Chapter 4

A High Precision Companion Matrix Method to Find Multiple Zeros of Polynomials

4.1 Introduction

As described in the previous chapter, the zeros of a polynomial can be calculated by computing the eigenvalues of the associated companion matrix. Stable and efficient method such as the QR method [18] can be used to calculate the eigenvalues accurately. Moreover, it is easy to validate the calculated zeros by using Gerschgorin's theorem [71].

One problem of such methods is that a small perturbation in the elements of a companion matrix may produce significant changes in the eigenvalues of the companion matrix calculated by the QR method. Fortune [16] proposed a method to compute the zeros of polynomials using Smith's companion matrix. This method calculate the companion matrix using multiple precision arithmetic and then calculate the eigenvalues of the companion matrix using floating point arithmetic. In this way, the increase of overall computing time can be contained. However, the method is not efficient in the case of multiple or closed zeros.

In the previous chapter we presented a new companion matrix to compute multiple zeros of polynomials. In this chapter, we show a method to compute the companion matrix to high precision, and then use the companion matrix to compute zeros of polynomials which have multiple zeros. A three-term recurrence algorithm based on Euclid's algorithm is used to calculate the values of the polynomial. This method can calculate the multiple zeros and their multiplicities accurately, and it is also efficient in calculating the center of the cluster of zeros and the number of zeros in the cluster. Numerical examples are presented to illustrate the efficiency of the method.

4.2 Companion Matrix Methods

4.2.1 Smith's Companion Matrix for Polynomials with Only Simple Zeros

Let $f(z)$ be a monic polynomial of degree n,

$$\begin{aligned} f(z) &:= z^n + a_{n-1}z^{n-1} + \cdots + a_0 \\ &= \prod_{k=1}^{m}(z-\xi_k)^{\nu_k}, \end{aligned} \tag{4.1}$$

where $\xi_1, \ldots, \xi_m$ are m distinct zeros of $f(z)$. ν_k is the multiplicity of ξ_k, $\sum_{k=1}^{m}\nu_k = n$.

If all the zeros of a polynomial are simple, then by a similarity transformation on the matrix calculated by setting $m = n$, $\nu_1 = \cdots = \nu_n = 1$ in the Smith's theorem 3.2.1, the following theorem can be obtained immediately [78].

Theorem 4.2.1 *Let all the zeros of polynomial* (4.1) *are simple zeros. Let* $z_1, \ldots, z_n \in \mathbb{C}$ *be approximations to the distinct zeros* $\xi_1, \ldots, \xi_n$ *of* $f(z)$ *and* $f(z_k) \neq 0$, $k = 1, \ldots, n$. *Then the zeros of* $f(z)$ *are the eigenvalues of the following matrix.*

$$W = \begin{pmatrix} z_1 - \frac{f(z_1)}{q'(z_1)} & -\frac{f(z_2)}{q'(z_2)} & \cdots & -\frac{f(z_n)}{q'(z_n)} \\ -\frac{f(z_1)}{q'(z_1)} & z_2 - \frac{f(z_2)}{q'(z_2)} & \cdots & -\frac{f(z_n)}{q'(z_n)} \\ \vdots & \vdots & & \vdots \\ -\frac{f(z_1)}{q'(z_1)} & -\frac{f(z_2)}{q'(z_2)} & \cdots & z_n - \frac{f(z_n)}{q'(z_n)} \end{pmatrix}, \tag{4.2}$$

where $q(z) = \prod_{k=1}^{n}(z - z_k)$.

Smith's method construct the companion matrix using the values of polynomials. If the approximate zeros are close to zeros, then companion matrix tends to a diagonal matrix. Smith's method can be used iteratively by using the calculated results as the new approximations.

4.2.2 High Precision Companion Matrix Methods

If the elements of companion matrix subject to errors in computation, the accuracy of the eigenvalues calculated by the QR method will be decreased [75, 76]. For this reason, a number of methods for finding polynomial zeros which use a high precision companion matrix were proposed. Fortune [16] proposed an iterative method based on Smith's companion matrix to calculate the polynomial zeros. This method computes the companion matrix using multiple precision arithmetic, then performs a floating-point eigenvalue computation of the matrix using the QR method, The computed eigenvalues are then used as the new approximate zeros. This process is used iteratively to compute the zeros of a polynomial to high precision. Fortune illustrated the convergency of the method and the efficiency of it in computing the zeros of polynomial of high degree. However, the method is not efficient in computing the multiple or close zeros of polynomials. Fortune's method is shown as below.

Algorithm 4.1(Fortune)

Input: approximate zeros $z_1, \cdots, z_n$.

(1) Compute $f(z_k)$, $k = 1, \ldots, n$ using multiple precision arithmetic.

(2) Compute the companion matrix W by (4.2).

(3) Compute the eigenvalues of W using the QR method in floating point arithmetic.

Since the most complicated computing processing—eigenvalue computation which needs a large number of arithmetic operations, is performed using floating point arithmetic, the increase of overall computing time can be contained.

Malek and Vaillancourt [31, 30] proposed a similar iterative method based on Fiedler's companion matrix [15]. This method uses multiple precision arithmetic to compute the companion matrix, then performs a floating point eigenvalue computation. Malek et al. illustrated the efficiency of the method in computing the zeros of polynomials of small to moderate degree. However, like Fortune's method, this method is not efficient in computing multiple or close zeros of polynomials.

4.3 Finding Multiple Zeros of Polynomials

4.3.1 Reduction to Computing Simple Zeros of a Polynomial

In the case that the polynomial $f(z)$ has multiple zeros, if we can find a new polynomial which has only simple zeros that are the distinct zeros of $f(z)$, then the zeros can be calculated by constructing a companion matrix of the new polynomial and then computing the eigenvalues of the companion matrix. Since the companion matrix has only simple eigenvalues, the eigenvalues will be calculated accurately. In this section, we show a method to compute the multiple zeros of $f(z)$ and their multiplicities separately. First we compute a polynomial $\varphi(z)$ which has simple zeros $\xi_1, \ldots, \xi_m$ that are the distinct zeros of $f(z)$. Then we compute the zeros of $\varphi(z)$ by computing the eigenvalues of its companion matrix. After that we calculate the multiplicities of the zeros using the calculated zeros. We assume that all the zeros of $f(z)$ are located inside the unit circle C in the complex plane. In the general cases, a circle which contains all the zeros can be obtained by using Aberth's initial value [1], and the zeros can be located in the unit circle by shifting and scaling to the origin point.

Consider the following contour integral μ_p.

$$\mu_p := \frac{1}{2\pi i}\int_C z^p \frac{f'(z)}{f(z)} dz, \; p = 0, 1, 2, \ldots. \tag{4.3}$$

From the residue theorem, we have

$$\mu_p = \sum_{k=1}^{m} \nu_k \xi_k^p, \; p = 0, 1, 2, \ldots. \tag{4.4}$$

Let H_m be the $m \times m$ Hankel matrix

$$H_m := \left(\mu_{p+q}\right)_{p,q=0}^{m-1} = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{m-1} \\ \mu_1 & \ddots & & \vdots \\ \vdots & & \ddots & \mu_{2m-3} \\ \mu_{m-1} & \cdots & \cdots & \mu_{2m-2} \end{pmatrix}. \tag{4.5}$$

From Lemma 3.3.1, H_m is nonsingular.

Let $\varphi(z)$ be the m degree monic polynomial

$$\varphi(z) := z^m + b_{m-1}z^{m-1} + \cdots + b_0, \tag{4.6}$$

where the coefficients of $\varphi(z)$ are calculated by the following equation.

$$\begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{m-1} \\ \mu_1 & \ddots & & \vdots \\ \vdots & & \ddots & \mu_{2m-3} \\ \mu_{m-1} & \cdots & \cdots & \mu_{2m-2} \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{m-1} \end{pmatrix} = - \begin{pmatrix} \mu_m \\ \mu_{m+1} \\ \vdots \\ \mu_{2m-1} \end{pmatrix}. \tag{4.7}$$

Since H_m is nonsingular, $b_0, b_1, \ldots, b_{m-1}$ are uniquely determined by (4.7). We have the following theorem.

Theorem 4.3.1 *Let m degree monic polynomial $\varphi(z)$ be defined by (4.6) and (4.7), and let $\xi_1, \ldots, \xi_m$ be m distinct zeros of $f(z)$. Then*

$$\varphi(\xi_k) = 0, \ k = 1, \ldots, m.$$

Proof. Let $\boldsymbol{b}_m$ and $\boldsymbol{\xi}_m$ be the vectors

$$\boldsymbol{b}_m = (b_0, \ldots, b_{m-1})^T, \tag{4.8}$$

$$\boldsymbol{\xi}_m = (\xi_1^m, \ldots, \xi_m^m)^T. \tag{4.9}$$

It follows from (3.22) and (4.4) that (4.7) can be factorized as follows:

$$V_m D_m V_m^T \boldsymbol{b}_m = -V_m D_m \boldsymbol{\xi}_m.$$

Since V_m and D_m are nonsingular,

$$V_m^T \boldsymbol{b}_m = -\boldsymbol{\xi}_m. \tag{4.10}$$

From $\boldsymbol{\xi}_m + V_m^T \boldsymbol{b}_m = 0$, we have

$$\varphi(\xi_k) = 0, \ k = 1, \ldots, m.$$

This proves the theorem. □

From theorem 4.3.1, $\varphi(z)$ and $f(z)$ have the same zeros. Therefore, the problem of computing the zeros of $f(z)$ is reduced to the problem of computing the zeros of $\varphi(z)$. Since $\varphi(z)$ has only simple zeros, the associated companion matrix has only simple eigenvalues.

The following theorem can be easily proved using the results of Theorem 4.3.1.

Theorem 4.3.2 *Let $z_1, \ldots, z_m \in \mathbb{C}$ be the m approximate zeros of the distinct zeros $\xi_1, \ldots, \xi_m$ of $f(z)$, $f(z_k) \neq 0$. Let $\varphi(z)$ be the m degree polynomial defined by (4.6) and (4.7). Then the zeros of $f(z)$ are the eigenvalues of the following matrix.*

$$N = \begin{pmatrix} z_1 - \frac{\varphi(z_1)}{q'(z_1)} & -\frac{\varphi(z_2)}{q'(z_2)} & \cdots & -\frac{\varphi(z_m)}{q'(z_m)} \\ -\frac{\varphi(z_1)}{q'(z_1)} & z_2 - \frac{\varphi(z_2)}{q'(z_2)} & \cdots & -\frac{\varphi(z_m)}{q'(z_m)} \\ \vdots & \vdots & & \vdots \\ -\frac{\varphi(z_1)}{q'(z_1)} & -\frac{\varphi(z_2)}{q'(z_2)} & \cdots & z_m - \frac{\varphi(z_m)}{q'(z_m)} \end{pmatrix}, \tag{4.11}$$

where $q(z) = \prod_{i=1}^m (z - z_i)$.

4.3.2 Euclid's Algorithm

Euclid's algorithm is a method to find the greatest common divisor(GCD) of two positive integers. It can be extended easily to find the greatest common divisor of two polynomials [2, 67].

Theorem 4.3.3 (Euclid) *For two polynomials $P_1(z)$ and $P_2(z)$ with* $\deg(P_1(z)) \geq \deg(P_2(z))$ *let $P_3(z)$ and $Q(z)$ be the remainder polynomial and quotient polynomial of $P_1(z)$ divided by $P_2(z)$, respectively:*

$$P_1(z) = Q(z)P_2(z) + P_3(z).$$

If $P_1(z)$ and $P_2(z)$ are primitive, then

$$\mathrm{GCD}(P_1(z), P_2(z)) = \begin{cases} \mathrm{GCD}(P_2(z), P_3(z)), & \text{if } P_3(z) \neq 0, \\ P_2(z), & \text{if } P_3(z) = 0. \end{cases} \tag{4.12}$$

The following extended Euclid's theorem can be obtained from Theorem 4.3.3.

Theorem 4.3.4 *Let $G(z)$ be the* GCD *of polynomials $P_1(z)$ and $P_2(z)$, then the polynomials $A(z)$ and $B(z)$ that satisfy the following equation are uniquely determined.*

$$\begin{aligned} &A(z)P_1(z) + B(z)P_2(z) = G(z), \\ &\deg(A(z)) < \deg(P_2(z)) - \deg(G(z)), \deg(B(z)) < \deg(P_1(z)) - \deg(G(z)). \end{aligned} \tag{4.13}$$

For $i \geq 1$, Let sequences of polynomials $Q_i(z)$ and $P_{i+2}(z)$ be the quotient and remainder polynomial of $P_i(z)$ divided by $P_{i+1}(z)$, respectively,

$$P_i(z) = Q_i(z)P_{i+1}(z) + P_{i+2}(z), \quad \deg(P_{i+1}(z)) < \deg(P_i(z)). \tag{4.14}$$

The polynomials $A_i(z)$ and $B_i(z)$ are defined by

$$A_i(z) = A_{i-2}(z) - Q_i(z)A_{i-1}(z), \tag{4.15}$$

$$B_i(z) = B_{i-2}(z) - Q_i(z)B_{i-1}(z), \tag{4.16}$$

where the initial conditions are

$$A_{-1}(z) = 1, \ A_0(z) = 0, \ B_{-1}(z) = 0, \ B_0(z) = 1.$$

Then the following equation holds [67].

$$A_i(z)P_1(z) + B_i(z)P_2(z) = P_{i+2}(z), \quad i = 1, 2, \ldots.$$

Since the degrees of the remainder $P_{i+2}(z)$ are strictly decreasing, there will be a last nonzero one, we call it $P_{n+2}(z)$. Then from Theorem 4.3.3, the GCD of $P_1(z)$ and $P_2(z)$ is $P_{n+2}(z)$, and the following equation holds.

$$A_n(z)P_1(z) + B_n(z)P_2(z) = P_{n+2}(z).$$

The above method to compute $A_n(z)$ and $B_n(z)$ that satisfy (4.13) is called the extended Euclid's algorithm.

4.3.3 Computation of $\varphi(z)$ by Euclid's Algorithm

In order to calculate the zeros of $f(z)$ using (4.11), in the following, we consider the problem of computing the values of $\varphi(z)$. Let polynomial $F_0(z)$, $F_1(z)$ be

$$\begin{aligned} F_0(z) &:= z^{2m}, \\ F_1(z) &:= \mu_0 z^{2m-1} + \mu_1 z^{2m-2} + \cdots + \mu_{2m-1}. \end{aligned}$$

The following theorem shows that the values of $\varphi(z)$ can be calculated by using the extended Euclid's algorithm.

Theorem 4.3.5 *For polynomials $F_0(z)$ and $F_1(z)$, the polynomials $\psi(z)$ and $\chi(z)$ of degree no more than $m-1$ and the monic polynomial $\phi(z)$ of degree m that satisfy the following equation are uniquely determined.*

$$\psi(z)F_0(z) + \phi(z)F_1(z) = \chi(z). \tag{4.17}$$

Moreover, the polynomial $\phi(z)$ is the same as the polynomial $\varphi(z)$ calculated by (4.6) *and* (4.7).

Proof. Let $m-1$ degree polynomial $\psi(z)$ and m degree monic polynomial $\phi(z)$ be

$$\begin{aligned} \psi(z) &:= \alpha_{m-1}z^{m-1} + \alpha_{m-2}z^{m-2} + \cdots + \alpha_0, \\ \phi(z) &:= z^m + \beta_{m-1}z^{m-1} + \cdots + \beta_0. \end{aligned}$$

Then for $F_0(z)$ and $F_1(z)$, the equation (4.17) is a polynomial of degree no more than $m-1$, that is, the coefficients of the terms from z^m to z^{3m-1} in the left side of (4.17) are

0. The following holds:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & \vdots & \mu_0 & 0 & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \mu_{m-2} & \cdots & \mu_0 & 0 \\ \hline & & & & \mu_{m-1} & \cdots & \mu_1 & \mu_0 \\ & \multicolumn{2}{c}{0} & & \vdots & \ddots & \ddots & \vdots \\ & & & & \mu_{2m-2} & \cdots & \cdots & \mu_{m-1} \end{array}\right) \begin{pmatrix} \alpha_{m-1} \\ \vdots \\ \alpha_0 \\ \beta_{m-1} \\ \vdots \\ \beta_0 \end{pmatrix} = - \begin{pmatrix} \mu_0 \\ \vdots \\ \mu_{m-1} \\ \mu_m \\ \vdots \\ \mu_{2m-1} \end{pmatrix}. \tag{4.18}$$

The left side of the above equation is a $2m \times 2m$ Sylvester matrix, we denote it by S_{2m}. It follows from (4.18) that

$$\det(S_{2m}) = (-1)^{[m/2]}\det(H_m), \tag{4.19}$$

where $[m/2]$ is the greatest integer that is less than or equal to $m/2$. Since H_m is nonsingular, S_{2m} is nonsingular. Therefore, polynomials $\psi(z)$, $\phi(z)$ and $\chi(z)$ that satisfy (4.17) are uniquely determined.

From (4.18), the coefficients of $\phi(z)$ can be calculated by

$$\begin{pmatrix} \mu_{m-1} & \cdots & \mu_1 & \mu_0 \\ \mu_m & \ddots & & \vdots \\ \vdots & & \ddots & \mu_{m-2} \\ \mu_{2m-2} & \cdots & \cdots & \mu_{m-1} \end{pmatrix} \begin{pmatrix} \beta_{m-1} \\ \beta_{m-2} \\ \vdots \\ \beta_0 \end{pmatrix} = - \begin{pmatrix} \mu_m \\ \mu_{m+1} \\ \vdots \\ \mu_{2m-1} \end{pmatrix}. \tag{4.20}$$

By comparing (4.20) with (4.7), we have

$$\phi(z) = \varphi(z). \tag{4.21}$$

The theorem is proved. □

From Theorem 4.3.4, the extended Euclid's algorithm can be used to calculate polynomials that satisfy the equation (4.17). From Theorem 4.3.5, $\varphi(z)$ can be calculated by using the extended Euclid's algorithm.

Let $Q_k(z)$, $F_{k+1}(z)$ be sequences of polynomials calculated by Euclid's algorithm as follows.

$$Q_k(z) = \mathrm{quo}\Big(\frac{\mathrm{lc}\,(F_k(z))}{\mathrm{lc}\,(F_{k-1}(z))}F_{k-1}(z), F_k(z)\Big), \tag{4.22}$$

$$F_{k+1}(z) = \mathrm{mod}\Big(\frac{\mathrm{lc}\,(F_k(z))}{\mathrm{lc}\,(F_{k-1}(z))}F_{k-1}(z), F_k(z)\Big). \tag{4.23}$$

$$k = 1, 2, \ldots,$$

where $\mathrm{quo}\,(F(z), G(z))$ and $\mathrm{mod}\,(F(z), G(z))$ denote the quotient polynomial and the remainder polynomialof $F(z)$ divided by $G(z)$, respectively. $\mathrm{lc}\,(F_k(z))$ is the leading coefficient: the coefficient of the term of highest degree of $F_k(z)$. Using $Q_k(z)$ and $F_k(z)$, we can calculate sequences of polynomials $A_k(z)$, $B_k(z)$ by the following recursive equations.

$$A_k(z) = -Q_k(z)A_{k-1}(z) + \frac{\mathrm{lc}\,(F_k(z))}{\mathrm{lc}\,(F_{k-1}(z))}A_{k-2}(z), \tag{4.24}$$

$$B_k(z) = -Q_k(z)B_{k-1}(z) + \frac{\mathrm{lc}\,(F_k(z))}{\mathrm{lc}\,(F_{k-1}(z))}B_{k-2}(z), \tag{4.25}$$

where $A_0(z) = 0$, $A_{-1}(z) = 1$, $B_0(z) = 1$, $B_{-1}(z) = 0$, $\mathrm{lc}\,(B_k(z)) = (-1)^k$. From the extended Euclid's algorithm, we have the following equations.

$$A_k(z)F_0(z) + B_k(z)F_1(z) = F_{k+1}(z), \; k = 1, 2, \ldots, \tag{4.26}$$

where $\deg(B_k(z)) \geq k$, $\deg(A_k(z)) \geq k-1$, $\deg(F_{k+1}(z)) \leq \deg(F_1(z)) - k$. In the case that the difference in the degrees of two successive polynomials in $F_k(z)$, $k = 1, 2, \ldots$ is one, the polynomial sequence $F_k(z)$ is called normal, otherwise it is called abnormal. When $F_k(z)$ is normal, we have $\deg(B_k(z)) = k$, $\deg(A_k(z)) = k-1$, $\deg(F_{k+1}(z)) = \deg(F_1(z)) - k$. By comparing (4.26) with (4.17) and from the uniqueness of $\varphi(z)$, we have

$$\varphi(z) = (-1)^m B_m(z). \tag{4.27}$$

If $F_k(z)$ is abnormal, the difference in the degrees of two successive polynomials in $F_k(z)$ is not one. In the case that $F_k(z)$ is near abnormal, that is, the absolute values of the leading coefficients of two successive polynomials in $F_k(z)$ differ substantially, $F_k(z)$ are numerically unstable. Ohsako et al. [45, 46, 47] proposed a method to calculate the sequence of remainder polynomials stably in such cases.

In order to compute $\varphi(z_k)$, the computation of $Q_1(z), \ldots, Q_m(z)$ is needed, the arithmetic operations of which is $3m^2 + 2m$. With these results, the arithmetic operations needed in the computation of $\varphi(z_k)$ is $2m$. Since $Q_1(z), \ldots, Q_m(z)$ need to be calculated only once when compute $\varphi(z_1), \ldots, \varphi(z_m)$, the total arithmetic operations needed is $3m^2 + 2m + 2m \times m = 5m^2 + 2m$.

4.3.4 Numerical Computation of μ_p

In this section, we consider the numerical computation of integral μ_p. K-point trapezoidal rule on the unit circle is used to calculate μ_p, where K is a positive integer. Let $\omega_0, \ldots, \omega_{K-1}$ be

$$\omega_j := \exp\Big(\frac{2\pi i}{K} j\Big),\ j = 0, 1, \ldots, K-1.$$

Then

$$\mu_p = \frac{1}{2\pi}\int_0^{2\pi} e^{ip\theta} e^{i\theta} \frac{f'(e^{i\theta})}{f(e^{i\theta})} d\theta,\ p = 0, 1, \ldots.$$

The approximations of μ_p can be calculated by the trapezoidal rule as follows.

$$\hat{\mu}_p = \frac{1}{K}\sum_{j=0}^{K-1} \frac{f'(\omega_j)}{f(\omega_j)} \omega_j^{p+1},\ p = 0, 1, \ldots, K-1. \tag{4.28}$$

From (4.7), μ_p for $p = 0, \ldots, 2m-1$ are needed in the computation, therefore we take $K = 2m$. The following theorem was presented in [63].

Theorem 4.3.6 *Let $\xi_1, \ldots, \xi_m$ be the m distinct zeros of $f(z)$, $\nu_1, \ldots, \nu_m$ be the multiplicities of the zeros. Then*

$$\sum_{i=1}^{m} \Big(\frac{\xi_i^k}{1-\xi_i^K}\Big)\nu_i = \hat{\mu}_k,\ k = 0, 1, \ldots, 2m-1. \tag{4.29}$$

Let $\hat{H}_m$ be the following $m \times m$ Hankel matrix

$$\hat{H}_m := \Big(\hat{\mu}_{p+q}\Big)_{p,q=0}^{m-1} = \begin{pmatrix} \hat{\mu}_0 & \hat{\mu}_1 & \cdots & \hat{\mu}_{m-1} \\ \hat{\mu}_1 & \ddots & & \vdots \\ \vdots & & \ddots & \hat{\mu}_{2m-3} \\ \hat{\mu}_{m-1} & \cdots & \cdots & \hat{\mu}_{2m-2} \end{pmatrix}. \tag{4.30}$$

Let Ψ_m be the following diagonal matrix

$$\Psi_m = \operatorname{diag}\left(\frac{1}{1-\xi_1^K}, \ldots, \frac{1}{1-\xi_m^K}\right).$$

Since $\xi_1, \ldots, \xi_m$ are located inside the unit circle, Ψ_m is nonsingular. We have the following Lemma.

Lemma 4.3.7 *Let $\xi_1, \ldots, \xi_m$ be the m distinct zeros of $f(z)$. Then $\hat{H}_m$ is nonsingular.*

Proof. From (4.29),

$$\hat{H}_m = V_m D_m \Psi_m V_m^T. \tag{4.31}$$

Since Ψ_m, V_m, D_m are nonsingular, $\hat{H}_m$ is nonsingular. □

Let $\hat{\varphi}(z)$ be the m degree polynomial,

$$\hat{\varphi}(z) = z^m + \hat{b}_{m-1} z^{m-1} + \cdots + \hat{b}_0, \tag{4.32}$$

where the coefficients of $\hat{\varphi}(z)$ are calculated by the following equation.

$$\begin{pmatrix} \hat{\mu}_0 & \hat{\mu}_1 & \cdots & \hat{\mu}_{m-1} \\ \hat{\mu}_1 & \ddots & & \vdots \\ \vdots & & \ddots & \hat{\mu}_{2m-3} \\ \hat{\mu}_{m-1} & \cdots & \cdots & \hat{\mu}_{2m-2} \end{pmatrix} \begin{pmatrix} \hat{b}_0 \\ \hat{b}_1 \\ \vdots \\ \hat{b}_{m-1} \end{pmatrix} = - \begin{pmatrix} \hat{\mu}_m \\ \hat{\mu}_{m+1} \\ \vdots \\ \hat{\mu}_{2m-1} \end{pmatrix}. \tag{4.33}$$

We have the following theorem.

Theorem 4.3.8 *Let $\hat{\varphi}(z)$ be defined by (4.32) and (4.33), then*

$$\varphi(z) = \hat{\varphi}(z).$$

Proof. By setting $k = m, \ldots, 2m-1$ in (4.29), we have

$$\begin{pmatrix} \hat{\mu}_m \\ \hat{\mu}_{m+1} \\ \vdots \\ \hat{\mu}_{2m-1} \end{pmatrix} = V_m D_m \Psi_m \boldsymbol{\xi}_m. \tag{4.34}$$

It follows from (4.33) that

$$\begin{pmatrix} \hat{b}_0 \\ \hat{b}_1 \\ \vdots \\ \hat{b}_{m-1} \end{pmatrix} = -\hat{H}_m^{-1} \begin{pmatrix} \hat{\mu}_m \\ \hat{\mu}_{m+1} \\ \vdots \\ \hat{\mu}_{2m-1} \end{pmatrix} = -(V_m D_m \Psi_m V_m^T)^{-1} V_m D_m \Psi_m \boldsymbol{\xi}_m \quad (4.35)$$
$$= -V_m^{-T} \boldsymbol{\xi}_m.$$

Hence, from(4.10),

$$\begin{pmatrix} \hat{b}_0 \\ \hat{b}_1 \\ \vdots \\ \hat{b}_{m-1} \end{pmatrix} = -V_m^{-T} \boldsymbol{\xi}_m = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{m-1} \end{pmatrix}. \quad (4.36)$$

It follows that $\hat{\varphi}(z)$ and $\varphi(z)$ have the same coefficients. □

From Theorem 4.3.8, the errors of the numerical computation of μ_p do not affect the accuracy of the zeros of $\varphi(z)$ calculated by using $\hat{\mu}_p$ and setting $K = 2m$.

4.3.5 Computation of Multiplicities

Let $\hat{\nu}_1, \ldots, \hat{\nu}_m$ be

$$\hat{\nu}_j := \frac{\nu_j}{1 - \xi_j^K}, \ j = 1, \ldots, m. \quad (4.37)$$

From (4.29), $\nu_1, \ldots, \nu_m$ can be calculated by

$$\nu_j = \hat{\nu}_j (1 - \xi_j^K), \ j = 1, \ldots, m, \quad (4.38)$$

where $\hat{\nu}_j, \ j = 1, \ldots, m$ are calculated by solving the following equation.

$$\begin{pmatrix} 1 & \cdots & 1 \\ \xi_1 & \cdots & \xi_m \\ \vdots & & \vdots \\ \xi_1^{m-1} & \cdots & \xi_m^{m-1} \end{pmatrix} \begin{pmatrix} \hat{\nu}_1 \\ \hat{\nu}_2 \\ \vdots \\ \hat{\nu}_m \end{pmatrix} = \begin{pmatrix} \hat{\mu}_0 \\ \hat{\mu}_1 \\ \vdots \\ \hat{\mu}_{m-1} \end{pmatrix}. \quad (4.39)$$

Since the coefficient matrix of (4.39) is Vandermonde matrix, it can be calculated using polynomial interpolation with arithmetic operations of $5m^2/2$ [18].

4.3.6 Cluster of Zeros

In the case of cluster of zeros, From the analysis in section 3.3.4, if the size of the cluster is small enough, then the center of the cluster and the number of zeros in the cluster can be regard as a multiple zero and the multiplicity of the zero, respectively. Therefore, the method proposed in this chapter can be used to calculate the center of the cluster and the number of zeros in the cluster.

4.3.7 Algorithm

From the above discussions, we give the following algorithm to compute the zeros of the polynomial (4.1).

Algorithm 4.2

Input parameters: Approximate zeros $z_1, \cdots, z_m$.

(1) Compute $\varphi(z_k)$, $k = 1, \ldots, m$ by (4.22)-(4.27) using multiple precision arithmetic.

(2) Compute companion matrix N by (4.11).

(3) Compute the eigenvalues of N by the QR method in floating point arithmetic.

(4) Compute the multiplicities of zeros by (4.39) and (4.38) in floating point arithmetic.

4.4 Numerical Examples

Numerical examples were calculated in Mathematica Ver. 4.0 using multiple precision arithmetic with 64-digit precision.

Example 4.1. Suppose that

$$f_1(z) = (z-0.1)^{15}(z-0.3)^{20}(z-0.5)^{25}(z-0.7)^{30}(z-0.9)(z+0.5-0.2i)(z+0.5+0.2i).$$

The approximate zeros are: $z_1 = 0.1+\varepsilon$, $z_2 = 0.3-\varepsilon$, $z_3 = 0.5+\varepsilon i$, $z_4 = 0.7-\varepsilon i$, $z_5 = 0.9+\varepsilon$, $z_6 = -0.5+0.2i+\varepsilon+\varepsilon i$, $z_7 = -0.5-0.2i-\varepsilon-\varepsilon i$, $\varepsilon = 0.001$.

The results calculated by Smith's method(Theorem 3.2.1) are shown in Table 4.1. The underline shows the different digits of the results compared with the exact results. The results calculated by the method proposed in this chapter are shown in Table 4.2.

Table 4.1 Zeros of $f_1(z)$ Calculated by Smith's method

Zeros	Multiplicities	Calculated results
0.1	15	$0.09998\underline{943587663} + 0.00000\underline{472632329}i$
0.3	20	$0.30000\underline{713224215} + 0.00003\underline{265231293}i$
0.5	25	$0.50008\underline{869399861} + 0.00\underline{134578609668}i$
0.7	30	$0.69\underline{846101612219} + 0.00\underline{352427425081}i$
0.9	1	$0.90001\underline{865994512} + 0.00004\underline{599931385}i$
$-0.5+0.2i$	1	$-0.49999\underline{263862377} + 0.19998\underline{854317185}i$
$-0.5-0.2i$	1	$-0.50000\underline{313487745} - 0.19998\underline{671246824}i$

Table 4.2 Zeros of $f_1(z)$ Calculated by our method

Zeros	Calculated results	Calculated multiplicities
0.1	$0.09999999999999 + 0.00000000000000i$	14.9999999999999
0.3	$0.30000000000000 + 0.00000000000000i$	19.9999999999999
0.5	$0.50000000000000 + 0.00000000000000i$	$25.000000000000\underline{1}$
0.7	$0.70000000000000 + 0.00000000000000i$	29.9999999999999
0.9	$0.90000000000000 + 0.00000000000000i$	0.99999999999999
$-0.5+0.2i$	$-0.50000000000000 + 0.20000000000000i$	0.99999999999999
$-0.5-0.2i$	$-0.50000000000000 - 0.20000000000000i$	0.99999999999999

Example 4.2. Suppose that

$$f_2(z) = (z-0.1)^{10}(z-0.2)^{10}(z-0.3)^{15}(z-0.4)^{20}(z-0.5)^{25}(z-0.6)(z-0.7)(z-0.8)(z-0.9).$$

The approximate zeros are: $z_1 = 0.1+\varepsilon$, $z_2 = 0.2-\varepsilon$, $z_3 = 0.3+2\varepsilon$, $z_4 = 0.4-2\varepsilon$, $z_5 = 0.5+3\varepsilon$, $z_6 = 0.6-3\varepsilon$, $z_7 = 0.7+4\varepsilon$, $z_8 = 0.8-4\varepsilon$, $z_9 = 0.9+5\varepsilon$, $\varepsilon = 0.001$.

The calculated results are shown in Table 4.3.

Table 4.3 Zeros of $f_2(z)$ Calculated by two methods

Zeros	Smith's method	Our method	Calculated multiplicities
0.1	0.10102225406462	0.10000000000000	9.99999999999972
0.2	0.19898498318075	0.20000000000000	10.0000000000005
0.3	0.30202777285857	0.29999999999999	15.0000000000011
0.4	0.39799999999999	0.40000000000000	19.9999999999945
0.5	0.50650158620460	0.50000000000000	25.0000000000095
0.6	0.59703859304823	0.60000000000000	0.99999999999899
0.7	0.70392818753897	0.69999999999999	1.00000000000506
0.8	0.79609164830036	0.80000000000000	0.99999999999845
0.9	0.90485559914874	0.90000000000000	1.00000000000019

From the results of example 4.1 and example 4.2, we see that in the case that polynomial has multiple zeros, our method can calculate the simple and multiple zeros accurately compared with the results calculated by Smith's method.

Example 4.3. Suppose that

$$\begin{aligned} f_3(z) &= (z-0.2-\delta)(z-0.2-2\delta)(z-0.2-3\delta)(z-0.2-4\delta)(z-0.2-5\delta) \\ &\quad \times(z-0.5)(z-0.8)(z+0.4+0.3i)(z+0.4-0.3i), \end{aligned}$$

where $\delta = 0.000001$. The approximate zeros are: $z_1 = 0.2+\varepsilon$, $z_2 = 0.5-\varepsilon$, $z_3 = 0.8+2\varepsilon$, $z_4 = -0.4-0.3i+\varepsilon i$, $z_5 = -0.4+0.3i-\varepsilon i$, $\varepsilon = 0.001$.

The results calculated by Smith's method are shown in Table 4.4. The results calculated by new method are shown in Table 4.5.

Table 4.4. Zeros of $f_3(z)$ Calculated by Smith's method

Zeros(Center of cluster)	Calculated results
0.200003	$0.20004927829148 - 0.00000003975207i$
0.5	$0.49917204961551 + 0.00000009061001i$
0.8	$0.80109763867191 + 0.00000003361876i$
$-0.4-0.3i$	$-0.40025065261406 - 0.30048998969831i$
$-0.4+0.3i$	$-0.40025061181716 + 0.30049006422992i$

Table 4.5. Zeros of $f_3(z)$ Calculated by our method

Zeros (Center of cluster)	Calculated results	Calculated multiplicities
0.200003	$0.20000299999\underline{066} + 0.00000000000000i$	$4.99999999\underline{72554}$
0.5	$0.49999999996\underline{669} + 0.00000000000000i$	$1.00000000\underline{24487}$
0.8	$0.79999999999\underline{871} + 0.00000000000000i$	$1.00000000000\underline{433}$
$-0.4 - 0.3i$	$-0.39999999999\underline{964} - 0.299999999999\underline{74}i$	$1.00000000000\underline{433}$
$-0.4 + 0.3i$	$-0.39999999999\underline{964} + 0.299999999999\underline{74}i$	$1.0000000000\underline{2099}$

From Table 4.4 and Table 4.5, we see that in the case that polynomial has cluster of zeros, our method can calculate the zeros and the center of the cluster accurately compared with the results calculated by Smith's method.

4.5 Conclusion

In this chapter, we proposed a new method to compute the companion matrix proposed in the previous chapter, and then use the companion matrix to calculate the multiple zeros of a polynomial. This method uses a recursive formula based on Euclid's algorithm to compute the polynomial values with arithmetic operations of $O(n^2)$ (n is the degree of the polynomial) and then computes the companion matrix. The zeros of the polynomial were calculated by computing the eigenvalue of the companion matrix. The eigenvalues were calculated using floating point arithmetic, and the other computations were performed using multiple precision arithmetic. Therefore, the increase of overall computing time were contained. This method is efficient in calculating multiple zeros and their multiplicities, and it is also efficient in computing the center of cluster of zeros and the number of zeros in the cluster. The effectiveness of the method has been illustrated with numerical examples.