Chapter 2

Interval Arithmetic

2.1 Introduction

Reliability of computational results is crucial in scientific computation and engineering. There are several types of errors that are generated in mathematical computations when using finite precision arithmetic on digital computers. These errors may be rounding errors due to finite representation of numbers in arithmetic units of computing machines, or errors due to uncertain values of parameters in mathematical models in physical and engineering sciences, or errors due to uncertain initial data. Interval arithmetic provides a tool for estimating and controlling these errors by providing upper and lower bounds on the effect of all these errors on a computed quantity.

When we perform computations on a computer, we can not carry out "exact" arithmetic because of the limited precision used in the computation. Applying interval arithmetic, we can compute intervals containing results of infinite precision, that is, we can find intervals containing the exact "real" arithmetic results.

Interval analysis was first introduced by Moore [35, 36] in 1966. The concept of interval analysis is to compute with intervals of real numbers in place of real numbers. While floating point arithmetic is affected by rounding errors, and can produce inaccurate results, interval arithmetic has the advantage of giving rigorous bounds for the exact solution. An application is when some parameters are not known exactly but are known to lie within a certain interval, algorithms may be implemented using interval arithmetic with uncertain parameters as intervals to produce an interval that bounds all possible results.

If the lower and upper bounds of the interval can be rounded down and rounded up respectively, then finite precision calculations can be performed using intervals to give an enclosure of the exact solution. Although it is not difficult to implement existing algorithms using intervals in place of real numbers, the result may be of no use if the interval obtained is too wide. If this is the case, other algorithms must be considered or new ones developed in order to make the interval result as narrow as possible.

2.2 Real Interval Arithmetic

In this section we give an introduction to real interval arithmetic [3, 35]. In this thesis the field of real number is denoted by $\mathbb{R}$. A subset of $\mathbb{R}$ of the form

$$A = [a_1, a_2] = \{x : a_1 \le x \le a_2,\ a_1, a_2 \in \mathbb{R}\}$$

is called a closed real interval or an interval. The set of all closed real intervals is denoted by $I(\mathbb{R})$. Real numbers $x \in \mathbb{R}$ may be considered special members $[x, x]$ from $I(\mathbb{R})$, and they will generally be called point intervals.

Two intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$ are called equal, that is, $A = B$, if $a_1 = b_1$ and $a_2 = b_2$.

The absolute value of an interval A is defined as

$$|A| := \max\{|x| : x \in A\}.$$

The arithmetic operations on elements from $I(\mathbb{R})$ are defined below.

Definition 2.2.1 *Let* $* \in \{+, -, \times, /\}$ *be a binary operation on the set of real numbers* $\mathbb{R}$. *If* $A, B \in I(\mathbb{R})$, *then*

$$A * B = \{x = a * b : a \in A,\ b \in B\}$$

defines a binary operation on $I(\mathbb{R})$.

It is assumed that $0 \notin B$ in the case of division. The operations on intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$ may be calculated explicitly as

$$
\begin{aligned}
A + B &= [a_1 + b_1, a_2 + b_2], \\
A - B &= [a_1 - b_2, a_2 - b_1], \\
A \times B &= [\min\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}, \max\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}], \\
A/B &= [a_1, a_2] \times [1/b_2, 1/b_1] \text{ if } 0 \notin B.
\end{aligned}
$$

For addition and multiplication the associative and commutative laws hold. However

$$A(B + C) \neq AB + AC,$$

except in special cases, therefore the distributive law does not hold. Instead there is the sub-distributive law

$$A(B + C) \subseteq AB + AC.$$

2.3 Inclusion Property

One of the fundamental properties of interval computations is inclusion monotonicity property, presented in the following theorem [3].

Theorem 2.3.1 *Let* $A_k, B_k \in I(\mathbb{R})$, $k = 1, 2$, *and let* $* \in \{+, -, \times, /\}$. *Then*

$$A_k \subseteq B_k (k = 1, 2) \Rightarrow A_1 * A_2 \subseteq B_1 * B_2. \tag{2.1}$$

More generally, if $F(X_1, \ldots, X_n)$ is a rational expression in the interval variables $X_1, \ldots, X_n$, then

$$Y_1 \subseteq X_1, \ldots, Y_n \subseteq X_n \Rightarrow F(Y_1, \ldots, Y_n) \subseteq F(X_1, \ldots, X_n)$$

for every set of intervals for which the interval arithmetic operations in F are defined.

2.4 Interval Extensions and Fundamental Property of Interval Arithmetic

One of the basic problems of numerical computations is the calculation of the range of a function on an interval X with a computer. In general, it is impossible to compute the range or safe bounds of it with floating point arithmetic because of the finite representation of numbers in the computations. This problem can be solved by using interval arithmetic [41, 58].

Let $f(x)$ be a function which may be written as an expression consists only of arithmetic operations and elementary functions, let X be an interval containing x, then the interval extension of $f(x)$ to X can be obtained by replacing each occurrence of the variable x by the interval X, the arithmetic operations of x by the corresponding interval arithmetic operations. The interval extension of $f(x)$ to X is denoted by $F(X)$.

It follows from the inclusion monotonicity property (2.1) that

$$x \in X \Rightarrow f(x) \in F(X). \tag{2.2}$$

Property (2.2) is very important to almost all interval arithmetic applications and results, it provides a method to compute validated range bounds for functions. It is called the fundamental property of interval arithmetic.

2.5 Complex Interval Arithmetic

Consideration and analysis of various problems in the complex plane which either involve "inexact" data, or require some information on upper error bound of the obtained result or solution, dictate the need for a structure which is referred to as complex interval arithmetic [52, 3]. Complex interval arithmetic is a natural extension of real interval arithmetic to the complex plane. There are two reasonable choices for complex intervals: circular regions(disks) and rectangles in the complex plane.

2.5.1 Rectangular Complex Arithmetic

Definition 2.5.1 *Let $A_1, A_2 \in I(\mathbb{R})$. Then the set*

$$A = \{a = a_1 + ia_2 : a_1 \in A_1, a_2 \in A_2\}$$

of complex numbers is called a complex interval.

Sets of complex numbers as Definition 2.5.1 constitute rectangles in the complex plane with sides parallel to the coordinate axes. The set of such complex intervals is denoted by $R(\mathbb{C})$, where $\mathbb{C}$ is the set of complex numbers. A complex number $a = a_1 + ia_2$ may be considered to be a complex point interval

$$A = [a_1, a_1] + i[a_2, a_2] \in R(\mathbb{C}).$$

Furthermore, every real interval $A_1 \in I(\mathbb{R})$ may be considered to be an element $A = A_1 + i[0, 0] \in R(\mathbb{C})$, which evidently implies $I(\mathbb{R}) \subset R(\mathbb{C})$.

Definition 2.5.2 *Let $A = A_1 + iA_2$ and $B = B_1 + iB_2$ be two members of $R(\mathbb{C})$. Then A and B are equal, written $A = B$, if*

$$A_1 = B_1 \text{ and } A_2 = B_2.$$

Definition 2.5.3 *Let $* \in \{+, -, \times, /\}$ be a binary operation on elements from $I(\mathbb{R})$. Then if*

$$A = A_1 + iA_2,\ B = B_1 + iB_2 \in R(\mathbb{C}),$$

we define

$$\begin{aligned} A \pm B &= A_1 \pm B_1 + i(A_2 \pm B_2), \\ A \times B &= A_1B_1 - A_2B_2 + i(A_1B_2 + A_2B_1), \\ A/B &= (A_1B_1 + A_2B_2)/(B_1^2 + B_2^2) + i(A_2B_1 - A_1B_2)/(B_1^2 + B_2^2). \end{aligned}$$

The case A/B requires special provisions since the division defined as above yields a complex interval(rectangle) that is generally far too large in comparison with the exact range $\{z_1/z_2 | z_1 \in A, z_2 \in B\}$. Therefore, it is preferable sometimes to apply in practice the definition of division introduced by Rokne and Lancaster [59] as follows

$$A/B = A \cdot \frac{1}{B},$$

where

$$\frac{1}{B} := \inf\{X \in R(\mathbb{C}) : \{1/b : b \in B\} \subseteq X\}.$$

In this way a smaller region is obtained, but the proposed set of formulas requires a considerable computational effort.

2.5.2 Circular Complex Arithmetic

Definition 2.5.4 *Let $c \in \mathbb{C}$ be a complex number and let $r \in \mathbb{R}$ be such that $r \geq 0$. The set*

$$Z = \{z \in \mathbb{C} : |z - c| \leq r\}$$

is called a circular interval or disk.

The center c and radius r of a disk Z will be denoted by $\mathrm{mid}(Z) = c$ and $\mathrm{rad}(Z) = r$. The set of circular disks is denoted by $K(\mathbb{C})$. A disk with center c and radius r is often written as $Z = \{c\,; r\}$. Complex numbers may be considered to be special elements from $K(\mathbb{C})$ of the form $\{c\,; 0\}$. Clearly $\mathbb{C} \subset K(\mathbb{C})$.

Definition 2.5.5 *Two circular disks $A = \{c_1\,; r_1\}$ and $B = \{c_2\,; r_2\}$ are called equal, that is $A = B$, if there is set theoretic equality between them. In this case $c_1 = c_2$ and $r_1 = r_2$.*

The operations on $K(\mathbb{C})$ are introduced as generalizations of operations on complex numbers in the following manner.

Definition 2.5.6 *Let $* \in \{+, -, \times, /\}$ be a binary operation on the complex numbers. Then if $A = \{c_1\,; r_1\}$ and $B = \{c_2\,; r_2\}$ we define*

$$\begin{aligned} A + B &= \{c_1 + c_2\,; r_1 + r_2\}, \\ A - B &= \{c_1 - c_2\,; r_1 + r_2\}, \end{aligned}$$

$$\begin{aligned} A \times B &= \{c_1c_2\,;|c_1|r_2 + |c_2|r_1 + r_1r_2\}, \\ \frac{1}{Z_2} &= \left\{\frac{\bar{c}_2}{c_2\bar{c}_2 - r_2^2}\,; \frac{r_2}{c_2\bar{c}_2 - r_2^2}\right\},\ (0 \notin B), \\ A/B &= A \times \frac{1}{B},\ (0 \notin B), \end{aligned}$$

where $\bar{c}$ denotes the complex conjugate of c.

Definition 2.5.7 *For a disk $Z = \{c\,;r\}$ and an analytic function $f(z)$ with $z \in Z$, the range of $f(z)$ in Z is denoted by $f(Z) = \{f(z) : z \in Z\}$. If function $F(Z) : K(\mathbb{C}) \to K(\mathbb{C})$ is a function of Z such that*

$$f(z) \subseteq F(Z) \tag{2.3}$$

for $z \in Z$, then we call $F(Z)$ the interval extension of $f(z)$ and (2.3) is called the fundamental property of interval arithmetic.

If $f(z)$ can be written as an expression consists of elementary functions and arithmetic operations, then $F(Z)$ can be calculated by replacing each occurrence of z in $f(z)$ by Z and arithmetic operations by the corresponding circular arithmetic operations [57].

The intersection disk of two disks A and B is defined to be the smallest disk that includes the intersection $A \cap B = \{z : z \in A \text{ and } z \in B\}$ of the disks A and B.

Circular complex arithmetic provides us with a means for the evaluation of upper and lower bounds on the ranges of values of real rational functions over a disk in the complex plane.

2.6 Interval Newton Method

Newton method to compute an approximation of a zero of a function $f(x)$ can be extended to interval arithmetic to compute an enclosure of the zero of $f(x)$. The interval Newton method [35, 19, 27] was derived by Moore in the following manner. For $f(x)$, from the Mean Value Theorem,

$$f(x) - f(x^*) = (x - x^*)f'(\xi), \tag{2.4}$$

where ξ is some point between x and x^*. If x^* is a zero of $f(x)$, then $f(x^*) = 0$ and from (2.4)

$$x^* = x - \frac{f(x)}{f'(\xi)}. \tag{2.5}$$

Let X be an interval containing both x and x^*. Since ξ is between x and x^*, it follows that $\xi \in X$. Let $F'(X)$ be the interval evaluation of $f'(x)$ over the interval X, then from (2.2), $f'(\xi) \in F'(X)$. Hence, $x^* \in N(x, X)$ where

$$N(x, X) = x - \frac{f(x)}{F'(X)}.$$

We assume that $0 \notin F'(X)$ so that $N(x, X)$ is a finite interval. Since any zero of $f(x)$ in X is also in $N(x, X)$, it is in the intersection $X \cap N(x, X)$.

Based on the above fact, the following algorithm can be used to find the zero x^* of $f(x)$.

Let X_0 be an interval containing x^*. For $n = 0, 1, 2, \ldots$, define

$$\begin{aligned} x_n &= \operatorname{mid}(X_n), & (2.6)\\ N(x_n, X_n) &= x_n - \frac{f(x_n)}{f'(X_n)}, & \\ X_{n+1} &= X_n \cap N(x_n, X_n). & (2.7) \end{aligned}$$

x_n can be any number that $x_n \in X_n$. It is usually convenient and efficient to choose x_n to be the midpoint of X_n. X_n is an interval which contains x^*. Because of the intersection with X_n the sequence

$$X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$$

is bounded. It can be shown that the sequence converges quadratically to x^* [3, 43]. The method can only be applied if $0 \notin F'(X_0)$. This guarantees that $f(x)$ has only a single zero in X_0.