Chapter 2

MOMENT METHOD FORMULATION

2.1 Theoretical Model and Boundary Conditions

The model used for the analysis and the coordinate system are shown in Fig. 2.1. A rectangularly bent slot is cut on an infinite ground plane located on the X-Y plane. It is assumed that the slot width W is very small compared with the wavelength λ_0 at a center frequency f_0. A rectangular cavity, which is made of a perfect electric conductor (PEC), is put under the ground plane. Its size is (Xc×Yc×Zc). The axial direction of each part of the slot is either the X direction or the Y direction and always normal or parallel to the sidewalls of the backing cavity. The slot is fed by a δ-function current source, which is located at (x_0, y_0, 0). The only power loss is from radiation since all the metallic surfaces are assumed to be perfect conductors and the region inside the cavity is assumed to be a vacuum.

The analysis approach is based on the application of the equivalence principle and the use of a generalized network formulation [34]. First, an equivalent magnetic current is introduced to the slot. Then by applying the equiv-

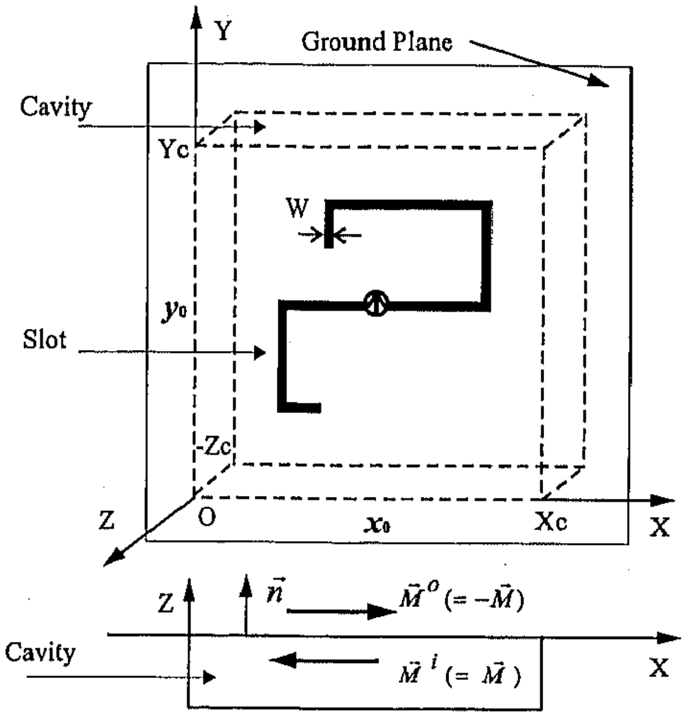

Fig. 2.1: A rectangular-cavity-backed slot antenna.

alence principle to the slot aperture, the problem can be separated into the upper half-space region and the interior of the cavity. The inside and outside equivalent magnetic current $\vec{M}^i(=\vec{M})$ and $\vec{M}^o(=-\vec{M})$ are placed just above and under the aperture region [I] respectively with the aperture shorted by a PEC [34]. As known, when the slot is very narrow compared with the wavelength, the magnetic current $\vec{M}$ has only the component along the slot and the slot can be approximated by a thin magnetically conducting wire, whose diameter is half of the slot width [31]. Thus the magnetic current can be expressed as

$$\vec{M}=\begin{cases} M\hat{x}=\vec{E}\times\vec{n} & \text{for slot segments along the X direction,} \\ M\hat{y}=\vec{E}\times\vec{n} & \text{for slot segments along the Y direction,} \end{cases} \quad (2.1)$$

where $\hat{x}$ and $\hat{y}$ are the unit vectors along the X and Y-axis, respectively, $\vec{n}$ is the unit normal vector to the ground plane pointing into the region outside

[I]The fact that the equivalent magnetic current under the aperture region is $+\vec{M}$ and above the aperture region is $-\vec{M}$ ensures that the tangential component of the electric field is continuous across the aperture.

the cavity and the electric field $\vec{E}$ is just above or under the aperture region. In order to ensure the continuity of the tangential component of the magnetic field across the slot aperture, it is required that

$$\vec{H}_t^i(\vec{M}) - \vec{H}_t^o(-\vec{M}) = i_0\delta(x-x_0)\delta(y-y_0)\delta(z)\vec{l} \tag{2.2}$$

where $\vec{l} = \hat{x}$ or $\hat{y}$, $\vec{H}_t^o$ and $\vec{H}_t^i$ are the tangential magnetic fields just over and under the aperture, respectively, and i_0 is the current source across the slot at $x = x_0, y = y_0$ and $z = 0$ (see Fig. 2.1).

The $\vec{H}_t$ component of the magnetic field in the $z = 0$ plane inside and outside the cavity can be related to the electric vector potential $\vec{F}$. In the $z = 0$ plane, the electric vector potential $\vec{F}$ has only F_x and F_y components, i.e.

$$\vec{F} = \vec{F}_x\hat{x} + \vec{F}_y\hat{y} \tag{2.3}$$

The magnetic field $\vec{H}$ can be related to the electric vector potential $\vec{F}$ by the Eq. (3-88) of Harrington as [31]

$$\vec{H} = -j\omega\varepsilon_0\vec{F} + \frac{1}{j\omega\mu_0}\nabla(\nabla\cdot)\vec{F} \tag{2.4}$$

Substitute (2.3) into (2.4) to obtain

$$\begin{aligned}\vec{H} = {} & \frac{1}{j\omega\mu_0}[(k^2+\frac{\partial^2}{\partial x^2})F_x + \frac{\partial^2}{\partial x\partial y}F_y]\hat{x} + \frac{1}{j\omega\mu_0}[(k^2+\frac{\partial^2}{\partial y^2})F_y + \frac{\partial^2}{\partial x\partial y}F_x]\hat{y} \\ & + \frac{1}{j\omega\mu_0}[\frac{\partial^2}{\partial z\partial x}F_x + \frac{\partial^2}{\partial z\partial y}F_y]\hat{z}\end{aligned} \tag{2.5}$$

where $k = \omega\sqrt{\mu_0\varepsilon_0}$.

In the $z = 0$ plane, $\vec{M}$ is independent of z, so that $\frac{\partial}{\partial z}F_x = 0$ and $\frac{\partial}{\partial z}F_y = 0$. And the following relation can be obtained:

$$\vec{H} = \frac{1}{j\omega\mu_0}[(k^2+\frac{\partial^2}{\partial x^2})F_x + \frac{\partial^2}{\partial x\partial y}F_y]\hat{x} + \frac{1}{j\omega\mu_0}[(k^2+\frac{\partial^2}{\partial y^2})F_y + \frac{\partial^2}{\partial x\partial y}F_x]\hat{y} \tag{2.6}$$

The magnetic fields inside and outside the cavity are given by the following equations:

$$\vec{H}_t^i(\vec{M}) = \frac{1}{j\omega\mu_0}\left[(k^2+\frac{\partial^2}{\partial x^2})F_x^i + \frac{\partial^2}{\partial x \partial y}F_y^i\right]\hat{x} + \frac{1}{j\omega\mu_0}\left[(k^2+\frac{\partial^2}{\partial y^2})F_y^i + \frac{\partial^2}{\partial x \partial y}F_x^i\right]\hat{y} \quad (2.7)$$

$$\vec{H}_t^o(-\vec{M}) = \frac{1}{j\omega\mu_0}\left[(k^2+\frac{\partial^2}{\partial x^2})F_x^o + \frac{\partial^2}{\partial x \partial y}F_y^o\right]\hat{x} + \frac{1}{j\omega\mu_0}\left[(k^2+\frac{\partial^2}{\partial y^2})F_y^o + \frac{\partial^2}{\partial x \partial y}F_x^o\right]\hat{y} \quad (2.8)$$

where $\vec{F}^i$ and $\vec{F}^o$ are electric vector potentials inside and outside the cavity, respectively. Note that for simplicity μ_0 and ε_0 are assumed outside and inside the cavity. As the magnetic current has only an x or y-component, the corresponding electric vector potential has also an x or y-component. The x and y-component are given by the following equations:

$$F_x^i = -\frac{1}{4\pi}\int G_x^i(\vec{r},\vec{r}\,')M_x(x')dx' \quad (2.9)$$

$$F_y^i = -\frac{1}{4\pi}\int G_y^i(\vec{r},\vec{r}\,')M_y(y')dy' \quad (2.10)$$

$$F_x^o = \frac{1}{4\pi}\int G^o(\vec{r},\vec{r}\,')M_x(x')dx' \quad (2.11)$$

$$F_y^o = \frac{1}{4\pi}\int G^o(\vec{r},\vec{r}\,')M_y(y')dy' \quad (2.12)$$

where M_x and M_y are the x and y-component of $\vec{M}$, respectively. $G^o = \frac{\exp(jkr)}{r}$ with $r = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$ is the scalar Green's function in free space. G_x^i and G_y^i are the scalar Green's functions inside the rectangular cavity and have to satisfy a scalar Hermholtz equation (Eq. (7.2.5) of P. M. Morse and H. Feshbach) [35], respectively.

$$\nabla^2 G^i + k^2 G^i = -4\pi\delta(x-x')\delta(y-y')\delta(z) \quad (2.13)$$

where G^i is G_x^i or G_y^i. The electric field inside the cavity must satisfy the following boundary conditions:

$$\hat{x} \times \vec{E}^i(\vec{M}) = 0, \; (x = 0, \mathrm{Xc})$$
$$\hat{y} \times \vec{E}^i(\vec{M}) = 0, \; (y = 0, \mathrm{Yc}) \tag{2.14}$$
$$\hat{z} \times \vec{E}^i(\vec{M}) = 0, \; (z = 0, \mathrm{Zc})$$

where

$$\vec{E}^i(\vec{M}) = -\nabla \times \vec{F}(\vec{M}) = \frac{\partial F_x^i}{\partial z}\hat{x} - \frac{\partial F_y^i}{\partial z}\hat{y} + \left[\frac{\partial F_x^i}{\partial y} - \frac{\partial F_y^i}{\partial x}\right]\hat{z} \tag{2.15}$$

and $\hat{z}$ is the unit vector along the Z-axis. Then G_x^i and G_y^i is obtained as (Reference to Eq. (11.3.10) of P. M. Morse and H. Feshbach) [35]

$$G_x^i = -\frac{4\pi}{\mathrm{XcYc}} \sum_{u=0}^{\infty} \sum_{v=1}^{\infty} \frac{\varepsilon_u \varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc}) \sin(\alpha_x x) \cos(\alpha_y y) \sin(\alpha_x x') \cos(\alpha_y y')$$
$$G_y^i = -\frac{4\pi}{\mathrm{XcYc}} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} \frac{\varepsilon_p \varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc}) \sin(\beta_y y) \cos(\beta_x x) \sin(\beta_y y') \cos(\beta_x x') \tag{2.16}$$

where

$$\alpha_x = \frac{v\pi}{\mathrm{Xc}}, \alpha_y = \frac{u\pi}{\mathrm{Yc}} \qquad \beta_x = \frac{p\pi}{\mathrm{Xc}}, \beta_y = \frac{q\pi}{\mathrm{Yc}}$$
$$\alpha_c^2 = k^2 - \alpha_x^2 - \alpha_y^2 \qquad \beta_c^2 = k^2 - \beta_x^2 - \beta_y^2 \tag{2.17}$$
$$\varepsilon_u = 1, \text{ if } u = 0. \qquad \varepsilon_p = 1, \text{ if } p = 0.$$
$$\varepsilon_u = 2, \text{ if } u \neq 0. \qquad \varepsilon_p = 2, \text{ if } p \neq 0.$$
$$\varepsilon_v = 2. \qquad \varepsilon_q = 2.$$

If (2.2) is satisfied exactly, we would have the true solution.

2.2 Moment Method Approximation

In this section, the method of moments [36] will be applied to (2.2) to obtain $\vec{M}$ which has been described in the last section. Before we apply the moment method to (2.2) to obtain $\vec{M}$, we must choose an expansion function for $\vec{M}$ and a weighting function for the boundary condition (2.2). There are two kinds of expansion functions:

1. an entire domain expansion function defined over the whole slot (polynomials, Fourier series, *et cetera*);

2. a subsectional expansion function defined only on a small subsection of the slot (pulses, triangles, piecewise sinusoids, *et cetera*).

There is a unique $\vec{M}$, which satisfies the boundary condition (2.2) exactly, but the $\vec{M}$ obtained by the moment method can approximately satisfy it. Thus, there are many approximate $\vec{M}$ values which depend on the choice of an expansion and weighting function. To satisfy (2.2) at discrete points on the side surface of the slot, Dirac's delta function is used for the weight. If we wish to satisfy (2.2) at continuous points on the surface, pulses, triangles, piecewise sinusoids, *et cetera* may be used as weighting functions.

The choice of an expansion and weighting function influences

1. the necessary number of expansion functions for satisfactory results;

2. the computation time;

3. the simplicity of the program.

For a thin slot antenna analysis, if a piecewise sinusoidal expansion function is used for the current $\vec{M}$ on the slot, a magnetic field $\vec{H}$ in (2.2) can be obtained in a close form. Then, the computer program becomes simpler and the computation time becomes shorter compared with other expansion functions.

Therefore, the piecewise sinusoidal expansion function is chosen for the magnetic current $\vec{M}$. The slot is divided into $N+1$ subsections, where N is the number of expansion functions. The end of each is numbered from 0 to $N+1$. Then, the magnetic current flowing on the slot is approximately written by the summation of N piecewise sinusoidal expansion functions as

$$\vec{M}(x',y') = \sum_{n=1}^{N} V_n \vec{M}_n(x',y') \tag{2.18}$$

$$\vec{M}_n(x',y') = \begin{cases} M_x^n(x')\hat{x},\ x'_{n-1} \le x' \le x'_{n+1}, & \text{for segments in the X direction,} \\ M_y^n(y')\hat{y},\ y'_{n-1} \le y' \le y'_{n+1}, & \text{for segments in the Y direction,} \\ 0, & \text{else where.} \end{cases} \tag{2.19}$$

where

$$M_x^n(x') = \begin{cases} \dfrac{\sin k(x'-x'_{n-1})}{\sin k(x'_n - x'_{n-1})}, & \text{if } x'_{n-1} \le x' \le x'_n, \\ \dfrac{\sin k(x'_{n+1}-x')}{\sin k(x'_{n+1} - x'_n)}, & \text{if } x'_n \le x' \le x'_{n+1}, \\ 0, & \text{else where.} \end{cases} \tag{2.20}$$

and

$$M_y^n(y') = \begin{cases} \dfrac{\sin k(y'-y'_{n-1})}{\sin k(y'_n - y'_{n-1})}, & \text{if } y'_{n-1} \le y' \le y'_n, \\ \dfrac{\sin k(y'_{n+1}-y')}{\sin k(y'_{n+1} - y'_n)}, & \text{if } y'_n \le y' \le y'_{n+1}, \\ 0, & \text{else where.} \end{cases} \tag{2.21}$$

Each $\vec{M}_n(x',y')$ occupies two subsections $x'_{n-1} - x'_n$ and $x'_n - x'_{n+1}$ for the segments in the X direction or $y'_{n-1} - y'_n$ and $y'_n - y'_{n+1}$ for the segments in the Y direction. Because the feed gap is assumed infinitesimal, it must correspond to one of the peaks of the expansion functions(see Fig. 2.2).

The coefficient V_n is an unknown complex constant to be determined from the boundary condition in (2.2). The coefficients $\{V_n, n = 1,2,\ldots,N\}$ are obtained by the Galerkin's procedure where the weighting function and the expansion function are the same. Once V_n is known, the magnetic current distribution can be obtained from (2.18). Note that $|V_n|$ becomes the magnitude of the magnetic current at $(x' = x'_n, y' = y'_n)$ because $|\vec{M}_n(x' = x'_n, y' = y'_n)| = |V_n|$.

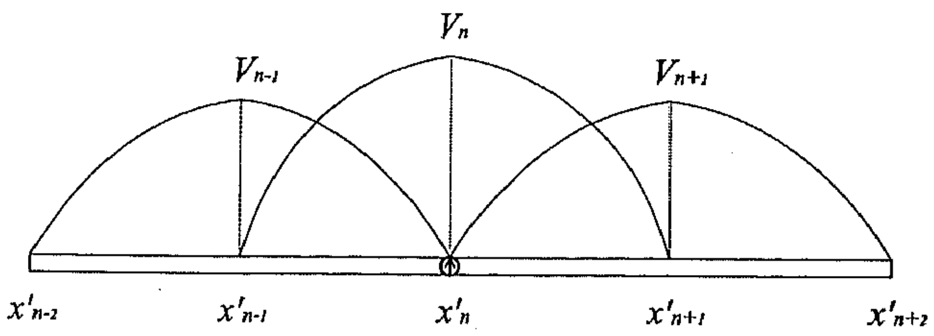

Fig. 2.2: Subsection and expansion function.

Substitute (2.18) into (2.2) and use the linearity of the $\vec{H}_t$ operators to obtain

$$\sum_{n=1}^{N} \vec{H}_t^i(\vec{M}_n) - \sum_{n=1}^{N} \vec{H}_t^o(-\vec{M}_n) = i_0\delta(x-x_0)\delta(y-y_0)\delta(z)\vec{l} \quad (2.22)$$

Next, define a symmetric product

$$\langle\vec{A},\vec{B}\rangle = \iint_{slot} \vec{A}\cdot\vec{B}ds \quad (2.23)$$

and a set of piecewise weighting function $\vec{W}_m$, which is chosen to be the same as the expansion function (2.19). We take the symmetric product of (2.22) with each weighting function $\vec{W}_m$, and use the linearity of symmetric product to obtain the set of equations

$$\sum_{n=1}^{N} V_n\langle\vec{W}_m,\vec{H}_t^i(\vec{M}_n)\rangle - \sum_{n=1}^{N} V_n\langle\vec{W}_m,\vec{H}_t^o(-\vec{M}_n)\rangle = \langle\vec{W}_m,i_0\delta(x-x_0)\delta(y-y_0)\delta(z)\vec{l}\rangle \quad (2.24)$$

$m = 1,2,\cdots,N$. The inner product in (2.24) exists only over two consecutive subsections. The solution of this set of linear equations determines the coefficients V_n and the magnetic current $\vec{M}$ according to (2.18). Once $\vec{M}$ is known, the fields and field-related parameters may be computed by standard methods.

2.3 Network Equivalence

Equation (2.24) comprises N-dimensional simultaneous linear equations, and can be put into matrix notation as follows.

Define an admittance matrix for the region inside the cavity as

$$[Y^i] = [\langle \vec{W}_m, \vec{H}_t^i(\vec{M}_n) \rangle]_{N \times N} \tag{2.25}$$

and an admittance matrix for the region outside the cavity as

$$[Y^o] = -[\langle \vec{W}_m, \vec{H}_t^o(-\vec{M}_n) \rangle]_{N \times N} \tag{2.26}$$

Define a source vector

$$[I^0] = [\langle \vec{W}_m, i_0 \delta(x - x_0) \delta(y - y_0) \delta(z) \vec{l} \rangle]_{N \times 1} \tag{2.27}$$

and a coefficient vector

$$[V] = [V_n]_{N \times 1}. \tag{2.28}$$

Now the matrix equation equivalent to (2.24) is

$$[Y^i + Y^o][V] = [I^0] \tag{2.29}$$

This can be interpreted in terms of generalized networks as two networks $[Y^i]$ and $[Y^o]$ in parallel with the current source $[I^0]$ as shown in Fig. 2.3.

The resultant voltage vector

$$[V] = [Y^i + Y^o]^{-1}[I^0] \tag{2.30}$$

is then the vector of coefficients which give $\vec{M}$ according to (2.18).

$[Y^i]$ and $[Y^o]$ are the $N \times N$ generalized admittance matrices inside and outside the cavity, respectively, and $[I^0]$ is a column matrix whose elements are zeros except the one corresponding to the feed point.

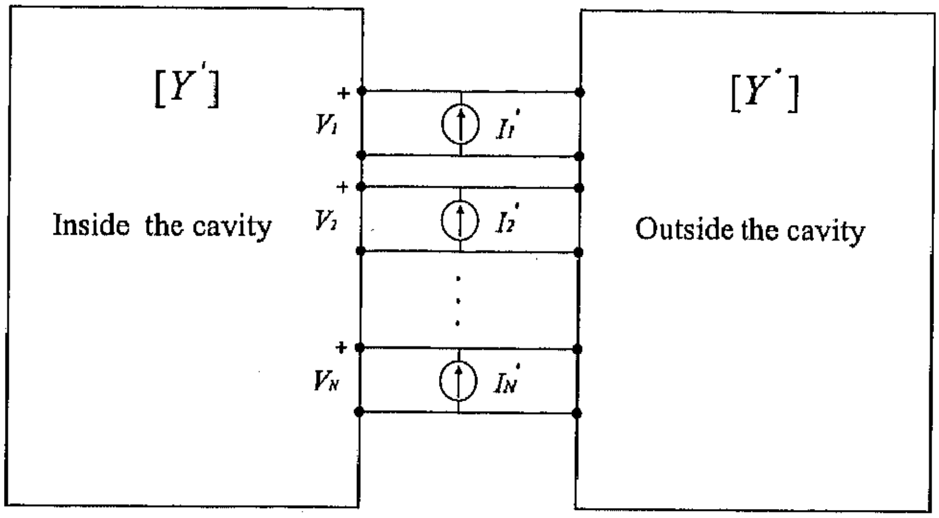

Fig. 2.3: Generalized network interpretation.

It is important to note that computation of $[Y^i]$ involves only region inside the cavity, and computation of $[Y^o]$ involves only region outside the cavity. Hence, we can divide the problem into two parts, each of which may be formulated independently. Once $[Y]$ is computed for one region, it may be combined with $[Y]$ for any other region which makes it useful for a wide range of problems. Because the feed gap is infinitesimal, $[I^0]$ becomes nonzero only at the feed gap, and one can assume that it is 1.

2.4 Admittance Matrix Inside the Cavity

Because the computation of $[Y^i]$ involves only the region inside the cavity, in this section, the expression for the matrix elements of (2.25) will be derived independently.

The analytical model used for the analysis is a two dimensional problem, and the magnetic current is in X or Y direction.

A two dimensional weighting function in the mth subsection is defined as

$$\vec{W}_m = [W_x(1p) + W_x(2p)]\hat{x} + [W_y(1p) + W_y(2p)]\hat{y} \tag{2.31}$$

where

$$W_x(1p) = \begin{cases} \dfrac{\sin k(x - x_{m-1})}{\sin k(x_m - x_{m-1})}, & \text{if } x_{m-1} \le x \le x_m, \\ 0, & \text{else where.} \end{cases} \tag{2.32}$$

$$W_x(2p) = \begin{cases} \dfrac{\sin k(x_{m+1} - x)}{\sin k(x_{m+1} - x_m)}, & \text{if } x_m \le x \le x_{m+1}, \\ 0, & \text{else where.} \end{cases} \tag{2.33}$$

$$W_y(1p) = \begin{cases} \dfrac{\sin k(y - y_{m-1})}{\sin k(y_m - y_{m-1})}, & \text{if } y_{m-1} \le y \le y_m, \\ 0, & \text{else where.} \end{cases} \tag{2.34}$$

$$W_y(2p) = \begin{cases} \dfrac{\sin k(y_{m+1} - y)}{\sin k(y_{m+1} - y_m)}, & \text{if } y_m \le x \le y_{m+1}, \\ 0, & \text{else where.} \end{cases} \tag{2.35}$$

and the magnetic field in X or Y direction owing to the nth subsection of the slot is

$$\begin{aligned} \vec{H}_n = & \left[H_x(M_x, 1p) + H_x(M_x, 2p) + H_x(M_y, 1p) + H_x(M_y, 2p)\right]\hat{x} \\ & + \left[H_y(M_x, 1p) + H_y(M_x, 2p) + H_y(M_y, 1p) + H_y(M_y, 2p)\right]\hat{y} \end{aligned} \tag{2.36}$$

where

$$H_x(M_x, 1p) = \begin{cases} \dfrac{1}{j\omega\mu_0}(k^2 + \dfrac{\partial^2}{\partial x^2})\left[-\dfrac{1}{4\pi}\displaystyle\int_{x'_{n-1}}^{x'_n} G_x^i(\vec{r}, \vec{r}\,')M_x^n(x')dx'\right], & \text{if } x'_{n-1} \le x' \le x'_n, \\ 0, & \text{else where.} \end{cases} \tag{2.37}$$

$$H_x(M_x, 2p) = \begin{cases} \dfrac{1}{j\omega\mu_0}(k^2 + \dfrac{\partial^2}{\partial x^2})\left[-\dfrac{1}{4\pi}\displaystyle\int_{x'_n}^{x'_{n+1}} G_x^i(\vec{r}, \vec{r}\,')M_x^n(x')dx'\right], & \text{if } x'_n \le x' \le x'_{n+1}, \\ 0, & \text{else where.} \end{cases} \tag{2.38}$$

$$H_x(M_y,1p) = \begin{cases} \frac{1}{j\omega\mu_0}(\frac{\partial^2}{\partial x \partial y})\left[-\frac{1}{4\pi}\int_{y'_{n-1}}^{y'_n} G_y^i(\vec{r},\vec{r}\,')M_y^n(y')dy'\right], & \text{if } y'_{n-1} \leq y' \leq y'_n, \\ 0, & \text{else where.} \end{cases} \tag{2.39}$$

$$H_x(M_y,2p) = \begin{cases} \frac{1}{j\omega\mu_0}(\frac{\partial^2}{\partial x \partial y})\left[-\frac{1}{4\pi}\int_{y'_n}^{y'_{n+1}} G_y^i(\vec{r},\vec{r}\,')M_y^n(y')dy'\right], & \text{if } y'_n \leq y' \leq y'_{n+1}, \\ 0, & \text{else where.} \end{cases} \tag{2.40}$$

$$H_y(M_y,1p) = \begin{cases} \frac{1}{j\omega\mu_0}(k^2+\frac{\partial^2}{\partial y^2})\left[-\frac{1}{4\pi}\int_{y'_{n-1}}^{y'_n} G_y^i(\vec{r},\vec{r}\,')M_y^n(y')dy'\right], & \text{if } y'_{n-1} \leq y' \leq y'_n, \\ 0, & \text{else where.} \end{cases} \tag{2.41}$$

$$H_y(M_y,2p) = \begin{cases} \frac{1}{j\omega\mu_0}(k^2+\frac{\partial^2}{\partial y^2})\left[-\frac{1}{4\pi}\int_{y'_n}^{y'_{n+1}} G_y^i(\vec{r},\vec{r}\,')M_y^n(y')dy'\right], & \text{if } y'_n \leq y' \leq y'_{n+1}, \\ 0, & \text{else where.} \end{cases} \tag{2.42}$$

$$H_y(M_x,1p) = \begin{cases} \frac{1}{j\omega\mu_0}(\frac{\partial^2}{\partial x \partial y})\left[-\frac{1}{4\pi}\int_{x'_{n-1}}^{x'_n} G_x^i(\vec{r},\vec{r}\,')M_x^n(x')dx'\right], & \text{if } x'_{n-1} \leq x' \leq x'_n, \\ 0, & \text{else where.} \end{cases} \tag{2.43}$$

$$H_y(M_x,2p) = \begin{cases} \frac{1}{j\omega\mu_0}(\frac{\partial^2}{\partial x \partial y})\left[-\frac{1}{4\pi}\int_{x'_n}^{x'_{n+1}} G_x^i(\vec{r},\vec{r}\,')M_x^n(x')dx'\right], & \text{if } x'_n \leq x' \leq x'_{n+1}, \\ 0, & \text{else where.} \end{cases} \tag{2.44}$$

The mutual admittance element Y^i_{mn}, which is derived from the inner product of (2.31) with (2.36), becomes

$$
\begin{aligned}
& Y^i_{mn} \\
= \;& \langle \vec{W}_m, \vec{H}_n \rangle \\
= \;& \langle W_x(1p), H_x(M_x,1p)\rangle + \langle W_x(1p), H_x(M_x,2p)\rangle \\
& + \langle W_x(1p), H_x(M_y,1p)\rangle + \langle W_x(1p), H_x(M_y,2p)\rangle \\
& + \langle W_x(2p), H_x(M_x,1p)\rangle + \langle W_x(2p), H_x(M_x,2p)\rangle \\
& + \langle W_x(2p), H_x(M_y,1p)\rangle + \langle W_x(2p), H_x(M_y,2p)\rangle \\
& + \langle W_y(1p), H_y(M_x,1p)\rangle + \langle W_y(1p), H_y(M_x,2p)\rangle \\
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& + \langle W_y(2p), H_y(M_x,1p)\rangle + \langle W_y(2p), H_y(M_x,2p)\rangle \\
& + \langle W_y(2p), H_y(M_y,1p)\rangle + \langle W_y(2p), H_y(M_y,2p)\rangle
\end{aligned}
\tag{2.45}
$$

Define

$$
\begin{aligned}
& \mathrm{Int1}p(n,\alpha_x,x') \\
= \;& \int_{x'_{n-1}}^{x'_n} \sin(\alpha_x x') \frac{\sin k(x'-x'_{n-1})}{\sin k(x'_n - x'_{n-1})} dx' \\
= \;& -\frac{1}{2\sin k(x'_n - x'_{n-1})} \int_{x'_{n-1}}^{x'_n} \Big(\cos\big((\alpha_x + k)x' - kx'_{n-1}\big) - \cos\big((\alpha_x - k)x' + kx'_{n-1}\big) \Big) dx' \\
= \;& -\frac{1}{2\sin k(x'_n - x'_{n-1})(\alpha_x + k)} \Big(sin\big(\alpha_x x'_n + k(x'_n - x'_{n-1})\big) - sin\big(\alpha_x x'_{n-1}\big) \Big) \\
& + \frac{1}{2\sin k(x'_n - x'_{n-1})(\alpha_x - k)} \Big(sin\big(\alpha_x x'_n - k(x'_n - x'_{n-1})\big) - sin\big(\alpha_x x'_{n-1}\big) \Big)
\end{aligned}
\tag{2.46}
$$

and

$$
\begin{aligned}
& \mathrm{Int2}p(n,\alpha_x,x') \\
= \ & \int_{x'_n}^{x'_{n+1}} \sin(\alpha_x x') \frac{\sin k(x'_{n+1}-x')}{\sin k(x'_{n+1}-x'_n)} dx' \\
= \ & -\frac{1}{2\sin k(x'_{n+1}-x'_n)} \int_{x'_n}^{x'_{n+1}} \Big(\cos\big((\alpha_x-k)x'+kx'_{n+1}\big) - \cos\big((\alpha_x+k)x'-kx'_{n+1}\big)\Big) dx' \\
= \ & -\frac{1}{2\sin k(x'_{n+1}-x'_n)(\alpha_x+k)} \Big(sin\big(\alpha_x x'_n - k(x'_{n+1}-x'_n)\big) - sin\big(\alpha_x x'_{n+1}\big)\Big) \\
& + \frac{1}{2\sin k(x'_{n+1}-x'_n)(\alpha_x-k)} \Big(sin\big(\alpha_x x'_n + k(x'_{n+1}-x'_n)\big) - sin\big(\alpha_x x'_{n+1}\big)\Big) \qquad (2.47)
\end{aligned}
$$

Substitute (2.20) and (2.21) into (2.37), (2.38), (2.39), (2.40), (2.41), (2.42), (2.43) and (2.44) we can obtain

$$
\begin{aligned}
& H_x(M_x,1p) \\
= \ & -\frac{V_n}{j4\pi\omega\mu_0} \int_{x'_{n-1}}^{x'_n} (k^2+\frac{\partial^2}{\partial x^2}) \\
& \Big(-\frac{4\pi}{\mathrm{XcYc}} \sum_{u=0}^{\infty}\sum_{v=1}^{\infty} \frac{\varepsilon_u\varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc}) \sin(\alpha_x x)\cos(\alpha_y y) \sin(\alpha_x x')\cos(\alpha_y y') \Big) \\
& \frac{\sin k(x'-x'_{n-1})}{\sin k(x'_n-x'_{n-1})} dx' \\
= \ & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{u=0}^{\infty}\sum_{v=1}^{\infty} \frac{\varepsilon_u\varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc})(k^2-\alpha_x^2)\sin(\alpha_x x)\cos(\alpha_y y)\cos(\alpha_y y') \\
& \Big(\int_{x'_{n-1}}^{x'_n} \sin(\alpha_x x') \frac{\sin k(x'-x'_{n-1})}{\sin k(x'_n-x'_{n-1})} dx' \Big) \\
= \ & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{u=0}^{\infty}\sum_{v=1}^{\infty} \frac{\varepsilon_u\varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc})(k^2-\alpha_x^2)\sin(\alpha_x x)\cos(\alpha_y y)\cos(\alpha_y y') \\
& \mathrm{Int1}p(n,\alpha_x,x') \qquad (2.48)
\end{aligned}
$$

$$
\begin{aligned}
& H_x(M_x, 2p) \\
= \; & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{u=0}^{\infty} \sum_{v=1}^{\infty} \frac{\varepsilon_u \varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc})(k^2 - \alpha_x^2) \sin(\alpha_x x) \cos(\alpha_y y) \cos(\alpha_y y') \\
& \left(\int_{x'_n}^{x'_{n+1}} \sin(\alpha_x x') \frac{\sin k(x'_{n+1} - x')}{\sin k(x'_{n+1} - x'_n)} dx' \right) \\
= \; & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{u=0}^{\infty} \sum_{v=1}^{\infty} \frac{\varepsilon_u \varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc})(k^2 - \alpha_x^2) \sin(\alpha_x x) \cos(\alpha_y y) \cos(\alpha_y y') \\
& \mathrm{Int}2p(n, \alpha_x, x')
\end{aligned}
\tag{2.49}
$$

$$
\begin{aligned}
& H_x(M_y, 1p) \\
= \; & -\frac{V_n}{j4\pi\omega\mu_0} \int_{y'_{n-1}}^{y'_n} \left(\frac{\partial^2}{\partial x \partial y}\right) \\
& \left(-\frac{4\pi}{\mathrm{XcYc}} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} \frac{\varepsilon_p \varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc}) \sin(\beta_y y) \cos(\beta_x x) \sin(\beta_y y') \cos(\beta_x x') \right) \\
& \frac{\sin k(y' - y'_{n-1})}{\sin k(y'_n - y'_{n-1})} dy' \\
= \; & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} \frac{\varepsilon_p \varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(-\beta_x \beta_y) \sin(\beta_x x) \cos(\beta_y y) \cos(\beta_x x')) \\
& \left(\int_{y'_{n-1}}^{y'_n} \sin(\beta_y y') \frac{\sin k(y' - y'_{n-1})}{\sin k(y'_n - y'_{n-1})} dy' \right) \\
= \; & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} \frac{\varepsilon_p \varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(-\beta_x \beta_y) \sin(\beta_x x) \cos(\beta_y y) \cos(\beta_x x')) \\
& \mathrm{Int}1p(n, \beta_y, y')
\end{aligned}
\tag{2.50}
$$

$$
\begin{aligned}
& H_x(M_y, 2p) \\
= \; & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} \frac{\varepsilon_p \varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(-\beta_x \beta_y) \sin(\beta_x x) \cos(\beta_y y) \cos(\beta_x x')) \\
& \left(\int_{y'_n}^{y'_{n+1}} \sin(\beta_y y') \frac{\sin k(y'_{n+1} - y')}{\sin k(y'_{n+1} - y'_n)} dy' \right) \\
= \; & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} \frac{\varepsilon_p \varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(-\beta_x \beta_y) \sin(\beta_x x) \cos(\beta_y y) \cos(\beta_x x')) \\
& \mathrm{Int}2p(n, \beta_y, y')
\end{aligned}
\tag{2.51}
$$

$$
\begin{aligned}
& H_y(M_y, 1p) \\
= & -\frac{V_n}{j4\pi\omega\mu_0}\int_{y'_{n-1}}^{y'_n}(k^2+\frac{\partial^2}{\partial y^2}) \\
& \left(-\frac{4\pi}{\mathrm{XcYc}}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}\frac{\varepsilon_p\varepsilon_q}{\beta_c}\cot(\beta_c \mathrm{Zc})\sin(\beta_y y)\cos(\beta_x x)\sin(\beta_y y')\cos(\beta_x x')\right) \\
& \frac{\sin k(y'-y'_{n-1})}{\sin k(y'_n-y'_{n-1})}dy' \\
= & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}\frac{\varepsilon_p\varepsilon_q}{\beta_c}\cot(\beta_c \mathrm{Zc})(k^2-\beta_y^2)\sin(\beta_y y)\cos(\beta_x x)\cos(\beta_x x') \\
& \left(\int_{y'_{n-1}}^{y'_n}\sin(\beta_y y')\frac{\sin k(y'-y'_{n-1})}{\sin k(y'_n-y'_{n-1})}dy'\right) \\
= & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}\frac{\varepsilon_p\varepsilon_q}{\beta_c}\cot(\beta_c \mathrm{Zc})(k^2-\beta_y^2)\sin(\beta_y y)\cos(\beta_x x)\cos(\beta_x x') \\
& \mathrm{Int1}p(n, \beta_y, y')
\end{aligned} \tag{2.52}
$$

$$
\begin{aligned}
& H_y(M_y, 2p) \\
= & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}\frac{\varepsilon_p\varepsilon_q}{\beta_c}\cot(\beta_c \mathrm{Zc})(k^2-\beta_y^2)\sin(\beta_y y)\cos(\beta_x x)\cos(\beta_x x') \\
& \left(\int_{y'_n}^{y'_{n+1}}\sin(\beta_y y')\frac{\sin k(y'_{n+1}-y')}{\sin k(y'_{n+1}-y'_n)}dy'\right) \\
= & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}\frac{\varepsilon_p\varepsilon_q}{\beta_c}\cot(\beta_c \mathrm{Zc})(k^2-\beta_y^2)\sin(\beta_y y)\cos(\beta_x x)\cos(\beta_x x') \\
& \mathrm{Int2}p(n, \beta_y, y')
\end{aligned} \tag{2.53}
$$

$$
\begin{aligned}
&H_y(M_x,1p)\\
=\ & -\frac{V_n}{j4\pi\omega\mu_0}\int_{x'_{n-1}}^{x'_n}\left(\frac{\partial^2}{\partial x\partial y}\right)\\
&\left(-\frac{4\pi}{\mathrm{XcYc}}\sum_{u=0}^{\infty}\sum_{v=1}^{\infty}\frac{\varepsilon_u\varepsilon_v}{\alpha_c}\cot(\alpha_c\mathrm{Zc})\sin(\alpha_x x)\cos(\alpha_y y)\sin(\alpha_x x')\cos(\alpha_y y')\right)\\
&\frac{\sin k(x'-x'_{n-1})}{\sin k(x'_n-x'_{n-1})}dx'\\
=\ & \frac{V_n}{j\omega\mu_0\mathrm{XcYc}}\sum_{u=0}^{\infty}\sum_{v=1}^{\infty}\frac{\varepsilon_u\varepsilon_v}{\alpha_c}\cot(\alpha_c\mathrm{Zc})(-\alpha_x\alpha_y)\sin(\alpha_y y)\cos(\alpha_x x)\cos(\alpha_y y')\\
&\left(\int_{x'_{n-1}}^{x'_n}\sin(\alpha_x x')\frac{\sin k(x'-x'_{n-1})}{\sin k(x'_n-x'_{n-1})}dx'\right)\\
=\ & \frac{V_n}{j\omega\mu_0\mathrm{XcYc}}\sum_{u=0}^{\infty}\sum_{v=1}^{\infty}\frac{\varepsilon_u\varepsilon_v}{\alpha_c}\cot(\alpha_c\mathrm{Zc})(-\alpha_x\alpha_y)\sin(\alpha_y y)\cos(\alpha_x x)\cos(\alpha_y y')\\
&\mathrm{Int1}p(n,\alpha_x,x') \qquad (2.54)
\end{aligned}
$$

$$
\begin{aligned}
&H_y(M_x,2p)\\
=\ & \frac{V_n}{j\omega\mu_0\mathrm{XcYc}}\sum_{u=0}^{\infty}\sum_{v=1}^{\infty}\frac{\varepsilon_u\varepsilon_v}{\alpha_c}\cot(\alpha_c\mathrm{Zc})(-\alpha_x\alpha_y)\sin(\alpha_y y)\cos(\alpha_x x)\cos(\alpha_y y')\\
&\left(\int_{x'_n}^{x'_{n+1}}\sin(\alpha_x x')\frac{\sin k(x'_{n+1}-x')}{\sin k(x'_{n+1}-x'_n)}dx'\right)\\
=\ & \frac{V_n}{j\omega\mu_0\mathrm{XcYc}}\sum_{u=0}^{\infty}\sum_{v=1}^{\infty}\frac{\varepsilon_u\varepsilon_v}{\alpha_c}\cot(\alpha_c\mathrm{Zc})(-\alpha_x\alpha_y)\sin(\alpha_y y)\cos(\alpha_x x)\cos(\alpha_y y')\\
&\mathrm{Int2}p(n,\alpha_x,x') \qquad (2.55)
\end{aligned}
$$

The inner products in (2.45) can be obtained as

$$
\begin{aligned}
&\langle W_x(1p),H_x(M_x,1p)\rangle\\
=\ & \frac{V_n}{j\omega\mu_0\mathrm{XcYc}}\sum_{u=0}^{\infty}\sum_{v=1}^{\infty}\frac{\varepsilon_u\varepsilon_v}{\alpha_c}\cot(\alpha_c\mathrm{Zc})(k^2-\alpha_x^2)\cos(\alpha_y y)\cos(\alpha_y y')\\
&\left(\int_{x_{m-1}}^{x_m}\sin(\alpha_x x)\frac{\sin k(x-x_{m-1})}{\sin k(x-x_{m-1})}dx\right)\left(\int_{x'_{n-1}}^{x'_n}\sin(\alpha_x x')\frac{\sin k(x'-x'_{n-1})}{\sin k(x'_n-x'_{n-1})}dx'\right)\\
=\ & \frac{V_n}{j\omega\mu_0\mathrm{XcYc}}\sum_{u=0}^{\infty}\sum_{v=1}^{\infty}\frac{\varepsilon_u\varepsilon_v}{\alpha_c}\cot(\alpha_c\mathrm{Zc})(k^2-\alpha_x^2)\cos(\alpha_y y)\cos(\alpha_y y')\\
&\mathrm{Int1}p(m,\alpha_x,x)\mathrm{Int1}p(n,\alpha_x,x') \qquad (2.56)
\end{aligned}
$$

$$
\begin{aligned}
&\langle W_x(1p), H_x(M_x, 2p)\rangle \\
= \; & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{u=0}^{\infty}\sum_{v=1}^{\infty} \frac{\varepsilon_u\varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc})(k^2-\alpha_x^2)\cos(\alpha_y y)\cos(\alpha_y y') \\
& \left(\int_{x_{m-1}}^{x_m} \sin(\alpha_x x)\frac{\sin k(x-x_{m-1})}{\sin k(x-x_{m-1})}dx\right)\left(\int_{x'_n}^{x'_{n+1}} \sin(\alpha_x x')\frac{\sin k(x'_{n+1}-x')}{\sin k(x'_{n+1}-x'_n)}dx'\right) \\
= \; & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{u=0}^{\infty}\sum_{v=1}^{\infty} \frac{\varepsilon_u\varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc})(k^2-\alpha_x^2)\cos(\alpha_y y)\cos(\alpha_y y') \\
& \mathrm{Int1}p(m,\alpha_x,x)\mathrm{Int2}p(n,\alpha_x,x') \qquad (2.57)
\end{aligned}
$$

$$
\begin{aligned}
&\langle W_x(1p), H_x(M_y, 1p)\rangle \\
= \; & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{\varepsilon_p\varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(-\beta_x\beta_y)\cos(\beta_y y)\cos(\beta_x x') \\
& \left(\int_{x_{m-1}}^{x_m} \sin(\beta_x x)\frac{\sin k(x-x_{m-1})}{\sin k(x-x_{m-1})}dx\right)\left(\int_{y'_{n-1}}^{y'_n} \sin(\beta_y y')\frac{\sin k(y'-y'_{n-1})}{\sin k(y'_n-y'_{n-1})}dy'\right) \\
= \; & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{\varepsilon_p\varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(-\beta_x\beta_y)\cos(\beta_y y)\cos(\beta_x x') \\
& \mathrm{Int1}p(m,\beta_x,x)\mathrm{Int1}p(n,\beta_y,y') \qquad (2.58)
\end{aligned}
$$

$$
\begin{aligned}
&\langle W_x(1p), H_x(M_y, 2p)\rangle \\
= \; & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{\varepsilon_p\varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(-\beta_x\beta_y)\cos(\beta_y y)\cos(\beta_x x') \\
& \left(\int_{x_{m-1}}^{x_m} \sin(\beta_x x)\frac{\sin k(x-x_{m-1})}{\sin k(x-x_{m-1})}dx\right)\left(\int_{y'_n}^{y'_{n+1}} \sin(\beta_y y')\frac{\sin k(y'_{n+1}-y')}{\sin k(y'_{n+1}-y'_n)}dy'\right) \\
= \; & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{\varepsilon_p\varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(-\beta_x\beta_y)\cos(\beta_y y)\cos(\beta_x x') \\
& \mathrm{Int1}p(m,\beta_x,x)\mathrm{Int2}p(n,\beta_y,y') \qquad (2.59)
\end{aligned}
$$

$$
\begin{aligned}
&\langle W_x(2p), H_x(M_x, 1p)\rangle \\
=\;& \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{u=0}^{\infty}\sum_{v=1}^{\infty} \frac{\varepsilon_u \varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc})(k^2-\alpha_x^2)\cos(\alpha_y y)\cos(\alpha_y y') \\
& \left(\int_{x_m}^{x_{m+1}} \sin(\alpha_x x)\frac{\sin k(x_{m+1}-x)}{\sin k(x_{m+1}-x_n)}dx\right)\left(\int_{x'_{n-1}}^{x'_n} \sin(\alpha_x x')\frac{\sin k(x'-x'_{n-1})}{\sin k(x'_n-x'_{n-1})}dx'\right) \\
=\;& \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{u=0}^{\infty}\sum_{v=1}^{\infty} \frac{\varepsilon_u \varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc})(k^2-\alpha_x^2)\cos(\alpha_y y)\cos(\alpha_y y') \\
& \mathrm{Int2}p(m,\alpha_x,x)\mathrm{Int1}p(n,\alpha_x,x')
\end{aligned}
\tag{2.60}
$$

$$
\begin{aligned}
&\langle W_x(2p), H_x(M_x, 2p)\rangle \\
=\;& \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{u=0}^{\infty}\sum_{v=1}^{\infty} \frac{\varepsilon_u \varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc})(k^2-\alpha_x^2)\cos(\alpha_y y)\cos(\alpha_y y') \\
& \left(\int_{x_m}^{x_{m+1}} \sin(\alpha_x x)\frac{\sin k(x_{m+1}-x)}{\sin k(x_{m+1}-x_n)}dx\right)\left(\int_{x'_n}^{x'_{n+1}} \sin(\alpha_x x')\frac{\sin k(x'_{n+1}-x')}{\sin k(x'_{n+1}-x'_n)}dx'\right) \\
=\;& \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{u=0}^{\infty}\sum_{v=1}^{\infty} \frac{\varepsilon_u \varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc})(k^2-\alpha_x^2)\cos(\alpha_y y)\cos(\alpha_y y') \\
& \mathrm{Int2}p(m,\alpha_x,x)\mathrm{Int2}p(n,\alpha_x,x')
\end{aligned}
\tag{2.61}
$$

$$
\begin{aligned}
&\langle W_x(2p), H_x(M_y, 1p)\rangle \\
=\;& \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{\varepsilon_p \varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(-\beta_x\beta_y)\cos(\beta_y y)\cos(\beta_x x') \\
& \left(\int_{x_m}^{x_{m+1}} \sin(\beta_x x)\frac{\sin k(x_{m+1}-x)}{\sin k(x_{m+1}-x_n)}dx\right)\left(\int_{y'_{n-1}}^{y'_n} \sin(\beta_y y')\frac{\sin k(y'-y'_{n-1})}{\sin k(y'_n-y'_{n-1})}dy'\right) \\
=\;& \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{\varepsilon_p \varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(-\beta_x\beta_y)\cos(\beta_y y)\cos(\beta_x x') \\
& \mathrm{Int2}p(m,\beta_x,x)\mathrm{Int1}p(n,\beta_y,y')
\end{aligned}
\tag{2.62}
$$

$$
\begin{aligned}
&\langle W_x(2p), H_x(M_y, 2p)\rangle \\
= \; & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{\varepsilon_p\varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(-\beta_x\beta_y)\cos(\beta_y y)\cos(\beta_x x') \\
& \left(\int_{x_m}^{x_{m+1}} \sin(\beta_x x)\frac{\sin k(x_{m+1}-x)}{\sin k(x_{m+1}-x_n)}dx\right)\left(\int_{y'_n}^{y'_{n+1}} \sin(\beta_y y')\frac{\sin k(y'_{n+1}-y')}{\sin k(y'_{n+1}-y'_n)}dy'\right) \\
= \; & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{\varepsilon_p\varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(-\beta_x\beta_y)\cos(\beta_y y)\cos(\beta_x x') . \\
& \mathrm{Int2}p(m,\beta_x,x)\mathrm{Int2}p(n,\beta_y,y')
\end{aligned} \tag{2.63}
$$

$$
\begin{aligned}
&\langle W_y(1p), H_y(M_y, 1p)\rangle \\
= \; & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{\varepsilon_p\varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(k^2-\beta_y^2)\cos(\beta_x x)\cos(\beta_x x') \\
& \left(\int_{y_{m-1}}^{y_m} \sin(\beta_y y)\frac{\sin k(y-y_{m-1})}{\sin k(y_m-y_{m-1})}dy\right)\left(\int_{y'_{n-1}}^{y'_n} \sin(\beta_y y')\frac{\sin k(y'-y'_{n-1})}{\sin k(y'_n-y'_{n-1})}dy'\right) \\
= \; & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{\varepsilon_p\varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(k^2-\beta_y^2)\cos(\beta_x x)\cos(\beta_x x') \\
& \mathrm{Int1}p(m,\beta_y,y)\mathrm{Int1}p(n,\beta_y,y')
\end{aligned} \tag{2.64}
$$

$$
\begin{aligned}
&\langle W_y(1p), H_y(M_y, 2p)\rangle \\
= \; & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{\varepsilon_p\varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(k^2-\beta_y^2)\cos(\beta_x x)\cos(\beta_x x') \\
& \left(\int_{y_{m-1}}^{y_m} \sin(\beta_y y)\frac{\sin k(y-y_{m-1})}{\sin k(y_m-y_{m-1})}dy\right)\left(\int_{y'_n}^{y'_{n+1}} \sin(\beta_y y')\frac{\sin k(y'_{n+1}-y')}{\sin k(y'_{n+1}-y'_n)}dy'\right) \\
= \; & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{\varepsilon_p\varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(k^2-\beta_y^2)\cos(\beta_x x)\cos(\beta_x x') \\
& \mathrm{Int1}p(m,\beta_y,y)\mathrm{Int2}p(n,\beta_y,y')
\end{aligned} \tag{2.65}
$$

$$
\begin{aligned}
&\langle W_y(1p), H_y(M_x, 1p)\rangle \\
= \;& \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{u=0}^{\infty}\sum_{v=1}^{\infty} \frac{\varepsilon_u\varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc})(-\alpha_x\alpha_y)\cos(\alpha_x x)\cos(\alpha_y y') \\
& \left(\int_{y_{m-1}}^{y_m} \sin(\alpha_y y)\frac{\sin k(y-y_{m-1})}{\sin k(y_m - y_{m-1})}dy\right)\left(\int_{x'_{n-1}}^{x'_n} \sin(\alpha_x x')\frac{\sin k(x'-x'_{n-1})}{\sin k(x'_n - x'_{n-1})}dx'\right) \\
= \;& \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{u=0}^{\infty}\sum_{v=1}^{\infty} \frac{\varepsilon_u\varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc})(-\alpha_x\alpha_y)\cos(\alpha_x x)\cos(\alpha_y y') \\
& \mathrm{Int1}p(m, \alpha_y, y)\mathrm{Int1}p(n, \alpha_x, y')
\end{aligned} \tag{2.66}
$$

$$
\begin{aligned}
&\langle W_y(1p), H_y(M_x, 2p)\rangle \\
= \;& \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{u=0}^{\infty}\sum_{v=1}^{\infty} \frac{\varepsilon_u\varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc})(-\alpha_x\alpha_y)\cos(\alpha_x x)\cos(\alpha_y y') \\
& \left(\int_{y_{m-1}}^{y_m} \sin(\alpha_y y)\frac{\sin k(y-y_{m-1})}{\sin k(y_m - y_{m-1})}dy\right)\left(\int_{x'_n}^{x'_{n+1}} \sin(\alpha_x x')\frac{\sin k(x'_{n+1}-x')}{\sin k(x'_{n+1} - x'_n)}dx'\right) \\
= \;& \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{u=0}^{\infty}\sum_{v=1}^{\infty} \frac{\varepsilon_u\varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc})(-\alpha_x\alpha_y)\cos(\alpha_x x)\cos(\alpha_y y') \\
& \mathrm{Int1}p(m, \alpha_y, y)\mathrm{Int2}p(n, \alpha_x, y')
\end{aligned} \tag{2.67}
$$

$$
\begin{aligned}
&\langle W_y(2p), H_y(M_y, 1p)\rangle \\
= \;& \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{\varepsilon_p\varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(k^2-\beta_y^2)\cos(\beta_x x)\cos(\beta_x x') \\
& \left(\int_{y_m}^{y_{m+1}} \sin(\beta_y y)\frac{\sin k(y_{m+1}-y)}{\sin k(y_{m+1} - y_m)}dy\right)\left(\int_{y'_{n-1}}^{y'_n} \sin(\beta_y y')\frac{\sin k(y'-y'_{n-1})}{\sin k(y'_n - y'_{n-1})}dy'\right) \\
= \;& \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{\varepsilon_p\varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(k^2-\beta_y^2)\cos(\beta_x x)\cos(\beta_x x') \\
& \mathrm{Int2}p(m, \beta_y, y)\mathrm{Int1}p(n, \beta_y, y')
\end{aligned} \tag{2.68}
$$

$$
\begin{aligned}
&\langle W_y(2p), H_y(M_y,2p)\rangle \\
= &\frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{\varepsilon_p\varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(k^2-\beta_y^2)\cos(\beta_x x)\cos(\beta_x x') \\
&\left(\int_{y_m}^{y_{m+1}} \sin(\beta_y y)\frac{\sin k(y_{m+1}-y)}{\sin k(y_{m+1}-y_m)}dy\right)\left(\int_{y'_n}^{y'_{n+1}} \sin(\beta_y y')\frac{\sin k(y'_{n+1}-y')}{\sin k(y'_{n+1}-y'_n)}dy'\right) \\
= &\frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{\varepsilon_p\varepsilon_q}{\beta_c} \cot(\beta_c \mathrm{Zc})(k^2-\beta_y^2)\cos(\beta_x x)\cos(\beta_x x') \\
&\mathrm{Int2}p(m,\beta_y,y)\mathrm{Int2}p(n,\beta_y,y')
\end{aligned}
\tag{2.69}
$$

$$
\begin{aligned}
&\langle W_y(2p), H_y(M_x,1p)\rangle \\
= &\frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{u=0}^{\infty}\sum_{v=1}^{\infty} \frac{\varepsilon_u\varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc})(-\alpha_x\alpha_y)\cos(\alpha_x x)\cos(\alpha_y y') \\
&\left(\int_{y_m}^{y_{m+1}} \sin(\alpha_y y)\frac{\sin k(y_{m+1}-y)}{\sin k(y_{m+1}-y_m)}dy\right)\left(\int_{x'_{n-1}}^{x'_n} \sin(\alpha_x x')\frac{\sin k(x'-x'_{n-1})}{\sin k(x'_n-x'_{n-1})}dx'\right) \\
= &\frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{u=0}^{\infty}\sum_{v=1}^{\infty} \frac{\varepsilon_u\varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc})(-\alpha_x\alpha_y)\cos(\alpha_x x)\cos(\alpha_y y') \\
&\mathrm{Int2}p(m,\alpha_y,y)\mathrm{Int1}p(n,\alpha_x,y')
\end{aligned}
\tag{2.70}
$$

$$
\begin{aligned}
&\langle W_y(2p), H_y(M_x,2p)\rangle \\
= &\frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{u=0}^{\infty}\sum_{v=1}^{\infty} \frac{\varepsilon_u\varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc})(-\alpha_x\alpha_y)\cos(\alpha_x x)\cos(\alpha_y y') \\
&\left(\int_{y_m}^{y_{m+1}} \sin(\alpha_y y)\frac{\sin k(y_{m+1}-y)}{\sin k(y_{m+1}-y_m)}dy\right)\left(\int_{x'_n}^{x'_{n+1}} \sin(\alpha_x x')\frac{\sin k(x'_{n+1}-x')}{\sin k(x'_{n+1}-x'_n)}dx'\right) \\
= &\frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{u=0}^{\infty}\sum_{v=1}^{\infty} \frac{\varepsilon_u\varepsilon_v}{\alpha_c} \cot(\alpha_c \mathrm{Zc})(-\alpha_x\alpha_y)\cos(\alpha_x x)\cos(\alpha_y y') \\
&\mathrm{Int2}p(m,\alpha_y,y)\mathrm{Int2}p(n,\alpha_x,y')
\end{aligned}
\tag{2.71}
$$

Note that (2.45) is a generalized form of the mutual admittance element Y^i_{mn}. For the analytical model in this analysis, among the sixteen inner products in (2.45), only four terms of the inner products are nonzero.

For Example, suppose the $\vec{W}_m$ is in the subsection (x_{m-1}, x_m) and (x_m, x_{m+1}) in the X direction, and $\vec{M}_n$ is in the subsection (y'_{n-1}, y'_n) and (y'_n, y'_{n+1}) in the Y direction(see Fig. 2.4).

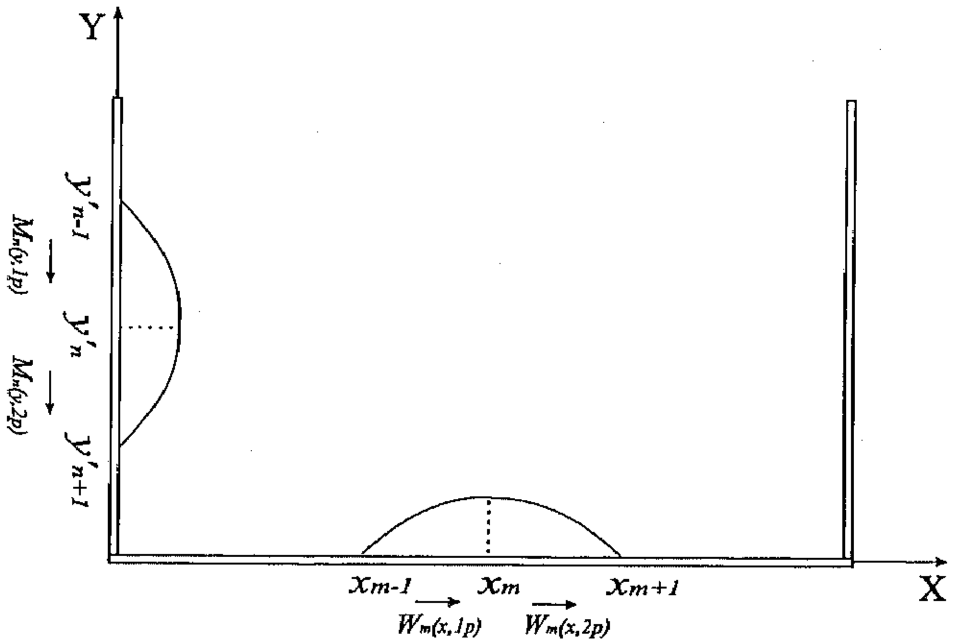

Fig. 2.4: Weighting function and expansion function.

Then, according to (2.31) and (2.36), we can obtain

$$\vec{W}_m = [W_x(1p) + W_x(2p)]\hat{x} \tag{2.72}$$

and

$$\vec{H}_n = [H_x(M_y, 1p) + H_x(M_y, 2p)]\hat{x} + [H_y(M_y, 1p) + H_y(M_y, 2p)]\hat{y} \tag{2.73}$$

We can obtain a four-term Y^i_{mn} as

$$
\begin{aligned}
& Y^i_{mn} \\
= & \langle W_x(1p)+W_x(2p), H_x(M_y,1p)\rangle + \langle W_x(1p)+W_x(2p), H_x(M_y,2p)\rangle \\
= & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{\varepsilon_p\varepsilon_q}{\beta_c}\cot(\beta_c \mathrm{Zc})(-\beta_x\beta_y)\cos(\beta_y y)\cos(\beta_x x') \\
& \left[\mathrm{Int1}p(m,\beta_x,x)+\mathrm{Int2}p(m,\beta_x,x)\right]\mathrm{Int1}p(n,\beta_y,y') \\
& + \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{\varepsilon_p\varepsilon_q}{\beta_c}\cot(\beta_c \mathrm{Zc})(-\beta_x\beta_y)\cos(\beta_y y)\cos(\beta_x x') \\
& \left[\mathrm{Int1}p(m,\beta_x,x)+\mathrm{Int2}p(m,\beta_x,x)\right]\mathrm{Int2}p(n,\beta_y,y')
\end{aligned}
\tag{2.74}
$$

Another example worth mentioning is

$$\vec{W}_m = W_x(2p)\hat{x} + W_y(1p)\hat{y} \tag{2.75}$$

and

$$\vec{H}_n = \left[H_x(M_x,1p)+H_x(M_y,2p)\right]\hat{x} + \left[H_y(M_x,1p)+H_y(M_y,2p)\right]\hat{y} \tag{2.76}$$

That represents the case when $\vec{W}_m$ is at the corner { subsection (y_{m-1}, y_m) in the Y direction, subsection (x_m, x_{m+1}) in the X direction } and $\vec{M}_n$ is also at the corner { subsection (x'_{n-1}, x'_n) in the X direction, subsection (y'_n, y'_{n+1}) in the Y direction } of the slot as shown in Fig. 2.5.

According to (2.45), we can obtain

$$
\begin{aligned}
& Y^i_{mn} \\
= & \langle W_y(1p), H_y(M_x,1p)+H_y(M_y,2p)\rangle + \langle W_x(2p), H_x(M_x,1p)+H_x(M_y,2p)\rangle \\
= & \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{u=0}^{\infty}\sum_{v=1}^{\infty} \frac{\varepsilon_u\varepsilon_v}{\alpha_c}\cot(\alpha_c \mathrm{Zc})(-\alpha_x\alpha_y)\cos(\alpha_x x)\cos(\alpha_y y') \\
& \left[\mathrm{Int1}p(m,\alpha_y,y)+\mathrm{Int2}p(m,\alpha_x,x)\right]\mathrm{Int1}p(n,\alpha_x,x') \\
& + \frac{V_n}{j\omega\mu_0 \mathrm{XcYc}} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \frac{\varepsilon_p\varepsilon_q}{\beta_c}\cot(\beta_c \mathrm{Zc})(-\beta_x\beta_y)\cos(\beta_y y)\cos(\beta_x x') \\
& \left[\mathrm{Int1}p(m,\beta_y,y)+\mathrm{Int2}p(m,\beta_x,x)\right]\mathrm{Int2}p(n,\beta_y,y')
\end{aligned}
\tag{2.77}
$$

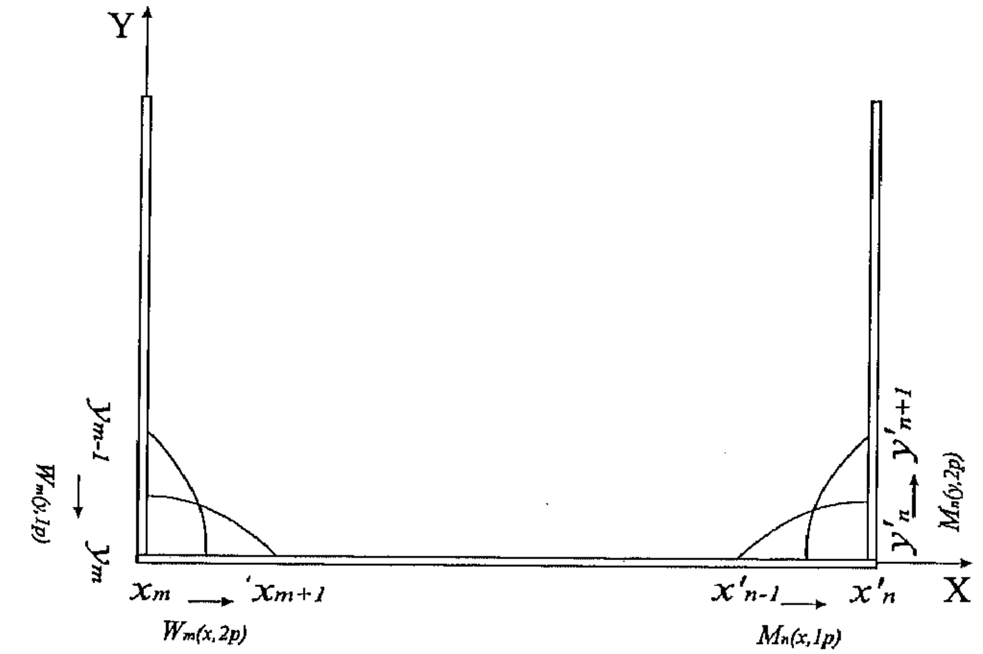

Fig. 2.5: Weighting function and expansion function at the corner of the slot.

2.5 Admittance Matrix Outside the Cavity

Now, we consider the admittance matrix of (2.26) which represents the admittance in the half-free space. Because $[Y^o]$ is independent of $[Y^i]$, we can handle it independently.

The equivalence principle plus duality can be used to transform an aperture type problem into a electric current type problem. In the calculation of the admittance outside the cavity, we can use the results of a wire antenna instead. Y^o_{mn}, the admittance of the slot radiating into half-space, and Z^o_{mn}, the impedance of the wire antenna (ribbon of current), can be related by Eq. (4.126) of R. F. Harrington [31]

$$Y^o_{mn} = \frac{2}{\eta_0^2} Z^o_{mn} \tag{2.78}$$

where

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \tag{2.79}$$

The impedance matrix element Z^o_{mn} of the wire antenna can be obtained from Eq. (2.15) of K. Hirasawa [37] as

$$Z^o_{mn} = -\int_{x_{m-1}}^{x_m} \frac{\sin k(z - x_{m-1})}{SN} E_z(z)dz - \int_{x_m}^{x_{m+1}} \frac{\sin k(z_{m+1} - x)}{SN} E_z(z)dz \tag{2.80}$$

where

$$\begin{aligned} E_z(z) &= -\frac{j30}{SN}\left[\frac{e^{-jkr_{n-1}}}{R_{n-1}} - CS\frac{e^{-jkr_n}}{R_n} + \frac{e^{-jkr_{n+1}}}{R_{n+1}}\right] \\ CS &= 2\cos(k\Delta z), \\ R^2_{n-1} &= a^2 + (z - z_{n-1})^2, R^2_n = a^2 + (z - z_n)^2, R_{n+1} = a^2 + (z - z_{n+1})^2 \end{aligned} \tag{2.81}$$

2.6 Total Admittance Matrix

Finally, the total admittance matrix element Y_{mn} can be expressed as the sum of the admittance matrix element Y^i_{mn} and the admittance matrix element Y^o_{mn} as follows:

$$Y_{mn} = Y^i_{mn} + Y^o_{mn} \tag{2.82}$$