4 Theorems on Double Eigenvalues and Their Computations

It is well-known that three-term recurrence relations with two parameters often arise in solving the zeros of special functions and the eigenvalue problems (EVP) of differential equations. One class of recurrence relations, which we define and focus on in this section, covers those obtained from solving (A) the zeros of $J_\nu(z)$ (where $J_\nu(z)$ denotes the Bessel function of the first kind of order ν); (B) the zeros of $zJ'_\nu(z) + HJ_\nu(z)$; (C) and ⓑ the EVP of the Mathieu differential equation; and ⓒ the EVP of the spheroidal wave equation.

It is shown that two types of eigenvalue problems for infinite complex symmetric tridiagonal matrices defined in this section are each equivalent to a two-parameter problem of recurrence relations of a certain class. Applying Theorem A and Theorem B to such matrix eigenvalue problems turns out to give a good method for obtaining approximate eigenvalues with good accuracy. This topic will appear in Section 4.2.

Furthermore, based on the newly proved theorem guaranteeing the necessary and sufficient condition for eigenvalues' being double, we propose an algorithm for the computation of double eigenvalues by a combination of Newton's method and Theorem A or B. This will be discussed in Section 4.3.

4.1 Setting of the Problem

The recent study shows that one of the effective ways for solving certain problems of special functions is to reformulate given infinite three-term recurrence relations, usually derived by the expansion of the related function by orthogonal basis, as eigenvalue problems for infinite matrices.

Examples are the computations of:

(A) the zeros of $J_\nu(z)$ [13],[15],

(B) the zeros of $zJ'_\nu(z)+HJ_\nu(z)$ (with H constant and " $'$ " representing the partial derivative with respect to z) [7],

(C) and ⓑ the eigenvalues of the Mathieu differential equation [13],[21],

ⓒ the eigenvalues of the spheroidal wave equation [22].

Although the three-term recurrence relations obtained from these problems are different in their forms, they had given the author the feeling that they have much in common. Besides, Theorem A and Theorem B are applicable to all those reformulated matrix eigenvalue problems, which show the methods for obtaining approximated eigenvalues with good accuracy. In this section, considering the fact that these 4 problems have been handled individually, a class of three-term recurrence relations is set up which covers all these cases and apply 3 powerful theorems, or Theorem A, Theorem B, and Theorem C. Also note that the class defined here is the subset of the one which Theorem A or Theorem B may apply. That's why the computation of the zeros of the Coulomb wave function $F_L(\eta, \rho)$ is not in the above list although it may be applied by Theorem A:

[**Hypothesis**] Consider the three-term recurrence relations with two complex parameters μ and λ of the following type:

$$\left\{\begin{array}{rcl} d_1 y_1 + f_2 y_2 & = & \lambda y_1, \\ f_k y_{k-1} + d_k y_k + f_{k+1} y_{k+1} & = & \lambda y_k \ (k = 2, 3, \ldots), \end{array}\right. \tag{4.1.1}$$

where

$$\begin{array}{rcl} d_n & = & a_n + b_n \cdot \mu \neq 0 \ (n = 1, 2, \ldots), \\ f_n & = & c_n \cdot \mu \neq 0 \ (n = 2, 3, \ldots), \end{array}$$

and a_n, b_n, and c_n are constants independent of μ and λ, and of the forms

$$a_n = an^\alpha[1 + o(1)] \ (n \to \infty), \ a \neq 0, \ \alpha > 0;$$
$$|b_n| \leq const \equiv B \ (n = 1, 2, \ldots); \ 0 \neq c_n = C[1 + o(1)] \ (n \to \infty),$$

and assume that $\{y_k\}$ $(k = 1, 2, \ldots)$ is the minimal solution of (4.1.1) and not the trivial solution, or $\mathbf{y} \equiv [y_1, y_2, \ldots]^T \neq \mathbf{0}$. Given these settings, the author tackles the solution of the following two problems:

- Problem I : Solve λ satisfying (4.1.1), given a parameter $\mu \neq 0$
- Problem II : Solve $\mu \neq 0$ satisfying (4.1.1), given a parameter λ.

4.2 Reformulation into the Eigenvalue Problem for Infinite Complex Symmetric Tridiagonal Matrices

In this section, the author focuses on the reformulation of the two-parameter problem of three-term recurrence relations (4.1.1) into eigenvalue problem for an infinite matrix, and also on the proposition of the method for computing approximate solutions of Problem I (and II). First, let the behavior of the solution of (4.1.1) be analyzed, applying Theorem C.

4.2.1 Analysis of the Behavior of $\{y_n\}$ Using Theorem C

First of all, consider the three-term recurrence relations having the same coefficients as (4.1.1), or

$$(4.2.1)\qquad \begin{cases} d_1 w_1 + f_2 w_2 &= \lambda w_1, \\ f_k w_{k-1} + d_k w_k + f_{k+1} w_{k+1} &= \lambda w_k \ (k = 2, 3, \ldots), \end{cases}$$

It has to be shown that (4.2.1) satisfies all the conditions on Theorem C. Transforming (4.2.1) into the form of (2.3) gives

$$(4.2.2)\qquad w_{n+1} + \tilde{p}_n w_n + \tilde{q}_n w_{n-1} = 0 \ (n = 2, 3, \ldots),$$

where

$$\begin{aligned} \tilde{p}_n &= \frac{d_n - \lambda}{f_{n+1}} = \frac{a}{c\mu} n^{\alpha}[1 + o(1)] \equiv \tilde{p} n^{\tilde{P}}[1 + o(1)] \ (n = 2, 3, \ldots \to \infty), \\ 0 \neq \tilde{q}_n &= \frac{f_n}{f_{n+1}} = 1 + o(1) \equiv \tilde{q} n^{\tilde{Q}}[1 + o(1)] \ (n = 2, 3, \ldots \to \infty), \end{aligned}$$

with $\tilde{p} = a/(c\mu) \neq 0, \tilde{P} = \alpha \in \mathrm{R}, \tilde{q} = 1 \neq 0, \tilde{Q} = 0 \in \mathrm{R}$, and $2\tilde{P} = 2\alpha > 0 = \tilde{Q}$. Since the conditions (2.4) are satisfied, Theorem C may apply. Therefore, it is guaranteed that (4.2.1) has two solutions $\{w_{n,1}\}$ and $\{w_{n,2}\}$ with the behaviors

$$(4.2.3)\qquad \frac{w_{n+1,1}}{w_{n,1}} = -\frac{a}{c\mu} n^{\alpha}[1 + o(1)], \quad \frac{w_{n+1,2}}{w_{n,2}} = -\frac{c\mu}{a} n^{-\alpha}[1 + o(1)] \to 0 \ (n \to \infty).$$

Now, taking into account the condition in **Hypothesis** that " $\{y_n\}$ is the minimal solution of (4.1.1)", one is given the behavior of $\{y_n\}$ in (4.1.1) as:

$$(4.2.4)\qquad \frac{y_{n+1}}{y_n} = \frac{w_{n+1,2}}{w_{n,2}} = -\frac{c\mu}{a} n^{-\alpha}[1 + o(1)] \to 0 \ (n \to \infty).$$

4.2.2 Applying Theorem A and B to Problem I and II

Let (4.1.1) be reformulated as eigenvalue problems for an infinite matrix first. The definitions of d_k, f_{k+1}, y_k $(k = 1, 2, \ldots)$ and μ, λ appearing in the theorems are all retained in Hypothesis. Here comes the theorem proving the equivalence between Problem I and the eigenvalue problem stated below:

[Theorem 4.2.1] Let complex $\mu \neq 0$ be given. Then, the problem of finding λ in (4.1.1) is equivalent to finding an eigenvalue λ of the next transformation

$$(4.2.5)\qquad \begin{aligned} \mathrm{T} &= \begin{bmatrix} d_1 & f_2 & & \mathbf{0} \\ f_2 & d_2 & f_3 & \\ & f_3 & d_3 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} : D(\mathrm{T}) \to \ell^2, \\ D(\mathrm{T}) &= \{[v_1, v_2, \ldots]^T : [d_1 v_1, d_2 v_2, \ldots]^T \in \ell^2\}. \end{aligned}$$

Moreover, letting an eigenvector of $\mathbf{T}$ corresponding to λ be $\mathbf{0} \neq \mathbf{y} \in \ell^2$, or

$$\mathbf{T}\mathbf{y} = \lambda \mathbf{y}, \tag{4.2.6}$$

one finds that $\mathbf{y}$ is a non-zero scalar multiple of

$$\mathbf{y} = [y_1, y_2, y_3, \ldots]^T \in D(\mathbf{T}) \subset \ell^2. \tag{4.2.7}$$

[**Proof**] To begin with, "Problem I $\Rightarrow$ the eigenvalue problem of (4.2.6)" shall be proved.

(4.2.6) is easily obtained by rewriting (4.1.1) into matrix form. Then, what is to be proved is only $\mathbf{0} \neq \mathbf{y} \in D(\mathbf{T})$. From the condition in **Hypothesis**, $\mathbf{y} \neq \mathbf{0}$. Next, $\mathbf{y} \in D(\mathbf{T})$ or

$$|d_1 y_1|^2 + |d_2 y_2|^2 + \cdots < \infty$$

needs to be proved. For this, it suffices to show

$$R \equiv \lim_{n\to\infty} \sup \left| \frac{d_{n+1}}{d_n} \cdot \frac{y_{n+1}}{y_n} \right| < 1,$$

from [6, Theorem 8.25]. Recalling the behavior of y_{n+1}/y_n by (4.2.4), one obtains $R \to 0 < 1$, leading $\mathbf{y} \in D(\mathbf{T})$.

Next, the proof for the converse, or "the eigenvalue problem of (4.2.6) $\Rightarrow$ Problem I" should be in order.

Letting the eigenvalue of $\mathbf{T}$ defined in (4.2.5) be λ and an eigenvector of $\mathbf{T}$ corresponding to λ be $\mathbf{0} \neq \tilde{\mathbf{y}} = [\tilde{y}_1, \tilde{y}_2, \ldots]^T \in D(\mathbf{T}) \subset \ell^2$, one is given $\tilde{y}_n \to 0$ $(n \to \infty)$ (since, otherwise, $\tilde{\mathbf{y}} \notin \ell^2$ is easily derived). With the expansion of $\mathbf{T}\tilde{\mathbf{y}} = \lambda\tilde{\mathbf{y}}$, the three-term relations with the same coefficients as (4.1.1) are given, which has two independent solutions. However, it is clear that only the minimal solution $\{w_{n,2}\}$ of (4.1.1) satisfies $w_{n,2}(= y_n) \to 0$ $(n \to \infty)$. Namely, one can write $\tilde{\mathbf{y}}$ as $\tilde{\mathbf{y}} = t\mathbf{y}$ $(t \neq 0)$. ∎

In succession, now the reformulation of the problem II is done:

[**Theorem 4.2.2**] Let complex λ be given. Then, the problem of finding $\mu \neq 0$ in (4.1.1) is equivalent to finding an eigenvalue $1/\mu$ of the next transformation

$$\mathbf{A} = \begin{bmatrix} \tilde{d}_1 & \tilde{f}_2 & & \mathbf{0} \\ \tilde{f}_2 & \tilde{d}_2 & \tilde{f}_3 & \\ & \tilde{f}_3 & \tilde{d}_3 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} : \ell^2 \to \ell^2 \text{ where} \tag{4.2.8}$$

$$\begin{aligned} \tilde{d}_i &= \frac{b_i}{\lambda - a_i} \ (i = 1, 2, \ldots) \to 0 \ (i \to \infty), \\ \tilde{f}_i &= \frac{c_i}{\sqrt{\lambda - a_{i-1}}\sqrt{\lambda - a_i}} \ (i = 2, 3, \ldots) \to 0 \ (i \to \infty). \end{aligned}$$

Moreover, letting an eigenvector of $\mathbf{A}$ corresponding to $1/\mu$ be $\mathbf{0} \neq \mathbf{x} \in \ell^2$, or

$$\mathbf{A}\mathbf{x} = \frac{1}{\mu}\mathbf{x}, \tag{4.2.9}$$

one finds that x is a non-zero scalar multiple of

$$\mathbf{x} \equiv [x_1, x_2, \ldots]^T = [\sqrt{\lambda - a_1}y_1, \sqrt{\lambda - a_2}y_2, \ldots]^T \in \ell^2. \tag{4.2.10}$$

[Proof] To begin with, let's prove "Problem II $\Rightarrow$ the eigenvalue problem of (4.2.9)". first the case of $\lambda \neq a_i$ $(i = 1, 2, \ldots)$. From the three-term relations (4.1.1), (4.2.6)

$$\mathbf{T}\mathbf{y} = \begin{bmatrix} a_1 + b_1\mu & c_2\mu & & \mathbf{0} \\ c_2\mu & a_2 + b_2\mu & c_3\mu & \\ & c_3\mu & a_3 + b_3\mu & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} \mathbf{y} = \lambda\mathbf{y}$$

is derived. With $\mu \neq 0$, one can also obtain

$$\begin{bmatrix} b_1 & c_2 & & \mathbf{0} \\ c_2 & b_2 & c_3 & \\ & c_3 & b_3 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} \mathbf{y} = \frac{1}{\mu} \cdot \mathrm{diag}(\lambda - a_1, \lambda - a_2, \ldots) \cdot \mathbf{y} \tag{4.2.11}$$

by the transformation of the matrix equation, where $\mathrm{diag}(s_1, s_2, \ldots)$ denotes an infi agonal matrix having $s_1, s_2, \ldots$ as its diagonal components. Also, operating $\mathrm{diag}(1/\sqrt{}$ $1/\sqrt{\lambda - a_2}, \ldots)$ from the left on both sides of (4.2.11), one can get (4.2.9). From sumptions $\lambda \neq a_i (i = 1, 2, \ldots)$ and $\mathbf{y} \neq \mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$ is obvious. Therefore, what is lef proved is $\mathbf{x} \in \ell^2$. This is true because $\lim_{n\to\infty} \sup |x_{n+1}/x_n| < 1$ is easily assured.

On the other hand, take the case where $\lambda = a_k$ holds for some natural number k. up the $k-1, k, k+1$th lines of (4.1.1) gives

$$f_{k-1}y_{k-2} + d_{k-1}y_{k-1} + f_k y_k = \lambda y_{k-1}, \tag{4.2.12}$$

$$f_k y_{k-1} + d_k y_k + f_{k+1}y_{k+1} = \lambda y_k, \text{ and} \tag{4.2.13}$$

$$f_{k+1}y_k + d_{k+1}y_{k+1} + f_{k+2}y_{k+2} = \lambda y_{k+1}. \tag{4.2.14}$$

Since $\lambda = a_k$, (4.2.13) is rewritten as

$$c_k y_{k-1} + b_k y_k + c_{k+1}y_{k+1} = 0.$$

Supposing $b_k \neq 0$, one gets

$$y_k = -\frac{c_k}{b_k}y_{k-1} - \frac{c_{k+1}}{b_k}y_{k+1}.$$

Substituting this into (4.2.12) and (4.2.14) yields

$$f_{k-2}y_{k-2} + \left\{a_{k-1} + \left(b_{k-1} - \frac{c_k^2}{b_k}\right)\mu\right\} y_{k-1} - \frac{c_{k+1}}{b_k} f_k y_{k+1} = \lambda y_{k-1}, \tag{4.2.15}$$

$$-\frac{c_k}{b_k} f_{k+1} y_{k-1} + \left\{a_{k+1} + \left(b_{k+1} - \frac{c_{k+1}^2}{b_k}\right)\mu\right\} y_{k+1} + f_{k+2}y_{k+2} = \lambda y_{k+1}. \tag{4.2.16}$$

If one regards the first to $k-2$th relation of (4.1.1), together with (4.2.15), (4.2.16), and the relations later than the $k+2$th lines again of (4.1.1) as newly defined three-term relations, one can take the same procedure as the last case where $\lambda \neq a_i$ $(i = 1, 2, \ldots)$. If there still exists another l which satisfies $a_k = \lambda = a_l$ $(l \neq k)$, you redo the same process. Note that this is finished in a finite number of times, considering the behavior of a_n $(n = 1, 2, \ldots)$.

On the contrary, suppose $b_k = 0$. By (4.2.13), $y_{k+1} = -\frac{c_k}{c_{k+1}} y_{k-1}$ is derived. Computing (4.2.12) $\times f_{k+1}-$ (4.2.14) $\times f_k$, with $y_{k+1} = -\frac{c_k}{c_{k+1}} y_{k-1}$ gives

$$\begin{aligned} f_{k-1}c_{k+1}^2 y_{k-2} + \left\{(a_{k+1}c_k^2 + a_{k-1}c_{k+1}^2) + (b_{k+1}c_k^2 + b_{k-1}c_{k+1}^2)\mu\right\} y_{k-1} \\ -f_{k+2}c_{k+1}c_k y_{k+2} = \lambda(c_{k+1}^2 + c_k^2) y_{k-1}, \end{aligned} \tag{4.2.17}$$

which is composed of y_{k-2}, y_{k-1}, and y_{k+2} terms. By regarding the first to $k-2$th, (4.2.17), and the equations later than the $k+2$th of (4.1.1) as new three-term relations, one only has to apply the same procedure as $\lambda \neq a_i$ $(i = 1, 2, \ldots)$ case.

The proof for "the eigenvalue problem of (4.2.9) $\Rightarrow$ Problem II".

Take first the case where 0 is an eigenvalue of $\mathbf{A}$. Then, assuming the corresponding eigenvector to be $\mathbf{u} = [u_1, u_2, \ldots]^T \in \ell^2$ gives the equation $\mathbf{A}\mathbf{u} = 0 \cdot \mathbf{u} = \mathbf{0}$. Expanding this gives

$$\begin{aligned} b_1 u_1 + c_2 u_2 &= 0, \\ c_k u_{k-1} + b_k u_k + c_{k+1} u_{k+1} &= 0 \ (k = 2, 3, \ldots). \end{aligned} \tag{4.2.18}$$

Under this setting, we shall show that the solutions $\{u_i\}$ $(i = 1, 2, \ldots)$ can't be the solutions of (4.1.1). (4.1.1) is transformed into

$$\begin{aligned} \frac{a_1 + b_1\mu - \lambda}{\mu} y_1 + c_2 y_2 &= 0, \\ c_k y_{k-1} + \frac{a_k + b_k\mu - \lambda}{\mu} y_k + c_{k+1} y_{k+1} &= 0 \ (k = 2, 3, \ldots). \end{aligned} \tag{4.2.19}$$

One will find that $\lambda = a_k$ for $k = 1, 2, \ldots$ have to hold, in order for the coefficients of u_i in (4.2.18) and the counterparts of y_i in (4.2.19) to be the same. This, however, turns out to be impossible since $a_k \to \infty$ $(k \to \infty)$. This means that if 0 is an eigenvalue of A, the solution $\{u_i\}$ $(i = 1, 2, \ldots)$ can't be a solution of (4.1.1). Thus, you may have the eigenvalue of $\mathbf{A}$ defined in the form of $1/\mu$, and successively, let $\tilde{\mathbf{x}} \in \ell^2$ be an eigenvector of $\mathbf{A}$ corresponding to $1/\mu$. Defining also $\tilde{x}_i$ $(i = 1, 2, \ldots)$ by $\tilde{\mathbf{x}} = [\sqrt{\lambda - a_1}\tilde{x}_1, \sqrt{\lambda - a_2}\tilde{x}_2, \ldots] \in \ell^2$ gives directly $\tilde{x}_n \to 0$ $(n \to \infty)$, since $\left|\sqrt{\lambda - a_n}\right| \to \infty$ $(n \to \infty)$. Expanding $\mathbf{A}\tilde{\mathbf{x}} = (1/\mu)\tilde{\mathbf{x}}$ yields three-term relations with the same coefficients as (4.1.1) which have two solutions in (4.2.3). However, the only minimal solution, $\{w_{n,2}\}$, satisfies $w_{n,2}(= y_n) \to 0$ $(n \to \infty)$. Therefore, one can write $\tilde{\mathbf{x}} = \tilde{t}\mathbf{x}$ $(\tilde{t} \neq 0)$. ∎

Thus, Problem I and II were successfully reformulated as the problems for the computations of eigenvalues for infinite matrices. In the following Theorem 4.2.3 and Theorem 4.2.4, the benefits by the application of Theorem A and Theorem B into (4.2.6) and (4.2.9), are unveiled.

[**Theorem 4.2.3**] Given $\mu \neq 0$, assume that $\lambda \neq 0$ is a simple eigenvalue of (4.2.5), $\mathbf{y}^T\mathbf{y} \neq 0$, and $\mathbf{T}^{-1}$ exists. Then, for each n, if one properly takes λ_n, one of the eigenvalues of $\mathbf{T}_n$, and $\mathbf{T}_n$ is the nth principal submatrix of $\mathbf{T}$, one has $\lambda_n \to \lambda$. Furthermore, the following error estimate is valid:

$$\lambda - \lambda_n = \frac{c\mu y_n y_{n+1}}{\mathbf{y}^T\mathbf{y}}[1 + o(1)] \to 0 \ (n \to \infty). \tag{4.2.20}$$

[**Proof**] It suffices to prove that the problem (4.2.6) may be applied by Theorem B. Then, one only has to show $f_{n+1}y_{n+1}/y_n \to 0$ $(n \to \infty)$. In fact, $f_{n+1}y_{n+1}/y_n \to 0$ $(n \to \infty)$ since $y_{n+1}/y_n \to 0$ $(n \to \infty)$ is guaranteed by (4.2.4). The substitution of each component of $\mathbf{T}, \mathbf{y}$ into (2.2) gives (4.2.20).

[**Lemma 4.2.1**] The condition "$\mathbf{T}^{-1}$ exists" in Theorem 4.2.3 can virtually be rid.

[**Proof**] Let it be proved by showing that $(\mathbf{T} + \alpha\mathbf{I})^{-1}$ exists with α taken appropriately. Namely, let us show that one always can choose α such that none of the Gershgorin Discs for $\mathbf{T} + \alpha\mathbf{I}$ includes the origin. With the facts that the center of each Gershgorin Disc d_n behaves as $d_n = a_n + b_n \cdot \mu$ $(n = 1, 2, \ldots), a_n = an^{\alpha}[1 + o(1)] \to \infty$ $(n \to \infty)$, and that the radius of each disc is bounded, it is enabled that all of the discs of $\mathbf{T} + \alpha\mathbf{I}$ with α thus taken are situated over the real axis of the complex plane (or below, depending on a). More concretely, α shall be taken as

$$\alpha \equiv \sup_{i=1,2,\ldots}\left\{\frac{(1 - \mathrm{sign}(\mathrm{Img}(d_i) \cdot \mathrm{Img}(a)))}{2} \cdot |\mathrm{Img}(d_i)|\right\} + \varepsilon + 2\bar{C},$$

where $\varepsilon > 0, \bar{C}$ is the upper bound of $|c_n|$ (or $|c_n| \leq \bar{C}$). ∎

[**Theorem 4.2.4**] Given λ, assume that $1/\mu$ is a simple eigenvalue of (4.2.8) and $\mathbf{x}^T\mathbf{x} \neq 0$. Then, for each n, if one properly takes $1/\mu_n$, one of the eigenvalues of $\mathbf{A}_n$, and $\mathbf{A}_n$ is the nth principal submatrix of $\mathbf{A}$, one has $\mu_n \to \mu$. Furthermore, the following error estimate is valid:

$$(4.2.21) \qquad \mu - \mu_n = -\frac{c\mu^2 y_n y_{n+1}}{\mathbf{x}^T\mathbf{x}}[1+o(1)] \to 0 \ (n \to \infty).$$

[**Proof**] What is required to show is that x_{n+1}/x_n are bounded for sufficiently large n, in order to guarantee that (4.2.9) may be applied by Theorem A. It has been proved that $x_n \neq 0$ for sufficiently large n, when

$$\left|\frac{x_{n+1}}{x_n}\right|^2 = \left|\frac{(\lambda - a_{n+1})y_{n+1}^2}{(\lambda - a_n)y_n^2}\right| \to 0 \ (\text{by } (4.2.4)).$$

Applying Theorem A gives

$$\begin{aligned}(1/\mu) - (1/\mu_n) &= \frac{\tilde{f}_{n+1}x_n x_{n+1}}{\mathbf{x}^T\mathbf{x}}[1+o(1)] \\ &= \frac{\frac{c_{n+1}}{\sqrt{\lambda - a_n}\sqrt{\lambda - a_{n+1}}} \cdot \sqrt{\lambda - a_n}y_n\sqrt{\lambda - a_{n+1}}y_{n+1}}{\mathbf{x}^T\mathbf{x}}[1+o(1)].\end{aligned}$$

This eventually turns (4.2.21). ∎

4.2.3 Acceleration of Approximate Eigenvalues (for $b_i = 0$ $(i = 1, 2, \ldots)$ Case)

Consider the case where $b_i = 0$ $(i = 1, 2, \ldots)$ are satisfied in **Hypothesis**, or all the diagonal components of $\mathbf{A}$ vanish. Taking advantage of this fact, we show that one can compute approximate eigenvalues for Problem II with better rate of convergence. Such a case subsumes (A) the zeros of $J_\nu(z)$, (B) the zeros of $zJ'_\nu(z) + HJ_\nu(z)$, (C) and ⓑ the eigenvalues of the Mathieu equation of ce_{2m}, se_{2m} types.

In Theorem 4.2.2, the substitution of $b_i = 0$ $(i = 1, 2, \ldots)$ or $\tilde{d}_i = 0$ $(i = 1, 2, \ldots)$ into (4.2.9) gives

$$(4.2.22) \qquad \mathbf{A}\mathbf{x} = \frac{1}{\mu}\mathbf{x}, \ \mathbf{A} = \begin{bmatrix} 0 & \tilde{f}_2 & & 0 \\ \tilde{f}_2 & 0 & \tilde{f}_3 & \\ & \tilde{f}_3 & 0 & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix},$$

where $\tilde{f}_i$ and $\mathbf{x}$ are the ones defined in (4.2.8), (4.2.10). Then,

[**Lemma 4.2.2**] The eigenvalue problem of (4.2.22) is equivalent to the two eigenvalue problems for the compact matrices $\mathbf{U}, \mathbf{V}$ defined below:

$$(4.2.23) \quad \mathbf{U}\mathbf{x}_1 = \frac{1}{\mu^2}\mathbf{x}_1, \text{ where}$$

$$(4.2.24)\quad \mathrm{U} = \begin{bmatrix} \tilde f_2^2 & \tilde f_2\tilde f_3 & & 0 \\ \tilde f_2\tilde f_3 & \tilde f_3^2+\tilde f_4^2 & \tilde f_4\tilde f_5 & \\ & \tilde f_4\tilde f_5 & \tilde f_5^2+\tilde f_6^2 & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix} : \ell^2 \to \ell^2, \mathrm{x}_1 = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ \vdots \end{bmatrix} \in \ell^2.$$

$$(4.2.25)\quad \mathrm{V}\mathrm{x}_2 = \frac{1}{\mu^2}\mathrm{x}_2, \text{ where}$$

$$(4.2.26)\quad \mathrm{V} = \begin{bmatrix} \tilde f_2^2+\tilde f_3^2 & \tilde f_3\tilde f_4 & & 0 \\ \tilde f_3\tilde f_4 & \tilde f_4^2+\tilde f_5^2 & \tilde f_5\tilde f_6 & \\ & \tilde f_5\tilde f_6 & \tilde f_6^2+\tilde f_7^2 & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix} : \ell^2 \to \ell^2, \mathrm{x}_2 = \begin{bmatrix} x_2 \\ x_4 \\ x_6 \\ \vdots \end{bmatrix} \in \ell^2.$$

[**Proof**] $\mathrm{A}^2\mathrm{x} = \frac{1}{\mu^2}\mathrm{x}$ is easily derived from $\mathrm{A}\mathrm{x} = \frac{1}{\mu}\mathrm{x}$, where

$$\mathrm{A}^2 = \begin{bmatrix} 0 & \tilde f_2 & & 0 \\ \tilde f_2 & 0 & \tilde f_3 & \\ & \tilde f_3 & 0 & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} 0 & \tilde f_2 & & 0 \\ \tilde f_2 & 0 & \tilde f_3 & \\ & \tilde f_3 & 0 & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix} = \begin{bmatrix} \tilde f_2^2 & 0 & \tilde f_2\tilde f_3 & & 0 \\ 0 & \tilde f_2^2+\tilde f_3^2 & 0 & \tilde f_3\tilde f_4 & \\ \tilde f_2\tilde f_3 & 0 & \tilde f_3^2+\tilde f_4^2 & 0 & \ddots \\ 0 & \tilde f_3\tilde f_4 & 0 & \ddots & \ddots \end{bmatrix}.$$

Expanding $\mathrm{A}^2\mathrm{x} = \frac{1}{\mu^2}\mathrm{x}$ and reformulating in matrix form from the set of odd- and even-numbered relations give the new eigenvalue problem for U, V each. ∎

[**Lemma 4.2.3**] The following relation is valid using x_1 and x_2 defined in Lemma 4.2.2:

$$\mathrm{x}_1^T\mathrm{x}_1 = \mathrm{x}_2^T\mathrm{x}_2.$$

[**Proof**] Expanding (4.2.22) gives

$$\begin{aligned} \tilde f_2 x_2 &= \frac{1}{\mu}x_1, \\ \tilde f_k x_{k-1} + \tilde f_{k+1}x_{k+1} &= \frac{1}{\mu}x_k \ (k = 2, 3, \ldots). \end{aligned}$$

Multiplying $(-1)^l x_l$ on both sides of the lth equation ($l = 1, 2, \ldots$) and adding them up derive

$$\text{LHS } = 0, \text{RHS } = \frac{1}{\mu}(-x_1^2 + x_2^2 - x_3^2 + x_4^2 - \cdots).$$

Therefore, $x_1^2 + x_3^2 + x_5^2 + \cdots = x_2^2 + x_4^2 + x_6^2 + \cdots$. Namely, $\mathrm{x}_1^T\mathrm{x}_1 = \mathrm{x}_2^T\mathrm{x}_2$. ∎

[**Lemma 4.2.4**] Assume in the eigenvalue problems (4.2.23) and (4.2.25) that $1/\mu^2$ are simple eigenvalues and $\mathrm{x}_1^T\mathrm{x}_1 (= \mathrm{x}_2^T\mathrm{x}_2) \neq 0$ holds. Then, the rate of convergence for approximate eigenvalues of (4.2.25) is faster than those of (4.2.9), (4.2.23).

[Proof] Let us skip the discussion that the rate of convergence for (4.2.9) is the slowes among the three since it's obvious by the formulas. Let the approximate eigenvalue by the eigenvalue problem of $\mathbf{U}$ be $\tilde{\mu}_n$, and likewise, the one of $\mathbf{V}$ be $\hat{\mu}_n$. It suffices to show $(\mu - \hat{\mu}_n)/(\mu - \tilde{\mu}_n) \to 0$. Applying Theorem A, one finds that each of the error estimate for $\{\tilde{\mu}_n\}, \{\hat{\mu}_n\}$ is

$$\begin{aligned}\mu - \tilde{\mu}_n &= -\frac{\mu^3}{2} \cdot \frac{\tilde{f}_{2n}\tilde{f}_{2n+1}x_{2n-1}x_{2n+1}}{\mathbf{x}_1^T\mathbf{x}_1},\\ \mu - \hat{\mu}_n &= -\frac{\mu^3}{2} \cdot \frac{\tilde{f}_{2n+1}\tilde{f}_{2n+2}x_{2n}x_{2n+2}}{\mathbf{x}_2^T\mathbf{x}_2}.\end{aligned}$$

The computation of $(\mu - \hat{\mu}_n)/(\mu - \tilde{\mu}_n)$ leads

$$\begin{aligned}\frac{\mu - \hat{\mu}_n}{\mu - \tilde{\mu}_n} &= \left(\frac{\mathbf{x}_2^T\mathbf{x}_2}{\mathbf{x}_1^T\mathbf{x}_1}\right) \cdot \frac{\tilde{f}_{2n+2}x_{2n}x_{2n+2}}{\tilde{f}_{2n}x_{2n-1}x_{2n+1}} = \frac{\tilde{f}_{2n+2}x_{2n}x_{2n+2}}{\tilde{f}_{2n}x_{2n-1}x_{2n+1}} \quad (\text{by } \mathbf{x}_1^T\mathbf{x}_1 = \mathbf{x}_2^T\mathbf{x}_2)\\ &= \left(\frac{c_{2n+2}}{c_{2n}}\right) \cdot \left(\frac{\lambda - a_{2n}}{\lambda - a_{2n+1}}\right) \cdot \left(\frac{y_{2n}}{y_{2n-1}}\right) \cdot \left(\frac{y_{2n+2}}{y_{2n+1}}\right) \to 0.\end{aligned}$$

$$\left(\left|\frac{\lambda - a_{2n}}{\lambda - a_{2n+1}}\right| \le \text{const by } |a_n| \to \infty, \text{and } \frac{y_{n+1}}{y_n} \to 0 \ (n \to \infty) \text{ by (4.2.4)}\right). \quad (4.2.27)$$ ∎

4.3 The Computation of Double Eigenvalues

4.3.1 The Theorems on Double Eigenvalues

In this section, let us regard μ and λ in (4.1.1) not as parameters (as previously set) but as variables and discuss the pairs (μ, λ) which satisfy $\mathrm{d}\lambda/\mathrm{d}\mu = 0$ or $\mathrm{d}\mu/\mathrm{d}\lambda = 0$. It will be soon proved that $\mathrm{d}\lambda/\mathrm{d}\mu = 0$ is equivalent to an eigenvalue of $\mathbf{A}$ being double (Likewise, $\mathrm{d}\mu/\mathrm{d}\lambda = 0$ is equivalent to an eigenvalue of $\mathbf{T}$ being double). This implies that Theorem 4.2.2, which only deals with the computation of simple eigenvalues of a matrix, may not solve Problem II when a pair of (μ, λ) satisfies $\mathrm{d}\lambda/\mathrm{d}\mu = 0$ (Again, likewise, Theorem 4.2.1 may not solve Problem I for (μ, λ) satisfying $\mathrm{d}\mu/\mathrm{d}\lambda = 0$. Therefore, an approach is indispensable for computing such pairs. Let's call such pairs (μ, λ) satisfying $\mathrm{d}\lambda/\mathrm{d}\mu = 0$ or $\mathrm{d}\mu/\mathrm{d}\lambda = 0$ as double pairs. Before proceeding to the discussion, let us define a double eigenvalue of a matrix in ℓ^2:

[Definition] Let an eigenvalue of matrix $\mathbf{X}$ be ν, and its corresponding eigenvector of $\mathbf{X}$ be $\mathbf{u} \in \ell^2$. The eigenvalue ν is double if and only if there exists a generalized eigenvector $\mathbf{v}$ satisfying

$$\begin{aligned}(\mathbf{X} - \nu\mathbf{I})\mathbf{v} &\equiv \mathbf{u} \neq \mathbf{0}, \quad \mathbf{u}, \mathbf{v} \in \ell^2,\\ \text{and } (\mathbf{X} - \nu\mathbf{I})^2\mathbf{v} &= (\mathbf{X} - \nu\mathbf{I})\mathbf{u} = \mathbf{0}.\end{aligned} \quad (4.3.1)$$

Then, let us proceed to the introduction of a new formula obtained by (4.2.6) and (

[**Lemma 4.3.1**] The following relation holds among λ, μ, and $\mathbf{y}, \mathbf{x}$ defined in Theoren and Theorem 4.2.2, respectively:

(4.3.2) $$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}\right) \cdot (\mathbf{y}^T\mathbf{y}) = \frac{1}{\mu} \cdot (\mathbf{x}^T\mathbf{x}),$$

when $|\mathrm{d}\lambda/\mathrm{d}\mu| < \infty$ is assumed.

[**Proof**] Differentiating (4.2.6) with respect to μ gives

$$\begin{aligned} \mathbf{T}'\mathbf{y} + \mathbf{T}\mathbf{y}' &= \lambda'\mathbf{y} + \lambda\mathbf{y}', \\ \lambda'\mathbf{y} &= (\mathbf{T} - \lambda\mathbf{I})\mathbf{y}' + \mathbf{T}'\mathbf{y}. \end{aligned}$$

Operating $\mathbf{y}^T$ on both sides from the left yields

$$\begin{aligned} \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}\right)\mathbf{y}^T\mathbf{y} &= \mathbf{y}^T(\mathbf{T} - \lambda\mathbf{I})\mathbf{y}' + \mathbf{y}^T\mathbf{T}'\mathbf{y} \\ &= \mathbf{y}^T\mathbf{T}'\mathbf{y} \quad (\mathbf{y}^T(\mathbf{T} - \lambda\mathbf{I})\mathbf{y}' = 0 \text{ by Appendix 3.2}). \end{aligned}$$

If one notices that

$$\mathbf{T}'\mathbf{y} = \begin{bmatrix} d_1' & f_2' & & \mathbf{0} \\ f_2' & d_2' & f_3' & \\ & f_3' & d_3' & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} \mathbf{y} = \begin{bmatrix} b_1 & c_2 & & \mathbf{0} \\ c_2 & b_2 & c_3 & \\ & c_3 & b_3 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} \mathbf{y}$$

is equal to the LHS of (4.2.11), $\mathbf{y}^T\mathbf{T}'\mathbf{y}$ is changed into

$$\begin{aligned} \mathbf{y}^T\mathbf{T}'\mathbf{y} &= \mathbf{y}^T \cdot \frac{1}{\mu} \cdot \mathrm{diag}(\lambda - a_1, \lambda - a_2, \ldots)\mathbf{y} \\ &= \frac{1}{\mu}\{(\lambda - a_1)y_1^2 + (\lambda - a_2)y_2^2 + \cdots\} = \frac{1}{\mu} \cdot (\mathbf{x}^T\mathbf{x}). \end{aligned}$$

Consequently, $$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}\right) \cdot (\mathbf{y}^T\mathbf{y}) = \frac{1}{\mu} \cdot (\mathbf{x}^T\mathbf{x}). \blacksquare$$

[**Theorem 4.3.1**] When $\mathbf{y}^T\mathbf{y} \neq 0$, the next 3 conditions are equivalent:

(*a*) an eigenvalue $1/\mu$ of $\mathbf{A}$ in (4.2.8) is double; (*b*) $\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = 0$; (*c*) $\mathbf{x}^T\mathbf{x} = 0$.

[**Proof**] (*b*) $\Leftrightarrow$ (*c*) is obvious from Lemma 4.3.1. What remains to be proved is (*a*) $\Leftrightarrow$ Let us begin with (*a*) $\Rightarrow$ (*c*) to prove (*a*) $\Rightarrow$ (*b*). Since the eigenvalue $1/\mu$ of $\mathbf{A}$ is dou there exists a generalized eigenvector $\mathbf{v} \in \ell^2$ which satisfies

(4.3.3) $$\begin{aligned} (\mathbf{A} - \frac{1}{\mu}\mathbf{I})\mathbf{v} &\equiv \mathbf{x} \neq \mathbf{0}, \quad \mathbf{x}, \mathbf{v} \in \ell^2, \\ \text{and } (\mathbf{A} - \frac{1}{\mu}\mathbf{I})^2\mathbf{v} &= (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}, \end{aligned}$$

from its definition. Now the computation of $x^T x$ leads

$$\begin{aligned} x^T x &= \{(A - \frac{1}{\mu}I)v\}^T (A - \frac{1}{\mu}I)v = v^T (A - \frac{1}{\mu}I)^2 v \text{ (by the symmetry of A)} \\ &= v^T 0 \text{ (by } (A - \frac{1}{\mu}I)^2 v = 0) = 0. \end{aligned}$$

Let us move on to the proof of $(b) \Rightarrow (a)$. With the supposition $d\lambda/d\mu = 0$, it suffices to show the existence of an eigenvector which corresponds to $v \in \ell^2$ in (4.3.3). Differentiating (4.2.8) with respect to μ gains

$$A'x + \frac{1}{\mu^2}x = -(A - \frac{1}{\mu}I)x'. \tag{4.3.4}$$

Since A is dependent only on λ, one finds that

$$A' = \frac{dA}{d\lambda} \cdot \frac{d\lambda}{d\mu} = \frac{dA}{d\lambda} \cdot 0 = 0,$$

when, it is led that

$$x = -\mu^2 (A - \frac{1}{\mu}I)x' \tag{4.3.5}$$

from (4.3.4). What only needs to be proved to show the existence of a generalized eigenvector is $0 \neq x' \in \ell^2$. (one can take $(-\mu^2 x')$ as w in (4.3.1)).

First, $x' \neq 0$ is easily shown, for, the assumption $x' = 0$ directly leads to $x = 0$ from (4.3.5) which contradicts x to be nonzero. Secondly, $x' \in \ell^2$ shall be proved. The differentiation of the both sides of $x = [\sqrt{\lambda - a_1}y_1, \sqrt{\lambda - a_2}y_2, \ldots]^T$ with respect to μ results in

$$\begin{aligned} x' &= \left[\frac{1}{2\sqrt{\lambda - a_1}}\left(\frac{d\lambda}{d\mu}\right)y_1 + \sqrt{\lambda - a_1}y_1', \frac{1}{2\sqrt{\lambda - a_2}}\left(\frac{d\lambda}{d\mu}\right)y_2 + \sqrt{\lambda - a_2}y_2', \ldots\right]^T \\ &= \left[\sqrt{\lambda - a_1}y_1', \sqrt{\lambda - a_2}y_2', \ldots\right]^T \quad \left(\text{since } \frac{d\lambda}{d\mu} = 0\right). \text{ Hence} \qquad (4.3.6) \\ \|x'\|^2 &= \left|\sqrt{\lambda - a_1}y_1'\right|^2 + \left|\sqrt{\lambda - a_2}y_2'\right|^2 + \cdots = \left|(\lambda - a_1)y_1'^2\right| + \left|(\lambda - a_2)y_2'^2\right| + \cdots. \end{aligned}$$

By differentiating the both sides of (4.1.1) with respect to μ, one is given, defining $y_0 = y_0' \equiv 0$,

$$c_n y_{n-1} + f_n y_{n-1}' + b_n y_n + d_n y_n' + c_{n+1} y_{n+1} + f_{n+1} y_{n+1}' = \left(\frac{d\lambda}{d\mu}\right) y_n + \lambda y_n', \tag{4.3.7}$$

for $n = 1, 2, \ldots$. Putting $d\lambda/d\mu = 0$ and multiplying y_n on both sides of (4.3.7) turn

$$(\lambda - a_n)y_n'^2 = c_n y_{n-1} y_n' + f_n y_{n-1}' y_n' + c_{n+1} y_n' y_{n+1} + f_{n+1} y_n' y_{n+1} + b_n y_n y_n' + b_n \mu y_n'^2.$$

This helps to prove $x' \in \ell^2$, for,

$$\begin{aligned}
\|x'\|^2 &= \left|c_2 y_1' y_2 + f_2 y_1' y_2' + b_1 y_1 y_1' + b_1 \mu y_1'^2\right| \\
&+ \left|c_2 y_1 y_2' + f_2 y_1' y_2' + c_3 y_2' y_3 + f_3 y_2' y_3 + b_2 y_2 y_2' + b_2 \mu y_2'^2\right| + \cdots \\
&\leq \{|c_2| \cdot |y_1| \cdot |y_2'| + |c_3| \cdot |y_2| \cdot |y_3'| + |c_4| \cdot |y_3| \cdot |y_4'| + \cdots\} \\
&+ \{|f_2| \cdot |y_1'| \cdot |y_2'| + |f_3| \cdot |y_2'| \cdot |y_3'| + |f_4| \cdot |y_3'| \cdot |y_4'| + \cdots\} \\
&+ \{|c_2| \cdot |y_1'| \cdot |y_2| + |c_3| \cdot |y_2'| \cdot |y_3| + |c_4| \cdot |y_3'| \cdot |y_4| + \cdots\} \\
&+ \{|f_2| \cdot |y_1'| \cdot |y_2'| + |f_3| \cdot |y_2'| \cdot |y_3'| + |f_4| \cdot |y_3'| \cdot |y_4'| + \cdots\} \\
&+ \{|b_1| \cdot |y_1| \cdot |y_1'| + |b_2| \cdot |y_2| \cdot |y_2'| + |b_3| \cdot |y_3| \cdot |y_3'| + \cdots\} \\
&+ \{|b_1| \cdot |\mu| \cdot |y_1'|^2 + |b_2| \cdot |\mu| \cdot |y_2'|^2 + |b_3| \cdot |\mu| \cdot |y_3'|^2 + \cdots\} \\
&\leq \sup_i |c_i| \cdot \{|y_1| \cdot |y_2'| + |y_2| \cdot |y_3'| + |y_3| \cdot |y_4'| + \cdots\} \\
&+ 2\sup_i |c_i| \cdot |\mu| \cdot \{|y_1'| \cdot |y_2'| + |y_2'| \cdot |y_3'| + |y_3'| \cdot |y_4'| + \cdots\} \\
&+ \sup_i |c_i| \cdot \{|y_1'| \cdot |y_2| + |y_2'| \cdot |y_3| + |y_3'| \cdot |y_4| + \cdots\} \\
&+ B \cdot \{|y_1| \cdot |y_1'| + |y_2| \cdot |y_2'| + |y_3| \cdot |y_3'| + \cdots\} \\
&+ B \cdot |\mu| \cdot \{|y_1'|^2 + |y_2'|^2 + |y_3'|^2 + \cdots\}.
\end{aligned}$$

This contributes to $\|x'\|^2 < \infty$ since, if we define $\tilde{y}_1 \equiv [|y_1|, |y_2|, \ldots]^T$, $\tilde{y}_2 \equiv [|y_2'|, |y_3'|, \ldots]^T$, for instance, then

$$\tilde{y}_1 \in \ell^2 \text{ and } \tilde{y}_2 \in \ell^2 \text{ (by } y' \in \ell^2\text{. See Appendix 3.2 for details),}$$

and by Cauchy-Schwarz's inequality,

$$|y_1|\,|y_2'| + |y_2|\,|y_3'| + \cdots = (\tilde{y}_1, \tilde{y}_2) \leq \|\tilde{y}_1\| \cdot \|\tilde{y}_2\| = \sqrt{|y_1|^2 + |y_2|^2 + \cdots} \cdot \sqrt{|y_2'|^2 + |y_3'|^2 + \cdots} < \infty$$

and

$$\sum_{i=1}^{\infty} |y_i'| \left|y_{i+1}'\right| < \infty, \ \sum_{i=1}^{\infty} |y_i'|\,|y_{i+1}| < \infty, \ \sum_{i=1}^{\infty} |y_i|\,|y_i'| < \infty,$$

in analogy. Therefore, $x' \in \ell^2$. ∎

[**Remark 4.3.1**] In (4.2.5), if $\mathbf{T}$ is a real matrix, the three conditions (a), (b), (c) in Theorem 4.3.1 are unconditionally equivalent.

[**Proof**] When $\mathbf{T}$ is real (symmetric), $\mathbf{y}$ is a real vector, obviously from (4.1.1) (or in the form of a real vector multiplied by a complex scalar). Since $\mathbf{y} \neq \mathbf{0}$, then $\mathbf{y}^T\mathbf{y} \neq 0$. This is enough to prove the proposition. ∎

[**Remark 4.3.2**] One may delete the condition "$1/\mu$ is assumed to be a simple eigenvalue" in Theorem 4.2.4.

[Proof] By (4.3.2), it is evident that if $x^T x \neq 0$, then $\frac{d\lambda}{d\mu} \neq 0$ holds, namely, $1/\mu$ has to be a simple eigenvalue. Thus, $x^T x \neq 0$ is enough. ∎

[**Remark 4.3.3**] The next relation also holds:

$$\mu \cdot (y^T y) = \left(\frac{d\mu}{d\lambda}\right) \cdot (x^T x) \quad (\text{assuming } |d\mu/d\lambda| < \infty). \tag{4.3.8}$$

Also, as Theorem 4.3.1 states, the conditions (a'),(b'),(c') below are equivalent if $x^T x \neq 0$:

(a') an eigenvalue λ of $\mathbf{T}$ (4.2.6) is double; (b') $d\mu/d\lambda = 0$; (c') $y^T y = 0$.

[**Proof**] Let the proof be omitted since they are shown as were in Lemma 4.3.1 and Theorem 4.3.1. ∎

4.3.2 The Algorithm for Computing Double Pairs

Now we are ready to propose the method for computing double pairs. First, the algorithm is shown for double pairs satisfying $d\lambda/d\mu = 0$. And the one for another type of double pairs (or satisfying $d\mu/d\lambda = 0$) follows.

***The Algorithm for Double Pairs ($d\lambda/d\mu = 0$ Type)** From Theorem 4.3.1, one finds that the pairs of (μ, λ) satisfying $d\lambda/d\mu = 0$ are gained by the computation of the zeros of $f(\mu, \lambda) = x^T x$, if μ and λ are assumed to be real (Let me notify that this method applies even to complex μ and λ if $y^T y \neq 0$, given those variables). The algorithm is shown, by the combination of the method in [13, Section 4] for approximate eigenvalues of infinite matrices and Newton-Raphson method (Let's say Newton method for short in the sequel).

Newton method is widely known as an iterative method for approximating zeros t of $F(t)$, given an initial value t_0, by

$$t_{n+1} = t_n - F(t_n)/F'(t_n) \quad (n = 0, 1, \ldots).$$

In this section, considering the parameter μ to be a variable, we let the iteration

$$\mu_{n+1} = \mu_n - f(\mu_n, \lambda)/f'(\mu_n, \lambda) \quad (n = 0, 1, \ldots) \tag{4.3.9}$$

be executed. From the appearance of (4.3.9), obviously one needs the values of $f(\mu, \lambda)$, $f'(\mu, \lambda)$ (or the ratio of $f(\mu, \lambda)$ to $f'(\mu, \lambda)$) for the iterations. The differentiation with respect to μ to

$$f(\mu, \lambda) \;=\; x^T x = (\lambda - a_1)y_1^2 + (\lambda - a_2)y_2^2 + (\lambda - a_3)y_3^2 + \cdots$$

turns out to

$$
\begin{aligned}
f'(\mu,\lambda)=\frac{\mathrm{d}}{\mathrm{d}\mu}f(\mu,\lambda) &= \left\{\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}\right)y_1^2+2(\lambda-a_1)y_1y_1'\right\}+\left\{\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}\right)y_2^2+2(\lambda-a_2)y_2y_2'\right\}+\cdots \\
&= \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}\right)\left\{y_1^2+y_2^2+y_3^2+\cdots\right\}+2\left((\lambda-a_1)y_1y_1'+(\lambda-a_2)y_2y_2'+\cdots\right) \\
&= \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}\right)(\mathbf{y}^T\mathbf{y})+2\left((\lambda-a_1)y_1y_1'+(\lambda-a_2)y_2y_2'+\cdots\right).
\end{aligned}
$$

This means that the computations of $\{y_n\}$ and $\{y_n'\}(=\{\mathrm{d}y_n/\mathrm{d}\mu\})$ $(n=1,2,\ldots)$ are required for (4.3.9). (4.1.1) and (4.3.7) help to compute the values.

Substituting

$$c_ny_{n-1}+b_ny_n+c_{n+1}y_{n+1}=\frac{1}{\mu}(\lambda-a_n)y_n\ (n=1,2,\ldots),$$

derived by (4.1.1), into (4.3.7) becomes

$$f_ny_{n-1}'+(d_n-\lambda)y_n'+f_{n+1}y_{n+1}'=\left\{\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}\right)-\frac{1}{\mu}(\lambda-a_k)\right\}y_n \tag{4.3.10}$$

$(n=1,2,\ldots)$. Letting $y_{N+1}'=y_N'=y_{N+1}=0$ and $y_N=1$ for sufficiently large $n\equiv N$, one can execute the computation of $\{y_n\}$ $(n=1,2,\ldots,N-1)$ by backward substitution (using (4.1.1)), followed by the computation of $\{y_n'\}$ $(n=1,2,\ldots,N-1)$, again by backward substitution (using (4.3.10)).

Also, $(\mathrm{d}\lambda/\mathrm{d}\mu)$ appearing in (4.3.10) and $f'(\mu,\lambda)$ is computed by

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}=\frac{1}{\mu}\cdot\frac{\mathbf{x}^T\mathbf{x}}{\mathbf{y}^T\mathbf{y}}. \tag{4.3.11}$$

With the above settings, the algorithm for the computation of approximate double pairs (μ,λ) for $\mathrm{d}\lambda/\mathrm{d}\mu=0$ is proposed:

[**Algorithm 4.3.1**]

① Give an initial value μ_0 (and set $n=0$).

② Compute the appropriate eigenvalue λ_n by the algorithm in [13, Section 4] for the given μ_n.

③ Compute $\{y_n\}$, $\{y_n'\}$ $(n=1,2,\ldots)$, by (4.1.1),(4.3.10),(4.3.11). Also, compute $f(\mu_n,\lambda_n)$ and $f'(\mu_n,\lambda_n)$.

④ Obtain the next approximate solution μ_{n+1} by the iteration

$$\mu_{n+1}=\mu_n-f(\mu_n,\lambda_n)/f'(\mu_n,\lambda_n)$$

⑤ Exit if $x^T x$ is sufficiently small. Otherwise, letting $n = n+1$, go back to ②.

The pros of Newton method, as well known, are the fast convergence, correspondence to the complex zeros of complex functions, and ease of implementation. On the other hand, as a con, the good choice of an initial value is necessary since otherwise, it might cause a great change of the result.

One more remarkable point of this algorithm is that one can compute the approximate *double* eigenvalues (μ, λ) with high precision. It is known empirically that the computation of double eigenvalues of $\mathbf{A}$ (in (4.2.8)) results in the insufficiency of precision. By the algorithm presented here, however, one can prevent such a phenomenon since eigenvalues are gained by a well-conditioned matrix $\mathbf{T}$(in (4.2.5)).

***The Algorithm for Double Pairs ($\mathrm{d}\mu/\mathrm{d}\lambda = 0$ Type)** Similarly as the previous case for $\mathrm{d}\lambda/\mathrm{d}\mu = 0$, the computation of the zeros of $g(\mu, \lambda) = y^T y$ is equivalent to computing double pairs for $\mathrm{d}\mu/\mathrm{d}\lambda = 0$, but supposing $x^T x \neq 0$ (from (4.3.8)). Of course this restriction (or the condition $x^T x \neq 0$) is not vital, since computed (μ, λ) for $g(\mu, \lambda) = 0$ may be regarded as double pairs after checking $x^T x \neq 0$ for such (μ, λ).

What is different from the previous case for $\mathrm{d}\lambda/\mathrm{d}\mu = 0$ is that it is nonsense to assume μ and λ are both real, for, with such an assumption, $y^T y \neq 0$ since y is real. Then, the algorithm to be incorporated is useful only when either μ or λ, or both is not real.

This time, let the iteration be

$$\lambda_{n+1} = \lambda_n - g(\mu, \lambda_n)/g'(\mu, \lambda_n) \quad (n = 0, 1, \ldots). \tag{4.3.12}$$

As the former case, the computations of $g(\mu, \lambda)$ and $g'(\mu, \lambda)$ (or the ratio of $g(\mu, \lambda)$ to $g'(\mu, \lambda)$) are necessary. The differentiation with respect to λ to

$$g(\mu, \lambda) \quad = \quad y^T y = y_1^2 + y_2^2 + y_3^2 + \cdots$$

turns out to

$$g'(\mu, \lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda} g(\mu, \lambda) = 2(y_1 y_1' + y_2 y_2' + y_3 y_3' + \cdots).$$

This means that the computations of $\{y_n\}$ and $\{y_n'\}(= \{\mathrm{d}y_n/\mathrm{d}\lambda\})$ $(n = 1, 2, \ldots)$ are required for the iterations. By differentiating the both sides of (4.1.1) with respect to λ, one is given, defining $y_0 = y_0' \equiv 0$,

$$\left(\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right)(c_n y_{n-1} + b_n y_n + c_{n+1} y_{n+1}) + f_n y_{n-1}' + d_n y_n' + f_{n+1} y_{n+1}' = y_n + \lambda y_n', \tag{4.3.13}$$

for $n = 1, 2, \ldots$.

Substituting

$$c_n y_{n-1} + b_n y_n + c_{n+1} y_{n+1} = \frac{1}{\mu}(\lambda - a_n) y_n \ (n = 1, 2, \ldots),$$

derived by (4.1.1), into (4.3.13) rewrites (4.3.13) as

$$(4.3.14)\ f_n y'_{n-1} + (d_n - \lambda) y'_n + f_{n+1} y'_{n+1} = \left(\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right)\left\{\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}\right) - \frac{1}{\mu}(\lambda - a_k)\right\} y_n \ (n = 1, 2, \ldots).$$

Letting $y'_{N+1} = y'_N = y_{N+1} = 0$ and $y_N = 1$ for sufficiently large $n \equiv N$, one can execute the computation $\{y_n\}$ $(n = 1, 2, \ldots, N-1)$ by backward substitution (using (4.1.1)), followed by the computation of $\{y'_n\}$ $(n = 1, 2, \ldots, N-1)$, again by backward substitution (using (4.3.14)).

Also, $(\mathrm{d}\mu/\mathrm{d}\lambda)$ appearing in (4.3.14) is computed by

$$(4.3.15) \qquad \frac{\mathrm{d}\mu}{\mathrm{d}\lambda} = \mu \cdot \frac{\mathbf{y}^T \mathbf{y}}{\mathbf{x}^T \mathbf{x}}.$$

With the above settings, the algorithm for the computation of approximate double pairs (μ, λ) for $\mathrm{d}\mu/\mathrm{d}\lambda = 0$ is again proposed:

[**Algorithm 4.3.2**]

①' Give an initial value λ_0 (and set $n = 0$).

②' Compute the appropriate eigenvalue μ_n by the algorithm in [21] for the given λ_n.

③' Compute $\{y_n\}$, $\{y'_n\}$ $(n = 1, 2, \ldots)$, by (4.1.1),(4.3.14),(4.3.15). Also, compute $g(\mu_n, \lambda_n)$ and $g'(\mu_n, \lambda_n)$.

④' Obtain the next approximate solution λ_{n+1} by the iteration

$$\lambda_{n+1} = \lambda_n - g(\mu_n, \lambda_n)/g'(\mu_n, \lambda_n)$$

⑤' Exit if $\mathbf{y}^T\mathbf{y}$ is sufficiently small. Otherwise, letting $n = n + 1$, go back to ②'.

Similarly, even in computing double eigenvalues of $\mathbf{T}$ (in (4.2.5)), one can prevent the insufficiency of precision since the computations of eigenvalues are performed by well-conditioned matrix $\mathbf{A}$ (in (4.2.8)).

4.4 Applications of Double Eigenvalue Computation

Three examples are chosen as the applications of Section 4.3. The first example is on proof that there exist no complex pairs of zero (z, ν) which satisfy $J'_\nu(z) = \mathrm{d}J_\nu(z)/\mathrm{d}z =$ with the aid of Theorem 4.4.1 (Section 4.4.1), the second on the computation of double ro of $zJ'_\nu(z) + HJ_\nu(z) = 0$ by Algorithm 4.3.1 (Section 4.4.2), and the third on the computat of double eigenvalues of the Mathieu differential equations, for both types of $\mathrm{d}\lambda/\mathrm{d}q = 0$ ε $\mathrm{d}q/\mathrm{d}\lambda = 0$ (Section 4.4.3).

4.4.1 Example 1 : the Computation on Double Zeros of $J_\nu(z)$

In this section, it is proved, using Theorem 4.4.1, that there are no pairs of complex ze (z, ν) of $J_\nu(z) = 0$ such that $\mathrm{d}\nu/\mathrm{d}z = 0$.

$J_\nu(z)$, the Bessel function of the first kind, is known as one of the independent solutic of the Bessel differential equation of order ν

$$z^2 f''(z) + zf'(z) + (z^2 - \nu^2)f(z) = 0.$$

Also, three-term relations on the Bessel functions are widely known:

$$J_\nu(z) - \frac{2(\nu+1)}{z} J_{\nu+1}(z) + J_{\nu+2}(z) = 0.$$

By these relations, [13] proved that the problem of computing the zeros of $J_\nu(z)$, giver ("inverse problem" of solving $J_\nu(z) = 0$ for given ν, so to speak) is equivalent to the one the computations of the eigenvalues of the following matrix:

$$(4.4.1) \qquad \begin{aligned} \mathrm{Ty} &= \nu\mathrm{y}, \text{ where} \\ \mathrm{T} &= \begin{bmatrix} -1 & z/2 & & \mathbf{0} \\ z/2 & -2 & z/2 & \\ & z/2 & -3 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} : D(\mathrm{T}) \to \ell^2, \\ 0 \neq \mathrm{y} &= [y_1, y_2, y_3, \ldots]^T = [J_{\nu+1}(z), J_{\nu+2}(z), \ldots]^T \in \ell^2, \\ D(\mathrm{T}) &= \{[w_1, w_2, \ldots]^T : [-1w_1, -2w_2, \ldots]^T \in \ell^2\}. \end{aligned}$$

Fig. 4.4.1 is the z-ν curve created by the algorithm in [13] for the matrix eigenvalue proble of (4.4.1).

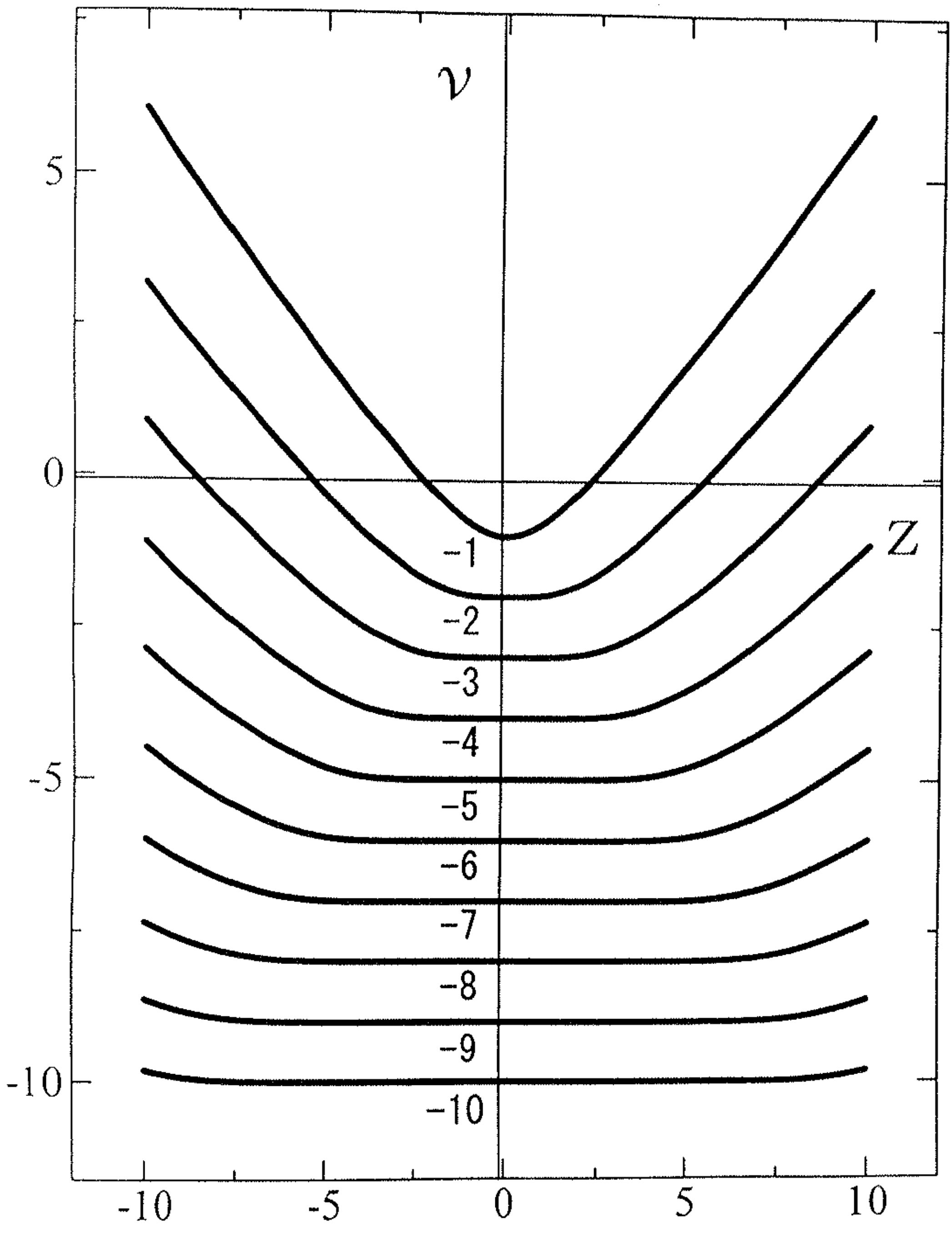

Fig. 4.4.1: z-ν curve ((z, ν) satisfying $J_\nu(z) = 0$)

From this figure, it is conjectured that given $z \neq 0$, there are no ν satisfying $\mathrm{d}\nu/\mathrm{d}z \neq 0$ When $\nu > -1$, this is easily proved from the matrix theory. However, the proof in this section covers the case for the general complex ν and $z \neq 0$.

One obtains the next relation if one applies Lemma 4.4.1 to this problem:

$$
\begin{aligned}
(4.4.2) \qquad \left(\frac{\mathrm{d}\nu}{\mathrm{d}z}\right)\left(\mathrm{y}^T\mathrm{y}\right) &= \frac{1}{z}\left(\mathrm{x}^T\mathrm{x}\right), \\
\text{where } \mathrm{y} &= [J_{\nu+1}(z), J_{\nu+2}(z), \ldots]^T \in \ell^2, \\
\mathrm{x} &= [\sqrt{\nu+1}J_{\nu+1}(z), \sqrt{\nu+2}J_{\nu+2}(z), \ldots]^T \in \ell^2.
\end{aligned}
$$

Since $\mathrm{y} \in \ell^2$, $\left|\mathrm{y}^T\mathrm{y}\right| < \infty$ is obvious. Then, it suffices to show $\mathrm{x}^T\mathrm{x} \neq 0$ to show $\mathrm{d}\nu/\mathrm{d}z \neq 0$. Let $\mathrm{x}^T\mathrm{x}$ be transformed as follows:

$$
\begin{aligned}
\mathrm{x}^T\mathrm{x} &= (\nu+1)J_{\nu+1}^2(z) + (\nu+2)J_{\nu+2}^2(z) + \cdots \\
&= \{(\nu+1)J_{\nu+1}^2(z) + (\nu+3)J_{\nu+3}^2(z) + \cdots\} + \{(\nu+2)J_{\nu+2}^2(z) + (\nu+4)J_{\nu+4}^2(z) + \cdots\} \\
&= \frac{z^2}{4}(J_\nu^2(z) - J_{\nu-1}(z)J_{\nu+1}(z)) + \frac{z^2}{4}(J_{\nu+1}^2(z) - J_\nu(z)J_{\nu+2}(z)) \text{ (by [29, page152])} \\
&= \frac{z^2}{4}J_{\nu+1}(z)(J_{\nu+1}(z) - J_{\nu-1}(z)) \text{ (by } J_\nu(z) = 0) \\
&= -z^2 J_{\nu+1}(z)J_\nu'(z)/2 \text{ (since } 2J_{\nu+1}' = J_\nu - J_{\nu+2} \text{ by [2])} \\
&\neq 0 \; (J_{\nu+1}(z) \neq 0 \text{ and } J_\nu'(z) \neq 0 \text{ when } J_\nu(z) = 0)
\end{aligned}
$$

As was conjectured, that all the complex $z \neq 0$ and ν satisfying $J_\nu(z) = 0$ have $\mathrm{d}\nu/\mathrm{d}z \neq 0$ is proved.

4.4.2 Example 2 : the Computation on Double Zeros of $zJ_\nu'(z) + HJ_\nu(z)$

It is known that the pairs of zero (z, ν) of $zJ_\nu'(z) + HJ_\nu(z)$ (in this example, let the value of H be fixed to be $H = 1$.) are distributed as in Fig. 4.4.2[7]:

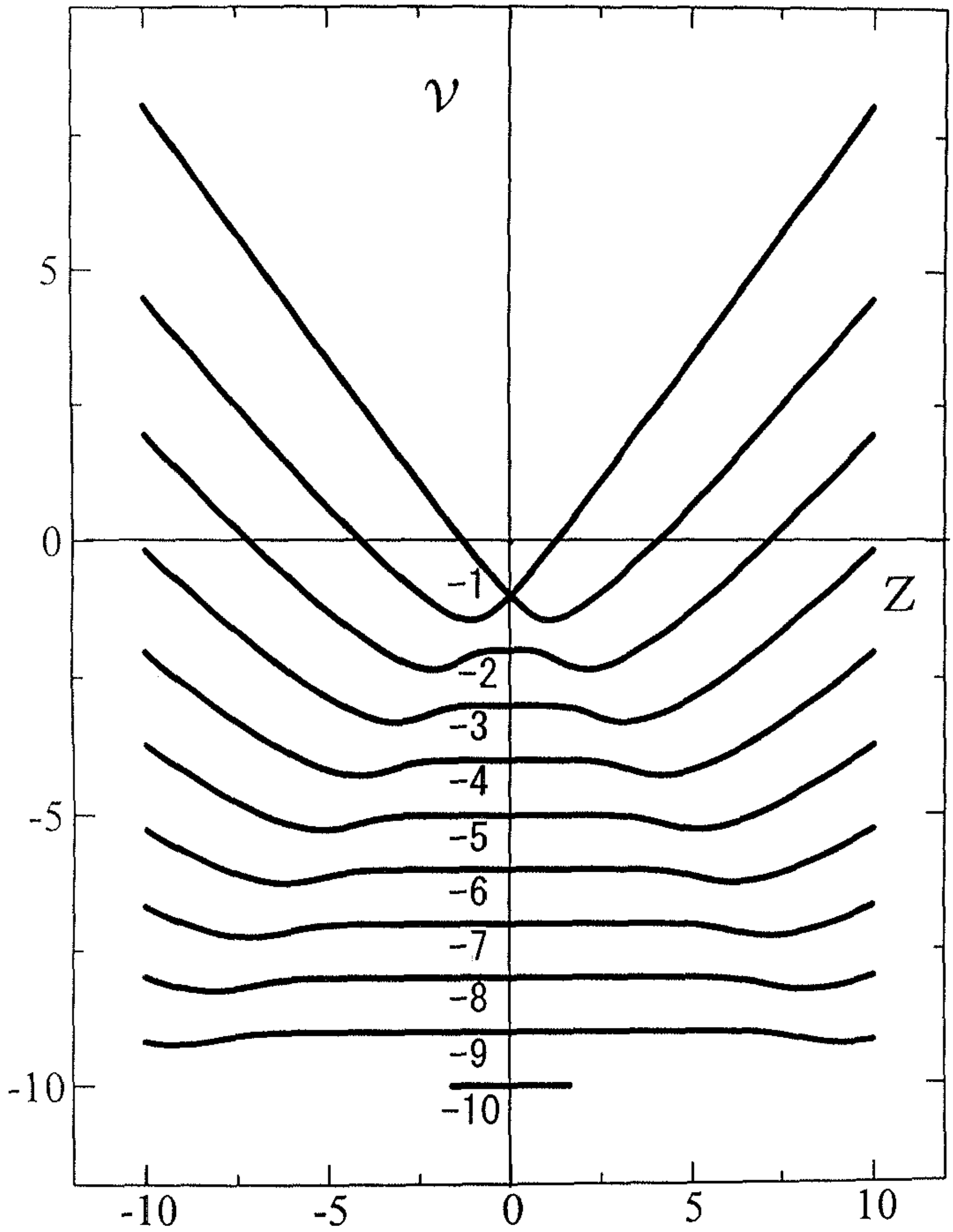

Fig. 4.4.2: (z, ν) satisfying $zJ'_\nu(z) + 1 \cdot J_\nu(z) = 0$

What this figure reads in contrast with Fig. 4.4.1 is that it is guessed that double pairs fc $d\nu/dz = 0$ type exist. Algorithm 4.3.1 was used, for the computation of such double pair: and locating them was successfully performed. The closest 20 points to the origin have bee shown with those 10 significant digits in Table 4.4.1. N's represent that the correspondin points are the Nth closest to the origin (if we regard two symmetrical points with respec to ν-axis as identical).

Table 4.4.1: First 20 double roots (z, ν) $(H = 1)$

N	z	ν	N	z	ν
1	$\pm$ 1.051328528 $\cdots$	-1.450962327 $\cdots$	11	$\pm$ 11.17415387 $\cdots$	-11.21881075 $\cdots$
2	$\pm$ 2.118898212 $\cdots$	-2.343017207 $\cdots$	12	$\pm$ 12.17470605 $\cdots$	-12.21570577 $\cdots$
3	$\pm$ 3.143427582 $\cdots$	-3.298656842 $\cdots$	13	$\pm$ 13.17512233 $\cdots$	-13.21301815 $\cdots$
4	$\pm$ 4.155369454 $\cdots$	-4.274002258 $\cdots$	14	$\pm$ 14.17543695 $\cdots$	-14.21066546 $\cdots$
5	$\pm$ 5.162139588 $\cdots$	-5.258106611 $\cdots$	15	$\pm$ 15.17567424 $\cdots$	-15.20858602 $\cdots$
6	$\pm$ 6.166349121 $\cdots$	-6.246908153 $\cdots$	16	$\pm$ 16.17585191 $\cdots$	-16.20673270 $\cdots$
7	$\pm$ 7.169132116 $\cdots$	-7.238539584 $\cdots$	17	$\pm$ 17.17598304 $\cdots$	-17.20506883 $\cdots$
8	$\pm$ 8.171052876 $\cdots$	-8.232017074 $\cdots$	18	$\pm$ 18.17607749 $\cdots$	-18.20356539 $\cdots$
9	$\pm$ 9.172420411 $\cdots$	-9.226770626 $\cdots$	19	$\pm$ 19.17614279 $\cdots$	-19.20219915 $\cdots$
10	$\pm$ 10.17341643 $\cdots$	-10.22244598 $\cdots$	20	$\pm$ 20.17618473 $\cdots$	-20.20095122 $\cdots$

4.4.3 Example 3 : the Computation on Double Eigenvalues of Mathieu Differential Equation

***Computation of Double Pairs ($d\lambda/dq = 0$ Type)** We shall first plot, in Fig. 4.4.3, the real pairs of points (q, λ) such that the solution of the Mathieu differential equation (se_{2m} type)

$$w''(z) + (\lambda - 2q\cos 2z)w(z) = 0$$

is π- or 2π- periodic.

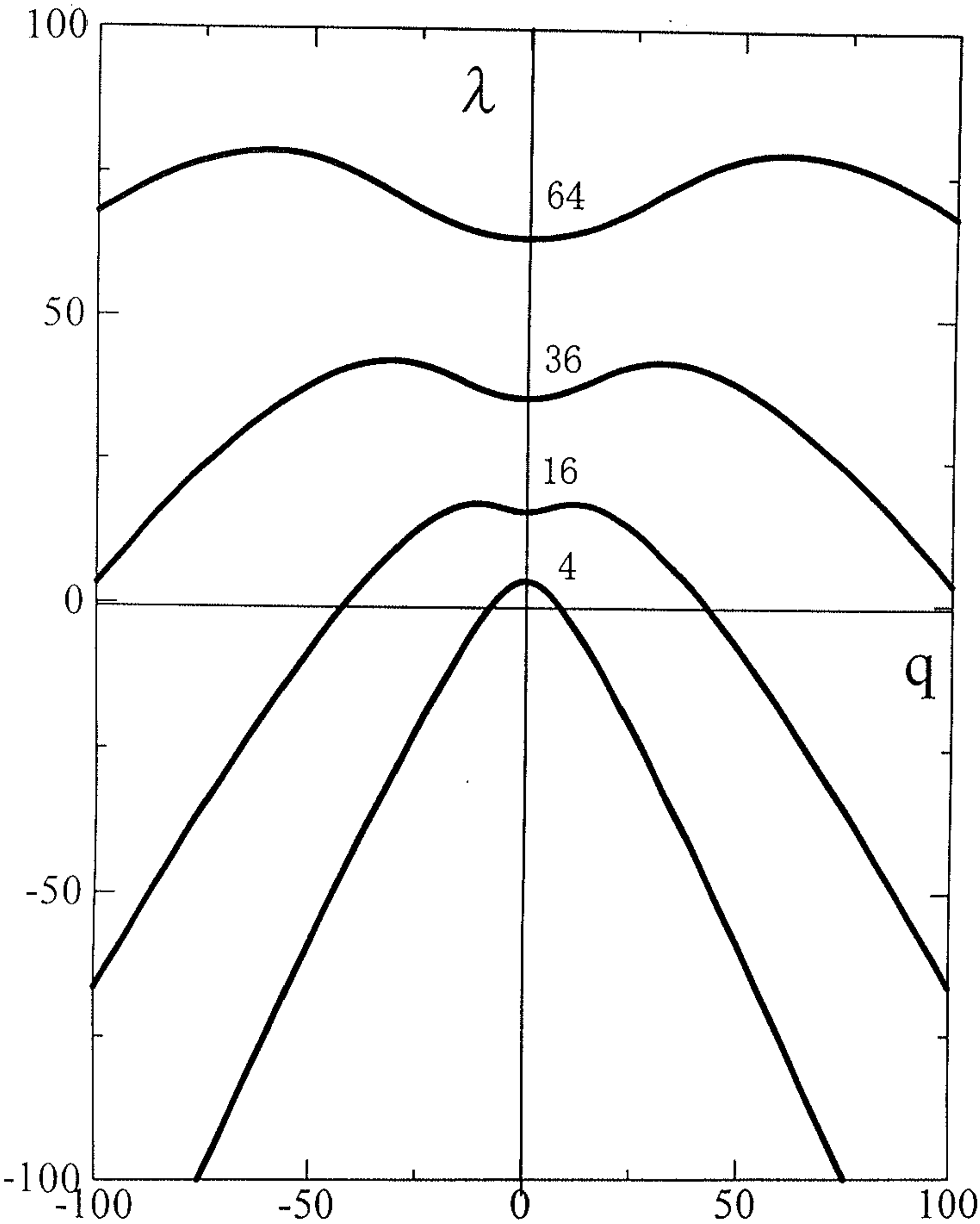

Fig. 4.4.3: Real pairs of (q, λ) of $se_{2m}(z, q)$ type

This plotting suggests that there are points where $\mathrm{d}\lambda/\mathrm{d}q = 0$ hold. In this section, let the author elaborate on the algorithm for computing such points.

Analogous to (4.3.9), the iteration to be executed in Mathieu's case is

$$q_{n+1} = q_n - f(q_n, \lambda)/f'(q_n, \lambda) \quad (n = 0, 1, \ldots), \tag{4.4.3}$$

where $f(q, \lambda) = \mathrm{x}^T\mathrm{x}$ with $x^{(i)}$ $(i = 1, 2, \ldots)$, each component of x, defined as (3.2.17). From (4.4.3), the computations of $f(q, \lambda)$ and $f'(q, \lambda)$ (or the ratio of $f(q, \lambda)$ and $f'(q, \lambda)$) are indispensable for this iteration.

Differentiating the both sides of

$$f(q, \lambda) \;=\; \mathrm{x}^T\mathrm{x} = r_4 B_4^2 + r_8 B_8^2 + \cdots = \frac{1}{2}(r_2 B_2^2 + r_4 B_4^2 + \cdots) \text{ (by (3.2.30))}$$

with respect to q gives

$$\begin{aligned} f'(q, \lambda) = \frac{\mathrm{d}}{\mathrm{d}q} f(q, \lambda) &= -\frac{1}{2}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}q}\right) B_2^2 + r_2 B_2 B_2' - \frac{1}{2}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}q}\right) B_4^2 + r_4 B_4 B_4' - \cdots \\ &= -\frac{1}{2}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}q}\right) \cdot (\mathrm{y}^T\mathrm{y}) + (r_2 B_2 B_2' + r_4 B_4 B_4' + \cdots). \end{aligned}$$

They imply that the computations of $\{B_{2n}\}$ and $\{B_{2n}'\}$ $(n = 1, 2, \ldots)$ are required for the iteration of (4.4.3). The three-term relations

$$q B_{2n-2} + r_{2n} B_{2n} + q B_{2n+2} = 0 \ (n = 1, 2, \ldots) \tag{4.4.4}$$

(the same as (3.2.7). Let $B_0 \equiv 0$.) turn out to be

$$B_{2n-2} + q B_{2n-2}' - \left(\frac{\mathrm{d}\lambda}{\mathrm{d}q}\right) B_{2n} + r_{2n} B_{2n}' + B_{2n+2} + q B_{2n+2}' = 0 \ (n = 1, 2, \ldots) \tag{4.4.5}$$

after the differentiation with respect to q (where $B_0' \equiv 0$).

Let us follow [4] for computing $\{B_{2n}\}, \{B_{2n}'\}$. In [4], the both sides of (4.4.4) are differentiated with respect to λ (where q and λ are regarded as independent variables) for computing eigenvalue λ by Newton-Raphson method. Let's call the resulting relations (R). Next, putting $B_{2N+2}' = B_{2N}' = B_{2N+2} = 0$, $B_{2N} = 1$ for sufficiently large $n \equiv N$, one obtains $\{B_{2n}\}$ $(n = 1, 2, \ldots, N-1)$ successively from the backward direction by (4.4.4), followed by the computation of $\{B_{2n}'\}$ $(n = 1, 2, \ldots, N-1)$ by (R). In this paper, we adopt this way for the computations of $\{B_{2n}\}, \{B_{2n}'\}$ by (4.4.4) and (4.4.5). Note that the value of $(\mathrm{d}\lambda/\mathrm{d}q)$ appearing in (4.4.5) and therefore $f'(q, \lambda)$ is obtained by (3.2.40):

$$\frac{\mathrm{d}\lambda}{\mathrm{d}q} = -\frac{2}{q} \cdot \frac{\mathrm{x}^T\mathrm{x}}{\mathrm{y}^T\mathrm{y}}. \tag{4.4.6}$$

With everything set, the algorithm for computing double pairs (q, λ) is proposed:

[Algorithm 4.4.1]

<1> Give an initial value q_0 ($n = 0$).

<2> For given q_n, compute certain approximate eigenvalue λ_n, by the algorithm in [13 Chapter 4].

<3> For q_n and λ_n gained in <2>, compute $\{B_{2n}\}$, $\{B'_{2n}\}$ ($n = 1, 2, \ldots$) by (4.4.4) to (4.4.6). Also, compute $f(q_n, \lambda_n), f'(q_n, \lambda_n)$.

<4> Compute the next approximate solution q_{n+1} by Newton-Raphson method:

$$q_{n+1} = q_n - f(q_n, \lambda_n)/f'(q_n, \lambda_n)$$

<5> Exit if $\mathrm{x}^T\mathrm{x}$ is sufficiently small. Otherwise, go back to <2> after setting $n = n + 1$.

The con of this method is, as already stated, that the choice of an initial value might influence the output greatly. Then, in order to solve this problem, the discussion is in order for the selection of initial values, considering the characteristics of q–λ graph.

[4, Chapter 3,(24b)&Fig.1] showed, with experiments, that a group of q–λ curves $\{a_k\}$ ($k = 1, 2, \ldots$) (a_k corresponds to the curve with q and λ real which is kth closest to the origin) have the following relation:

$$a_n(h) \sim \left(\frac{n}{m}\right)^2 a_m\left(\frac{m}{n} \cdot h\right) \quad (\text{where } h^2 = 4q) \tag{4.4.7}$$

This of course is true of the double pairs (q, λ) on those curves too. Then, once a double pair is gained, one can utilize the previous value for the selection of the initial value for the next double pair. The algorithm for computing the first M double eigenvalues closest to the origin is given next:

[Algorithm 4.4.2]

<a> $N = 1$; Give an initial value q_0.

<b> Exit if $N > M$. Otherwise, execute the following three:

a) Execute <2> to <5> of Algorithm 4.4.1. (for the double eigenvalue of a_N curve. Name the q-value $\bar{q}$).

b) Decide the initial value for the next double eigenvalue by $q_0 = (\frac{N+1}{N})^2\bar{q}$.

c) $N = N + 1$; Go back to <b>.

Lastly, let us show the results of experiments for the program we implemented for A
gorithm 4.4.1 and 4.4.2. Table 4.4.2 is for the list of the first 30 double pairs closest to t
origin with the display of 10 significant digits (Note that if (q, λ) in the table is a double pa
so is $(-q, \lambda)$). Table 4.4.3 tabulates the initial value and the convergence of q, only for t
computations of the double pairs closest to 15, 16, and 17th to the origin, by Algorithm 4.4.
Only the first 12 digits are displayed after rounding. n represents the number of iteratio
by Newton method ($n = 0$ represents that the corresponding q_n are the initial values to l
used by Newton method). In each case (or $N = 15, 16, 17$), at the stage each initial value
determined, as many as three digits (roughly speaking) are already in agreement with th
true value q. This helps the approximate values converge only at around $n = 4$. Also, in Fi
4.4.4, Fig. 4.4.3 overlapped with the zeros of $f(q, \lambda) = \mathbf{x}^T\mathbf{x}$ is provided, in order to visuali
the behavior of the zeros of $f(q, \lambda)$ and how $f(q, \lambda)$ and a_k ($k = 1, 2, \ldots$) intersect.

Table 4.4.2. The first 30 double pairs (q, λ)

N	q	λ	N	q	λ
1	11.14606106 ···	17.41358458 ···	16	1208.542746 ···	1575.832363 ···
2	31.48781870 ···	42.39762508 ···	17	1357.118729 ···	1769.643863 ···
3	60.12377598 ···	78.78937721 ···	18	1514.307842 ···	1974.690807 ···
4	97.10095598 ···	126.4898649 ···	19	1680.110082 ···	2190.973195 ···
5	142.5125993 ···	185.4563602 ···	20	1854.525445 ···	2418.491029 ···
6	196.4363376 ···	255.6705920 ···	21	2037.553929 ···	2657.244308 ···
7	258.9220170 ···	337.1250126 ···	22	2229.195531 ···	2907.233033 ···
8	329.9970023 ···	429.8166325 ···	23	2429.450251 ···	3168.457205 ···
9	409.6747517 ···	533.7443161 ···	24	2638.318087 ···	3440.916823 ···
10	497.9613680 ···	648.9076487 ···	25	2855.799037 ···	3724.611887 ···
11	594.8594513 ···	775.3064854 ···	26	3081.893101 ···	4019.542399 ···
12	700.3700537 ···	912.9407786 ···	27	3316.600278 ···	4325.708357 ···
13	814.4935803 ···	1061.810514 ···	28	3559.920568 ···	4643.109763 ···
14	937.2301783 ···	1221.915690 ···	29	3811.853970 ···	4971.746616 ···
15	1068.579896 ···	1393.256306 ···	30	4072.400483 ···	5311.618916 ···

Table 4.4.3. The intermediate result of Algorithm 4.4.2 (Only $N = 15, 16, 17$ cases are shown)

N	n	q_n	N	n	q_n	N	n	q_n
15	0	1066.35966961	16	0	1206.32652339	17	0	1354.90605474
	1	1068.18352388		1	1208.19445836		1	1356.81189137
	2	1068.56512348		2	1208.53285101		2	1357.11204100
	3	1068.57987488		3	1208.54273814		3	1357.11872624
	4	1068.57989615		4	1208.54274635		4	1357.11872949

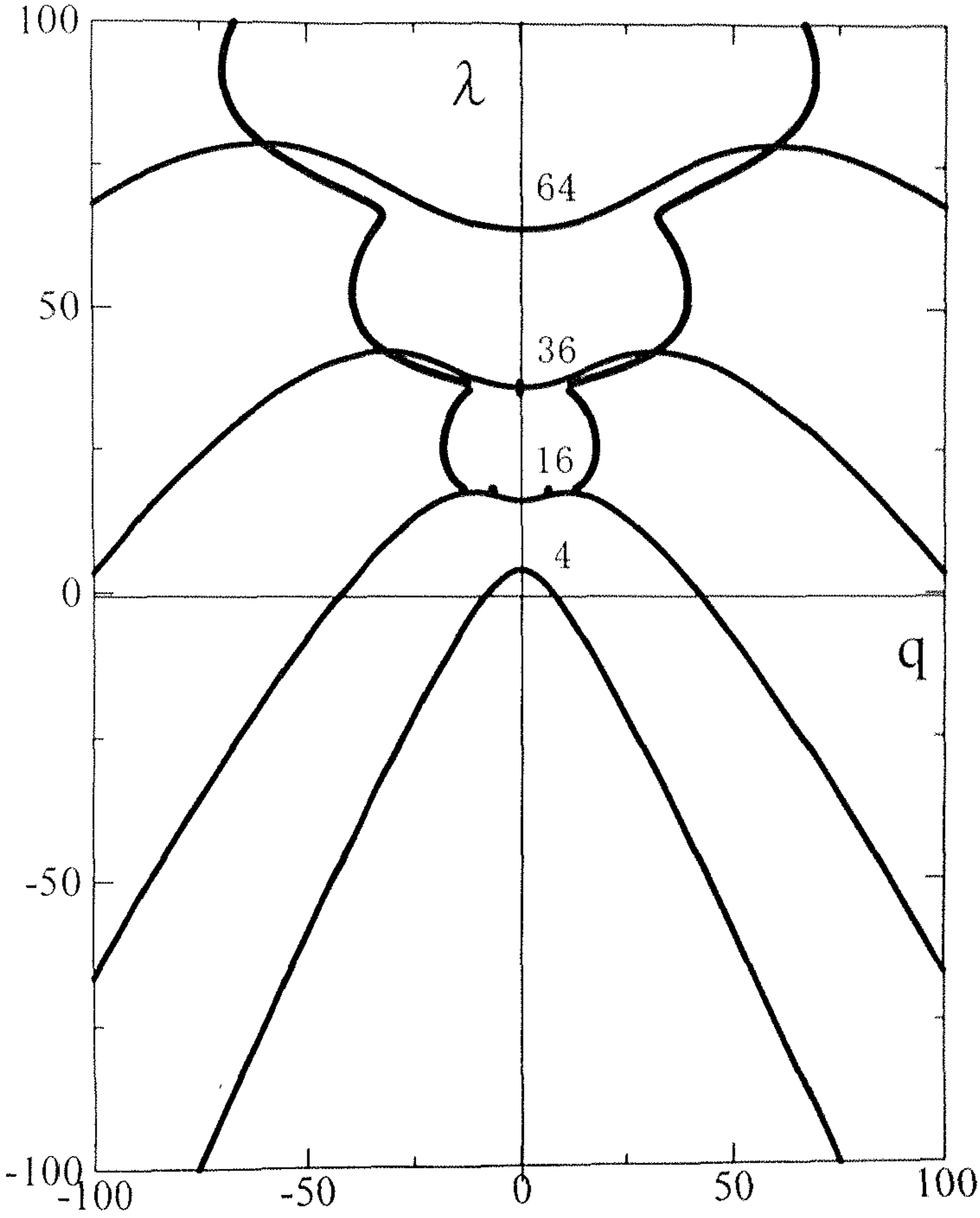

Fig. 4.4.4: Real pairs of (q, λ) of $se_{2m}(z, q)$ type with the zeros of $f(q, \lambda) = \mathrm{x}^T \mathrm{x}$

*Computation of Double Pairs ($dq/d\lambda = 0$ Type) In the previous case, graph was created, for real q and real λ. Next, let the new graph be plotted, this real λ and pure imaginary q. Since the real part of q is zero (constant), the grap displayed in 2-dimensions (in Fig. 4.4.5).

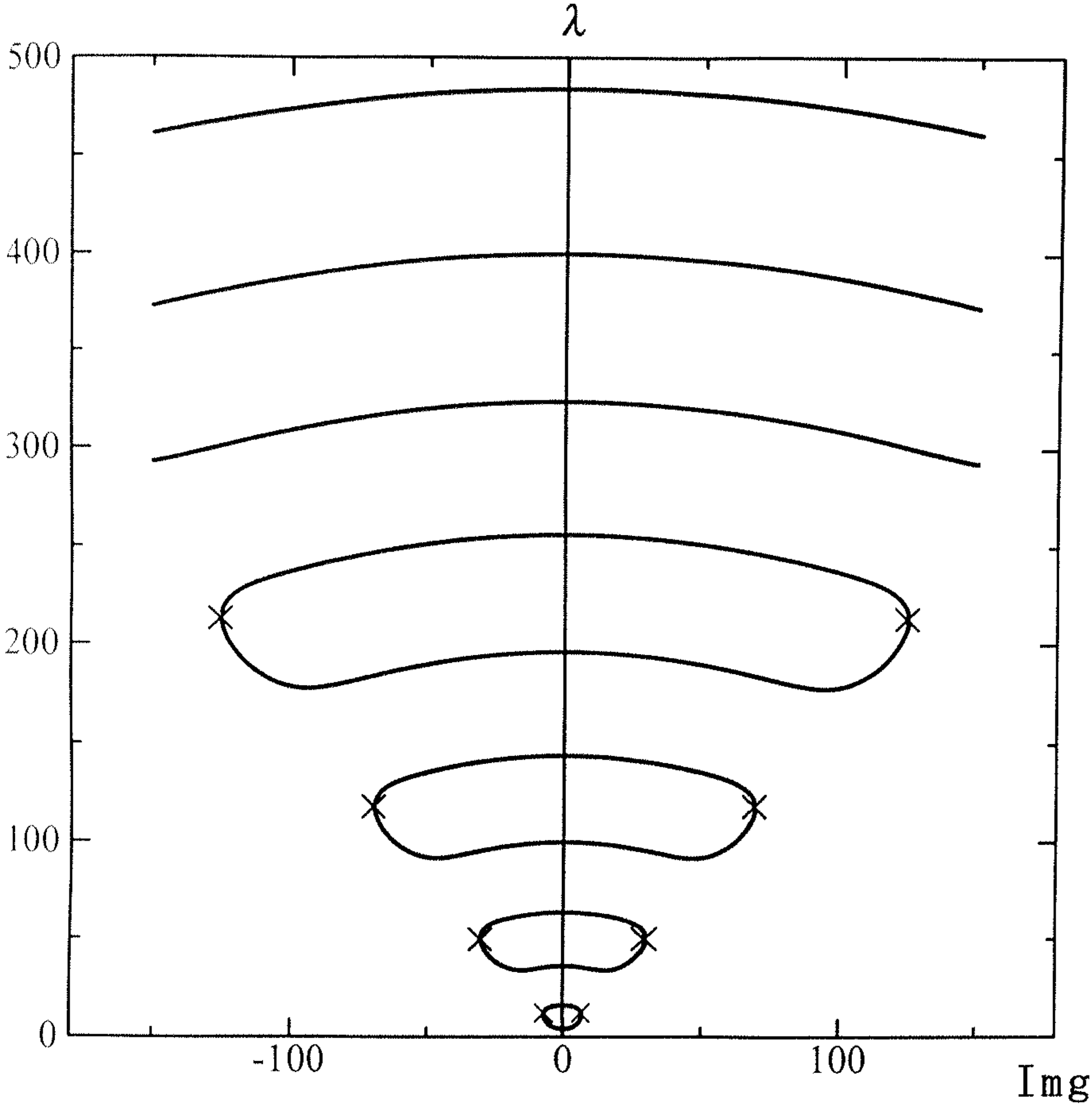

Fig. 4.4.5: Pairs of (q, λ) of $se_{2m}(z, q)$ type
(λ real and q pure imaginary)

It is observed in Fig. 4.4.5 that there should exist pairs of λ (real) and q (pure imaginary) such that $\mathrm{d}q/\mathrm{d}\lambda = 0$, at the points marked "×". Then let the computation of such pairs be attempted by using Algorithm 4.3.2. The detailed procedure, however, is omitted in this case since no further discussion is required but to apply Algorithm 4.3.2, and the procedure is very analogous to the former case where Algorithm 4.3.1 was used to compute double pairs for $\mathrm{d}\lambda/\mathrm{d}q = 0$ type.

The first 10 double pairs satisfying $\mathrm{d}q/\mathrm{d}\lambda = 0$ closest to the origin with the display of 10 digits (Note that if (q, λ) in the table is a double pair, so is $(-q, \lambda)$) are listed in Table 4.4.4.

Table 4.4.4. The first 10 double pairs (q, λ)

N	q	λ
1	$i \cdot 6.928954758 \cdots$	$11.19047359 \cdots$
2	$i \cdot 30.09677283 \cdots$	$50.47501615 \cdots$
3	$i \cdot 69.59879327 \cdots$	$117.8689241 \cdots$
4	$i \cdot 125.4354113 \cdots$	$213.3725686 \cdots$
5	$i \cdot 197.6066786 \cdots$	$336.9860439 \cdots$
6	$i \cdot 286.1126087 \cdots$	$488.7093844 \cdots$
7	$i \cdot 390.9532062 \cdots$	$668.5426056 \cdots$
8	$i \cdot 512.1284733 \cdots$	$876.4857154 \cdots$
9	$i \cdot 649.6384116 \cdots$	$1112.538718 \cdots$
10	$i \cdot 803.4830180 \cdots$	$1376.701616 \cdots$

In the former ($\mathrm{d}\lambda/\mathrm{d}q = 0$) case, Algorithm 4.4.2 was used, to determine initial values for Newton's method based on the property (4.4.7). It is naturally conjectured that (4.4.7) applies for real λ and pure imaginary q as well. The fact is that Table 4.4.4 was thus created, incorporating the method for selecting initial values used in Algorithm 4.4.2. Table 4.4.5 describes the initial values chosen and how fast the approximate values converged, only for the 5th, 6th, and 7th closest double pairs to the origin. Only the first 12 digits are displayed after rounding. n represents the number of iterations by Newton method ($n = 0$ represents the corresponding λ_n are the initial values). In each case (or $N = 5, 6, 7$), at the stage of determining the initial value, about two digits are already in agreement with the true value λ. Since the given initial values are already very close to the true values, the approximate values converge only at around $n = 4$ or $n = 5$.

Table 4.4.5. The intermediate result (only the cases $N = 5, 6, 7$ are shown)

N	n	λ_n	N	n	λ_n	N	n	λ_n
5	0	333.394638496	6	0	485.259903273	7	0	665.187773314
	1	335.859145525		1	487.942276449		1	665.011914901
	2	336.895370234		2	488.679867317		2	667.921219915
	3	336.985593670		3	488.709346955		3	668.527491472
	4	336.986043939		4	488.709384476		4	668.542597529
							5	668.542605652

4.5 Summary of Section 4

In this section, we set a class of three-term recurrence relations and showed that Theorem A and Theorem B may apply to the eigenvalue problem for the infinite complex symmetric tridiagonal matrices obtained by the set of the relations. Furthermore, we proposed the method for computing double pairs by the newly proved theorems concerning the eigenvalue problem and Newton method. Three examples were demonstrated and the solutions for double pairs were achieved.

The following two future problems may be listed. One of them is to find more properties on the three-term relations defined in this section, and further widen the coverage of applications. The other is to generalize this research with the conjecture that what was obtained in Section 4.3 should be true of the wider class of relations set in this section. Another possible future plan is how to locate the double pairs (and determine the initial value for Newton method) when we deal with two complex variables μ and λ where difficulty of visualization might also occur. This is opposite to the case of the two variables both being real, where locating them is relatively easier.