Chapter 3

A Method for Finding the Zeros of Polynomials Using a Companion Matrix

3.1 Introduction

In computing zeros of polynomials using the companion matrix method, Smith's method [69] and Fiedler's method [15] use the values of a polynomial at approximate zeros to construct a companion matrix and can provide very accurate results in calculating simple zeros, but the approximations are not sharp in the case of multiple or close zeros. In this chapter, we consider the problem of finding polynomial zeros using a companion matrix. First we review several related companion matrices. We then propose a new companion matrix, which has a high efficiency in finding multiple zeros of a polynomial. It is also efficient when calculating the arithmetic mean of a cluster of zeros and the total number of zeros in the cluster. With this method we can obtain good approximations for multiple zeros as well as for simple zeros, and we can obtain the multiplicities of the zeros with a high accuracy. Numerical examples are presented to illustrate the effectiveness of the method.

3.2 Finding Polynomial Zeros Using a Companion Matrix

3.2.1 Fiedler's Companion Matrix

Fiedler proposed a method to construct a symmetric matrix to be the companion matrix of a given polynomial [15]. The matrix can be real for polynomials with only real roots.

For monic polynomial

$$p(z) := z^n + a_{n-1}z^{n-1} + \cdots + a_0, \tag{3.1}$$

let $z_1, \ldots, z_n \in \mathbb{C}$ be n distinct approximate zeros such that $p(z_k) \neq 0$ for $k = 1, \ldots, n$. Set

$$\begin{aligned} q(z) &:= \prod_{k=1}^{n}(z - z_k), \\ B &:= \mathrm{diag}(z_1, \ldots, z_n) \in \mathbb{C}^{n\times n}, \\ d &:= (d_1, \ldots, d_n)^T,\ d_k := \sqrt{p_k}, \end{aligned} \tag{3.2}$$

where p_k is the Weierstrass correction:

$$p_k := p(z_k)/q'(z_k),\ k = 1, \ldots, n. \tag{3.3}$$

Then Fiedler's companion matrix F is defined by

$$F := B - dd^T. \tag{3.4}$$

Since a real symmetric matrix has many beneficial algebraic properties and it is a simple process to solve its characteristic equation, Fiedler's method works well when the polynomial only has real zeros. However, this requires explicit knowledge of the set of points separating the zeros.

3.2.2 Smith's Companion Matrix

Smith proposed another method to construct the companion matrix of a polynomial [69], which can be described as the following theorem.

Theorem 3.2.1 (Smith) *For monic polynomial $p(z)$ of degree n, suppose m distinct approximate zeros $z_1, \ldots, z_m \in \mathbb{C}$ and their respective multiplicities ν_k, $k = 1, 2, \ldots, m$ be given. For each $k = 1, 2, \ldots, m$ and each $j = 1, 2, \ldots, \nu_k$, let*

$$p_{kj} := \frac{1}{(j-1)!}\left(\frac{d}{dz}\right)^{j-1} p(z)\Big|_{z=z_k}$$

and

$$h_{kj} := \frac{1}{(\nu_k - j)!}\left(\frac{d}{dz}\right)^{\nu_k - j} \prod_{i \neq k}(z - z_i)^{-\nu_i}\Big|_{z=z_k}.$$

In addition, let $\mathbf{p}^T$ and $\mathbf{h}^T$ be the row vectors

$$(p_{11}, p_{12}, \cdots, p_{1\nu_1}, p_{21}, \cdots, p_{m\nu_m}) \text{ and } (h_{11}, h_{12}, \cdots, h_{1\nu_1}, h_{21}, \cdots, h_{m\nu_m}),$$

then the zeros of polynomial $p(z)$ are the eigenvalues of the matrix

$$\boldsymbol{R} = \boldsymbol{J} - \boldsymbol{p}\boldsymbol{h}^T, \tag{3.5}$$

where

$$\boldsymbol{J} = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_L \end{pmatrix}_{n \times n} \quad \text{and} \quad J_k = \begin{pmatrix} z_k & & & \\ 1 & z_k & & \\ & \ddots & \ddots & \\ & & 1 & z_k \end{pmatrix}_{\nu_k \times \nu_k}.$$

If the polynomial $p(z)$ has only simple zeros, then from Smith's theorem, it is easy to verify that the zeros of $p(z)$ are the eigenvalues of the following matrix

$$R = \begin{pmatrix} z_1 - \dfrac{p(z_1)}{q'(z_1)} & -\dfrac{p(z_2)}{q'(z_2)} & \cdots & -\dfrac{p(z_n)}{q'(z_n)} \\ -\dfrac{p(z_1)}{q'(z_1)} & z_2 - \dfrac{p(z_2)}{q'(z_2)} & \cdots & -\dfrac{p(z_n)}{q'(z_n)} \\ \vdots & \vdots & & \vdots \\ -\dfrac{p(z_1)}{q'(z_1)} & -\dfrac{p(z_2)}{q'(z_2)} & \cdots & z_n - \dfrac{p(z_n)}{q'(z_n)} \end{pmatrix}. \tag{3.6}$$

Fiedler's method and Smith's method work well when the polynomial has only simple zeros. In cases where the polynomial has multiple zeros or clusters of zeros, the multiplicities or the number of zeros in the clusters are needed to be known in advance and the zeros can not be calculated accurately. Fiedler's method and Smith's method use the values of the polynomial at approximate zeros to construct the companion matrices, these companion matrices can be regarded as the combination of a diagonal matrix and a symmetric perturbation matrix(Fiedler's method) or a rank-one perturbation matrix(Smith's method). When the approximate zeros are very close to the zeros, the companion matrices are approximate to the diagonal matrix of zeros. Thereby Fiedler's method and Smith's method can be used iteratively by taking the newly calculated eigenvalues as the new approximate zeros and all the zeros of the polynomial can be found simultaneously. As an iterative procedure, Newton method needs the initial value to be near the zeros of the polynomial to get good results. Conversely, when using Fiedler's method recursively, the starting values can be taken randomly or equally distant on a large circle in the complex plane or provided by the eigenvalues of other companion matrices [68].

A recursive application of Fiedler's method and Smith's method converge rapidly to simple zeros. In the case of multiple zeros, the iterations reached a saturation point after several steps. Malek and Vaillancourt [31] proposed an iterative method based on Fiedler's companion matrix which used an extended-precision arithmetic in the calculation, the method can produce fast convergence to simple zeros, but in the case of multiple zeros, the method can not work well unless multiple precision arithmetic was used in eigenvalue computation and an averaging procedure was taken in the computation.

Fortune [16] proposed another iterative method to approximate the zeros of a polynomial to floating-point accuracy. The method is based on Smith's companion matrix and uses extended-precision arithmetic to construct the companion matrix and then performs a floating-point eigenvalue computation of the matrix. This method works well when the polynomial has only simple zeros, but the results are not accurate in the case of multiple zeros or clusters of zeros.

It is noted that the approximations of a multiple root tend to approach that root from uniformly spaced positions around a circle with center at the multiple zero [75]. Hence, we can take points equally distributed on a circle to be the initial value when iteratively calculating the multiple zeros, where the radius of the circle can take the maximum absolute value of the roots. We can get the estimate of this value from formula [50]

$$|x| \leq 2 \max_{k=1,\ldots,n} |a_{n-k}|^{1/k}, \tag{3.7}$$

where a_k, $k = 0, 1, \ldots, n-1$ are coefficients of (3.1).

Malek and Vaillancourt [30] proposed a three-stage algorithm to find the zeros of a polynomial, which is based on Fiedler's method and calculates the zeros and their multiplicities separately. Malek's algorithm is summarized as below.

Algorithm 3.1(Malek's algorithm)

(1) Find the greatest common divisor(GCD) of $p(z)$ and $p'(z)$. Reduce the polynomial $p(z)$ to a polynomial $p_1(z)$ having only simple roots.

(2) Compute the simple roots of $p_1(z)$ by applying Fiedler's method recursively.

(3) Calculate the multiplicity of each root of $p(z)$ by means of Lagouanelle's modified limiting formula.

Theoretically, in step (1), the GCD of $p(z)$ and $p'(z)$can be calculated accurately. In practical computation, however, it may be difficult to determine it. The method also can not give a good result in the case of clusters of zeros.

3.3 A New Method for Finding Multiple Zeros of Polynomials

3.3.1 A Reduced Polynomial with Only Simple Zeros

In this section, we present a new companion matrix with which we can obtain good approximations for multiple zeros. Some classical methods fail in calculating multiple zeros because they treat a multiple zero as simple zeros and hence always lead to ill-conditioning when solving characteristic equations. This is considered to be the wrong way to set the computational objective, that is, we should compute the distinct zeros of the polynomials and their multiplicities separately. The companion matrix proposed in this section has only simple eigenvalues, which are given by the distinct zeros of the given polynomial, and the multiplicities of the zeros can be calculated separately. This method has the advantage of the efficiency of other companion matrices for finding simple zeros, and both zeros and their multiplicities can be computed. Let

$$f(z) := c_n z^n + c_{n-1} z^{n-1} + \cdots + c_0 = c_n \prod_{k=1}^{m} (z - \xi_k)^{\nu_k}, \quad \sum_{k=1}^{m} \nu_k = n, \; c_n \neq 0, \tag{3.8}$$

be a polynomial of degree n with all n zeros located inside the circle $\Gamma : \{z : |z - \gamma| < \rho\}$ in the complex plane, and $\xi_1, \ldots, \xi_m$ be $m (m \leq n)$ mutually distinct zeros of $f(z)$ with multiplicities $\nu_1, \ldots, \nu_m$, respectively. Since it is possible for us to shift and scale the polynomial so that all the zeros are located inside the unit circle, then for the sake of simplicity, we consider the case that all the zeros are located inside the unit circle C. It follows from the theory of complex variables that $\nu_1, \ldots, \nu_m$ are residues of $f'(z)/f(z)$ at ξ_k. Let

$$\mu_p := \frac{1}{2\pi i} \int_C z^p \frac{f'(z)}{f(z)} dz, \; p = 0, 1, 2, \ldots. \tag{3.9}$$

The residue theorem implies that the μ_p's are equal to the *Newton sums* of the unknown zeros,

$$\mu_p = \sum_{k=1}^{m} \nu_k \xi_k^p, \; p = 0, 1, 2, \ldots. \tag{3.10}$$

Let H_m be the $m \times m$ Hankel matrix

$$H_m := \left(\mu_{p+q}\right)_{p,q=0}^{m-1} = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{m-1} \\ \mu_1 & \ddots & & \vdots \\ \vdots & & \ddots & \mu_{2m-3} \\ \mu_{m-1} & \cdots & \cdots & \mu_{2m-2} \end{pmatrix}, \tag{3.11}$$

and let $H_m^<$ be the $m \times m$ shifted Hankel matrix

$$H_m^< := \left(\mu_{p+q}\right)_{p,q=1}^{m} = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_m \\ \mu_2 & \ddots & & \vdots \\ \vdots & & \ddots & \mu_{2m-2} \\ \mu_m & \cdots & \cdots & \mu_{2m-1} \end{pmatrix}. \tag{3.12}$$

Lemma 3.3.1 *If $\xi_1, \ldots, \xi_m$ are mutually distinct zeros of $f(z)$, then H_m is nonsingular.*

Proof. Let V_m be the Vandermonde matrix

$$V_m := \left(\xi_k^p\right)_{p=0,k=1}^{m-1,m} = \begin{pmatrix} 1 & \cdots & 1 \\ \xi_1 & \cdots & \xi_m \\ \vdots & & \vdots \\ \xi_1^{m-1} & \cdots & \xi_m^{m-1} \end{pmatrix}, \tag{3.13}$$

and let

$$D_m = \mathrm{diag}(\nu_1, \ldots, \nu_m). \tag{3.14}$$

Then it follows from (3.10) that

$$H_m = V_m D_m V_m^T.$$

Since $\xi_1, \ldots, \xi_m$ are distinct and $\nu_1, \ldots, \nu_m \neq 0$, V_m and D_m are nonsingular, H_m is nonsingular. □

Let $\varphi_m(z)$ be the polynomial

$$\varphi_m(z) := z^m + b_{m-1}z^{m-1} + \cdots + b_0 = \prod_{k=1}^{m}(z - \xi_k) , \tag{3.15}$$

where $\xi_1, \ldots, \xi_m$ are the distinct zeros of $f(z)$, then the problem of finding the n zeros of $f(z)$ reduces to the problem of finding the m simple zeros of $\varphi_m(z)$ and then computing the multiplicities of these zeros. Usually, in practical computation, the polynomial $\varphi_m(z)$ may be ill-conditioned such as the well-known Wilkinson's polynomial, which means that the small changes of the coefficients of the polynomial may produce large changes in the zeros of the polynomial. This ill-conditioning of the map between the coefficients of a polynomial and its zeros is discussed in [75]. Therefore, instead of computing the coefficients of $\varphi_m(z)$, we use a companion matrix to compute the zeros of it.

We have the following theorems.

Theorem 3.3.2 *Let $\xi_1, \ldots, \xi_m$ be m distinct zeros of $f(z)$ and $C_m \in \mathbb{C}^{m\times m}$ be the Frobenius companion matrix of $\varphi_m(z)$, then*

$$H_m^{-1} H_m^{<} = C_m . \tag{3.16}$$

Proof. Let

$$I_k := \mu_{k+m} + b_{m-1}\mu_{k+m-1} + \cdots + b_0\mu_k = \mu_{k+m} + \sum_{i=0}^{m-1} b_i\mu_{k+i}, \quad k = 0, 1, \ldots, m-1 \ .$$

Then it follows from (3.10) that

$$\begin{aligned} I_k &= \sum_{i=1}^{m} \nu_i \xi_i^k (\xi_i^m + b_{m-1}\xi_i^{m-1} + \cdots + b_0) \\ &= \sum_{i=1}^{m} \nu_i \xi_i^k \varphi_m(\xi_i) = 0, \quad k = 0, 1, \ldots, m-1. \end{aligned}$$

Hence

$$\mu_{k+m} = -\sum_{i=0}^{m-1} b_i \mu_{k+i}, \ k = 0, 1, \ldots, m-1.$$

Therefore, from (3.11) and (3.15) we have

$$\begin{aligned} H_m C_m &= \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{m-1} \\ \mu_1 & \ddots & & \vdots \\ \vdots & & \ddots & \mu_{2m-3} \\ \mu_{m-1} & \cdots & \cdots & \mu_{2m-2} \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & -b_0 \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & -b_{m-2} \\ 0 & \cdots & 1 & -b_{m-1} \end{pmatrix} \\ &= \begin{pmatrix} \mu_1 & \mu_2 & \cdots & -\sum_{k=0}^{m-1} \mu_k b_k \\ \mu_2 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \mu_{m-1} & \cdots & \cdots & -\sum_{k=0}^{m-1} \mu_{m+k-1} b_k \end{pmatrix} \\ &= \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_m \\ \mu_2 & \ddots & & \vdots \\ \vdots & & \ddots & \mu_{2m-2} \\ \mu_{m-1} & \cdots & \cdots & \mu_{2m-1} \end{pmatrix} = H_m^< \ . \end{aligned}$$

Since H_m is nonsingular, the theorem is proved. □

3.3.2 A New Companion Matrix

Theorem 3.3.3 *Let $z_1, \ldots, z_m$ be m distinct approximate zeros of $\varphi_m(z)$, and define*

$$p_m(z) := \det(H_m^< - zH_m)/\det(H_m), \qquad q_m(z) := \prod_{k=1}^{m}(z - z_k).$$

Then the m distinct zeros of $f(z)$ are the eigenvalues of the matrix

$$A = \begin{pmatrix} z_1 - \dfrac{p_m(z_1)}{q'_m(z_1)} & -\dfrac{p_m(z_2)}{q'_m(z_2)} & \cdots & -\dfrac{p_m(z_m)}{q'_m(z_m)} \\ -\dfrac{p_m(z_1)}{q'_m(z_1)} & z_2 - \dfrac{p_m(z_2)}{q'_m(z_2)} & \cdots & -\dfrac{p_m(z_m)}{q'_m(z_m)} \\ \vdots & \vdots & \cdot & \vdots \\ -\dfrac{p_m(z_1)}{q'_m(z_1)} & -\dfrac{p_m(z_2)}{q'_m(z_2)} & \cdots & z_m - \dfrac{p_m(z_m)}{q'_m(z_m)} \end{pmatrix}. \tag{3.17}$$

Proof. Since

$$\begin{aligned} p_m(z) &= \det(H_m^< - zH_m)/\det(H_m) \\ &= \det(H_m^{-1})\det(H_m^< - zH_m) \\ &= \det(H_m^{-1}H_m^< - zI), \end{aligned}$$

then it follows from Theorem 3.3.2 that

$$p_m(z) = \det(C_m - zI) = (-1)^m \varphi_m(z),$$

where C_m is the Frobenius companion matrix of $\varphi_m(z)$. Therefore, the zeros of $p_m(z)$ are given by $\xi_1, \ldots, \xi_m$. From (3.6), the m distinct zeros of $\varphi_m(z)$ are the eigenvalues of the matrix

$$S = \begin{pmatrix} z_1 - \dfrac{\varphi_m(z_1)}{q'_m(z_1)} & -\dfrac{\varphi_m(z_2)}{q'_m(z_2)} & \cdots & -\dfrac{\varphi_m(z_m)}{q'_m(z_m)} \\ -\dfrac{\varphi_m(z_1)}{q'_m(z_1)} & z_2 - \dfrac{\varphi_m(z_2)}{q'_m(z_2)} & \cdots & -\dfrac{\varphi_m(z_m)}{q'_m(z_m)} \\ \vdots & \vdots & & \vdots \\ -\dfrac{\varphi_m(z_1)}{q'_m(z_1)} & -\dfrac{\varphi_m(z_2)}{q'_m(z_2)} & \cdots & z_m - \dfrac{\varphi_m(z_m)}{q'_m(z_m)} \end{pmatrix}. \tag{3.18}$$

This proves the theorem. □

Matrix A is considered to be a new companion matrix, and all the eigenvalues of A are simple. The distinct zeros $\xi_1, \ldots, \xi_m$ of $f(z)$ can be calculated from the eigenvalues of A.

3.3.3 Numerical Computation of μ_p and an Error Analysis

The integrals μ_p can be approximated via a numerical integration method. Here we calculate it using the K-point trapezoidal rule on the unit circle. Let K be a positive integer and $\omega_0, \ldots, \omega_{K-1}$ be the Kth roots of unity, i.e.,

$$\omega_j := \exp\Big(\frac{2\pi i}{K} j\Big), \ j = 0, 1, \ldots, K-1.$$

Then from (3.9), we have

$$\mu_p = \frac{1}{2\pi} \int_0^{2\pi} e^{ip\theta} e^{i\theta} \frac{f'(e^{i\theta})}{f(e^{i\theta})} d\theta, \ p = 0, 1, \ldots.$$

We obtain the approximations of μ_p via the trapezoidal rule,

$$\hat{\mu}_p = \frac{1}{K} \sum_{j=0}^{K-1} \frac{f'(\omega_j)}{f(\omega_j)} \omega_j^{p+1}, \ p = 0, 1, \ldots, K-1. \tag{3.19}$$

Define

$$\hat{H}_m := \left[\hat{\mu}_{k+l}\right]_{k,l=0}^{m-1} \ \ and \ \ \hat{H}_m^{<} := \left[\hat{\mu}_{1+k+l}\right]_{k,l=0}^{m-1}, \tag{3.20}$$

then $\hat{\mu}_p$, $\hat{H}_m$, $\hat{H}_m^{<}$ are approximations of μ_p, H_m, $H_m^{<}$, respectively. A more detailed error analysis was presented in [26, 63].

Then we have the following theorem.

Theorem 3.3.4 *Let $\xi_1, \ldots, \xi_m$ be m distinct zeros of $f(z)$, then the corresponding multiplicities $\nu_1, \ldots, \nu_m$ are the solutions of the linear system of equations*

$$\sum_{k=1}^{m} \Big(\frac{\xi_k^p}{1-\xi_k^K}\Big) \nu_k = \hat{\mu}_p, \ p = 0, 1, \ldots, m-1. \tag{3.21}$$

Proof. Since

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{m} \frac{\nu_k}{z - \xi_k} = \frac{\mu_0}{z} + \frac{\mu_1}{z^2} + \cdots,$$

then it follows from (3.19) that

$$\begin{aligned}\hat{\mu}_p &= \frac{1}{K}\sum_{j=0}^{K-1}\omega_j^{p+1}\left(\sum_{l=0}^{+\infty}\frac{\mu_l}{\omega_j^{l+1}}\right)\\ &= \sum_{l=0}^{+\infty}\mu_l\left(\frac{1}{K}\sum_{j=0}^{K-1}\omega_j^{p-l}\right)\\ &= \sum_{r=0}^{+\infty}\mu_{p+rK}.\end{aligned}$$

The last step follows from the fact that

$$\frac{1}{K}\sum_{j=0}^{K-1}\omega_j^{p-l} = \begin{cases} 1, & \text{if } p-l = rK \text{ for } r\in\mathbb{Z};\\ 0, & \text{otherwise.}\end{cases}$$

It follows that

$$\begin{aligned}\hat{\mu}_p &= \sum_{k=1}^{m}\nu_k\xi_k^p(1+\xi_k^K+\xi_k^{2K}+\cdots)\\ &= \sum_{k=1}^{m}\left(\frac{\nu_k}{1-\xi_k^K}\right)\xi_k^p,\end{aligned}$$

for $p = 0, 1, \ldots$. The theorem is proved. □

From Theorem 3.3.4 we have the following lemma.

Lemma 3.3.5 *If $\xi_1, \ldots, \xi_m$ are distinct zeros of $f(z)$, then $\hat{H}_m$ is nonsingular.*

Proof. Let

$$U_m := \left(\frac{\xi_k^p}{1-\xi_k^K}\right)_{p=0,k=1}^{m-1,m} = \begin{pmatrix} \frac{1}{1-\xi_1^K} & \cdots & \frac{1}{1-\xi_m^K}\\ \frac{\xi_1}{1-\xi_1^K} & \cdots & \frac{\xi_m}{1-\xi_m^K}\\ \vdots & & \vdots\\ \frac{\xi_1^{m-1}}{1-\xi_1^K} & \cdots & \frac{\xi_m^{m-1}}{1-\xi_m^K}\end{pmatrix}.$$

Then it follows from (3.21) that

$$\hat{H}_m = U_m D_m V_m^T, \tag{3.22}$$

where V_m and D_m are defined by (3.13) and (3.14), respectively.

Since $\xi_1, \ldots, \xi_m$ are distinct and inside the unit circle, U_m, V_m and D_m are nonsingular, $\hat{H}_m$ is nonsingular. □

From Theorem 3.3.4 and Lemma 3.3.5, we have the following theorem.

Theorem 3.3.6 *Let $\xi_1, \ldots, \xi_m$ be m distinct zeros of $f(z)$, and let $C_m \in \mathbb{C}^{m\times m}$ be the Frobenius companion matrix of $\varphi_m(z)$. Let $\hat{H}_m$ and $\hat{H}_m^<$ be the $m \times m$ matrices defined by (3.20), then*

$$\hat{H}_m^{-1}\hat{H}_m^< = C_m.$$

Proof. Let

$$\hat{I}_k := \hat{\mu}_{k+m} + b_{m-1}\hat{\mu}_{k+m-1} + \cdots + b_0\hat{\mu}_k = \hat{\mu}_{k+m} + \sum_{i=0}^{m-1} b_i\hat{\mu}_{k+i}, \quad k = 0, 1, \ldots, m-1 \ ,$$

where $\hat{\mu}_p$, $p = 0, 1, \ldots, 2m-1$ are defined by (3.19). Then it follows from (3.21) that

$$\begin{aligned} \hat{I}_k &= \sum_{i=1}^{m} \frac{\nu_i\xi_i^k}{1-\xi_i^K}(\xi_i^m + b_{m-1}\xi_i^{m-1} + \cdots + b_0) \\ &= \sum_{i=1}^{m} \frac{\nu_i\xi_i^k}{1-\xi_i^K}\varphi_m(\xi_i) = 0, \quad k = 0, 1, \ldots, m-1. \end{aligned}$$

Hence

$$\hat{\mu}_{k+m} = -\sum_{i=0}^{m-1} b_i\hat{\mu}_{k+i}, \ k = 0, 1, \ldots, m-1.$$

Then by a similar way as the proof of Theorem 3.3.2, we have

$$\hat{H}_m C_m = \hat{H}_m^<.$$

Since $\hat{H}_m$ is nonsingular, the theorem is proved. □

Theorem 3.3.6 implies that the error of numerical integration of $\hat{\mu}_p$ does not affect the results.

3.3.4 Cluster of Zeros

In the case that the polynomial has one or several clusters of zeros, suppose that $z_1, z_2, \cdots, z_\nu$ form a cluster of zeros. Let z_G be the arithmetic mean of the cluster, then $z_j = z_G + \epsilon_j$, $j =$

$1, 2, \cdots, \nu$, if we set $\epsilon = \max_{1\le j\le\nu} |\epsilon_j|$, then from [26], we have

$$\sum_{j=1}^{\nu} \frac{z_j^p}{1 - z_j^K} = \nu \frac{z_G^p}{1 - z_G^K} + O(\epsilon^2).$$

If the size of the cluster is sufficiently small, then we can take the center of the cluster as one multiple zero and the number of zeros in the cluster as the multiplicity of the multiple zero. This makes it possible to calculate the center of the cluster and the total number of zeros in the cluster by our method.

3.3.5 Algorithm

From the discussions in the above sections, we give the following algorithm to calculate the distinct zeros of a polynomial and their respective multiplicities. From the definition of $\hat{H}_m$ and $\hat{H}_m^<$, we have to calculate $2m$ integrals $\hat{\mu}_0, \ldots, \hat{\mu}_{2m-1}$. Therefore we take $K = 2m$ in the calculation. To apply the algorithm to a general case that all the zeros of $f(z)$ are located inside Γ, we set $F(z) := f(\gamma + \rho z)$ and use $F(e^{i\theta})$ in the computation. Let λ_k and ξ_k be zeros of $F(z)$ and $f(z)$, respectively. If the eigenvalues of the companion matrix A associated with $p_m(z)$ are $\lambda_1, \ldots, \lambda_m$, then $\xi_1, \ldots, \xi_m$ can be calculated by $\xi_j = \gamma + \rho\lambda_j,\ j = 1, 2, \ldots, m$. Since

$$\hat{\mu}_k = \frac{1}{2m} \sum_{j=0}^{2m-1} \frac{F'(\omega_j)}{F(\omega_j)} \omega_j^{k+1} = \frac{1}{2m} \sum_{j=0}^{2m-1} \frac{\rho f'(\gamma + \rho\omega_j)}{f(\gamma + \rho\omega_j)} \omega_j^{k+1},\ k = 0, 1, \ldots, 2m-1,$$

and the zeros of $F(z)$ have the same multiplicities as the zeros of $f(z)$, the multiplicities of ξ_k can be calculated by solving the following equations

$$\sum_{k=1}^{m} \left(\frac{\lambda_k^p}{1 - \lambda_k^{2m}} \right) \nu_k = \hat{\mu}_p,\ p = 0, \ldots, m-1.$$

Algorithm 3.2

Input: $a_0, \ldots, a_n$, polynomial coefficients.

m, the number of distinct zeros.

$\gamma,\ \rho$, the center and radius of the circle which contains all the zeros.

$z_1, \ldots, z_m$, mutually distinct approximate zeros.

Output: distinct zeros and their respective multiplicities.

(1)set $\omega_j = e^{2\pi ji/2m}$, $j = 0, \ldots, 2m-1$.

(2)set $\hat{\mu}_k = \frac{1}{2m}\sum_{j=0}^{2m-1} \frac{\rho f'(\gamma + \rho\omega_j)}{f(\gamma + \rho\omega_j)}\omega_j^{k+1}, k = 0, \ldots, 2m-1$.

(3)set $\hat{H}_m = \left(\hat{\mu}_{p+q}\right)_{p,q=0}^{m-1}$, $\hat{H}_m^{<} = \left(\hat{\mu}_{p+q}\right)_{p,q=1}^{m}$.

(4)**compute** the companion matrix A by (3.17).

(5)**compute** $\lambda_1, \ldots, \lambda_m$ by evaluating the eigenvalues of A.

(6)**set** $\xi_j = \gamma + \rho\lambda_j$, $j = 1, \ldots, m$.

(7)**compute** $\nu_1, \ldots, \nu_m$ by solving the system $\sum_{k=1}^{m}\left(\frac{\lambda_k^p}{1-\lambda_k^{2m}}\right)\nu_k = \hat{\mu}_p$, $p = 0, \ldots, m-1$.

In step (4), in order to compute $p_m(z)$, we have to calculate the determinant of Hankel matrix H_m, generally this can be performed with $O(n^3)$ arithmetic operations. Because of the special structure possessed by the Hankel matrix, recently a number of fast methods were proposed [28, 56, 74] which reduced the number of arithmetic operations to $O(n^2)$ or $O(n\log^2 n)$.

3.4 Numerical Examples

As described in the previous sections, some classical methods can be applied only for simple zeros, even when good starting values are given. In this section, we present examples to show the efficiency of our method in computing multiple and close zeros compared with some existed methods.

The computations presented in this section were performed in Matlab Ver. 6.5. Floating point arithmetic of double precision was used to evaluate the given polynomial and to perform the calculations. The eigenvalues of the companion matrix were solved by the eig function in Matlab based on the QR method. The computer environment used in the numerical test of this thesis was a Dell INSPIRON I8200 with a Pentium4 1.8GHz CPU and 1.0GBytes memory.

Example 3.1. Suppose that

$$\begin{aligned} f_1(z) &= (z-0.2)^3(z-0.5)^4(z-0.8)(z+0.5+0.2i)(z+0.5-0.2i) \\ &= z^{10}-2.4z^9+1.79z^8+0.01z^7-0.6061z^6+0.18844z^5+0.087101z^4 \\ &\quad -0.071927z^3+0.019439z^2-0.002413z+0.000116 \end{aligned}$$

The approximate zeros are: $z_1=0.2-\varepsilon,\ z_2=0.2+\varepsilon,\ z_3=0.2-2\varepsilon,\ z_4=0.5-\varepsilon,\ z_5=0.5+\varepsilon,\ z_6=0.5-2\varepsilon,\ z_7=0.5+2\varepsilon,\ z_8=0.8+\varepsilon,\ z_9=-0.5-0.2i-\varepsilon,\ z_{10}=-0.5+0.2i-\varepsilon,\ \varepsilon=0.001$.

The results calculated by Fiedler's method are shown in Table 3.1.

Table 3.1 Zeros of $f_1(z)$ calculated by Fiedler's method

Zeros	Calculated results
0.2	$0.20000251239661+0.000004354419709i$
0.2	$0.20000251239642-0.000004354419719i$
0.2	$0.19999497526450+0.00000000000011i$
0.5	$0.50016879383330-0.00000000000000i$
0.5	$0.49999987594171-0.000168599503395i$
0.5	$0.49999987594172+0.000168599503395i$
0.5	$0.49983144600920+0.00000000000000i$
0.8	$0.79999999997270+0.00000000000000i$
$-0.5+0.2i$	$-0.49999999999209+0.20000000000156i$
$-0.5-0.2i$	$-0.49999999999209-0.20000000000156i$

The results calculated by Smith's method are shown in Table 3.2.

The results can be used as the initial value for the iterative computation, which converges quickly to simple zeros. Malek and Vaillancourt [31] and Fortune [16] use this method to obtain good approximations to simple zeros, but the results are not accurate in the case of multiple zeros.

Table 3.2 Zeros of $f_1(z)$ calculated by Smith's method

Zeros	Calculated results
0.2	$0.19999703285594 + 0.00000000000020i$
0.2	$0.20000148362804 - 0.00000259161887i$
0.2	$0.20000148362841 + 0.00000259161867i$
0.5	$0.49983057667040 + 0.00000000000000i$
0.5	$0.49999969596006 - 0.00016983127829i$
0.5	$0.49999969596005 + 0.00016983127829i$
0.5	$0.50017004248709 + 0.00000000000000i$
0.8	$0.79999999996334 - 0.00000000000000i$
$-0.5 + 0.2i$	$-0.49999999998918 + 0.20000000000221i$
$-0.5 - 0.2i$	$-0.49999999998918 - 0.20000000000221i$

We calculated the zeros of $f_1(z)$ by our method, the approximate zeros are: $z_1 = 0.2 + \varepsilon,\ z_2 = 0.5 + \varepsilon,\ z_3 = 0.8 + \varepsilon,\ z_4 = -0.5 + 0.2i + \varepsilon,\ z_5 = -0.5 - 0.2i + \varepsilon,\ \varepsilon = 0.001,\ \gamma = 0,\ \rho = 1$. The results are shown in Table 3.3.

Table 3.3 Zeros of $f_1(z)$ calculated by new method

Zeros	Calculated results	Multiplicities
0.2	$0.19999999999988 - 0.00000000000000i$	2.99999999999888
0.5	$0.49999999999996 + 0.00000000000000i$	4.00000000000131
0.8	$0.80000000000001 + 0.00000000000000i$	0.99999999999990
$-0.5 + 0.2i$	$-0.49999999999998 + 0.20000000000004i$	0.99999999999990
$-0.5 - 0.2i$	$-0.49999999999998 - 0.20000000000004i$	1.00000000000001

Example 3.2. Suppose that

$$\begin{aligned} f_2(z) &= (z+0.5)^3(z-2)^4(z-4.5)(z-2-2.5i)(z-2+2.5i) \\ &= z^{10} - 15z^9 + 96.25z^8 - 340z^7 + 660.9375z^6 - 510.5625z^5 - 354.765625z^4 \\ &\quad +710.625z^3 + 70.625z^2 - 312.5z - 92.25 \end{aligned}$$

The approximate zeros are: $z_1 = -0.5+\varepsilon,\ z_2 = 2+\varepsilon,\ z_3 = 4.5+\varepsilon,\ z_4 = 2+2.5i+\varepsilon,\ z_5 = 2 - 2.5i + \varepsilon,\ \varepsilon = 0.001,\ \gamma = 2,\ \rho = 5$. The results are shown in Table 3.4.

Table 3.4 Zeros of $f_2(z)$ calculated by new method

Zeros	Calculated results	Multiplicities
-0.5	$-0.49999999999998+0.00000000000000i$	2.99999999999656
2	$1.99999999999521-0.00000000000026i$	4.00000000000624
4.5	$4.50000000000199+0.00000000000009i$	0.99999999999838
$2-2.5i$	$2.00000000000202-2.50000000000090i$	1.00000000000033
$2+2.5i$	$2.00000000000192+2.50000000000106i$	0.99999999999849

Example 3.3. Suppose that

$$\begin{aligned} f_3(z) &= (z-1.2)(z-1.2-\delta)(z-1.2+\delta)(z-1.2-2\delta)(z-1.2-3\delta)(z-0.5) \\ &\quad \times(z-1.5-i)(z-1.5+i)(z-2.4) \\ &= z^9+a_8z^8+a_7z^7+a_6z^6+a_5z^5+a_4z^4+a_3z^3+a_2z^2+a_1z+a_0 \end{aligned}$$

where $\delta=10^{-8}$ and

$$\begin{aligned} a_0 &= -9.7044484043520002 \\ a_1 &= 72.84557069827201 \\ a_2 &= -235.1566155496320 \\ a_3 &= 432.2522997191521 \\ a_4 &= -501.8515311445200 \\ a_5 &= 383.7300067016500 \\ a_6 &= -194.1650025055000 \\ a_7 &= 62.95000053499999 \\ a_8 &= -11.90000005000000 \end{aligned}$$

Note that $f_3(z)$ has five clusters of zeros, each cluster includes 5, 1, 1, 1, and 1 zeros, respectively. The approximate zeros are: $z_1=1.5+i+\varepsilon$, $z_2=1.5-i+\varepsilon$, $z_3=0.5+\varepsilon$, $z_4=2.4+\varepsilon$, $z_5=1.2+\varepsilon$, $\varepsilon=0.001$, $\gamma=1$, $\rho=1.5$.

The results calculated by Fiedler's method are shown in Table 3.5.

Table 3.5 Zeros of $f_3(z)$ calculated by Fiedler's method

Zeros(Center of cluster)	Calculated results
$1.5+i$	$1.50000000000001 + 0.99999999999987i$
$1.5-i$	$1.49999999999999 - 0.99999999999991i$
0.5	$0.50000000000000 + 0.00000000000000i$
2.4	$2.40000000000012 + 0.00000000000000i$
1.20000001	$1.20012113286826 - 0.00015690888448i$

The results calculated by our method are shown in Table 3.6.

Table 3.6 Zeros of $f_3(z)$ calculated by new method

Zeros (Center of cluster)	Calculated results	Multiplicities
$1.5+i$	$1.49999999999999 + 1.00000000000009i$	1.00000000000030
$1.5-i$	$1.49999999999995 - 1.00000000000008i$	1.00000000000023
0.5	$0.50000000000006 + 0.00000000000005i$	0.99999999999965
2.4	$2.39999999999944 - 0.00000000000000i$	1.00000000000094
1.20000001	$1.20000000999968 - 0.00000000000004i$	4.99999999999873

In Table 3.7, we also show the results with $\delta = 10^{-4}$ and $\delta = 10^{-6}$.

Table 3.7 Center of the cluster of $f_3(z)$ calculated by two methods

δ	Fiedler's method	New method
10^{-4}	$1.20065747466531 + 0.00012796206631i$	$1.20010000178466 - 0.00000000000003i$
10^{-6}	$1.20018232636525 - 0.00021919219477i$	$1.20000100000026 + 0.00000000000001i$

From the results, we see that our method can calculate the center of the cluster and the number of the zeros in the cluster more accurately compared with the results calculated by Fiedler's method.

In practical computation with the method presented in this chapter, we consider a combined algorithm which is composed of two stages [9]. In the first stage, we use some existed methods to perform the initial calculation with the starting values required. After some iterations, we can obtain some improved approximate zeros and from these we know more about the zeros of the polynomial, that is, by checking the distance between the approximate zeros, we can know more clearly about the distribution of the zeros such as the existence of multiple zeros or cluster of zeros. Then in the second stage we take these improved results as the initial values, and apply our new method to compute the zeros of the polynomial to obtain better approximations.

3.5 Conclusion

The zeros of a polynomial can be calculated by constructing a companion matrix and then perform the eigenvalue computation by the QR method. Fiedler's method and Smith's method can be used recursively and the results converge rapidly to simple zeros. In cases where the polynomial has multiple zeros, the multiplicities are needed to be known in advance and the iterations reached a saturation point after several steps. In this chapter, we presented a method to construct a new companion matrix which has only simple eigenvalues. Using this method, we can calculate the distinct zeros of a polynomial and then calculate their multiplicities to a high accuracy. The method is also efficient in calculating the center of the cluster of zeros and the number of zeros in the cluster. The applicability of the method has been demonstrated with numerical examples.