l_1 \le l - 1)$, then, $t = \frac{M_1 - l_1}{l} = \frac{M_1 + l - l_1}{l} - 1 = \lceil M_1/l \rceil - 1$.

From Theorem 1, according to the generality of ensuring M_l as large as possible, we have the following conclusions:

If $M_1 = tl$, then, $M_l = \frac{tl}{l} = t = \frac{M_1}{l}$.

If $M_1 \neq tl$, then, $M_l = \frac{j-i}{l} + \frac{tl+l+i-j}{l} = \frac{(t+1)l}{l} = t + 1 = \lceil M_1/l \rceil$.

That is to say: $M_l = \lceil M_1/l \rceil$.

In general, we have the conclusion as: $K_l \le \lceil (K_1 - 1)/l \rceil + 1$. □

From Theorem 2, we have the upper bound of constraint length for high rate $\frac{l}{nl-m}$ PCC, which is related to the constraint length K_1 of an original $R = \frac{1}{n}$ CC and the periodical length l, but has no relationship with n and m.

For example, the constraint length of a high rate $\frac{3}{4}$ CC punctured from low rate $\frac{1}{3}$ CC with $K_1 = 7$ is : $K_3 = \lceil (7-1)/3 \rceil + 1 = 3$, which is in good agreement with (3.5).

3.4 Puncturing Realization of the Known Good High Rate CCs

Up to now, many good PCCs have been obtained on the basis of one general constructing method, which includes two steps as follows:

- **Step 1** Selecting a known good $\frac{1}{n}$ CC with a given constraint length as an original code.

- **Step 2** Determining the perforation matrices that will yield good PCCs for different coding rates.

But, by this method, all good PCCs may not always correspond with the known good high rate CCs. In order to produce the same PCCs as the known good high rate CCs by available perforation matrices, the original CCs should be selected. In the following, the generator polynomial matrices of systematic and nonsystematic original CCs are given, which can produce the known good high rate CCs by the available perforation matrices.

3.4.1 Puncturing Realization of the Known Good Nonsystematic CCs

Suppose that an orthogonal perforation matrix[47] is used, where only one entry "1" is in each row, the puncturing bits $m = nl - n$, and $R = \frac{l}{n}$ PCC is obtained. Then, the corresponding generator polynomial matrix $Q(D)$ can be expressed by:

Table 3.3: $R = \frac{1}{3}$ **original codes that yield the known good** $R = \frac{2}{3}$ **CCs**

$R=\frac{1}{3}$ Original CCs		$R=\frac{2}{3}$ PCCs, i.e. the known good $R=\frac{2}{3}$ CCs	
K_1	$G_1(D)$ $G_2(D)$ $G_3(D)$	$Q_{11}(D)$ $Q_{12}(D)$ $Q_{13}(D)$	$Q_{21}(D)$ $Q_{22}(D)$ $Q_{23}(D)$
7	$1+D+D^2+D^3+D^5$ $1+D^2+D^3+D^4+D^5+D^6$ $1+D+D^3+D^5$	$1+D$ $D+D^2$ $1+D+D^2$	$D+D^2+D^3$ $1+D+D^2+D^3$ 1
9	$1+D^4+D^6$ $1+D^3+D^5+D^6+D^8$ $1+D+D^2+D^3+D^6+D^7+D^8$	$1+D^2+D^3$ $D+D^2$ $1+D+D^3$	0 $1+D^3+D^4$ $1+D+D^3+D^4$
11	$1+D+D^4+D^5+D^6+D^9$ $1+D^2+D^4+D^5+D^7+D^9+D^{10}$ $1+D+D^2+D^3+D^7+D^9$	$1+D^2+D^3$ $D^2+D^3+D^4$ $1+D+D^3+D^4$	$D+D^3+D^5$ $1+D+D^2+D^5$ $1+D$

$$Q(D) = \begin{pmatrix} Q_{1,1}(D) & Q_{1,2}(D) & \dots & Q_{1,n}(D) \\ Q_{2,1}(D) & Q_{2,2}(D) & \dots & Q_{2,n}(D) \\ \vdots & \vdots & \ddots & \vdots \\ Q_{l,1}(D) & Q_{l,2}(D) & \dots & Q_{l,n}(D) \end{pmatrix}.$$

From Theorem 1, we know $Q_{j,i}(D)$ is one of n entries: $J_{j,(i-1)n+1}(D)$, $J_{j,(i-1)n+2}(D)$, $\dots$, $J_{j,in}(D)$, which is only related to $G_i(D)$, for $i = 1,\dots,n$ and $j = 1,\dots,l$. This makes the original code be used fully (i.e., each generator polynomial is used). On the other hand, this makes it possible to yield the determinate original codes from the known good high rate codes by some orthogonal perforation matrices.

By virtue of these conclusions, we have obtained the low rate $\frac{1}{3}$ original codes corresponding to the known good $R = \frac{2}{3}$ CCs and $R = \frac{3}{4}$ original codes corresponding to the known good $R = \frac{3}{4}$ CCs[50], as listed in Tables 3.3 and 3.4. It is worth noting that Tables 3.3 and 3.4 are identical to the conclusions in [47], which are based on the general PCC constructing method, as mentioned at the beginning of this Section. However, Tables 3.3 and 3.4 are induced by different methods, which are based on how the generator polynomial matrices of an original low rate CC are constructed from a good high rate CC. In Table 3.3, K_1 is the constraint length of $R = \frac{1}{3}$ CC and $m = 3 \times 2 - 3 = 3$, the orthogonal perforation matrix is:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

In Table 3.4, K_1 is the constraint length of $R = \frac{1}{4}$ CC and $m = 4 \times 3 - 4 = 8$, the orthogonal perforation matrix is:

Table 3.4: $R = \frac{1}{4}$ **original codes that yield the known good** $R = \frac{3}{4}$ **CCs**

$R = \frac{1}{4}$ Original CCs		$R = \frac{3}{4}$ PCCs, i.e. the known good $R = \frac{3}{4}$ CCs		
	$G_1(D)$	$Q_{11}(D)$	$Q_{21}(D)$	$Q_{31}(D)$
K_1	$G_2(D)$	$Q_{12}(D)$	$Q_{22}(D)$	$Q_{32}(D)$
	$G_3(D)$	$Q_{13}(D)$	$Q_{23}(D)$	$Q_{33}(D)$
	$G_4(D)$	$Q_{14}(D)$	$Q_{24}(D)$	$Q_{34}(D)$
	1	1	0	0
7	$1+D+D^2+D^3$	1	$1+D$	D
	$1+D^2+D^4+D^6$	1	D	$1+D^2$
	$1+D+D^2+D^6$	1	1	$1+D^2$
	$1+D+D^2+D^3+D^5$	$1+D$	$D+D^2$	D
9	$1+D^2+D^5+D^7$	D^2	1	$D+D^2$
	$1+D^3+D^6+D^7$	0	D^2	$1+D+D^2$
	$1+D+D^2+D^4+D^5+D^8$	$1+D+D^2$	$1+D$	1
	$1+D^3+D^5+D^6+D^7+D^8$	$1+D+D^2$	D^2+D^3	D^3
11	$1+D^4+D^5+D^6+D^7$	$D+D^2$	$1+D^2$	D^2
	$1+D+D^3+D^5+D^6+D^9$	D	1	$1+D+D^2+D^3$
	$1+D+D^2+D^7+D^{10}$	1	$1+D^2+D^3$	1

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

By this method, we can construct any known good high rate $\frac{l}{n}$ CCs with original $R = \frac{1}{n}$ CCs and available orthogonal perforation matrices.

3.4.2 Puncturing Realization of the Known Good Systematic CCs

Since there is a large body of research devoted to the class of $R = \frac{1}{2}$ CCs, the constructing of PCCs from the original $R = \frac{1}{2}$ CCs is very useful. The determinate generator polynomial matrix of the original $R = \frac{1}{2}$ CCs can not be obtained from the generator polynomial matrices of $R = \frac{2}{3}$ and $R = \frac{3}{4}$ PCCs as before. But the good systematic PCCs can be found [39] from systematic $R = \frac{1}{2}$ original CCs for their determinate relationship as follows.

Suppose the generator polynomial matrix of systematic $R = \frac{1}{2}$ CCs is:

$$G(D) = [1, G_2(D)]. \tag{3.9}$$

If the generator polynomial matrix of systematic $R = \frac{2}{3}$ CCs is:

$$\begin{pmatrix} 1 & 0 & Q_{13}(D) \\ 0 & 1 & Q_{23}(D) \end{pmatrix},$$

then, from Theorem 1, its determinate relationship with $G(D)$ is:

$$G_2(D) = Q_{23}(D^2) + DQ_{13}(D^2). \tag{3.10}$$

If the generator polynomial matrix of systematic $R = \frac{3}{4}$ CCs is:

$$\begin{pmatrix} 1 & 0 & 0 & Q_{14}(D) \\ 0 & 1 & 0 & Q_{24}(D) \\ 0 & 0 & 1 & Q_{34}(D) \end{pmatrix},$$

then, from Theorem 1, its determinate relationship with $G(D)$ is:

$$G_2(D) = Q_{34}(D^3) + DQ_{24}(D^3) + D^2Q_{14}(D^3). \tag{3.11}$$

Therefore, Tables 3.5 and 3.6 give the systematic original codes that yield good systematic $R = \frac{2}{3}$ and $R = \frac{3}{4}$ CCs. For Table 3.5, K_2 is the constraint length of $R = \frac{2}{3}$ CCs, m=1, the corresponding perforation matrix is:

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

For Table 3.6, K_3 is the constraint length of $R = \frac{3}{4}$ CCs, m=2, the corresponding perforation matrix is:

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Table 3.5: Systematic original codes that yield the known good systematic $R = \frac{2}{3}$ CCs

Systematic original $R = \frac{1}{2}$ CCs	Systematic punctured $R = \frac{2}{3}$ CCs		
$G_2(D)$	K_2	$Q_{13}(D)$	$Q_{23}(D)$
$1+D+D^3$	2	$1+D$	1
$1+D+D^3+D^4$	3	$1+D$	$1+D^2$
$1+D+D^3+D^4+D^6$	4	$1+D$	$1+D^2+D^3$
$1+D+D^3+D^4+D^6+D^8$ $+D^9$	5	$1+D+D^4$	$1+D^2+D^3+D^4$
$1+D+D^3+D^4+D^6+D^8$ $+D^9+D^{10}+D^{11}$	6	$1+D+D^4+D^5$	$1+D^2+D^3+D^4+D^5$
$1+D+D^3+D^4+D^6+D^8$ $+D^9+D^{10}+D^{11}+D^{13}$	7	$1+D+D^4+D^5+D^6$	$1+D^2+D^3+D^4+D^5$

Table 3.6: Systematic original codes that yield the known good systematic $R = \frac{3}{4}$ CCs

Systematic original $R = \frac{1}{2}$ CCs	Systematic punctured $R = \frac{3}{4}$ CCs	
$G_2(D)$	K_3	$Q_{14}(D)$ $Q_{24}(D)$ $Q_{34}(D)$
$1+D+D^2+D^3+D^5+D^6+D^7$	3	$1+D$ $1+D^2$ $1+D+D^2$
$1+D+D^2+D^3+D^5+D^6+D^7+D^9+D^{10}$ $+D^{11}$	4	$1+D+D^3$ $1+D^2+D^3$ $1+D+D^2+D^3$
$1+D+D^2+D^3+D^5+D^6+D^7+D^9+D^{10}$ $+D^{11}+D^{13}+D^{14}$	5	$1+D+D^3+D^4$ $1+D^2+D^3+D^4$ $1+D+D^2+D^3$
$1+D+D^2+D^3+D^5+D^6+D^7+D^9+D^{10}$ $+D^{11}+D^{13}+D^{14}+D^{19}$	7	$1+D+D^3+D^4$ $1+D^2+D^3+D^4+D^6$ $1+D+D^2+D^3$
$1+D+D^2+D^3+D^5+D^6+D^7+D^9+D^{10}$ $+D^{11}+D^{13}+D^{14}+D^{19}+D^{21}$	8	$1+D+D^3+D^4$ $1+D^2+D^3+D^4+D^6$ $1+D+D^2+D^3+D^7$