Chapter 5

A Verified Method for Finding Polynomial Factors of Analytic Functions

5.1 Introduction

In this chapter, we consider the problem of locating cluster of zeros of analytic functions. For the determination of multiple or close zeros of an analytic function $f(z)$, iterative methods such as Newton method usually require large number of iterations, or fail by a jump of an approximation. Furthermore, when performing validation on the approximate zeros of $f(z)$, it is always needed to assume that each interval contains only one zero, this makes it difficult to validate the approximate zeros in the case of close zeros. On the other hand, multiple or cluster of zeros can be calculated by factoring methods as a polynomial. The computation of the coefficients of a polynomial of which zeros are close is more stable than the determination of locations of close zeros.

Sakurai and Sugiura [64, 65] proposed a factoring method with validation by considering a fixed point iteration for a polynomial factor $p^*(z)$ of $f(z)$. The validated Taylor coefficients of $f(z)$ are needed in the computation, moreover, a circular region with radius ρ in which $f(z)$ is analytic and the maximum value M of $|f(z)|$ on the boundary

$|z| = \rho$ of the region are assumed to be given in the computation. In this chapter, we show a method to compute ρ and M using circular complex arithmetic. We also propose a method to perform the validated computation of the Taylor coefficients of $f(z)$. Based on these results, a validated computation is performed to calculate an interval coefficient polynomial which contains a polynomial factor of $f(z)$.

5.2 Validated Computation of Factors of Polynomials

5.2.1 An Iterative Method to Find a Factor of a Polynomial

In this section, we consider the validated computation of a factor of a polynomial. Let $f(z)$ be a polynomial of degree $m+n$.

$$f(z) := \sum_{k=0}^{m+n} c_k z^k = c_{m+n} \prod_{i=1}^{m+n} (z - \zeta_i), \quad c_{m+n} \neq 0. \tag{5.1}$$

Let the first m zeros $\{\zeta_i\}_{1 \leq i \leq m}$ of $f(z)$ are distinctly separated from other zeros $\{\zeta_i\}_{m+1 \leq i \leq m+n}$ and form a cluster around the origin. Let $p^*(z)$ be the m degree monic polynomial factor of $f(z)$,

$$p^*(z) = \prod_{i=1}^{m} (z - \zeta_i). \tag{5.2}$$

We consider the problem of computing $p^*(z)$. Let $f(z)$ be factored as

$$f(z) = p^*(z) q^*(z), \quad q^*(z) = c_{m+n} \prod_{i=m+1}^{m+n} (z - \zeta_i).$$

Let the zeros of $p^*(z)$ be located inside a circle centered at the origin with a radius $\delta < 1$ and let the zeros of $q^*(z)$ be located outside the unit circle $|z| = 1$. These conditions can be satisfied by shifting the zeros $\{\zeta_i\}_{1 \leq i \leq m}$ in the cluster to the point $z = z_0$ near the origin and scaling $z \longleftarrow \alpha z$. Let $p(z) = z^m$, if δ is small, then $p(z)$ can be regarded as a good initial approximation for $p^*(z)$. Let

$$r(z) = \sum_{k=0}^{m-1} c_k z^k, \quad q(z) = \sum_{k=m}^{m+n} c_k z^{k-m}, \tag{5.3}$$

where $n \geq m$. Then

$$f(z) = p^*(z)q^*(z) = r(z) + p(z)q(z). \tag{5.4}$$

Define

$$s^{(k)}(z) := \sigma_0^{(k)} + \sigma_1^{(k)} z + \cdots + \sigma_{m-1}^{(k)} z^{m-1},$$

and

$$t^{(k)}(z) := \tau_0^{(k)} + \tau_1^{(k)} z + \cdots + \tau_{n-1}^{(k)} z^{n-1}.$$

In [63], $s^{(k)}(z)$ and $t^{(k)}(z)$ that satisfy the following equation were calculated by taking $s^{(0)}(z) \equiv 0,\ t^{(0)}(z) \equiv 0$ as the initial functions.

$$s^{(k)}(z)(q(z) + t^{(k-1)}(z)) + t^{(k)}(z)p(z) = r(z), \quad k = 1, 2, \ldots. \tag{5.5}$$

By comparing the coefficients in (5.5), we have the following relations.

$$\begin{pmatrix} c_m^{(k-1)} & & & \\ c_{m+1}^{(k-1)} & c_m^{(k-1)} & & \\ \vdots & & \ddots & \\ c_{2m-1}^{(k-1)} & \cdots & \cdots & c_m^{(k-1)} \end{pmatrix} \begin{pmatrix} \sigma_0^{(k)} \\ \sigma_1^{(k)} \\ \vdots \\ \sigma_{m-1}^{(k)} \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{m-1} \end{pmatrix}, \tag{5.6}$$

$$\begin{pmatrix} \tau_0^{(k)} \\ \tau_1^{(k)} \\ \vdots \\ \vdots \\ \tau_{n-1}^{(k)} \end{pmatrix} = - \begin{pmatrix} c_{2m}^{(k-1)} & c_{2m-1}^{(k-1)} & \cdots & c_{m+1}^{(k-1)} \\ \vdots & \vdots & \cdots & c_{m+2}^{(k-1)} \\ c_{m+n}^{(k-1)} & & & \vdots \\ & \ddots & & \vdots \\ & & \ddots & c_{m+n}^{(k-1)} \end{pmatrix} \begin{pmatrix} \sigma_0^{(k)} \\ \sigma_1^{(k)} \\ \vdots \\ \vdots \\ \sigma_{m-1}^{(k)} \end{pmatrix}, \tag{5.7}$$

where $c_{m+j}^{(k-1)} = c_{m+j} + \tau_j^{(k-1)}$, $0 \leq j \leq n-1$ and $c_{m+n}^{(k-1)} = c_{m+n}$.

Let $p^{(k)}(z)$ be the approximate factors of $p^*(z)$,

$$p^{(k)}(z) = p(z) + s^{(k)}(z),\ k = 1, 2, \ldots. \tag{5.8}$$

Then the polynomial $p^{(k)}(z)$ converges to $p^*(z)$ provided the starting factor $p(z)$ is sufficiently near $p^*(z)$.

5.2.2 Verification of the Factor

In this section we show a method to give a validation for coefficients of $p^{(k)}(z)$ obtained in the previous section. Let $v(z)$ be the remainder polynomial of polynomial $g(z)$ divided by polynomial $p(z)$, we denote $v(z)$ by $v(z) = \mathrm{mod}(g(z), p(z))$.

It follows from (5.4) and (5.5) that

$$\begin{aligned}&(p^{(k)}(z) - p^*(z))(q(z) + t^{(k)}(z)) + (q(z) + t^{(k)}(z) - q^*(z))p^*(z) \\ &= s^{(k)}(z)(t^{(k)}(z) - t^{(k-1)}(z)).\end{aligned} \tag{5.9}$$

Then

$$p^*(z) = p^{(k)}(z) - \mathrm{mod}\Big(\frac{s^{(k)}(z)(t^{(k)}(z) - t^{(k-1)}(z))}{q(z) + t^{(k)}(z)}, p^*(z)\Big). \tag{5.10}$$

Let $\boldsymbol{q}(z)$, $\boldsymbol{r}(z)$ be sets of polynomials so that $q(z) \in \boldsymbol{q}(z)$, $r(z) \in \boldsymbol{r}(z)$, respectively. Let $\boldsymbol{p}(z)$ be a set of polynomials so that $p^*(z) \in \boldsymbol{p}(z)$. Let $\boldsymbol{s}(z)$, $\boldsymbol{t}(z)$ be sets of polynomials satisfying the following equation.

$$\boldsymbol{s}(z)(\boldsymbol{q}(z) + t^{(k)}(z)) + \boldsymbol{t}(z)\boldsymbol{p}(z) = \boldsymbol{r}(z). \tag{5.11}$$

Then it follows from (5.10) that

$$p^*(z) \in p(z) + s(z) - \mathrm{mod}\Big(\frac{\boldsymbol{s}(z)(\boldsymbol{t}(z) - t^{(k)}(z))}{\boldsymbol{q}(z) + \boldsymbol{t}(z)}, \boldsymbol{p}(z)\Big). \tag{5.12}$$

Therefore, a set of polynomials which includes $p^*(z)$ can be calculated by (5.12).

5.3 Validated Computation of Polynomial Factors of Analytic Functions

In this section, we show a verified method to calculate a polynomial factor of an analytic function $f(z)$ using circular complex arithmetic [3, 50].

5.3.1 Computation of a Polynomial Factor

Let $f(z)$ be analytic in $|z| < R$, where R is a positive real number. Then

$$f(z) = \sum_{k=0}^{\infty} c_k z^k, \quad |z| < R. \tag{5.13}$$

Let $p^*(z)$ be a m degree polynomial factor of $f(z)$ and the zeros of $p^*(z)$ be located inside a small disk with a radius $\delta < 1$ around the origin. Let $q^*(z)$ be an analytic function such that $f(z) = p^*(z)q^*(z)$. Let $p(z) = z^m$ and polynomial $r(z)$, $q(z)$ be defined as

$$r(z) := \sum_{k=0}^{m-1} c_k z^k, \quad q(z) := \sum_{k=m}^{m+n} c_k z^{k-m}. \tag{5.14}$$

Moreover, let analytic function $h(z)$ be

$$h(z) := \sum_{k=m+n+1}^{\infty} c_k z^k. \tag{5.15}$$

Then

$$f(z) = p^*(z)q^*(z) = r(z) + p(z)q(z) + h(z). \tag{5.16}$$

Let polynomial $f_{m+n}(z)$ of degree $m+n$ be

$$f_{m+n}(z) = r(z) + p(z)q(z).$$

Then $f_{m+n}(z)$ is a polynomial approximation of $f(z)$. From the discussions in the previous section, for polynomial $f_{m+n}(z)$, (5.6)–(5.8) can be used to calculate a sequence of polynomials $p^{(k)}(z)$ which converge to a factor $\tilde{p}^*(z)$ of $f_{m+n}(z)$. If $p(z)$ is sufficiently near $p^*(z)$ and the degree of $q(z)$ is large enough, then $\tilde{p}^*(z)$ is sufficiently near $p^*(z)$ [64]. Therefore, in the case that $f(z)$ is an analytic function, we can obtain an approximate polynomial $p^{(k)}(z)$ of $p^*(z)$. In order to give a validation for the coefficients of $p^{(k)}(z)$, we should take account of the influence of $h(z) = f(z) - f_{m+n}(z)$.

5.3.2 Verification of the Factor

Let $v(z)$ be the interpolate polynomial of degree at most $\deg(p(z))-1$ of analytic function $g(z)$ on the zeros of polynomial $p(z)$. Then $g(z) - v(z) = 0$ holds on the zeros of $p(z)$, i.e., $g(z)-v(z)$ is divisible by $p(z)$. We denote $v(z)$ by $v(z) = \bmod\ (g(z), p(z))$. When $p(z)$ has multiple zeros, $v(z)$ represents the appropriate Hermite interpolant of $g(z)$ over the zeros of $p(z)$. Let $\boldsymbol{g}(z)$ be set of analytic functions and $\boldsymbol{p}(z)$ be set of polynomials. Let $\boldsymbol{v}(z)$ be set of polynomials such that all the elements in it are calculated by the above mentioned method for all the elements in $\boldsymbol{g}(z)$ and $\boldsymbol{p}(z)$. We denote $\boldsymbol{v}(z)$ by $\boldsymbol{v}(z) = \text{ mod } (\boldsymbol{g}(z), \boldsymbol{p}(z))$, i.e.,

$$\boldsymbol{v}(z) = \{\text{mod}(g(z), p(z)) : g(z) \in \boldsymbol{g}(z),\ p(z) \in \boldsymbol{p}(z)\}.$$

Let

$$w(z) = \text{mod}(h(z), p^*(z)). \tag{5.17}$$

Since $h(z)$ is analytic for $|z| < R$, and all the zeros of $p^*(z)$ lie in the disk with the radius $\delta < 1$, $h(z)$ takes finite values on zeros of $p^*(z)$. Let $\boldsymbol{w}(z)$ be set of polynomials so that $w(z) \in \boldsymbol{w}(z)$, and $\mathbf{p}(z)$ be set of polynomials so that $p^*(z) \in \mathbf{p}(z)$. The following theorem was established in [64].

Theorem 5.3.1 *Let $\| r(z) \| \leq \varepsilon$ with sufficiently small $\varepsilon > 0$. If $q(z)$ has no zeros in common with any polynomial in $\mathbf{p}(z)$, then*

$$p^*(z) \in p^{(k)}(z) - \text{mod}\Big(\frac{s^{(k)}(z)(t^{(k)}(z) - t^{(k-1)}(z)) - \boldsymbol{w}(z)}{q^{(k)}(z)}, \boldsymbol{p}(z)\Big), \tag{5.18}$$

where $\| \cdot \|$ for a polynomial denotes the vector 1-norm for a vector of coefficients of the polynomial.

From Theorem 5.3.1, in the case that $f(z)$ is analytic function, if we can calculate $\boldsymbol{w}(z)$ and can also calculate $\boldsymbol{s}^{(k)}(z)$ and $\boldsymbol{t}^{(k)}(z)$ that satisfying

$$\boldsymbol{s}^{(k)}(z)q^{(k)}(z) + \boldsymbol{t}^{(k)}(z)\boldsymbol{p}(z) = s^{(k)}(z)(t^{(k)}(z) - t^{(k-1)}(z)) - \boldsymbol{w}(z),$$

then

$$p^*(z) \in \mathbf{p}^{(k)}(z) := \mathbf{p}(z) + \boldsymbol{s}^{(k)}(z).$$

This means that a set of polynomials which contains the polynomial factor $p^*(z)$ of $f(z)$ can be obtained.

Now we consider the problem of computing $\boldsymbol{w}(z)$. Let

$$C_\varphi = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0/a_m \\ 1 & 0 & \cdots & 0 & -a_1/a_m \\ 0 & 1 & \cdots & 0 & -a_2/a_m \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{m-1}/a_m \end{pmatrix} \tag{5.19}$$

denote the companion matrix of a polynomial

$$\varphi(z) = a_0 + a_1 z + \cdots + a_m z^m \ (a_m \neq 0).$$

The notation $\hat{\varphi}$ implies the vector $(a_0, a_1, \ldots, a_m)^T$ of the coefficients of $\varphi(z)$.

The following theorem was established in [70].

Theorem 5.3.2 *Let $\varphi(z)$ be a polynomial of degree m, and let $z_1, \ldots, z_m$ be distinct zeros of $\varphi(z)$. Let $g(z)$ be a rational function defined at the z_i. Let the coefficients of the polynomial $v(z)$ be defined by*

$$\hat{v} = g(C_\varphi)e_1,$$

where $e_1 = (1, 0, \ldots, 0)^T$. Then

$$v(z) = \mathrm{mod}(g(z), \varphi(z)).$$

Here we shall use the following notations. For matrices $A = (\alpha_{ij})$ and $B = (\beta_{ij})$, the matrix $|A|$ has elements $|\alpha_{ij}|$. The notation $A \leq B$ implies $\alpha_{ij} \leq \beta_{ij}$ for every i and j. We also define $|\boldsymbol{\alpha}| := \max_{\alpha \in \boldsymbol{\alpha}} |\alpha|$ for a closed set $\boldsymbol{\alpha}$ of complex numbers. We have the following theorem.

Theorem 5.3.3 *Let $h(z) = \sum_{k=m+n+1}^{\infty} c_k z^k$ with c_k satisfying*

$$|c_k| \leq \frac{M}{\rho^k}, \quad 1 < \rho < R, \quad k \geq m + n + 1. \tag{5.20}$$

Let $w(z) = \mathrm{mod}(h(z), p^*(z))$, and let $\hat{w}$ denote the vector of the coefficients of $w(z)$. If the spectral radius of $|C_p|$ is smaller than ρ, then

$$|\hat{w}| \leq M\rho^{-(m+n+1)} \left|C_p\right|^{m+n+1} (I - \rho^{-1}\left|C_p\right|)^{-1} e_1.$$

Proof. Let $w_k(z) = \mathrm{mod}(z^k, p^*(z))$. By Theorem 5.3.2 we have the vector $\hat{w}_k$ of the coefficients of $w_k(z)$ by

$$\hat{w}_k = (C_{p^*})^k e_1.$$

Therefore

$$\hat{w} = \sum_{k=m+n+1}^{\infty} c_k \hat{w}_k = \sum_{k=m+n+1}^{\infty} c_k (C_{p^*})^k e_1.$$

Since $|c_k| < M\rho^{-k}$ for every k, and $|C_{p^*}| \leq |C_{\boldsymbol{p}}|$ for $p^*(z) \in \boldsymbol{p}(z)$,

$$|\hat{w}| \leq \sum_{k=m+n+1}^{\infty} M\rho^{-k} |C_{p^*}|^k e_1 \quad \leq \sum_{k=m+n+1}^{\infty} M\rho^{-k} |C_{\boldsymbol{p}}|^k e_1$$
$$= M\rho^{-(m+n+1)} |C_{\boldsymbol{p}}|^{m+n+1} \left(\sum_{k=0}^{\infty} \rho^{-k} |C_{\boldsymbol{p}}|^k \right) e_1.$$

By the hypothesis that the spectral radius of $|C_{\boldsymbol{p}}|$ is smaller than ρ, $\sum_{k=0}^{\infty} \rho^{-k} |C_{\boldsymbol{p}}|^k$ is well defined, hence we obtain the result of the theorem. □

Let $|\zeta_i| < \delta (1 \leq i \leq m)$ and $p(z) \in \boldsymbol{p}(z)$ be defined as

$$p(z) := \prod_{i=1}^{m} (z - \zeta_i) = \sum_{k=0}^{m} a_k z^k, \quad a_m = 1. \tag{5.21}$$

Then for the coefficients of $p(z)$, the following relation holds

$$\begin{aligned} |a_{m-k}/a_m| &= \left| \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} \zeta_{i_1} \zeta_{i_2} \cdots \zeta_{i_k} \right| \leq \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} |\zeta_{i_1} \zeta_{i_2} \cdots \zeta_{i_k}| \\ &< \binom{m}{k} \delta^k, \quad 1 \leq k \leq m. \end{aligned} \tag{5.22}$$

Therefore

$$|a_{m-k}/a_m|^{1/k} < \binom{m}{k}^{1/k} \delta, \quad 1 \leq k \leq m. \tag{5.23}$$

Moreover, we have

$$\binom{m}{k}^{1/k} = \left(\frac{m}{k} \cdot \frac{m-1}{k-1} \cdot \cdots \cdot \frac{m-k+1}{1} \right)^{1/k} \leq m - k + 1 \leq m, \quad 1 \leq k \leq m. \tag{5.24}$$

Let

$$R = 2 \max_{1 \leq k \leq m} |a_{m-k}/a_m|^{1/k}. \tag{5.25}$$

Then all the zeros of $p(z)$ lie in the disk around the origin with a radius R [50]. Hence we have $R < 2m\delta$. From the above results, we see that if $2m\delta < \rho$, then $R < \rho$ and the spectral radius of $|C_{\boldsymbol{p}}|$ is smaller than ρ. Therefore, if all the zeros of any polynomial that belongs to $\boldsymbol{p}(z)$ lie in a small disk then we can expect that $|C_{\boldsymbol{p}}|$ is also small.

From Theorem 5.3.3, $w(z)$ can be calculated by using circular complex arithmetic as follows.

For a circular coefficients polynomial $\boldsymbol{p}(z)$, let $\hat{\boldsymbol{p}}$ denote the vector of the circular coefficients of $\boldsymbol{p}(z)$. Let $\mathrm{mid}(\hat{\boldsymbol{p}})$ denote the vector of the center of the coefficients of $\boldsymbol{p}(z)$ and $\mathrm{rad}(\hat{\boldsymbol{p}})$ denote the vector of the radius of the coefficients of $\boldsymbol{p}(z)$. From theorem 5.3.3, if the coefficients of $h(z)$ satisfy (5.20), then the coefficient $\hat{\boldsymbol{w}}$ of $\boldsymbol{w}(z)$ can be calculated by

$$\mathrm{mid}(\hat{\boldsymbol{w}}) = (0,\ldots,0)^T, \tag{5.26}$$

$$\mathrm{rad}(\hat{\boldsymbol{w}}) = M\rho^{-(m+n+1)}(I - \rho^{-1}\left|C_{\boldsymbol{p}}\right|)^{-1}e_1. \tag{5.27}$$

Then

$$|\hat{\boldsymbol{w}}| = M\rho^{-(m+n+1)}\left|C_{\boldsymbol{p}}\right|^{m+n+1}(I - \rho^{-1}\left|C_{\boldsymbol{p}}\right|)^{-1}e_1.$$

Therefore if we have $\boldsymbol{p}(z)$, then Theorem 5.3.3 implies that $w(z) \in \boldsymbol{w}(z)$.

Based on the result of $\boldsymbol{w}(z)$, by Theorem 5.3.1 and from the definition of circular complex disk, $\boldsymbol{p}^{(k)}(z)$ can be calculated by the following equation such that $p^*(z) \in \boldsymbol{p}^{(k)}(z)$.

$$\boldsymbol{p}^{(k)}(z) = p(z) + s(z) - \mathrm{mod}\Big(\frac{s(z)(t(z) - t^{(k)}(z)) - \boldsymbol{w}(z)}{q(z) + t(z)}, \boldsymbol{p}(z)\Big). \tag{5.28}$$

Where $s(z)$, $t(z)$ are calculated by (5.11).

In order to apply Theorem 5.3.3, the validated $m+n$ degree Taylor polynomial of $f(z)$ is needed in the computation. Moreover, a disk around the origin with radius ρ in which $f(z)$ is analytic and the upper bound M of the absolute value of the Taylor coefficients are needed in the computation. In [64], M and ρ were assumed to be given in the computation. In this chapter, we use circular complex arithmetic to compute M and ρ.

From Cauchy's inequality [20], we have the following theorem.

Theorem 5.3.4 *Let ρ be a positive real number and $1 < \rho < R$. Let $D = \{0\,;\rho\}$ be a disk centered at the origin with a radius of ρ. Let*

$$f(z) = \sum_{j=0}^{\infty} c_j z^j.$$

If $f(z)$ is analytic in D, then

$$|c_j| \le \frac{M_\rho}{\rho^j}, \quad j \ge 0,$$

where $M_\rho := \max_{|z|=\rho} |f(z)|$.

From Theorem 5.3.4, we have

$$h(z) = \sum_{k=m+n+1}^{\infty} c_k z^k, \quad |c_k| \le \frac{M_\rho}{\rho^k}, \quad k \ge m+n+1.$$

Therefore Theorem 5.3.3 can be applied by setting $M = M_\rho$.

5.3.3 Determination of the Analyticity of Analytic Functions

Since function $f(z)$ can be extended in the Taylor series on the disk in which $f(z)$ is analytic, in order to apply Theorem 5.3.3 and 5.3.4, a real number ρ which is smaller than the analytic radius R of $f(z)$ should be calculated. This can be done by using circular complex arithmetic.

Since the distribution of poles of elementary functions are well known, the analyticity of the elementary functions in input disks can be determined. In the case that $f(z)$ can be written as an expression of arithmetic operations of elementary functions, if the input disk does not contain poles of any elementary functions of $f(z)$, then $f(z)$ is analytic in the input disk.

Based on this principle, a circular interval arithmetic system Cdomain which has the function of determining the analyticity of a function in a disk was developed by Sugiura and Katou [72]. Cdomain deals with a complex disk and a flag as a pair in computation, the flag is used to indicate the analyticity of the computational results. Thus the analyticity of the results can be determined while circular arithmetic operations are carrying out. The definition of complex disks and arithmetic operations on the disks are performed using interval arithmetic system INTLAB [61, 62], and the function to determine the analyticity of computed results is implemented. This system can be used to determine the analyticity of a function on a disk.

5.3.4 Validated Computation of the Maximum Value of $|f(z)|$ on a Circle

From Theorem 5.3.4, in order to calculate the upper bound of the absolute value of Taylor coefficient of $f(z)$, the computation of the maximum value of $|f(z)|$ on a circle is needed. In this section, we show a method to calculate the maximum absolute value of $f(z)$ on a circle $\Gamma : \{z : |z| = \rho\}$ centered at the origin with a radius $\rho < R$. In [13, 39], rectangular complex interval arithmetic was used in the computation. In this section, we use circular complex arithmetic.

Let $F(Z)$ be the interval extension of $f(z)$. Let $\overline{F}(Z)$ denote the upper bound of $F(Z)$ and be calculated by

$$\overline{F}(Z) = |\mathrm{mid}(F(Z))| + \mathrm{rad}(F(Z)). \tag{5.29}$$

Let $\underline{F}(Z)$ denote the lower bound of $F(z)$ and be calculated by

$$\underline{F}(Z) = \begin{cases} |\mathrm{mid}(F(Z))| - \mathrm{rad}(F(Z)), & \text{if } |\mathrm{mid}(F(Z))| > \mathrm{rad}(F(Z)). \\ 0, & \text{if } |\mathrm{mid}(F(Z))| \leq \mathrm{rad}(F(Z)). \end{cases} \tag{5.30}$$

Then

$$\underline{F}(Z) \leq |f(z)| \leq \overline{F}(Z), \quad z \in Z. \tag{5.31}$$

The maximum absolute value of $f(z)$ on Γ can be calculated by the following method. First we divide Γ into l parts $\Gamma_1, \dots, \Gamma_l$, then cover these l parts with l disks $Z_1, \dots, Z_l$ centered at the center points $z_1, \dots, z_l$ of these arcs. Next we calculate the interval extension $F(Z_1), \dots, F(Z_l)$ of $f(z)$ on these disks. The upper bound $\overline{F}(Z_1), \dots, \overline{F}(Z_l)$ and lower bound $\underline{F}(Z_1), \dots, \underline{F}(Z_l)$ of $F(Z_1), \dots, F(Z_l)$ are computed by (5.29) and (5.30), respectively. From the fundamental property of interval arithmetic, $|f(z)|$ is located between the upper bounds and the lower bounds of $F(Z_1), \dots, F(Z_l)$ as shown in Fig. 1.

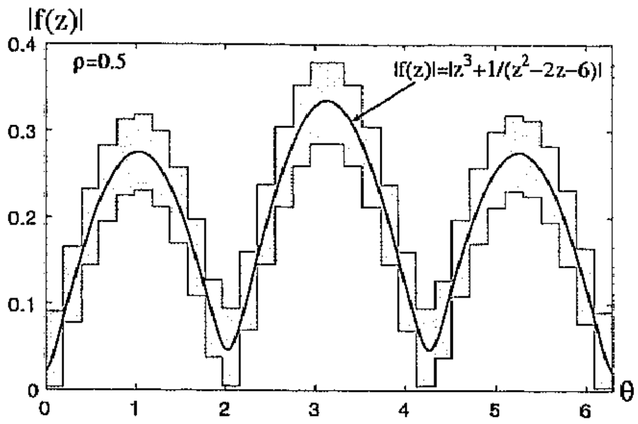

Fig. 1 Computation of maximum value of $|f(z)|$.

Let $\overline{F}_{\max} := \max_{1\le i\le l}(\overline{F}(Z_i))$, then $\overline{F}_{\max}$ is a validated upper bound of the maximum value of $|f(z)|$. Let $\boldsymbol{f}(z_1),\ldots,\boldsymbol{f}(z_l)$ be the disks containing $f(z_1),\ldots,f(z_l)$, respectively and be calculated by

$$\boldsymbol{f}(z_i) = F(\{z_i\,;0\}), \quad i = 1,\ldots,l.$$

Then $f(z_i) \in \boldsymbol{f}(z_i)$. Let $\underline{\boldsymbol{f}}_{\max}$ be the maximum value of the lower bound of $\boldsymbol{f}(z_i)$: $\underline{\boldsymbol{f}}_{\max} := \max_{1\le i\le l}(\underline{\boldsymbol{f}}(z_i))$. Since the disks $F(Z_i)$ that satisfy $\overline{F}(Z_i) < \underline{\boldsymbol{f}}_{\max}$ do not contain the maximum value of $f(z)$, the disks Z_i are removed from all disks and the procedure described above is repeated on the arcs contained in the remaining disks until a desired accuracy is obtained.

5.3.5 Validated Computation of Taylor Coefficients

From (5.28), in order to calculate the set of polynomials $p^{(k)}(z)$ which contains the polynomial factor $p^*(z)$ of $f(z)$, the coefficients of $m+n$ degree Taylor polynomial of $f(z)$ should be calculated. In this section, we present a verified method to compute the Taylor coefficients of a function using circular complex arithmetic.

Let $f(z)$ be analytic in disk $Z = \{0\,;\rho\}$, $1 \le \rho < R$, then $f(z)$ has the following Taylor expansion.

$$f(z) = \sum_{k=0}^{\infty} c_k z^k, \quad z \in Z.$$

The coefficient c_k can be calculated by the following integrals.

$$c_k = \frac{1}{2\pi i}\int_{|z|=1} f(z)z^{-(k+1)}dz, \quad k = 0, 1, \ldots.$$

By setting $z = e^{it}$, we have

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}f(e^{it})dt, \quad k = 0, 1, \ldots.$$

This implies that c_k is the Fourier coefficient of $f(e^{it})$ and can be calculated by FFT. Let $\tilde{c}_k$ be the approximate value of c_k calculated by FFT, then

$$\tilde{c}_k - c_k = \sum_{l=1}^{\infty} c_{k+lN},$$

where N is the number of sampling points of FFT. From Theorem 5.3.4, the error of $\tilde{c}_k$ can be calculated by

$$|\tilde{c}_k - c_k| \le M_\rho \frac{\rho^{-N-k}}{1-\rho^{-N}}, \quad M_\rho = \max_{|z|=\rho} |f(z)|. \tag{5.32}$$

Let $\tilde{\boldsymbol{c}}_k$ be interval extension of $\tilde{c}_k$ calculated using circular arithmetic. From (5.32), the disk $\boldsymbol{c}_k$ which contains c_k can be calculated by

$$\mathrm{mid}(\boldsymbol{c}_k) = \mathrm{mid}(\tilde{\boldsymbol{c}}_k), \tag{5.33}$$

$$\mathrm{rad}(\boldsymbol{c}_k) = \mathrm{rad}(\tilde{\boldsymbol{c}}_k) + M_k, \quad M_k = M_\rho \frac{\rho^{-N-k}}{1-\rho^{-N}}, \quad k = 0, 1, \ldots. \tag{5.34}$$

Therefore the validated Taylor coefficients c_k of $f(z)$ can be calculated. If $\mathrm{rad}(\boldsymbol{c}_k)$ is sufficiently small, then $\mathrm{mid}(\boldsymbol{c}_k)$ can be regarded as a good approximation to c_k.

5.4 Algorithm

From the above discussions, we give the following algorithm to calculate an interval coefficient polynomial which contains a polynomial factor of $f(z)$. We assume that the analytic radius of the function is larger than 1, $R > 1$, this condition can be satisfied by scaling. Suppose that the radius δ of a disk around the origin which includes m zeros of $f(z)$ is given. For two disks Z_1 and Z_2, $Z_1 \cap Z_2$ returns a disk contains the common part of Z_1 and Z_2.

Algorithm

Input: N, δ, m, n, $\rho(\rho > 2m\delta)$, ε, j_{max}, k_{max}.

Output: $\boldsymbol{p}^{(k)}(z)$.

1. Determine the analyticity of $f(z)$ in a disk centered at the origin and of radius ρ.
2. Compute the maximum absolute value M_ρ of $f(z)$ on the circle $|z| = \rho$.
3. Compute $\{\boldsymbol{c}_k\}_{k=0}^{m+n}$ by (5.33), (5.34).
4. Compute $t^{(j)}(z)$ using floating point arithmetic.

$p(z) = z^m$

$q(z) = \sum_{k=m}^{m+n} \mathrm{mid}(\boldsymbol{c}_k) z^{k-m}, \quad r(z) = \sum_{k=0}^{m-1} \mathrm{mid}(\boldsymbol{c}_k) z^k$

$s^{(0)} = 0, \ t^{(0)} = 0$

for $j = 1, 2, \ldots, j_{max}$

  Compute $s^{(j)}(z)$, $t^{(j)}(z)$ by (5.6) and (5.7).

  if $\| s^{(j)}(z) - s^{(j-1)}(z) \| \leq \varepsilon$ **then** exit for loop

end for

5. Compute $\boldsymbol{p}^{(k)}(z)$ using circular complex arithmetic.

$\boldsymbol{p}^{(0)}(z) = (z - \{0\,;\delta\})^m$

$\boldsymbol{q}(z) = \sum_{k=m}^{m+n} \boldsymbol{c}_k z^{k-m}, \quad \boldsymbol{r}(z) = \sum_{k=0}^{m-1} \boldsymbol{c}_k z^k$

Compute $\boldsymbol{s}(z)$ and $\boldsymbol{t}(z)$ such that

$\boldsymbol{s}(z)(\boldsymbol{q}(z) + t^{(j)}(z)) + \boldsymbol{t}(z)p(z) = \boldsymbol{r}(z)$

for $k = 1, 2, \ldots, k_{max}$

  $\mathrm{mid}(\hat{\boldsymbol{w}}) = (0, \ldots, 0)^T$

  $\mathrm{rad}(\hat{\boldsymbol{w}}) = \rho^{-(m+n+1)} |C_{\boldsymbol{p}^{(k-1)}}|^{m+n+1} M_\rho (I - \rho^{-1} |C_{\boldsymbol{p}^{(k-1)}}|)^{-1} e_1$

  $\boldsymbol{u}_k(z) = \mathrm{mod}\Big(\dfrac{\boldsymbol{s}(z)(\boldsymbol{t}(z) - t^{(j)}(z)) - \boldsymbol{w}(z)}{\boldsymbol{q}(z) + \boldsymbol{t}(z)}, \boldsymbol{p}^{(k-1)}(z) \Big)$

  $\boldsymbol{p}^{(k)}(z) = (p(z) + \boldsymbol{s}(z) - \boldsymbol{u}_k(z)) \cap \boldsymbol{p}^{(k-1)}(z)$

end for

Where N is the number of sampling points of FFT, ε is a value to determine the convergence of the iteration. The computation of $\tilde{\boldsymbol{c}}_k$ in step 3 is performed using a verified FFT algorithm. j_{max} and k_{max} are the maximum number of iterations in step 4 and step 5, respectively. If $\rho > 2m\delta$, then the prerequisite of Theorem 5.3.3 is satisfied, therefore the coefficient disks of the polynomial calculated by the above algorithm contains the true coefficients of the polynomial factor.

5.5 Numerical Examples

Numerical examples are calculated in Matlab Ver. 6.5 with interval arithmetic package INTLAB [61, 62] which provides circular arithmetic facilities for Matlab. In order to perform a validated computation of the radius of $\hat{w}$ by (5.27), function verifylss is used to perform a validated computation of the inverse matrix $(I - \rho^{-1}|C_{\boldsymbol{p}}|)^{-1}$. The analyticity of function is determined by Cdomain circular arithmetic system [72]. The verified FFT is implemented in Matlab using INTLAB interval arithmetic system.

Example 5.1. Let

$$p^*(z) = (z - 0.01)^2(z - 0.02),$$
$$q^*(z) = (z - 3)(z - 9)(z - 15)\exp(\frac{1}{3}z^2 + 2).$$

Parameters were $m = 3$, $\delta = 0.1$, $\varepsilon = 10^{-10}$. The maximum absolute value of $f(z)$ on a circle centered at the origin and of radius ρ was calculated. Let $\tilde{M} := M_\rho/(m + n + 1)$. The results of M_ρ, $\tilde{M}$ and the analyticity of function for different ρ are shown in Table 5.1. The center mid(c_k) and radius rad(c_k) of validated Taylor coefficients c_k of $f(z)$ are shown in Table 5.2. Underlines show the difference with the results calculated in Mathematica Ver. 4.0 using multiple precision arithmetic with 32-digit precision. N is the number of sampling points of FFT.

Table 5.1 Maximum value of $|f(z)|$ and analyticity of $f(z)$.

ρ	M_ρ	$\tilde{M}(n = 10)$	$\tilde{M}(n = 16)$	Analyticity
2	2.1×10^5	1.3×10^1	2.0×10^{-1}	yes
4	1.7×10^8	6.3×10^{-1}	1.5×10^{-4}	yes
6	7.4×10^{11}	9.4×10^0	2.0×10^{-4}	yes
8	3.0×10^{16}	6.8×10^3	2.6×10^{-2}	yes
16	5.1×10^{45}	7.1×10^{28}	4.2×10^{21}	yes

Table 5.2 Taylor coefficients of $f(z)$. $(N = 32)$

k	$\mathrm{mid}(c_k)$	$\mathrm{rad}(c_k)$
0	$5.985135440\underline{393919} \times 10^{-3}$	1.1×10^{-11}
1	$-1.499342929258\underline{345} \times 10^{0}$	5.4×10^{-12}
2	$1.2046987016309\underline{26} \times 10^{2}$	2.2×10^{-12}
3	$-3.054348652577926 \times 10^{3}$	8.0×10^{-12}
4	$1.57767477847371\underline{3} \times 10^{3}$	4.0×10^{-12}
⋮	⋮	⋮
20	$6.499284\underline{877224497} \times 10^{-6}$	3.8×10^{-12}

From Table 5.1, we see that for large ρ, $\tilde{M}$ is also large. We set parameters to $\rho = 4$, $n = 16$ so that $\tilde{M}$ is small. The result after 2 iterations is shown below.

$$\begin{aligned} \boldsymbol{p}^{(2)}(z) &= z^3 - \{4.0000000000000 \times 10^{-2}\,; 4.75 \times 10^{-15}\}z^2 \\ &\quad + \{4.999999999999\underline{3} \times 10^{-4}\,; 3.91 \times 10^{-15}\}z \\ &\quad - \{2.0000000000\underline{869} \times 10^{-6}\,; 3.74 \times 10^{-15}\}. \end{aligned}$$

Let $\mathrm{err}(\boldsymbol{p}^{(k)}(z)) = (\delta a_m, \ldots, \delta a_0)^T$ denote the absolute value of the difference between $\mathrm{mid}(\boldsymbol{p}^{(k)}(z))$ and the coefficients of $p^*(z)$, where δa_k denotes the absolute value of the difference of the coefficient of z^k. We have the result

$$\mathrm{err}(\boldsymbol{p}^{(2)}(z)) = (0, 1.84 \times 10^{-16}, 6.80 \times 10^{-17}, 8.69 \times 10^{-17})^T.$$

This implies that $p^*(z)$ is included in $\boldsymbol{p}^{(2)}(z)$.

Example 5.2 Let

$$\begin{aligned} p^*(z) &= z^4 + 0.05z^3 + 0.05^2 z^2 + 0.05^3 z + 0.05^4, \\ q^*(z) &= (z-4)(z-4.01)(z-6)(z-6.01)(z-8) \\ &\quad \times \exp(2z-1)/(\tfrac{1}{2}z^2 - 3z + 7). \end{aligned}$$

Parameters were $m = 4$, $\delta = 0.1$, $\varepsilon = 10^{-10}$. The results of the maximum absolute value of $f(z)$ on a circle and the analyticity of $f(z)$ are shown in Table 5.3. The center $\mathrm{mid}(\boldsymbol{c}_k)$ and radius $\mathrm{rad}(\boldsymbol{c}_k)$ of the validated Taylor coefficient $\boldsymbol{c}_k$ of $f(z)$ are shown in Table 5.4.

Table 5.3 Maximum value of $|f(z)|$ and analyticity of $f(z)$.

ρ	M_ρ	$\tilde{M}(n=17)$	$\tilde{M}(n=22)$	Analyticity
2	4.7×10^4	1.1×10^{-2}	3.5×10^{-4}	yes
3	1.3×10^6	4.1×10^{-5}	1.7×10^{-7}	yes
4	4.5×10^7	2.6×10^{-6}	2.5×10^{-9}	yes
5	–	–	–	no

Table 5.4 Taylor coefficients of $f(z)$. $(N = 32)$

k	$\mathrm{mid}(\boldsymbol{c}_k)$	$\mathrm{rad}(\boldsymbol{c}_k)$
0	$-1.519873939\underline{435534} \times 10^{-3}$	2.8×10^{-12}
1	$-3.2633424416\underline{31641} \times 10^{-2}$	1.1×10^{-12}
2	$-6.54195421315\underline{7634} \times 10^{-1}$	4.9×10^{-13}
3	$-1.3084513620281\underline{32} \times 10^{1}$	4.0×10^{-13}
4	$-2.616903979665127 \times 10^{2}$	6.6×10^{-13}
$\vdots$	$\vdots$	$\vdots$
27	$-7.86\underline{2036577321874} \times 10^{-10}$	3.5×10^{-13}

The result for $\rho = 4,\ n = 22$ after 2 iterations is shown below.

$$\begin{aligned} \boldsymbol{p}^{(2)}(z) &= z^4 + \{4.9999999999\underline{89} \times 10^{-2}; 1.21 \times 10^{-10}\} z^3 \\ &\quad + \{2.49999999999\underline{0} \times 10^{-3}; 1.25 \times 10^{-11}\} z^2 \\ &\quad + \{1.2500000000\underline{59} \times 10^{-4}; 7.19 \times 10^{-13}\} z \\ &\quad + \{6.24999999\underline{5190} \times 10^{-6}; 2.79 \times 10^{-14}\}. \end{aligned}$$

The errors of the coefficients are

$$\mathrm{err}(\boldsymbol{p}^{(2)}(z)) = (0, 1.11 \times 10^{-13}, 1.04 \times 10^{-14}, 5.91 \times 10^{-15}, 4.81 \times 10^{-15})^T.$$

This implies that $p^*(z)$ is included in $\boldsymbol{p}^{(2)}(z)$.

Example 5.3 Let

$$p^*(z) = (z - 10^{-3})(z + 10^{-3}/2)(z - 10^{-3}/4)$$

$$= z^3 - 7.5 \times 10^{-4} z^2 - 3.75 \times 10^{-7} z + 1.25 \times 10^{-10},$$

$$q^*(z) = \prod_{k=1}^{5} (z-k)\log(z+6).$$

Parameters were $m = 3$, $\delta = 0.01$, $\varepsilon = 10^{-10}$. The results of the maximum absolute value of $f(z)$ on a circle and the analyticity of $f(z)$ are shown in Table 5.5. The center $\mathrm{mid}(c_k)$ and radius $\mathrm{rad}(c_k)$ of the validated Taylor coefficient c_k of $f(z)$ are shown in Table 5.6.

Table 5.5 Maximum value of $|f(z)|$ and analyticity of $f(z)$.

ρ	M_ρ	$\tilde{M}(n=10)$	$\tilde{M}(n=16)$	Analyticity
2	2.8×10^4	1.7×10^0	2.7×10^{-2}	yes
3	2.1×10^5	4.4×10^{-2}	6.0×10^{-5}	yes
4	1.1×10^6	4.1×10^{-3}	1.0×10^{-6}	yes
5	4.9×10^6	8.0×10^{-4}	5.1×10^{-8}	yes
6	–	–	–	no

Table 5.6 Taylor coefficients of $f(z)$. ($N = 32$)

k	$\mathrm{mid}(c_k)$	$\mathrm{rad}(c_k)$
0	$-2.687639\underline{766296704} \times 10^{-8}$	4.4×10^{-13}
1	$8.0688043\underline{91208366} \times 10^{-5}$	5.4×10^{-13}
2	$1.61081704468\underline{5495} \times 10^{-1}$	3.4×10^{-13}
3	$-2.15364209434736\underline{8} \times 10^{2}$	6.8×10^{-13}
4	$4.7120891242447\underline{20} \times 10^{2}$	9.4×10^{-13}
$\vdots$	$\vdots$	$\vdots$
20	$-2.42\underline{4940248602070} \times 10^{-10}$	9.4×10^{-13}

The result for $\rho = 5$, $n = 16$ after 1 iteration is shown below.

$$\begin{aligned} p^{(1)}(z) &= z^3 - \{7.499999\underline{988071} \times 10^{-4}; 5.51 \times 10^{-11}\} z^2 \\ &\quad - \{3.749999\underline{995093} \times 10^{-7}; 5.47 \times 10^{-13}\} z \\ &\quad + \{1.250000\underline{259605} \times 10^{-10}; 3.85 \times 10^{-15}\}. \end{aligned}$$

The errors of the coefficients are

$$\mathrm{err}(\boldsymbol{p}^{(1)}(z)) = (0, 1.19 \times 10^{-12}, 4.91 \times 10^{-16}, 2.60 \times 10^{-17})^T$$

This implies that $p^*(z)$ is included in $\boldsymbol{p}^{(1)}(z)$.

5.6 Conclusion

In this chapter, the analyticity of an analytic function $f(z)$ was calculated by using circular arithmetic, and the upper bound of the maximum absolute value of the function on a circle is calculated. Based on these results, the validated computation of the Taylor coefficeints of $f(z)$ was performed and a verified polynomial factor of $f(z)$ was calculated. The effectiveness of the method proposed in this chapter has been illustrated with numerical examples.