Appendix A
Proof of Theorem 1

Theorem 1 is proved in the following three steps:

Step1. The relationship of inputs of two encoders

Let inputs of the $R = \frac{1}{n}$ CCs and $R = \frac{l}{nl}$ CCs be $X(D)$ and $X^*(D) = [X_1^*(D), X_2^*(D), \cdots, X_l^*(D)]$, respectively. When same input sequences are encoded by these CCs, the successive l information bits of $R = \frac{1}{n}$ CCs are equal to one information byte of $R = \frac{l}{nl}$ CCs. So, we have:

$$X(D) = X_1^*(D^l) + DX_2^*(D^l) + \cdots + D^{l-1}X_l^*(D^l). \tag{A.1}$$

Step2. The relationship of outputs of two encoders

Let outputs of the $R = \frac{1}{n}$ CCs and $R = \frac{l}{nl}$ CCs be $Y(D) = [Y_1(D), Y_2(D), \cdots, Y_n(D)]$, and,

$$\begin{aligned} Y^*(D) \;=\; & [Y_1^*(D), Y_2^*(D), \cdots, Y_n^*(D), Y_{n+1}^*(D), Y_{n+2}^*(D), \cdots, Y_{2n}^*(D), \cdots, \\ & Y_{(l-1)n+1}^*(D), Y_{(l-1)n+2}^*(D), \cdots, Y_{nl}^*(D)], \end{aligned}$$

respectively, then the successive l encoded information bytes of $R = \frac{1}{n}$ CCs are equal to one encoded information byte of $R = \frac{l}{nl}$ CCs. So, we have:

$$\begin{aligned} Y_1(D) &= Y_1^*(D^l) + DY_{n+1}^*(D^l) + \cdots + D^{l-1}Y_{(l-1)n+1}^*(D^l), \\ Y_2(D) &= Y_2^*(D^l) + DY_{n+2}^*(D^l) + \cdots + D^{l-1}Y_{(l-1)n+2}^*(D^l), \\ &\cdots \\ Y_n(D) &= Y_n^*(D^l) + DY_{2n}^*(D^l) + \cdots + D^{l-1}Y_{nl}^*(D^l). \end{aligned} \tag{A.2}$$

Step3. The relationship of $G(D)$ and $J(D)$

Since $Y(D) = X(D)G(D)$, we immediately have $Y_s(D) = X(D)G_s(D)$ for $R = \frac{1}{n}$ CCs and $s = 1, 2, \cdots, n$. Since $Y^*(D) = X^*(D)J(D)$, we have $Y_i^*(D) = X_1^*(D)J_{1,i}(D) + \cdots + X_l^*(D)J_{l,i}(D)$ for $R = \frac{l}{nl}$ CCs and i=1, 2, ..., nl. Therefore, $Y_1(D)$ can be expressed by:

$$\begin{aligned} & (X_1^*(D^l) + DX_2^*(D^l) + \cdots + D^{l-1}X_l^*(D^l))G_1(D) \\ = \;& Y_1^*(D^l) + DY_{n+1}^*(D^l) + \cdots + D^{l-1}Y_{(l-1)n+1}^*(D^l) \end{aligned}$$

$$
\begin{aligned}
&= X_1^*(D^l)J_{1,1}(D^l) + X_2^*(D^l)J_{2,1}(D^l) + \cdots + X_l^*(D^l)J_{l,1}(D^l)\\
&+ DX_1^*(D^l)J_{1,n+1}(D^l) + DX_2^*(D^l)J_{2,n+1}(D^l) + \cdots + DX_l^*(D^l)J_{l,n+1}(D^l) + \cdots\\
&+ \cdots\\
&+ D^{l-1}X_1^*(D^l)J_{1,(l-1)n+1}(D^l) + D^{l-1}X_2^*(D^l)J_{2,(l-1)n+1}(D^l) + \cdots\\
&+ D^{l-1}X_l^*(D^l)J_{l,(l-1)n+1}(D^l).
\end{aligned}
$$

Obviously,

$$
\begin{aligned}
G_1(D) &= J_{1,1}(D^l) + DJ_{1,n+1}(D^l) + \cdots + D^{l-1}J_{1,(l-1)n+1}(D^l),\\
DG_1(D) &= J_{2,1}(D^l) + DJ_{2,n+1}(D^l) + \cdots + D^{l-1}J_{2,(l-1)n+1}(D^l),\\
&\cdots\\
D^{l-1}G_1(D) &= J_{l,1}(D^l) + DJ_{l,n+1}(D^l) + \cdots + D^{l-1}J_{l,(l-1)n+1}(D^l).
\end{aligned}
$$

So,

$$
\begin{aligned}
&G_{1,0}(D) + G_{1,1}(D) + \cdots + G_{1,l-1}(D)\\
=\ &J_{1,1}(D^l) + DJ_{1,n+1}(D^l) + \cdots + D^{l-1}J_{1,(l-1)n+1}(D^l),\\
&D[G_{1,0}(D) + G_{1,1}(D) + \cdots + G_{1,l-1}(D)]\\
=\ &J_{2,1}(D^l) + DJ_{2,n+1}(D^l) + \cdots + D^{l-1}J_{2,(l-1)n+1}(D^l),\\
&\cdots\\
&D^{l-1}[G_{1,0}(D) + G_{1,1}(D) + \cdots + G_{1,l-1}(D)]\\
=\ &J_{l,1}(D^l) + DJ_{l,n+1}(D^l) + \cdots + D^{l-1}J_{l,(l-1)n+1}(D^l).
\end{aligned}
$$

Therefore, we have:

$$
\begin{aligned}
J_{1,1}(D) &= G_{1,0}(D^{\frac{1}{l}}),\\
J_{1,n+1}(D) &= D^{-\frac{1}{l}}G_{1,1}(D^{\frac{1}{l}}),\\
&\cdots\\
J_{1,(l-1)n+1}(D) &= D^{-\frac{l-1}{l}}G_{1,l-1}(D^{\frac{1}{l}}),\\
J_{2,1}(D) &= D^{\frac{1}{l}}G_{1,l-1}(D^{\frac{1}{l}}),\\
J_{2,n+1}(D) &= G_{1,0}(D^{\frac{1}{l}}),\\
&\cdots\\
J_{2,(l-1)n+1}(D) &= D^{-\frac{l-2}{l}}G_{1,l-2}(D^{\frac{1}{l}}),\\
&\cdots\\
J_{l,1}(D) &= D^{\frac{l-1}{l}}G_{1,1}(D^{\frac{1}{l}}),\\
J_{l,n+1}(D) &= D^{\frac{l-2}{l}}G_{1,2}(D^{\frac{1}{l}}),
\end{aligned}
$$

$$\cdots$$

$$J_{l,(l-1)n+1}(D) = G_{1,0}(D^{\frac{1}{l}}).$$

For $i, j = 1, \cdots, l$ and $h = (l + i - j) \bmod l$, then the above equations can be expressed generally by: $J_{j,(i-1)n+1}(D) = D^{\frac{j-i}{l}} G_{1,h}(D^{\frac{1}{l}})$.

By the same process, we have:

$$J_{j,(i-1)n+2}(D) = D^{\frac{j-i}{l}} G_{2,h}(D^{\frac{1}{l}}),$$

$$\cdots$$

$$J_{j,in}(D) = D^{\frac{j-i}{l}} G_{n,h}(D^{\frac{1}{l}}).$$

That is to say,

$$J_{j,(i-1)n+s}(D) = D^{\frac{j-i}{l}} G_{s,h}(D^{\frac{1}{l}}), \tag{A.3}$$

where $s = 1, 2, \cdots, n$. □