Appendix 1

1. Matrix form and error estimate for $se_{2m}(z,\lambda), ce_{2m}(z,\lambda)$ type

$$\mathbf{P}\mathbf{z} = \frac{1}{q^2}\mathbf{z},\ \mathbf{P} = \begin{bmatrix} d_1 & f_2 & & \mathbf{0} \\ f_2 & d_2 & f_3 & \\ & f_3 & d_3 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} : \ell^2 \to \ell^2,\ \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \end{bmatrix} \in \ell^2$$

is the common form. The components of d_i, f_{i+1}, z_i $(i = 1, 2, \ldots)$, and error estimate (EE) are shown below (Note that case (a) ($\lambda \neq (2k)^2$ $(k = 1, 2, \ldots)$ case) for $w(z) = se_{2m}(z,\lambda)$ type is omitted since that's elaborated in Section 3.2).

- $w(z) = se_{2m}(z,\lambda)$ type:

  (b) $\lambda = (2k)^2$ $(k = 1, 2, \ldots)$ case

  (i) when k is an even number,

$$(d_i, f_{i+1}) = \begin{cases} \left(\frac{1}{r_{4i}}\left(\frac{1}{r_{4i-2}} + \frac{1}{r_{4i+2}}\right), \frac{1}{r_{4i+2}\sqrt{r_{4i}}\sqrt{r_{4i+4}}}\right) & (\text{for } 1 \le i \le (k-4)/2) \\ \left(\frac{1}{r_{4i}}\left(\frac{1}{r_{4i-2}} + \frac{1}{8}\right), \frac{1}{8\sqrt{r_{4i}}\sqrt{r_{4i+8}}}\right) & (\text{for } i = (k-2)/2) \\ \left(\frac{1}{r_{4i+4}}\left(\frac{1}{8} + \frac{1}{r_{4i+6}}\right), \frac{1}{r_{4i+6}\sqrt{r_{4i+4}}\sqrt{r_{4i+8}}}\right) & (\text{for } i = k/2) \\ \left(\frac{1}{r_{4i+4}}\left(\frac{1}{r_{4i+2}} + \frac{1}{r_{4i+6}}\right), \frac{1}{r_{4i+6}\sqrt{r_{4i+4}}\sqrt{r_{4i+8}}}\right) & (\text{for } (k+2)/2 \le i) \end{cases}$$

$$z_i = \begin{cases} -\sqrt{r_{4i}}B_{4i} & (\text{for } 1 \le i \le (k-2)/2) \\ \sqrt{r_{4i+4}}B_{4i+4} & (\text{for } k/2 \le i) \end{cases}$$

  (ii) when k is an odd number,

$$(d_i, f_{i+1}) = \begin{cases} \left(\frac{1}{r_{4i}}\left(\frac{1}{r_{4i-2}} + \frac{1}{r_{4i+2}}\right), \frac{1}{r_{4i+2}\sqrt{r_{4i}}\sqrt{r_{4i+4}}}\right) & (\text{for } 1 \le i \le (k-5)/2) \\ \left(\frac{1}{r_{4i}}\left(\frac{1}{r_{4i-2}} + \frac{1}{r_{4i+2}}\right), \frac{1}{r_{4i+2}\sqrt{r_{4i}}\sqrt{8}}\right) & (\text{for } i = (k-3)/2) \\ \left(\frac{1}{8}\left(\frac{1}{r_{4i-2}} + \frac{1}{r_{4i+6}}\right), \frac{1}{r_{4i+6}\sqrt{8}\sqrt{r_{4i+8}}}\right) & (\text{for } i = (k-1)/2) \\ \left(\frac{1}{r_{4i+4}}\left(\frac{1}{r_{4i+2}} + \frac{1}{r_{4i+6}}\right), \frac{1}{r_{4i+6}\sqrt{r_{4i+4}}\sqrt{r_{4i+8}}}\right) & (\text{for } (k+1)/2 \le i) \end{cases}$$

$$z_i = \begin{cases} -\sqrt{r_{4i}}B_{4i} & (\text{for } 1 \le i \le (k-3)/2) \\ \sqrt{8}B_{4i+4} & (\text{for } i = (k-1)/2) \\ \sqrt{r_{4i+4}}B_{4i+4} & (\text{for } (k+1)/2 \le i) \end{cases}$$

$$(EE)\ q - q_n = -\frac{q^3}{2} \cdot \frac{B_{4n+4}B_{4n+8}}{r_{4n+6} \cdot (\mathbf{z}^T\mathbf{z})}[1 + o(1)]\ (n \to \infty).$$

- $w(z) = ce_{2m}(z,\lambda)$ type:

(a) $\lambda \neq (2k)^2$ case $(k = 0, 1, 2, \ldots)$

$$\begin{aligned} d_1 &= \frac{1}{r_2}\left(\frac{2}{r_0} + \frac{1}{r_4}\right), d_i = \frac{1}{r_{4i-2}}\left(\frac{1}{r_{4i-4}} + \frac{1}{r_{4i}}\right) \text{ (for } 2 \leq i); \\ f_i &= \frac{1}{r_{4i-4}\sqrt{r_{4i-6}}\sqrt{r_{4i-2}}} \text{ (for } 2 \leq i); z_i = \sqrt{r_{4i-2}}A_{4i-2} \text{(for } 1 \leq i) \\ (EE)\ q - q_n &= -\frac{q^3}{2} \cdot \frac{A_{4n-2}A_{4n+2}}{r_{4n} \cdot (\mathbf{z}^T\mathbf{z})}[1 + o(1)]\ (n \to \infty). \end{aligned}$$

(b) $\lambda = (2k)^2$ case $(k = 0, 1, 2, \ldots)$

Define the following two functions $g_1(l)$ and $g_2(l)$.

$$g_1(l) = \begin{cases} 2 & (l = 1) \\ 1 & (l \neq 1) \end{cases}, \quad g_2(l) = \begin{cases} \frac{2}{r_0 + 2r_4} & (l = 1) \\ \frac{1}{8} & (l \neq 1) \end{cases}.$$

(i) when k is an even number,

$$(d_i, f_{i+1}) = \begin{cases} \left(\frac{1}{r_{4i-2}}\left(\frac{g_1(i)}{r_{4i-4}} + \frac{1}{r_{4i}}\right), \frac{1}{r_{4i}\sqrt{r_{4i-2}}\sqrt{r_{4i+2}}}\right) & \text{(for } 1 \leq i \leq (k-4)/2) \\ \left(\frac{1}{r_{4i-2}}\left(\frac{g_1(i)}{r_{4i-4}} + \frac{1}{r_{4i}}\right), \frac{1}{r_{4i}\sqrt{r_{4i-2}}\sqrt{8}}\right) & \text{(for } i = (k-2)/2) \\ \left(\frac{1}{8}\left(\frac{g_1(i)}{r_{4i-4}} + \frac{1}{r_{4i+4}}\right), \frac{1}{r_{4i+4}\sqrt{8}\sqrt{r_{4i+6}}}\right) & \text{(for } i = k/2) \\ \left(\frac{1}{r_{4i+2}}\left(\frac{1}{r_{4i}} + \frac{1}{r_{4i+4}}\right), \frac{1}{r_{4i+4}\sqrt{r_{4i+2}}\sqrt{r_{4i+6}}}\right) & \text{(for } (k+2)/2 \leq i) \end{cases}$$

$$z_i = \begin{cases} -\sqrt{r_{4i-2}}A_{4i-2} \text{ (for } 1 \leq i \leq (k-2)/2) \\ \sqrt{8}A_{4i+2} \text{ (for } i = k/2) \\ \sqrt{r_{4i+2}}A_{4i+2} \text{ (for } (k+2)/2 \leq i) \end{cases}$$

(ii) when k is an odd number,

$$(d_i, f_{i+1}) = \begin{cases} \left(\frac{1}{r_{4i-2}}\left(\frac{g_1(i)}{r_{4i-4}} + \frac{1}{r_{4i}}\right), \frac{1}{r_{4i}\sqrt{r_{4i-2}}\sqrt{r_{4i+2}}}\right) & \text{(for } 1 \leq i \leq (k-3)/2) \\ \left(\frac{1}{r_{4i-2}}\left(\frac{g_1(i)}{r_{4i-4}} + \frac{1}{8}\right), \frac{1}{8\sqrt{r_{4i-2}}\sqrt{r_{4i+6}}}\right) & \text{(for } i = (k-1)/2) \\ \left(\frac{1}{r_{4i+2}}\left(g_2(i) + \frac{1}{r_{4i+4}}\right), \frac{1}{r_{4i+4}\sqrt{r_{4i+2}}\sqrt{r_{4i+6}}}\right) & \text{(for } i = (k+1)/2) \\ \left(\frac{1}{r_{4i+2}}\left(\frac{1}{r_{4i}} + \frac{1}{r_{4i+4}}\right), \frac{1}{r_{4i+4}\sqrt{r_{4i+2}}\sqrt{r_{4i+6}}}\right) & \text{(for } (k+3)/2 \leq i) \end{cases}$$

$$z_i = \begin{cases} -\sqrt{r_{4i-2}}A_{4i-2} \text{ (for } 1 \leq i \leq (k-1)/2) \\ \sqrt{r_{4i+2}}A_{4i+2} \text{ (for } (k+1)/2 \leq i) \end{cases}$$

$$(EE)\ q - q_n = -\frac{q^3}{2} \cdot \frac{A_{4n+2}A_{4n+6}}{r_{4n+4} \cdot (\mathbf{z}^T\mathbf{z})}[1 + o(1)]\ (n \to \infty).$$

2. Matrix form and error estimate for $se_{2m+1}(z,\lambda), ce_{2m+1}(z,\lambda)$ type

$$\mathbf{Q}\mathrm{w} = -\frac{1}{q}\mathrm{w},\ \mathbf{Q} = \begin{bmatrix} d_1 & f_2 & & \mathbf{0} \\ f_2 & d_2 & f_3 & \\ & f_3 & d_3 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} : \ell^2 \to \ell^2,\ \mathrm{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \end{bmatrix} \in \ell^2$$

is the common form. The components of d_i, f_{i+1}, w_i $(i = 1, 2, \ldots)$, and error estimate (EE) are shown below.

- $w(z) = se_{2m+1}(z, \lambda)$ type:

(a) $\lambda \neq (2k+1)^2$ case $(k = 0, 1, 2, \ldots)$

$$\begin{aligned} d_1 &= -\frac{1}{r_1}, d_i = 0 \text{ (for } 2 \leq i); f_i = \frac{1}{\sqrt{r_{2i-3}}\sqrt{r_{2i-1}}} \text{ (for } 2 \leq i); \\ w_i &= \sqrt{r_{2i-1}} B_{2i-1} \text{(for } 1 \leq i); (EE) \;\; q - q_n = \frac{q^2 B_{2n-1} B_{2n+1}}{\mathbf{w}^T \mathbf{w}} [1 + o(1)] \;\; (n \to \infty). \end{aligned}$$

(b) $\lambda = 1^2$ case

$$\begin{aligned} d_1 &= \frac{1}{r_3}, d_i = 0 \text{ (for } 2 \leq i); f_i = \frac{1}{\sqrt{r_{2i-1}}\sqrt{r_{2i+1}}} \text{ (for } 2 \leq i); \\ w_i &= \sqrt{r_{2i+1}} B_{2i+1} \text{(for } 1 \leq i); (EE) \;\; q - q_n = \frac{q^2 B_{2n+1} B_{2n+3}}{\mathbf{w}^T \mathbf{w}} [1 + o(1)] \;\; (n \to \infty). \end{aligned}$$

(c) $\lambda = (2k+1)^2$ case $(k = 1, 2, \ldots)$

$$\begin{aligned} d_1 &= -\frac{1}{r_1}, d_i = 0 \text{ (for } 2 \leq i); f_i = \begin{cases} \frac{1}{\sqrt{r_{2i-3}}\sqrt{r_{2i-1}}} & \text{(for } 2 \leq i \leq k-1) \\ \frac{1}{\sqrt{r_{2i-3}}\sqrt{8}} & \text{(for } i = k) \\ \frac{1}{\sqrt{8}\sqrt{r_{2i+3}}} & \text{(for } i = k+1) \\ \frac{1}{\sqrt{r_{2i+1}}\sqrt{r_{2i+3}}} & \text{(for } k+2 \leq i) \end{cases} \\ w_i &= \begin{cases} -\sqrt{r_{2i-1}} B_{2i-1} & \text{(for } 1 \leq i \leq k-1) \\ \sqrt{r_{2i+3}} B_{2i+3} & \text{(for } k \leq i) \end{cases} ; (EE) \;\; q - q_n = \frac{q^2 B_{2n+3} B_{2n+5}}{\mathbf{w}^T \mathbf{w}} [1 + o(1)] \;\; (n \to \end{aligned}$$

- $w(z) = ce_{2m+1}(z, \lambda)$ type:

(a) $\lambda \neq (2k+1)^2$ case $(k = 0, 1, 2, \ldots)$

$$\begin{aligned} d_1 &= \frac{1}{r_1}, d_i = 0 \text{ (for } 2 \leq i); f_i = \frac{1}{\sqrt{r_{2i-3}}\sqrt{r_{2i-1}}} \text{ (for } 2 \leq i); \\ w_i &= \sqrt{r_{2i-1}} A_{2i-1} \text{(for } 1 \leq i); (EE) \;\; q - q_n = \frac{q^2 A_{2n-1} A_{2n+1}}{\mathbf{w}^T \mathbf{w}} [1 + o(1)] \;\; (n \to \infty). \end{aligned}$$

(b) $\lambda = 1^2$ case

$$\begin{aligned} d_1 &= -\frac{1}{r_3}, d_i = 0 \text{ (for } 2 \leq i); f_i = \frac{1}{\sqrt{r_{2i-1}}\sqrt{r_{2i+1}}} \text{ (for } 2 \leq i); \\ w_i &= \sqrt{r_{2i+1}} A_{2i+1} \text{(for } 1 \leq i); (EE) \;\; q - q_n = \frac{q^2 A_{2n+1} A_{2n+3}}{\mathbf{w}^T \mathbf{w}} [1 + o(1)] \;\; (n \to \infty). \end{aligned}$$

(c) $\lambda = (2k+1)^2$ case $(k = 1, 2, \ldots)$

$$d_1 = \frac{1}{r_1}, d_i = 0 \text{ (for } 2 \leq i); f_i = \begin{cases} \frac{1}{\sqrt{r_{2i-3}}\sqrt{r_{2i-1}}} & \text{(for } 2 \leq i \leq k-1) \\ \frac{1}{\sqrt{r_{2i-3}}\sqrt{8}} & \text{(for } i = k) \\ \frac{1}{\sqrt{8}\sqrt{r_{2i+3}}} & \text{(for } i = k+1) \\ \frac{1}{\sqrt{r_{2i+1}}\sqrt{r_{2i+3}}} & \text{(for } k+2 \leq i) \end{cases}$$

$$w_i = \begin{cases} -\sqrt{r_{2i-1}}A_{2i-1} \text{ (for } 1 \leq i \leq k-1) \\ \sqrt{r_{2i+3}}A_{2i+3} \text{ (for } k \leq i) \end{cases} ;(EE) \quad q - q_n = \frac{q^2 A_{2n+3}A_{2n+5}}{\mathbf{w}^T\mathbf{w}}[1 + o(1)$$

Appendix 2

[**Appendix 2.1**] $\mathbf{V} = \mathbf{S}^T\mathbf{S}$ holds for $\mathbf{V}$ and $\mathbf{S}$ defined in (3.3.39), (3.3.40) each, and furthermore, $\mathbf{V}$ is positive definite.

[**Proof**] Let us focus on the proof for $\mathbf{V} = \mathbf{S}^T\mathbf{S}$ first. The computation of $\mathbf{S}^T\mathbf{S}$ gives

$$\mathbf{S}^T\mathbf{S} = \begin{bmatrix} e_1^2 + e_2^2 & e_2e_3 & & \mathbf{0} \\ e_2e_3 & e_3^2 + e_4^2 & e_4e_5 & \\ & e_4e_5 & e_5^2 + e_6^2 & \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix}.$$

Then, one can check component-wise now.

- It is obvious that $e_1 = 0, e_2 = \frac{1}{\sqrt{-d_0}}\frac{1}{\sqrt{2m+3}}$.
- For $n = 2, 3, \ldots$, supposing that

$$\begin{cases} e_{2n-1} &= \frac{1}{\sqrt{-d_{P_{2n-1}}}} \cdot \sqrt{\frac{(2n-2)(2m+2n-2)}{(2m+4n-5)(2m+4n-3)}} \\ &= \frac{1}{\sqrt{-d_{2n-2}}} \cdot \sqrt{\frac{(2n-2)(2m+2n-2)}{(2m+4n-5)(2m+4n-3)}}, \\ e_{2n} &= \frac{1}{\sqrt{-d_{P_{2n}}}} \cdot \sqrt{\frac{(2n-1)(2m+2n-1)}{(2m+4n-3)(2m+4n-1)}} \\ &= \frac{1}{\sqrt{-d_{2n-2}}} \cdot \sqrt{\frac{(2n-1)(2m+2n-1)}{(2m+4n-3)(2m+4n-1)}} \end{cases}$$

  hold, one may only have to show that e_{2n+1}, e_{2n+2} both satisfy (3.3.40).

- e_{2n+1} is obtained by solving $e_{2n}e_{2n+1} = \frac{\sqrt{a_{2n-2}}\sqrt{r_{2n}}}{\sqrt{-d_{2n-2}}\sqrt{-d_{2n}}}$ with respect to e_{2n+1}.

$$\begin{aligned} \sqrt{a_{2n-2}}\sqrt{r_{2n}} &= \sqrt{\frac{(2m+2n)(2m+2n-1)}{(2m+4n-1)(2m+4n+1)} \cdot \frac{(2n)(2n-1)}{(2m+4n-3)(2m+4n-1)}}, \\ e_{2n+1} &= \frac{1}{e_{2n}} \cdot \frac{\sqrt{a_{2n-2}}\sqrt{r_{2n}}}{\sqrt{-d_{2n-2}}\sqrt{-d_{2n}}} \\ &= \frac{\sqrt{-d_{2n-2}}}{\sqrt{-d_{2n-2}}\sqrt{-d_{2n}}} \cdot \sqrt{\frac{(2m+4n-3)(2m+4n-1)}{(2n-1)(2m+2n-1)}} \\ &\times \sqrt{\frac{(2m+2n)(2m+2n-1)}{(2m+4n-1)(2m+4n+1)} \cdot \frac{(2n)(2n-1)}{(2m+4n-3)(2m+4n-1)}} \\ &= \sqrt{\frac{2n(2m+2n)}{(2m+4n+1)(2m+4n-1)}} \cdot \frac{1}{\sqrt{-d_{2n}}}. \end{aligned}$$

Next, e_{2n+2} is given by $e_{2n+1}^2 + e_{2n+2}^2 = \frac{b_{2n}}{-d_{2n}}$. By this, e_{2n+2}^2 turns out to be

$$e_{2n+2}^2 = \frac{(2n+1)(2m+2n+1)}{-d_{2n}(2m+4n+1)(2m+4n+3)}.$$

Therefore, e_{2n+2} may be $e_{2n+2} = \sqrt{\frac{(2n+1)(2m+2n+1)}{-d_{2n}(2m+4n+1)(2m+4n+3)}}$.

This is enough to determine the concrete form of e_k.

Secondly, let the positive definiteness of $\mathbf{V} = \mathbf{S}^T\mathbf{S}$ be proved. One needs to show that $\mathrm{x}^T\mathrm{V}\mathrm{x} \geq 0$ holds for all $\mathrm{x} = [x_1, x_2, \ldots]^T \in \ell^2$ and the equality is valid only when $\mathrm{x} = \mathbf{0}$.

Since $\mathrm{V} = \mathbf{S}^T\mathbf{S}$,

$$\mathrm{x}^T\mathrm{V}\mathrm{x} = \mathrm{x}^T\mathrm{S}^T\mathrm{S}\mathrm{x} = (\mathrm{S}\mathrm{x})^T(\mathrm{S}\mathrm{x}) = \|\mathrm{S}\mathrm{x}\|^2 \geq 0.$$

Next, $\mathrm{x}^T\mathrm{V}\mathrm{x} = 0 \Longleftrightarrow \mathrm{x} = \mathbf{0}$ has to be shown.

It is obvious that $\mathrm{x}^T\mathrm{V}\mathrm{x} = 0$ holds when $\mathrm{x} = \mathbf{0}$. Conversely, if one assumes $\mathrm{x}^T\mathrm{V}\mathrm{x} = \|\mathrm{S}\mathrm{x}\|^2 = 0$, one finds

$$\mathrm{S}\mathrm{x} = \begin{bmatrix} e_1 & & & \mathbf{0} \\ e_2 & e_3 & & \\ & e_4 & e_5 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} e_1x_1 \\ e_2x_1 + e_3x_2 \\ e_4x_2 + e_5x_3 \\ e_6x_3 + e_7x_4 \\ \vdots \end{bmatrix} = \mathbf{0}, \text{ where } e_1 = 0,\ e_i \neq 0\ (i = 2, 3, \ldots).$$

Suppose $x_1 \neq 0$. Then,

$$x_2 = -\frac{e_2}{e_3}x_1, x_3 = \frac{e_4}{e_5}\frac{e_2}{e_3}x_1, \ldots, x_n = (-1)^{(n+1)}\left(\Pi_{i=1}^{n-1}\frac{e_{2i}}{e_{2i+1}}\right)x_1.$$

On the other hand, with $\lim_{i\to\infty}\frac{e_{2i}}{e_{2i+1}} = 1$,

$$\|\mathrm{x}\|^2 = x_1^2 + \left(\frac{e_2}{e_3}\right)^2 x_1^2 + \left(\frac{e_2}{e_3}\right)^2\left(\frac{e_4}{e_5}\right)^2 x_1^2 + \cdots \to \infty.$$

This contradicts the premise that $\mathrm{x} \in \ell^2$. Then, $x_1 = 0$, which leads to $x_2 = x_3 = \ldots = 0$ or $\mathrm{x} = 0$. This proves the proposition. ∎

Appendix 3

[**Appendix 3.1**] The class of three-term relations in (4.1.1) is equivalent to the class s below in (A3.1).

$$\text{(A3.1)} \qquad \begin{cases} d_1 y_1 + f_2 y_2 &= \lambda y_1, \\ g_k y_{k-1} + d_k y_k + f_{k+1} y_{k+1} &= \lambda y_k \ (k = 2, 3, \ldots), \end{cases}$$

where $g_n = \tilde{c}_n \cdot \mu$ $(n = 2, 3, \ldots), 0 \neq \tilde{c}_n = \tilde{C}[1 + o(1)]$ $(n \to \infty)$ with $\tilde{C}$ constant. definitions of the other symbols are retained in **Hypothesis**.

[**Proof**] The three-term relations in (A3.1) are rewritten as the next relations (A3.2). is easily found to be the same type with (4.1.1):

$$\text{(A3.2)} \qquad \begin{cases} d_1 \hat{y}_1 + \hat{f}_2 \hat{y}_2 &= \lambda \hat{y}_1, \\ \hat{f}_k \hat{y}_{k-1} + d_k \hat{y}_k + \hat{f}_{k+1} \hat{y}_{k+1} &= \lambda \hat{y}_k \ (k = 2, 3, \ldots), \end{cases}$$

where $\hat{f}_k \equiv \sqrt{f_k}\sqrt{g_k}$, $\hat{y}_k \equiv \left(\Pi_{i=2}^{k} \sqrt{f_i}/\sqrt{g_i}\right) y_k$ $(k = 2, 3, \ldots)$, and $\hat{y}_1 = y_1$. ▮

[**Appendix 3.2**] Letting " $'$ " be the differentiation with respect to μ, one finds that following equality holds:

$$\mathrm{y}^T(\mathrm{T} - \lambda \mathrm{I})\mathrm{y}' = 0.$$

[**Proof**]

$$\begin{aligned} \mathrm{y}^T(\mathrm{T} - \lambda\mathrm{I})\mathrm{y}' &= [y_1, y_2, y_3, \ldots] \begin{bmatrix} (d_1 - \lambda) y_1' + f_2 y_2' \\ f_2 y_1' + (d_2 - \lambda) y_2' + f_3 y_3' \\ f_3 y_2' + (d_3 - \lambda) y_3' + f_4 y_4' \\ \vdots \end{bmatrix} \\ &= \{(d_1 - \lambda) y_1 y_1' + f_2 y_1 y_2'\} + \{f_2 y_1' y_2 + (d_2 - \lambda) y_2 y_2' + f_3 y_2 y_3'\} \\ &\quad + \{f_3 y_2' y_3 + (d_3 - \lambda) y_3 y_3' + f_4 y_3 y_4'\} + \cdots \\ &= \{-f_2 y_1' y_2 + f_2 y_1 y_2'\} + \{f_2 y_1' y_2 - f_2 y_1 y_2' - f_3 y_2' y_3 + f_3 y_2 y_3'\} \\ &\quad + \{f_3 y_2' y_3 - f_3 y_2 y_3' - f_4 y_3' y_4 + f_4 y_3 y_4'\} + \cdots \ \text{(by (4.1.1))} \\ &= \lim_{n \to \infty} \{-f_{n+1} y_n' y_{n+1} + f_{n+1} y_n y_{n+1}'\}. \end{aligned}$$

Since $\mathrm{y} \in \ell^2$, $y_i \to 0$ $(i \to \infty)$. Then, it suffices to show $\mathrm{y}' \in \ell^2$ in order to prove $\mathrm{y}^T($' $\lambda\mathrm{I})\mathrm{y}' = 0$, for, $y_i' \to 0$ $(i \to \infty)$ directly follows. Differentiating (4.2.5) with respect t gives

$$\text{(A3.3)} \qquad (\mathbf{T} - \lambda \mathbf{I})\mathrm{y}' = \lambda' \mathrm{y} - \mathbf{T}' \mathrm{y}, \text{ where}$$

$$\mathbf{T}' = \begin{bmatrix} d_1' & f_2' & & \mathbf{0} \\ f_2' & d_2' & f_3' & \\ & f_3' & d_3' & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} = \begin{bmatrix} b_1 & c_2 & & \mathbf{0} \\ c_2 & b_2 & c_3 & \\ & c_3 & b_3 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix}.$$

Since there exist upper bounds for each of $|b_i|$ and $|c_i|$, then $\mathrm{T}'\mathrm{y} \in \ell^2$ is assured. Also, by the assumption of $|\mathrm{d}\lambda/\mathrm{d}\mu| < \infty$, $\lambda'\mathrm{y} \in \ell^2$. This means the RHS of (A3.3) $\in \ell^2$. Thus,

$$(\mathbf{T} - \lambda\mathbf{I})\mathbf{y}' \in \ell^2. \tag{A3.4}$$

So, what one only needs to prove is $D(\tilde{\mathrm{y}}) \subset \ell^2$, defining $D(\tilde{\mathrm{y}}) \equiv \{\tilde{\mathrm{y}} : (\mathrm{T}-\lambda\mathrm{I})\tilde{\mathrm{y}} \in \ell^2\}$. Namely, for $\mathrm{w} \equiv [w_1, w_2, \ldots]^T \in D(\tilde{\mathrm{y}})$, $(\mathbf{T} - \lambda\mathbf{I})\mathrm{w} \in \ell^2$ holds. Now, for

$$\mathbf{T} = \begin{bmatrix} d_1 & f_2 & & \mathbf{0} \\ f_2 & d_2 & f_3 & \\ & f_3 & d_3 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix},$$

let S and U be

$$\mathbf{S} = \begin{bmatrix} d_1 & & & \mathbf{0} \\ & d_2 & & \\ & & d_3 & \\ \mathbf{0} & & & \ddots \end{bmatrix}, \mathbf{U} = \begin{bmatrix} -\lambda & f_2 & & \mathbf{0} \\ f_2 & -\lambda & f_3 & \\ & f_3 & -\lambda & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix}.$$

Then, $\mathbf{T} - \lambda\mathbf{I} = \mathrm{S} + \mathbf{U}$. This means that $\mathrm{Sw} \in \ell^2$ (or $\|\mathrm{Sw}\|^2 < \infty$) is to hold. Equivalently,

$$\begin{aligned} \|\mathrm{Sw}\|^2 < \infty \;\Leftrightarrow\; & |d_1 w_1|^2 + |d_2 w_2|^2 + |d_3 w_3|^2 + \cdots \\ = \; & |d_1|^2 |w_1|^2 + |d_2|^2 |w_2|^2 + |d_3|^2 |w_3|^2 + \cdots < \infty. \end{aligned} \tag{A3.5}$$

Now, let $d_{\min}$ be the smallest value in $|d_i|$ $(i = 1, 2, \ldots)$. It is easily shown that $d_{\min} \neq 0$, since $d_i \neq 0$ $(i = 1, 2, \ldots)$ from the assumption. Thus, computing $d_{\min}^2 \|\mathrm{w}\|^2$ gives

$$d_{\min}^2 \|\mathrm{w}\|^2 = d_{\min}^2 |w_1|^2 + d_{\min}^2 |w_2|^2 + \cdots \leq |d_1|^2 |w_1|^2 + |d_2|^2 |w_2|^2 + \cdots < \infty,$$

by (A3.5). Therefore, $\mathrm{w} \in \ell^2$. This concludes $\mathrm{y}^T(\mathbf{T} - \lambda\mathbf{I})\mathrm{y}' = 0$. ∎