3 Applied Problems with Proofs and Results

As stated in Section 1, the author proved that three problems listed below may be applied by either Theorem A or Theorem B, or both:

ⓐ the computation of the zeros of Coulomb wave function $F_L(\eta, \rho)$ and its first derivative (with an explicit and a closed error formula) [23],[25],

ⓑ the inverse EVP of the Mathieu differential equation [21],

ⓒ the ordinary and inverse EVP of the spheroidal wave equation [22],

The solutions of each problem either have hardly been found* or have not been satisfactory yet in the sense that they have some limitations (For the details, see each section).

Since Theorem A and Theorem B have strong usefulness and uniqueness as was explained in Section 2, the problems ⓐ - ⓒ are solved with certain superiorities over other methods. That the matrix method with Theorem A or Theorem B gives precise error estimates in *equation form* is one of them.

Each section consists roughly of three parts. First, the main part, or the method for obtaining approximate solutions (zeros or eigenvalues) by Theorem A or Theorem B is stated. Secondly, newly found relations or properties specific to each problem are shown, most of which were proved by matrix theory. And at the end of each problem, experimental results close the section to show the validity of the used method, along with its relevant graphs, also created by using the computed data by the method.

3.1 The Computation of the Zeros of Coulomb Wave Function and Its First Derivative

In 1975, the author of [12] showed that the problem of computing the zeros of the regular Coulomb wave functions and of their derivatives may be reformulated as the eigenvalue problem for infinite matrices. Approximation by truncation is justified but no error estimates are given there.

*The author investigated two of the major database on academic journals, COMPENDEX PLUS (the database dealing with academic papers on engineering, since 1976, carrying approximately 2.34 million articles) and INSPEC (the database dealing with academic papers on engineering and physics, since 1969, carrying approximately 4.7 million articles)

The class of eigenvalue problems studied there turns out to be subsumed in a more general problem studied by the same author et. al in 1993[15], where an extremely accurate asymptotic error estimate is shown.

In this section, we apply this error formula to the former case to obtain error formulas in a closed, explicit form.

3.1.1 Description of the Problem

The second-order linear differential equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\rho^2} + \left[1 - \frac{2\eta}{\rho} - \frac{L(L+1)}{\rho^2}\right] w = 0, \tag{3.1.1}$$

where $\rho > 0$, $-\infty < \eta < \infty$, and L is a non-negative integer, has two independent solutions defined as Coulomb wave functions, one called the regular Coulomb wave function $w = F_L(\eta, \rho)$, and the other the irregular Coulomb wave function $w = G_L(\eta, \rho)$ (For more details on $F_L(\eta, \rho)$ and $G_L(\eta, \rho)$, refer to [2]). (3.1.1) appears in atomic and nuclear physics, and is obtained when we deal with the scattering problems with charged particles† or the separation of Schrödinger's wave equation for a Coulomb force field. One will find that there is abundant literature for the computation of the function value $F_L(\eta, \rho)$. Nevertheless, when it comes to the computation of the zeros ρ of $F_L(\eta, \rho)$, no previous research but [1]‡ and [12] was found, according to the author's investigation.

In 1975, the author of [12] showed that the problem of computing the zeros of $F_L(\eta, \rho)$ and of their derivatives may be reformulated as a matrix eigenvalue problem by rewriting the three-term recurrence relations satisfied by $F_L(\eta, \rho)$, which represents a minimal solution of the recurrence relations (a second-order linear homogeneous difference equation) in the sense of [11]. Here are the main theorems of [12]:

[12, Theorem 2.1] Let L and η be given. Then $\rho \neq 0$ is a zero of $F_L(\eta, \rho)$ if and only if $1/\rho$ is an eigenvalue of $\mathbf{T}_{L,\eta}$ defined as follows:

$$\mathbf{T}_{L,\eta} = \begin{bmatrix} -\eta d_{L+1} & e_{L+1} & & 0 \\ e_{L+1} & -\eta d_{L+2} & e_{L+2} & \\ & e_{L+2} & -\eta d_{L+3} & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix} : \ell^2 \to \ell^2, \text{ with} \tag{3.1.2}$$

†In this problem, L represents the orbital angular momentum quantum number, $\eta = ZZ'e^2/\hbar v$, and $\rho = \mu v r/\hbar$, where $Ze, Z'e$ are the charges of the two particles, v is their relative velocity, r is the distance between them, and μ is the reduced mass.

‡[1] proposes a method for computing the zeros ρ of $F_0(\eta, \rho)$, and the first three positive zeros are computed for given $\eta = 0.0, 0.5, \ldots, 3.0$. However, no error estimate is presented for the approximate zeros.

$$(3.1.3)\quad d_k = \frac{1}{k(k+1)}\ (k=1,2,\ldots),\ e_k = \frac{1}{k+1}\sqrt{\frac{(k+1)^2+\eta^2}{(2k+1)(2k+3)}}\ (k=0,1,2,\ldots)\,.$$

Moreover, one finds that an eigenvector of $\mathbf{T}_{L,\eta}$ corresponding to $1/\rho$ is a non-zero scalar multiple of $0 \neq \mathrm{u} \equiv [u^{(1)}, u^{(2)}, \ldots]^T = [\sqrt{2L+3}F_{L+1}(\eta,\rho), \sqrt{2L+5}F_{L+2}(\eta,\rho), \ldots]^T \in \ell^2$. Approximate zeros may be computed by truncation to any degree of accuracy.

[12, **Theorem 3.1**] Let L and η be given. Then $\rho \neq 0$ is a zero of $F_L'(\eta,\rho)$ if and only if $1/\rho$ is an eigenvalue of $\tilde{\mathbf{T}}_{L,\eta}$ defined as follows:

$$(3.1.4)\qquad \tilde{\mathbf{T}}_{L,\eta} = \begin{bmatrix} \frac{-\eta}{(L+1)^2} & \sqrt{\frac{2L+1}{L+1}}e_L & & & 0 \\ \sqrt{\frac{2L+1}{L+1}}e_L & -\eta d_{L+1} & e_{L+1} & & \\ & e_{L+1} & -\eta d_{L+2} & e_{L+2} & \\ & & e_{L+2} & -\eta d_{L+3} & \ddots \\ 0 & & & \ddots & \ddots \end{bmatrix} : \ell^2 \to \ell^2,$$

where the definitions of d_k, e_k are retained as (3.1.3). Furthermore, an eigenvector of $\tilde{\mathbf{T}}_{L,\eta}$ corresponding to $1/\rho$ is a non-zero scalar multiple of $0 \neq \tilde{\mathrm{u}} = [\sqrt{L+1}F_L(\eta,\rho), \sqrt{2L+3}F_{L+1}(\eta,\rho), \sqrt{2L+5}F_{L+2}(\eta,\rho), \ldots]^T \in \ell^2$. Approximate zeros may again be computed by truncation to any degree of accuracy.

What is missing from these two theorems is the precise error estimation. In fact, the derivation of the explicit error estimates for the numerical procedure in [12, Theorem 2.1] and [12, Theorem 3.1] is the concern of this paper. Our main results in this regard are stated in the next section (See Theorem 3.1.1 and Theorem 3.1.2 in Section 3.1.2).

The derivation of [12, Theorem 2.1] and [12, Theorem 3.1] is nothing but a formal matrix reformulation of the recurrence relations satisfied by $u_L \equiv F_L(\eta,\rho)$, found in [2, Chapter14], which are (3.1.7) and (3.1.8) in the below. For our purpose, we need two more recurrence relations (3.1.5) and (3.1.6) also found in [2, Chapter14] (, where "$'$" represents ρ-derivative).

$$(3.1.5)\qquad L\cdot u_L' = \sqrt{L^2+\eta^2}\, u_{L-1} - \left(\frac{L^2}{\rho}+\eta\right)u_L,$$

$$(3.1.6)\qquad (L+1)\cdot u_L' = \left[\frac{(L+1)^2}{\rho}+\eta\right]u_L - \sqrt{(L+1)^2+\eta^2}\, u_{L+1},$$

and by (3.1.5) and (3.1.6),

$$(3.1.7)\quad L\sqrt{(L+1)^2+\eta^2}\, u_{L+1} = (2L+1)\left[\frac{L(L+1)}{\rho}+\eta\right]u_L - (L+1)\sqrt{L^2+\eta^2}\, u_{L-1},$$

$$(3.1.8)\ (L+n+1)\sqrt{(L+n)^2+\eta^2}u_{L+n-1} - (2L+2n+1)\left[\frac{(L+n)(L+n+1)}{\rho}+\eta\right]u_{L+n}$$
$$+(L+n)\sqrt{(L+n+1)^2+\eta^2}u_{L+n+1} = 0,\ n=0,1,2,\ldots,$$

which is obtained from (3.1.7) by replacing L by $L+n$.

In [12], the asymptotic behavior of u_{L+n} is also derived from (3.1.8), using Theorem C:

$$\frac{u_{L+n+1}}{u_{L+n}} = \frac{\rho}{2n}[1+o(1)] \to 0 \ (n \to \infty). \tag{3.1.9}$$

The relation (3.1.8) is also satisfied by $G_L(\eta,\rho)$ (i.e., (3.1.1) still holds true when we substitute $G_L(\eta,\rho)$ for u_L). The point is that $u_L = F_L(\eta,\rho)$ represents a minimal solution of (3.1.8). See Gautschi[11].

In 1993, the same author(Ikebe) et. al[15] studies a more general problem subsuming the former cases, not only justifying the approximation by truncation but also deriving an asymptotic error formula, and it is Theorem A, especially (2.1), that we use in this section to derive the error estimates in Section 3.1.2.

As seen later, the eigenvalues of $\mathrm{T}_{L,\eta}$ (and $\tilde{\mathrm{T}}_{L,\eta}$, too) are all real and simple, the matrices under consideration being compact, real, symmetric, and tridiagonal, where all super-&sub-diagonal elements are nonzero (Such a matrix operator is diagonalizable. See [26]) and for any given eigenvalue the corresponding eigenvector is uniquely determined (since super-&sub-diagonal elements are nonzero, the recurrence relations yield a unique solution up to constant multiplication).

3.1.2 Error Formulas & Their Proofs

We now state two main theorems of this section, the error formulas ((3.1.10)–(3.1.13) in the below), followed by their proofs.

***Error Formulas** First, the error formulas shall be shown. Theorem 3.1.1 deals with the approximate zero of $F_L(\eta,\rho)$, while Theorem 3.1.2 with $F_L'(\eta,\rho)$:

[**Theorem 3.1.1**] For each k, let $\mathrm{T}_{L,\eta}^{(k)}$ be the kth principal submatrix of $\mathrm{T}_{L,\eta}$ defined in (3.1.2). Then, one can choose each λ_k, an eigenvalue of $\mathrm{T}_{L,\eta}^{(k)}$, such that $1/\lambda_k \equiv \rho_k \to \rho$. And the following error estimate (3.1.10) and the rate of convergence (3.1.11) are valid.

$$\rho - \rho_k = -\frac{\sqrt{(L+k+1)^2+\eta^2}}{L+k+1} \cdot \frac{(L+1)^2}{(L+1)^2+\eta^2} \cdot \frac{F_{L+k}(\eta,\rho)F_{L+k+1}(\eta,\rho)}{F_{L+1}^2(\eta,\rho)}[1+o(1)], \tag{3.1.10}$$

$$\frac{\rho-\rho_{k+1}}{\rho-\rho_k} = \left(\frac{\rho}{2k}\right)^2 [1+o(1)] \text{ (they hold as } k \to \infty). \tag{3.1.11}$$

[**Theorem 3.1.2**] For each k, let $\tilde{\mathrm{T}}_{L,\eta}^{(k)}$ be the kth principal submatrix of $\tilde{\mathrm{T}}_{L,\eta}$ defined in (3.1.4). Then, one can choose each $\tilde{\lambda}_k$, an eigenvalue of $\tilde{\mathrm{T}}_{L,\eta}^{(k)}$, such that $1/\tilde{\lambda}_k \equiv \tilde{\rho}_k \to \rho$. And the following error estimate (3.1.12) and (3.1.13) are valid.

$$(3.1.12)\qquad \rho-\tilde{\rho}_k = -\rho^2\cdot\frac{\sqrt{(L+k)^2+\eta^2}}{L+k}\cdot\frac{F_{L+k-1}(\eta,\rho)F_{L+k}(\eta,\rho)}{\left\{\rho^2-2\eta\rho-L(L+1)\right\}F_L^2(\eta,\rho)}[1+o(1)]$$

$$= \frac{\sqrt{(L+k)^2+\eta^2}}{L+k}\cdot\frac{F_{L+k-1}(\eta,\rho)F_{L+k}(\eta,\rho)}{F_L''(\eta,\rho)F_L(\eta,\rho)}[1+o(1)],$$

$$(3.1.13)\qquad \frac{\rho-\tilde{\rho}_{k+1}}{\rho-\tilde{\rho}_k} = \left(\frac{\rho}{2k}\right)^2[1+o(1)]\ \text{(they hold as } k\to\infty).$$

***The Proofs of Theorem 3.1.1 and Theorem 3.1.2** After the introduction of a few more well-known relations by the Coulomb wave functions, we will show newly found relations which are to help the simplification of error formulas, and the proofs of the theorems. First, Wronskian relations and the concrete form of $F_L(\eta,\rho)$ are known[2, Chapter14]:

$$(3.1.14)\qquad F_L'(\eta,\rho)G_L(\eta,\rho)-F_L(\eta,\rho)G_L'(\eta,\rho) = 1,$$

$$(3.1.15)\qquad F_{L-1}(\eta,\rho)G_L(\eta,\rho)-F_L(\eta,\rho)G_{L-1}(\eta,\rho) = L(L^2+\eta^2)^{-1/2},$$

$$(3.1.16)\qquad F_L(\eta,\rho) = C_L(\eta)\rho^{L+1}\sum_{k=L+1}^{\infty}A_k^L(\eta)\rho^{k-L-1},\ \text{with } C_L(\eta)=\frac{2^Le^{-\frac{\pi\eta}{2}}|\Gamma(L+1+i\eta)|}{\Gamma(2L+2)},$$

$$A_{L+1}^L = 1,\ A_{L+2}^L = \frac{\eta}{L+1},\ (k+L)(k-L-1)A_k^L = 2\eta A_{k-1}^L-A_{k-2}^L\ (k>L+2).$$

Next, newly obtained relations shall be shown.

[**Lemma 3.1.1**] In general, the following relation holds:

$$(3.1.17)\qquad \left(u_L^2\left(\frac{u_{L+1}}{u_L}\right)'\right)' = \left(u_{L+1}'u_L-u_L'u_{L+1}\right)' = \frac{2(L+1)}{\rho^2}u_Lu_{L+1}.$$

[**Proof**] The first equality is obvious. Replacing L by $L+1$ in (3.1.1), one is given

$$(3.1.18)\qquad u_{L+1}''+\left[1-\frac{2\eta}{\rho}-\frac{(L+1)(L+2)}{\rho^2}\right]u_{L+1}=0.$$

(3.1.1) $\times u_{L+1}-$ (3.1.18) $\times u_L$ yields

$$u_L''u_{L+1}-u_{L+1}''u_L+\frac{2(L+1)}{\rho^2}u_{L+1}u_L=0.$$

Hence, the second equality also holds, since $u_{L+1}''u_L-u_L''u_{L+1}=(u_{L+1}'u_L-u_L'u_{L+1})'$. ∎

[**Lemma 3.1.2**] Let $y(\rho)\equiv(2L+3)u_{L+1}^2+(2L+5)u_{L+2}^2+\cdots=\sum_{i=1}^{\infty}(2L+2i+1)u_{L+i}^2$. Then

$$(3.1.19)\qquad y(\rho)=\rho^2\frac{\sqrt{(L+1)^2+\eta^2}}{L+1}(u_{L+1}'u_L-u_L'u_{L+1}).$$

[**Proof**] $\{(3.1.5)+(3.1.6)\} \times u_L$ gives

$$(2L+1)u_L' u_L = \sqrt{L^2+\eta^2}u_{L-1}u_L + \frac{2L+1}{\rho}u_L^2 - \sqrt{(L+1)^2+\eta^2}u_L u_{L+1}.$$

Replacing L by $L+1, L+2, \ldots$ and adding both sides of each equation yield

$$\begin{aligned}(2L+3)u_{L+1}'u_{L+1} + (2L+5)u_{L+2}'u_{L+2} + \ldots &= \\ \sqrt{(L+1)^2+\eta^2}u_L u_{L+1} &+ \frac{1}{\rho}\left\{(2L+3)u_{L+1}^2 + (2L+5)u_{L+2}^2 + \ldots\right\}, \text{or}\end{aligned}$$

$$y'(\rho) - \frac{2}{\rho}y(\rho) = 2\sqrt{(L+1)^2+\eta^2}u_L u_{L+1}. \qquad (3.1.20)$$

The LHS of (3.1.20) is equal to $\rho^2\left(\frac{y(\rho)}{\rho^2}\right)'$, while the RHS turns $\frac{\rho^2\sqrt{(L+1)^2+\eta^2}}{L+1}\times \left(u_{L+1}'u_L - u_L'u_{L+1}\right)'$ by (3.1.17). Equating them gives

$$\frac{y(\rho)}{\rho^2} - \frac{\sqrt{(L+1)^2+\eta^2}}{L+1}\left(u_{L+1}'u_L - u_L'u_{L+1}\right) = c \text{ (constant)}. \qquad (3.1.21)$$

What is left now is to show $c=0$. Consider the asymptotic behavior of the LHS of (3.1.21) as $\rho \to 0$. (3.1.16) informs that u_L is a power series with its initial term $C_L(\eta)\rho^{L+1}$. That means the order of $u_{L+1}'u_L - u_L'u_{L+1}$ is at least $O(\rho^{2L+2})$. On the other hand, $y(\rho)/\rho^2 = O(\rho^{2L+2})$, directly from the definition of $y(\rho)$. Consequently, the conceivable least order of the LHS of (3.1.21) is $O(\rho^{2L+2})$. Since $L \geq 0$, the LHS of (3.1.21) $\to 0$ ($\rho \to 0$). Therefore, $0 = c$. ∎

Finally, we are ready to proceed to the proofs of Theorem 3.1.1 and Theorem 3.1.2.

[**Proof of Theorem 3.1.1**] Let us first show that the eigenvalue problem in question satisfies the conditions imposed on Theorem A, part (*i*). The form of $\mathbf{T}_{L,\eta}$ obviously meets the requirements since $d_k \to 0, e_k \to 0$ ($k \to \infty$) and $e_k \neq 0$ ($k = 0, 1, 2, \ldots$). We only have to show that all the eigenvalues of $\mathbf{T}_{L,\eta}$ are simple. In order to prove this, two facts

- There are no generalized eigenvectors of rank 2 or more corresponding to eigenvalues for an infinite real symmetric matrix in the Hilbert space (See standard books on functional analysis, e.g. [26])
- Once the first component of an eigenvector of $\mathbf{T}_{L,\eta}$ is given, all the others are uniquely determined, since $e_k \neq 0$. That is, there is only one linearly independent eigenvector

are enough, since they are the definition of an eigenvalue being simple in themselves.

The derivation of an error estimate (Theorem A, part (*ii*)) follows. First, let us evaluate $\mathbf{u}^T\mathbf{u}$. Using $y(\rho)$ defined in Lemma 3.1.2, we have $\mathbf{u}^T\mathbf{u} = y(\rho)$ and

$$\mathbf{u}^T\mathbf{u} = \rho^2\frac{\sqrt{(L+1)^2+\eta^2}}{L+1}(-u_L'u_{L+1}) \text{ (by } u_L = 0)$$

$$(3.1.22) \qquad = \rho^2 \frac{(L+1)^2+\eta^2}{(L+1)^2} u_{L+1}^2$$

$$\left(u_L' = -\frac{\sqrt{(L+1)^2+\eta^2}}{L+1} u_{L+1} \text{ is given by (3.1.6) and } u_L = 0\right).$$

Next, let's check the conditions. By (3.1.15), $u_{L+1} \neq 0$ when $u_L = 0$, leading $\mathrm{u}^T\mathrm{u} \neq 0$. And it is obvious by (3.1.9) that $u^{(n+1)}/u^{(n)}$ is bounded for all sufficiently large n. Now that all are cleared, one can put the components of $\mathbf{T}_{L,\eta}$, u and (3.1.22) into (2.1) and obtains (3.1.10). (3.1.11) is easily derived by (3.1.10) and (3.1.9). ∎

[**Proof of Theorem 3.1.2**] Let us skip the proof for the part (i) of Theorem A, since they are shown in nearly the same way as Theorem 3.1.1. Let the derivation of an error estimate be shown to the details instead. Substituting (2.1) with the components of $\tilde{\mathbf{T}}_{L,\eta}$ and $\tilde{\mathrm{u}}$, one obtains

$$\rho - \tilde{\rho}_k = -\rho^2 \cdot \frac{\sqrt{(L+k)^2+\eta^2}}{L+k} \cdot \frac{u_{L+k-1}u_{L+k}}{\tilde{\mathrm{u}}^T\tilde{\mathrm{u}}}[1+o(1)] \ (k \to \infty).$$

In order to achieve an error estimate in a closed form (3.1.12), what is still to be proved is

$$(3.1.23) \qquad \tilde{\mathrm{u}}^T\tilde{\mathrm{u}} = \left\{\rho^2 - 2\eta\rho - L(L+1)\right\} u_L^2 \ (= -\rho^2 u_L'' u_L).$$

By the definition of $\tilde{\mathrm{u}}$ and Lemma 3.1.2,

$$(3.1.24) \tilde{\mathrm{u}}^T\tilde{\mathrm{u}} = (L+1)u_L^2 + y(\rho) = (L+1)u_L^2 + \rho^2 \frac{\sqrt{(L+1)^2+\eta^2}}{L+1} u_{L+1}' u_L \ (\text{by } u_L' = 0).$$

Replacing L by $L+1$ in (3.1.5), which gives

$$(L+1) \cdot u_{L+1}' = \sqrt{(L+1)^2+\eta^2} u_L - \left(\frac{(L+1)^2}{\rho} + \eta\right) u_{L+1},$$

and putting $u_L' = 0$ into (3.1.6), which also gives

$$\left(\frac{(L+1)^2}{\rho} + \eta\right) u_L = \sqrt{(L+1)^2+\eta^2} u_{L+1},$$

yield, with $L+1 \geq 1$, $u_{L+1}' = \dfrac{(L+1)u_L}{\sqrt{(L+1)^2+\eta^2}} \left\{1 - \dfrac{2\eta}{\rho} - \dfrac{(L+1)^2}{\rho^2}\right\}$.

Substituting this into (3.1.24), one finally obtains (3.1.23) (the second equality is simply by (3.1.1)).

The proof of $\tilde{\mathrm{u}}^T\tilde{\mathrm{u}} = (L+1)u_L^2 + y(\rho) \neq 0$ has no difficulty since $y(\rho) \geq 0$ and $(L+1)u_L^2 > 0$ ($u_L \neq 0$ by (3.1.14)). (3.1.13) is derived directly by the error estimate (3.1.12) and (3.1.9).

∎

3.1.3 Numerical Experiments

The author executed the numerical experiments for the presented methods in Theorem 3.1.1 and Theorem 3.1.2. The computations were done on Hitachi parallel computer SR2001, using double precision floating-point arithmetic by FORTRAN77[§]. We used the FORTRAN subroutine COMQR[¶] in EISPACK [28] for the computation of eigenvalues.

We first computed ρ_m ($\tilde{\rho}_m$) by sufficiently large mth order principal submatrix of (3.1.2) ((3.1.4)), and regarded as the true value ρ. Then, for each k, we computed the reciprocals of all the eigenvalues of $\mathrm{T}_{L,\eta}^{(k)}$ ($\tilde{\mathrm{T}}_{L,\eta}^{(k)}$) and chose the closest to ρ to be ρ_k. The values of u_{L+n} ($n = 0, 1, 2, \ldots$) were obtained by back-substitution.[‖]

Experiment 3.1.1

Results of error estimate ($F_L(\eta, \rho) = 0$)

Given $L = 1, \eta = 1.0$,

compute $\rho = 6.566570903 \cdots$.

Table 3.1.1. Actual errors & estimates of (3.1.10)

k	(A.E.)	(T.E.)
8	-5.01E-05	-5.71E-05
9	-4.93E-06	-5.47E-06
10	-3.99E-07	-4.35E-07
11	-2.72E-08	-2.92E-08
12	-1.58E-09	-1.68E-09
13	-7.94E-11	-8.36E-11
14	-3.49E-12	-3.65E-12

Experiment 3.1.2

Results of error estimate ($F'_L(\eta, \rho) = 0$)

Given $L = 0, \eta = 0.0$,

compute $\rho = \pi/2$.

Table 3.1.2. Actual errors & estimates of (3.1.12)

k	(A.E.)	(T.E.)
2	-1.03E-01	-8.75E-02
3	-6.58E-03	-6.93E-03
4	-2.78E-04	-2.90E-04
5	-7.36E-06	-7.56E-06
6	-1.32E-07	-1.34E-07
7	-1.72E-09	-1.74E-09
8	-1.70E-11	-1.71E-11

[§]The experiments in this paper, however, do not include parallel computations.

[¶]A subroutine IMTQL1 in the same package is replaceable, as was used in [12].

[‖]By (3.1.8) and the behavior $u_{L+n} \to 0$ ($n \to \infty$), we let $u_{L+N} = 0$ and $u_{L+N-1} = \varepsilon$ ($\neq 0, \varepsilon$ shall be taken appropriately so that an overflow doesn't occur) for sufficiently large N, and computed u_{L+n} ($n = N-1, N-2, \ldots, 0$) successively in a decreasing order.

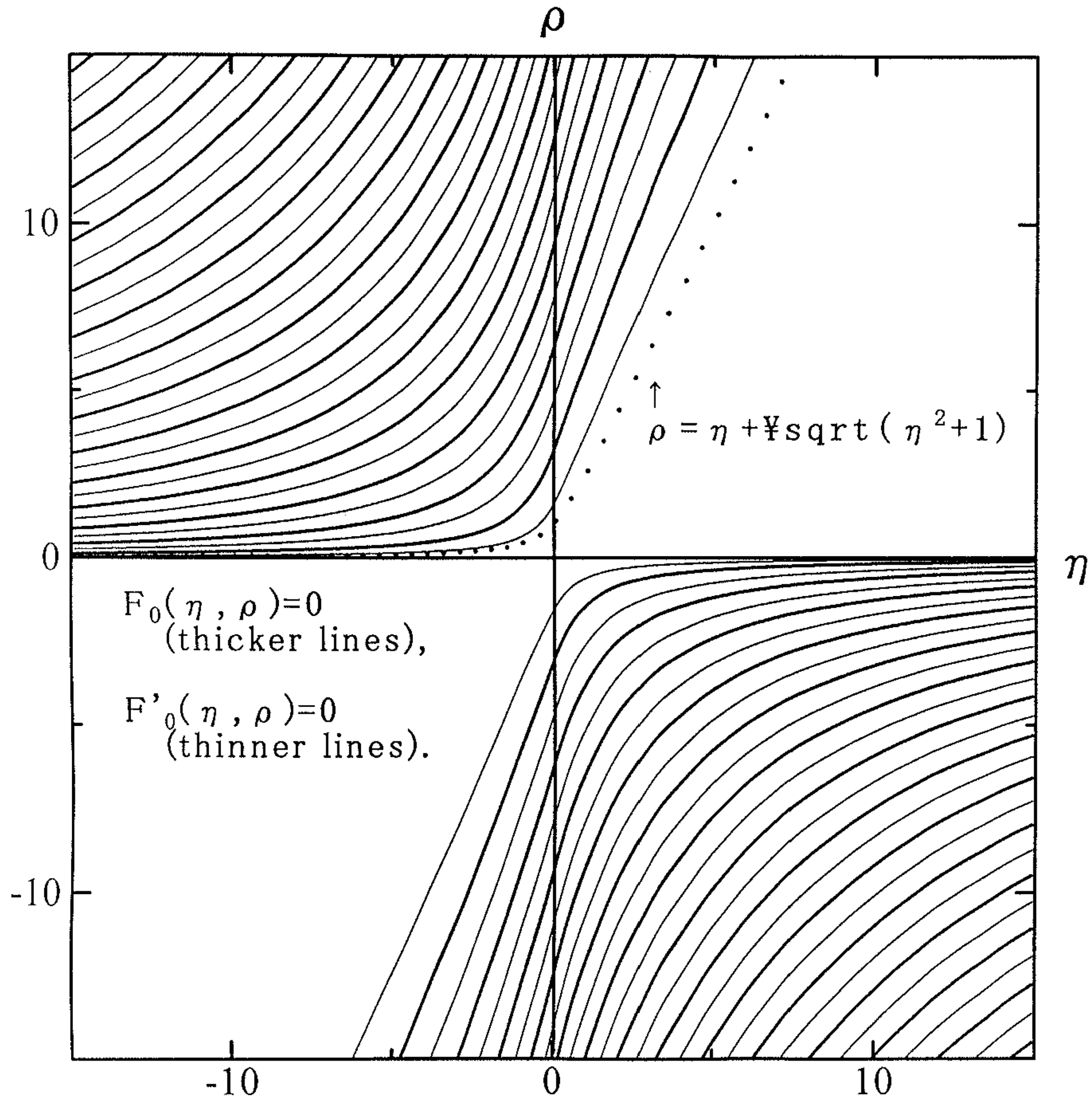

Fig. 3.1.1. (η, ρ)-pairs satisfying $F_0(\eta, \rho) = 0$, $F_0'(\eta, \rho) = 0$ and $\rho = \eta + \sqrt{\eta^2 + 1}$

Table 3.1.1 is the result of the numerical computations for a zero of $F_L(\eta, \rho)$, and Table 3.1.2 of $F_L'(\eta, \rho)$. In the tables, actual errors (A.E.) represent the LHS of (3.1.10) ((3.1.12)) divided by ρ while theoretical errors (T.E.) the RHS of (3.1.10) ((3.1.12)) without $[1 + o(1)]$, again divided by ρ, and 3 significant figures are displayed after rounding. One can observe that (A.E.) and (T.E.) get closer and each error gets smaller acceleratively as k becomes larger. Those figures are in agreement for the first digit in Table 3.1.1, and for the first two digits in Table 3.1.2.

Let us show another result in Fig. 3.1.1, or the (η, ρ)-plots satisfying $F_0(\eta, \rho) = 0$ and $F_0'(\eta, \rho) = 0$. For given each η, the computation of ρ was performed by the stated procedure. To visualize (3.1.25) (appearing in the next section), $\rho = \eta + \sqrt{\eta^2 + (L+1)^2}$ (with $L = 0$) is also plotted.

3.1.4 Remarks on the Zeros of $F_L(\eta, \rho)$ and $F_L'(\eta, \rho)$

This final section focuses on some remarks on the zeros ρ of $F_L(\eta, \rho)$ and $F_L'(\eta, \rho)$.

[Remark 3.1.1] For given L and η, the region of zeros of $F_L'(\eta, \rho)$ is determined by the inequality:

$$\rho > \eta + \sqrt{\eta^2 + (L+1)^2}. \tag{3.1.25}$$

[Proof] By (3.1.23), $u_L^2 \{\rho^2 - 2\eta\rho - (L+1)^2\} = (2L+3)u_{L+1}^2 + (2L+5)u_{L+2}^2 + \cdots$. The RHS is obviously positive, then so is the LHS. That means $\rho^2 - 2\eta\rho - (L+1)^2 > 0$. Considering $\rho > 0$, one has $\rho > \eta + \sqrt{\eta^2 + (L+1)^2}$. ∎

[Remark 3.1.2] For given L and η, the region of zeros of $F_L(\eta, \rho)$ is also confined to (3.1.25).

[Proof] Denoting the smallest zero of u_L (u_L') by ρ_0^{1st} (ρ_1^{1st}), we will show $\rho_1^{\text{1st}} < \rho_0^{\text{1st}}$. Noting that $u_L \to 0$ as $\rho \to 0$ (by the form of u_L in (3.1.16)) and $u_L(\eta, \rho_0^{\text{1st}}) = 0$, one finds, from the Rolle's theorem, that there exists at least one ρ satisfying $u_L' = 0$ in $(0, \rho_0^{\text{1st}})$. Therefore, $\rho_1^{\text{1st}} < \rho_0^{\text{1st}}$. This and Remark 3.1.1 are sufficient to prove the proposition. ∎

[Remark 3.1.3] There is one and only one zero of $F_L'(\eta, \rho)$ between two adjacent zeros of $F_L(\eta, \rho)$.

[Proof] Let us prove "there is one and only one zero of $F_L(\eta, \rho)$ between two adjacent zeros of $F_L'(\eta, \rho)$", which is equivalent to the proposition. By (3.1.23), $\rho^2 - 2\eta\rho - L(L+1) > 0$ holds when $u_L' = 0$. Recalling (3.1.1), which is $u_L'' + \{\rho^2 - 2\eta\rho - L(L+1)\} u_L/\rho^2 = 0$, we find that u_L and u_L'' have different signs then. Also note $u_L \neq 0$ (and so is u_L'') when $u_L' = 0$ by (3.1.14).

Suppose ρ_1, ρ_2 ($\rho_1 < \rho_2$) are two adjacent zeros of u_L. Then, u_L is of definite sign in (ρ_1, ρ_2). Now, without the loss of generality, we may assume $u_L > 0$. If there are more than one zeros of u'_L in (ρ_1, ρ_2), there is at least one pair of a maximal and a minimal point of u_L there, which is absurd since $u''_L > 0$ at a minimal point, or u_L and u''_L are of the same sign. This contradicts that u_L and u''_L have different signs at the zeros of u'_L. ∎

[**Remark 3.1.4**] There is one and only one zero of $F_{L+1}(\eta, \rho)$ between two adjacent zeros of $F_L(\eta, \rho)$.

[**Proof**] The next equation is obtained if one lets $u_L = F_L(\eta, \rho) = 0$ in (3.1.6):

$$(L+1) \cdot u'_L = -\sqrt{(L+1)^2 + \eta^2}\, u_{L+1}. \tag{3.1.26}$$

This shows that u'_L and u_{L+1} have different signs at the zeros of u_L. Now, let ρ_1, ρ_2 $(\rho_1 < \rho_2)$ be the adjacent zeros of u_L. Since u_L does not have double roots, obviously the signs of $u'_L(\rho_1)$ and $u'_L(\rho_2)$ are opposite. This means that $u_{L+1}(\rho_1)$ and $u_{L+1}(\rho_2)$ are of different sign too. Then, it is guaranteed, from the intermediate theorem, that there exists at least one point ρ which satisfies $u_{L+1}(\rho) = 0$ in the interval of (ρ_1, ρ_2).

The next to be proved in order is that there is only one ρ in (ρ_1, ρ_2) which is the zero of u_{L+1}. Replacing L by $L+1$ in (3.1.5) gives

$$(L+1) \cdot u'_{L+1} = \sqrt{(L+1)^2 + \eta^2}\, u_L - \left(\frac{(L+1)^2}{\rho} + \eta\right) u_{L+1}.$$

Suppose that there are more than one such points in (ρ_1, ρ_2), or that satisfies $u_{L+1} = 0$. Substituting $u_{L+1} = 0$ into the above equation yields $(L+1) \cdot u'_{L+1} = \sqrt{(L+1)^2 + \eta^2}\, u_L$. Since u_L is of definite sign in (ρ_1, ρ_2), so does u'_{L+1}. This obviously leads a contradiction, since $u_{L+1} = 0$ has no double roots and u'_{L+1} has to be of opposite signs at two adjacent zeros of u_{L+1}. ∎

3.1.5 Summary of Section 3.1

In this paper, the author derived an error estimate for approximate zeros of the regular Coulomb wave function $F_L(\eta, \rho)$ and $F'_L(\eta, \rho)$ for given L and η, as a successor of [12]. Some numerical results enough to show the validity of the error estimates were also presented. Furthermore, the author proved some properties regarding the zeros of $F_L(\eta, \rho)$ and $F'_L(\eta, \rho)$, as well as the new relations such as (3.1.19). The distributions of (η, ρ)-pairs satisfying $F_0(\eta, \rho) = 0$ and $F'_0(\eta, \rho) = 0$ were investigated too.

And let this paper be finished with some future plans. What are thought to be desirable for further research are: the computation of "other" zeros (such as the zeros η of $F_L(\eta, \rho)$ for given L, ρ); the computation of complex zeros when there are no restrictions on parameters L, η, and ρ; and the investigation of the distribution of those complex (L, η, ρ)-sets.

3.2 The (Inverse) Eigenvalue Problem of Mathieu Differential Equation

Given a complex number λ, we consider the problem of finding those values of q for which the Mathieu's equation

$$w''(z) + (\lambda - 2q\cos 2z)w(z) = 0$$

admits π- or 2π- periodic solutions. This is an inverse problem to the usual one where q is given and λ, an *eigenvalue* of the equation, is unknown.

In this section, we propose to solve the inverse problem by a matrix method. We will give an extremely accurate asymptotic error estimate.

In addition, we present a theorem (Theorem 3.2.3) which is fundamental to the method that computes the points (q, λ) such that λ is an eigenvalue satisfying $\mathrm{d}\lambda/\mathrm{d}q = 0$, with a good rate of convergence.

3.2.1 Description of the Problem

In this section, the solution of the "inverse eigenvalue problem" of the following second-order linear differential equation known as Mathieu differential equation

$$w''(z) + (\lambda - 2q\cos 2z)w(z) = 0 \qquad (0 \le z < 2\pi), \tag{3.2.1}$$

where q, λ are both parameters, is given. The stated "inverse eigenvalue problem" of Mathieu differential equation is defined together with the "(ordinary) eigenvalue problem" which equally should be treated as a pair:

[**Definition**] Let q be given. Then, λ in (3.2.1) is called an "eigenvalue" if the solution $w(z)$ of (3.2.1) is either π- or 2π- periodic, and further, the problem of obtaining $w(z)$ (called Mathieu function of the first kind) as well as an eigenvalue, is defined as the "(ordinary) eigenvalue problem".

On the other hand, we think of the inverse problem to the above stated one.

Let λ be given first. Then, we call q in (3.2.1) an "inverse eigenvalue" if the solution $w(z)$ of (3.2.1) is either π- or 2π- periodic, and further we define the problem of obtaining the Mathieu function of the first kind $w(z)$ as well as an inverse eigenvalue as the "inverse eigenvalue problem".

Let me then show the result of investigation as for the literature of the two eigenvalue problems defined above first, before proceeding to the actual solution of the inverse eigenvalue.

As for the ordinary eigenvalue problem, there is much literature dealing with it. The solution of the problem was initiated by using a continued fraction in [8]. Later, [32] proposes the method for its solution by dividing the range of q into four intervals to obtain λ fast and securely. In [19], the algorithm for obtaining the maximum 24 values of λ with the accuracy correct to the 9 decimal points is shown. [31] succeeded in computing eigenvalues by Newton's method. The newly published [13, Chapter 4] deals with the problem of obtaining complex eigenvalues λ for given complex parameters q by matrix method. The books to be recommended to read are [8] and [20].

Specifically, [13, Chapter 4] shares a similar methodology with the one presented in this section. Theorem B is applied to give a solution for the ordinary eigenvalue problem. As a result, this paper enables the solution for the complex parameter q, and the error estimation for its approximate eigenvalues in equation form. For the later reference, let us extract one of the core theorems in [13, Chapter 4]:

[13, **Chapter 4**] (ordinary eigenvalue problem, $se_{2m}(z,\lambda)$ type. As for $se_{2m}(z,\lambda)$, refer to Section 3.2.2) Let q be given. Then, $\lambda \neq 0$ is an eigenvalue of (3.2.1) if and only if λ is an eigenvalue of T defined below, or

$$\mathrm{T}\mathrm{y} = \lambda \mathrm{y}, \text{ with} \tag{3.2.2}$$

$$\mathrm{T} = \begin{bmatrix} 2^2 & q & & \mathbf{0} \\ q & 4^2 & q & \\ & q & 6^2 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} : D(\mathrm{T}) \to \ell^2,\ \mathbf{0} \neq \mathrm{y} = [B_2, B_4, B_6, \ldots]^T \in \ell^2$$

(where $D(\mathrm{T}) = \{[u^{(1)}, u^{(2)}, \ldots]^T : [2^2 \cdot u^{(1)}, 4^2 \cdot u^{(2)}, \ldots]^T \in \ell^2\}$, and again, the undefined term $B_{2k}(k = 1, 2, \ldots)$ appears in Section 3.2.2 with its definition). Furthermore, assuming the existence of T^{-1}, we have $\lambda_n \to \lambda$, where λ_n is an appropriate eigenvalue of T_n, and T_n the nth principal submatrix of T. Moreover, with the assumption that λ is a simple eigenvalue of T and $\mathrm{y}^T\mathrm{y} \neq 0$, the following estimate is valid:

$$\lambda - \lambda_n = \frac{qB_{2n}B_{2n+2}}{\mathrm{y}^T\mathrm{y}}[1 + o(1)]\ (n \to \infty).$$

In [13], it is also shown that $\mathbf{T} + \alpha\mathbf{I}$ has an inverse for an appropriate α. Then one need not be anxious about the existence of $\mathbf{T}^{-1}$.

The ordinary eigenvalue problem is applied to various fields. One of the examples is the application to the ship stability problem [5].

In [5], the ship stability problem is discussed with the analysis of the Mathieu differential equation with the term involving a first-order derivative $\mathrm{d}y/\mathrm{d}z$

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + \left(\frac{B_r}{\omega_e I_{xx}}\right)\frac{\mathrm{d}y}{\mathrm{d}z} + (a - 2q\sin 2z)y = 0.$$

In this equation, B_r represents the total hydrodynamic roll damping, ω_e encounter frequency, and I_{xx} roll virtual mass moment of inertia. And a and q are used as parameters. In this paper, three cases, undamped, linearly and non-linearly damped have been studied and the corresponding Mathieu charts are created.

Opposed to so many literature and applications for the ordinary eigenvalue problem, no literature for the inverse eigenvalue problem has been found. This indicates that this problem may hardly have been researched, although two eigenvalue problems (ordinary and inverse, of course) should be handled equally.

In this section, we will show the solution for the computation of inverse eigenvalues of Mathieu differential equations as one of the applications of Theorem A. We deal with the computation of an approximate (complex) inverse eigenvalue q for given (complex) λ (but the computation of $w(z)$, the Mathieu function of the first kind, is excluded in this paper). Since this solution is based on Theorem A, the superior characteristics of the theorem directly lead to the uniqueness and the usefulness of this solution presented here, which are:

1.) approximate inverse eigenvalues (say, q_n) can be close to the true value q with arbitrary precision,

2.) an asymptotic error estimate of q_n to q is achieved,

3.) the same algorithm applies to the computations for given real and complex λ,

4.) the simple procedure makes it easier to implement the algorithm into a variety of computers.

3.2.2 Three-Term Recurrence Relations Regarding Fourier Expansion Coefficients

When Mathieu differential equation (3.2.1) has a π- or 2π- periodic solution $w(z)$, $w(z)$ is called Mathieu function of the first kind, and is classified into 4 categories, depending on the type of expansion ((Fourier) sine or cosine series expansion and the period π or 2π) [2]. They are shown in Table 3.2.1:

Table 3.2.1. Four types of Mathieu functions of the first kind

Mathieu function	even or odd	period
$w(z) = ce_{2m}(z,\lambda) \quad = \sum_{k=0}^{\infty} A_{2k}\cos(2k)z, \ (m = 0,1,2,\ldots)$	even	π
$w(z) = ce_{2m+1}(z,\lambda) \ = \sum_{k=0}^{\infty} A_{2k+1}\cos(2k+1)z, \ (m = 0,1,2,\ldots)$	even	2π
$w(z) = se_{2m}(z,\lambda) \quad = \sum_{k=1}^{\infty} B_{2k}\sin(2k)z, \ (m = 1,2,3,\ldots)$	odd	π
$w(z) = se_{2m+1}(z,\lambda) \ = \sum_{k=0}^{\infty} B_{2k+1}\sin(2k+1)z, \ (m = 0,1,2,\ldots)$	odd	2π

In Table 3.2.1, "even", "odd" represent $w(z)$ being an even, odd function, respectively, an "π", "2π" $w(z)$ being π-periodic, 2π-periodic, again respectively.

It is found that the expansion coefficients $A_{2k}, A_{2k+1}, B_{2k+2}, B_{2k+1}$ $(k = 0,1,2,\ldots)$ hav the following relations, by the integrability of $w(z)$ and the Bessel's formula in [6]:

(3.2.3)
$$\sum_{k=0}^{\infty} |A_{2k}|^2 < \infty, \qquad \sum_{k=0}^{\infty} |A_{2k+1}|^2 < \infty,$$
$$\sum_{k=1}^{\infty} |B_{2k}|^2 < \infty, \qquad \sum_{k=0}^{\infty} |B_{2k+1}|^2 < \infty.$$

Also, the formulas regarding the expansion coefficients are known as below [2]. First, defir

(3.2.4)
$$r_n \equiv n^2 - \lambda \ (n = 0,1,2,\ldots).$$

(i) Formulas regarding $\{A_{2k}\}$ $(k = 0,1,2,\ldots)$ ($ce_{2m}(z,\lambda)$ type)

(3.2.5)
$$\begin{aligned} r_0A_0 + qA_2 &= 0 \\ 2qA_0 + r_2A_2 + qA_4 &= 0 \\ qA_{2k-2} + r_{2k}A_{2k} + qA_{2k+2} &= 0 \quad (k \geq 2) \end{aligned}$$

(ii) Formulas regarding $\{A_{2k+1}\}$ $(k = 0,1,2,\ldots)$ ($ce_{2m+1}(z,\lambda)$ type)

(3.2.6)
$$\begin{aligned} qA_1 + r_1A_1 + qA_3 &= 0 \\ qA_{2k-1} + r_{2k+1}A_{2k+1} + qA_{2k+3} &= 0 \quad (k \geq 1) \end{aligned}$$

(iii) Formulas regarding $\{B_{2k}\}$ $(k = 1,2,3,\ldots)$ ($se_{2m}(z,\lambda)$ type)

(3.2.7)
$$\begin{aligned} r_2B_2 + qB_4 &= 0 \\ qB_{2k-2} + r_{2k}B_{2k} + qB_{2k+2} &= 0 \quad (k \geq 2) \end{aligned}$$

(iv) Formulas regarding $\{B_{2k+1}\}$ ($k = 0, 1, 2, \ldots$) ($se_{2m+1}(z, \lambda)$ type)

$$
\begin{aligned}
-qB_1 + r_1 B_1 + qB_3 &= 0 \\
qB_{2k-1} + r_{2k+1} B_{2k+1} + qB_{2k+3} &= 0 \ \ (k \geq 1).
\end{aligned}
\tag{3.2.8}
$$

(3.2.5)$\sim$(3.2.8) are the fundamentals to proposing the method for the inverse eigenvalues in Section 3.2.3.

Applying Theorem C (as defined in Section 2), a theorem on second-order linear difference equations, to (3.2.5)$\sim$(3.2.8) yields the behaviors of the solutions of these recurrence relations. Let us apply the theorem to the case (iii). And let the other three cases be skipped since those solutions essentially have the same behaviors. The recurrence relations with the same coefficients as (3.2.7),

$$
\begin{aligned}
r_2 h_1 + qh_2 &= 0 \\
qh_{n-1} + r_{2n} h_n + qh_{n+1} &= 0 \ \ (n \geq 2),
\end{aligned}
\tag{3.2.9}
$$

are guaranteed to have two independent solutions $\{h_{n,1}\}$, $\{h_{n,2}\}$ with the behaviors

$$
\frac{h_{n+1,1}}{h_{n,1}} = -\frac{(2n)^2}{q}[1 + o(1)], \ \frac{h_{n+1,2}}{h_{n,2}} = -\frac{q}{(2n)^2}[1 + o(1)] \to 0 \ (n \to \infty)
\tag{3.2.10}
$$

since the conditions corresponding to (2.4) are met.

It is easily found that the solution $\{h_{n,1}\}$ diverges while $\{h_{n,2}\}$ converges to 0 (both as $n \to \infty$). Considering (3.2.3), it is clear that the solution $\{B_{2n}\}$ of (3.2.7) has to have the behavior of $\{h_{n,2}\}$, the minimal solution of (3.2.9), or

$$
\frac{B_{2(n+1)}}{B_{2n}} = -\frac{q}{(2n)^2}[1 + o(1)] \ \ (n \to \infty).
\tag{3.2.11}
$$

Furthermore, if we may assume that $r_n = n^2 - \lambda \neq 0$ ($n = 2, 4, 6, \ldots$), the recurrence relations with the terms $\{h_{2n-1}\}$ ($n = 1, 2, \ldots$) vanished,

$$
\begin{aligned}
\left(\frac{1}{r_2} + \frac{1}{r_6}\right) h_2 + \frac{1}{r_6} h_4 &= \frac{1}{q^2} r_4 h_2, \\
\frac{1}{r_{4n+2}} h_{2n} + \left(\frac{1}{r_{4n+2}} + \frac{1}{r_{4n+6}}\right) h_{2n+2} + \frac{1}{r_{4n+6}} h_{2n+4} &= \frac{1}{q^2} r_{4n+4} h_{2n+2} \ (n = 1, 2, \ldots)
\end{aligned}
\tag{3.2.12}
$$

are obtained by (3.2.9). Likewise, with the application of Theorem C, the above relations are found to have the following two solutions $\{h_{2n,1}\}$, $\{h_{2n,2}\}$:

$$
\frac{h_{2(n+1),1}}{h_{2n,1}} = \frac{(4n)^4}{q^2}[1 + o(1)], \ \frac{h_{2(n+1),2}}{h_{2n,2}} = \frac{q^2}{(4n)^4}[1 + o(1)] \to 0 \ (n \to \infty).
\tag{3.2.13}
$$

Again, with the consideration of (3.2.3),

$$
\frac{B_{4(n+1)}}{B_{4n}} = \frac{q^2}{(4n)^4}[1 + o(1)] \ \ (n \to \infty).
\tag{3.2.14}
$$

Note that (3.2.14) is derived from (3.2.11), too.

Then, we shall propose the method for the approximate solutions of inverse eigenvalues from the next section. Since the four cases (3.2.5)~(3.2.8) are essentially the same problems, only the problem for the case (*iii*) ($se_{2m}(z,\lambda)$ type) will be discussed to the details. As for $ce_{2m}(z,\lambda), se_{2m+1}(z,\lambda), ce_{2m+1}(z,\lambda)$ types, only the final results are shown.

3.2.3 The Solution of Inverse Eigenvalues

This section deals with the main theme, or the algorithm for the computation of approximate inverse eigenvalues is proposed. In addition, the error formula is derived for the approximate eigenvalue obtained by the algorithm.

***The Computation of Inverse Eigenvalues of $w(z) = se_{2m}(z,\lambda)$ Type** Let the author first categorize the problem into two cases: (a) $\lambda \neq (2k)^2$ $(k = 1, 2, \ldots)$, and (b) $\lambda = (2k)^2$ (for some natural number k) and discuss each solution.

(a) the case of $\lambda \neq (2k)^2$ or $r_{2k} \neq 0$ $(k = 1, 2, \ldots)$:

First, let us show the theorem concerning the reformulation of the problem for the computation of inverse eigenvalues, as a matrix eigenvalue problem.

[**Theorem 3.2.1**] Let a complex number λ be given. Then, $q \neq 0$ is the inverse eigenvalue of Mathieu differential equation (3.2.1) of $se_{2m}(z,\lambda)$ type if and only if $1/q^2$ is the eigenvalue of a compact operator $\mathbf{V}$ in ℓ^2

$$\mathbf{V} = \begin{bmatrix} \frac{1}{r_4}(\frac{1}{r_2}+\frac{1}{r_6}) & \frac{1}{r_6\sqrt{r_4}\sqrt{r_8}} & & \mathbf{0} \\ \frac{1}{r_6\sqrt{r_4}\sqrt{r_8}} & \frac{1}{r_8}(\frac{1}{r_6}+\frac{1}{r_{10}}) & \frac{1}{r_{10}\sqrt{r_8}\sqrt{r_{12}}} & \\ & \frac{1}{r_{10}\sqrt{r_8}\sqrt{r_{12}}} & \frac{1}{r_{12}}(\frac{1}{r_{10}}+\frac{1}{r_{14}}) & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix}. \tag{3.2.15}$$

Note that r_k is dependent only on k and λ as defined in (3.2.4). Let $\mathbf{0} \neq \mathbf{x} \equiv [x^{(1)}, x^{(2)}, \ldots]^T \in \ell^2$ be a corresponding eigenvector, or

$$\mathbf{V}\mathbf{x} = \frac{1}{q^2}\mathbf{x} \tag{3.2.16}$$

be satisfied. Then, $x^{(i)}$ $(i = 1, 2, \ldots)$ are uniquely expressed as follows, up to scalar multiplications:

$$x^{(i)} = \sqrt{r_{4i}}B_{4i} \ (i = 1, 2, \ldots). \tag{3.2.17}$$

[**Proof**] Suppose that $q \neq 0$ is the inverse eigenvalue of Mathieu differential equation of $se_{2m}(z,\lambda)$ type. Namely, (3.2.7) holds. Deleting the terms $\{B_{4k-2}\}$ $(k=1,2,\ldots)$ of (3.2.7), one obtains the recurrence relations only with $\{B_{4k}\}$ $(k=1,2,\ldots)$, or

$$\begin{aligned} \left(\frac{1}{r_2}+\frac{1}{r_6}\right)B_4+\frac{1}{r_6}B_8 &= \frac{1}{q^2}r_4B_4, \\ \frac{1}{r_{4k+2}}B_{4k}+\left(\frac{1}{r_{4k+2}}+\frac{1}{r_{4k+6}}\right)B_{4k+4}+\frac{1}{r_{4k+6}}B_{4k+8} &= \frac{1}{q^2}r_{4k+4}B_{4k+4}\ (k=1,2,\ldots). \end{aligned} \tag{3.2.18}$$

Reformulating (3.2.18) into matrix form with symmetry gives (3.2.16). In order to show that $1/q^2$ is the eigenvalue of $\mathbf{V}$, it suffices to prove $\mathbf{0} \neq \mathbf{x} \in \ell^2$.

$\mathbf{x}=\mathbf{0}$ means that $w(z)$ of (3.2.1) is a trivial solution, then we can assume $\mathbf{x} \neq \mathbf{0}$. Then, what is left to be proved is $\mathbf{x} \in \ell^2$, or $\|\mathbf{x}\|^2 < \infty$ from its definition. Computing $\|\mathbf{x}\|^2$ gives

$$\|\mathbf{x}\|^2 = \left|x^{(1)}\right|^2+\left|x^{(2)}\right|^2+\cdots=|\sqrt{r_4}B_4|^2+|\sqrt{r_8}B_8|^2+\cdots. \tag{3.2.19}$$

[6, Theorem 8.25] guarantees that

$$R \equiv \lim_{n\to\infty}\sup\left|\frac{x^{(n+1)}}{x^{(n)}}\right|^2 < 1$$

is equivalent to the convergence of $\sum_{n=1}^{\infty}|x^{(n)}|^2=\|\mathbf{x}\|^2$. By (3.2.14), the behavior

$$\left|\frac{x^{(n+1)}}{x^{(n)}}\right|=\left|\frac{\sqrt{r_{4n+4}}}{\sqrt{r_{4n}}}\right|\cdot\left|\frac{B_{4n+4}}{B_{4n}}\right|=\left|\frac{\sqrt{(4n+4)^2-\lambda}}{\sqrt{(4n)^2-\lambda}}\right|\cdot\left|\frac{q^2}{256n^4}\right|\cdot[1+o(1)]\to 0\ (n\to\infty) \tag{3.2.20}$$

is obtained, which leads obviously to $R<1$. Then, $\|\mathbf{x}\|^2 < \infty$ is satisfied and that $1/q^2$ is the eigenvalue of $\mathbf{V}$ is guaranteed.

Conversely, we shall show that letting the eigenvalues of $\mathbf{V}$ be $1/q^2$, one obtains the solution of (3.2.1) of $se_{2m}(z,\lambda)$ type.

Before putting the eigenvalue of $\mathbf{V}$ as $1/q^2$, the proof that 0 can never be an eigenvalue of $\mathbf{V}$ is in order. Supposing the contrary and we shall lead a contradiction. If 0 is an eigenvalue of $\mathbf{V}$, there exists an eigenvector $\mathbf{0} \neq \mathbf{a} \in \ell^2$ of $\mathbf{V}$ corresponding to the eigenvalue 0. Namely,

$$\mathbf{Va}=0\cdot\mathbf{a}=\mathbf{0}$$

holds. Let us define a_i $(i=1,2,\ldots)$ by $\mathbf{a}=[\sqrt{r_4}a_1,\sqrt{r_8}a_2,\ldots]^T$, and for convenience, $a_0 \equiv 0$. Expanding the above matrix equation gives

$$\frac{1}{r_{4k-2}}a_{k-1}+\left(\frac{1}{r_{4k-2}}+\frac{1}{r_{4k+2}}\right)a_k+\frac{1}{r_{4k+2}}a_{k+1} = 0\ (k=1,2,3,\ldots).$$

Again, defining $A_k \equiv a_{k-1} + a_k$ gives

$$\frac{1}{r_{4k-2}} A_k + \frac{1}{r_{4k+2}} A_{k+1} = 0 \ (k = 1, 2, \ldots),$$

which leads to

$$A_k = (-1)^{k+1} \frac{r_{4k-2}}{r_2} a_1 \ (k = 1, 2, \ldots).$$

$a_1 = 0$ easily derives $\mathbf{a} = \mathbf{0}$, therefore, $a_1 \neq 0$. Since $|A_k| = |a_{k-1} + a_k| \to \infty \ (k \to \infty)$,

$$\infty \leftarrow |a_{k-1} + a_k|^2 \leq 2\left(|a_{k-1}|^2 + |a_k|^2\right) \ (k \to \infty) \tag{3.2.21}$$

is obtained. This however contradicts $\mathbf{a} \in \ell^2$. Taking a sufficiently large natural number N leads

$$\begin{aligned} \|\mathbf{a}\|^2 &= |r_4| \cdot |a_1|^2 + |r_8| \cdot |a_2|^2 + |r_{12}| \cdot |a_3|^2 + \cdots \\ &\geq |r_{4N}| \cdot |a_N|^2 + |r_{4N+4}| \cdot |a_{N+1}|^2 + \cdots \geq |a_N|^2 + |a_{N+1}|^2 + \cdots \\ &\geq \frac{|a_N|^2 + |a_{N+1}|^2}{2} + \frac{|a_{N+1}|^2 + |a_{N+2}|^2}{2} + \cdots \to \infty \ (\text{by } (3.2.21)). \end{aligned}$$

This contradicts the assumption.

Now one is allowed to write the eigenvalue of $\mathbf{V}$ as $1/q^2$. The next step is to define $\{\hat{h}_{2k}\}$ $(k = 1, 2, \ldots)$ by $\hat{\mathbf{h}} = [\sqrt{r_4}\hat{h}_2, \sqrt{r_8}\hat{h}_4, \sqrt{r_{12}}\hat{h}_6, \ldots]^T \in \ell^2$, assuming that $\hat{\mathbf{h}}$ is an eigenvector of $\mathbf{V}$ corresponding to the eigenvalue $1/q^2$. It follows that $\hat{h}_{2k} \to 0 \ (k \to \infty)$, since $\hat{\mathbf{h}} \in \ell^2$. Now, expanding $\mathbf{V}\hat{\mathbf{h}} = \frac{1}{q^2}\hat{\mathbf{h}}$ gives the recurrence relations with the same coefficients as (3.2.12). However, as in (3.2.13), only the minimal solution $\{h_{2n,2}\}$ converges to 0, although there are two independent solutions $\{h_{2k,1}\}, \{h_{2k,2}\}$ $(k = 1, 2, \ldots)$ of (3.2.12). This implies that the solution $\{\hat{h}_{2k}\}$ is a scalar multiple of $\{h_{2k,2}\}$, or $\hat{h}_{2k} = cB_{4k}$ $(c \neq 0, k = 1, 2, \ldots)$. Therefore, (3.2.18) holds. Namely, $q \neq 0$ is the inverse eigenvalue of Mathieu differential equation (3.2.1) of $se_{2m}(z, \lambda)$ type.∎

Theorem 3.2.1 represents that the problem of computing inverse eigenvalues q of (3.2.1) is reformulated as the problem of computing q obtained by the eigenvalues of the matrix defined in (3.2.15). Using Theorem 3.2.1, the next lemma is also proved easily:

[**Lemma 3.2.1**] For given $\lambda \leq 4$, the inverse eigenvalue q is always real.

[**Proof**] First, consider the case of $\lambda < 4$. It suffices to show that $\mathbf{V}$ in (3.2.16) is a positive definite matrix. $r_{2n} = (2n)^2 - \lambda \geq 2^2 - \lambda = 4 - \lambda > 0$ $(n = 1, 2, \ldots)$ shows that $\mathbf{V}$ is real symmetric. Defining a real matrix $\mathbf{S}$ as

$$\mathbf{S} = \begin{bmatrix} e_1 & & & \mathbf{0} \\ e_2 & e_3 & & \\ & e_4 & e_5 & \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} : \ell^2 \to \ell^2, e_i = \frac{1}{\sqrt{r_{2i}}\sqrt{r_{2i+2}}} \ (i = 1, 2, \ldots),$$

one can verify that $\mathbf{V} = \mathbf{S}^T\mathbf{S}$ holds. This guarantees that $\mathbf{V}$ is positive definite.

In case of $\lambda = 4$ (, when $r_2 = 0$), one only has to follow the case (b) (when $k = 1$), to be discussed soon. After this, this can likewise be proved as $\lambda < 4$ case. Therefore, $\lambda \leq 4$ leads that q is real. ∎

One could say that Lemma 3.2.1 gives a sufficient condition for all q's to be real when λ is given.

Next, with the application of Theorem A to (3.2.16), we shall derive Theorem 3.2.2.

[**Theorem 3.2.2**] Given complex λ, consider the eigenvalue problem (3.2.16) with $q \neq 0$. The definitions of $\mathbf{V}, \mathbf{x}$ in (3.2.16) are retained in (3.2.15),(3.2.17). Then, for each n, if one takes q_n by $\xi_n = 1/q_n^2$ properly, where ξ_n is the eigenvalue of $\mathbf{V}_n$, and $\mathbf{V}_n$ is the nth principal submatrix of $\mathbf{V}$, one has a sequence $\{q_n\}$ converging to q. Furthermore, with the assumption of $\mathbf{x}^T\mathbf{x} \neq 0$, the following error estimate is valid:

$$q - q_n = -\frac{q^3}{2} \cdot \frac{B_{4n}B_{4n+4}}{r_{4n+2} \cdot (\mathbf{x}^T\mathbf{x})}[1 + o(1)] \ (n \to \infty). \tag{3.2.22}$$

Also, directly from (3.2.22) and (3.2.14),

$$\frac{q - q_{n+1}}{q - q_n} = \left(\frac{q}{16}\right)^4 \cdot \frac{1}{n^8} [1 + o(1)] \ (n \to \infty). \tag{3.2.23}$$

[**Proof**] That Theorem A may apply to the eigenvalue problem (3.2.16) will be proved. The components of $\mathbf{V}$ obviously meet the conditions in Theorem A. Besides, $\mathbf{x} \in \ell^2$ is already proved in Theorem 3.2.1. Then, what remains to be proved here is the next two:

(i) If we let $\mathbf{x} = [x^{(1)}, x^{(2)}, \ldots]^T = [\sqrt{r_4}B_4, \sqrt{r_8}B_8, \ldots]^T$, $x^{(n+1)}/x^{(n)}$ is bounded for all sufficiently large n.

(ii) If $\mathbf{x}^T\mathbf{x} \neq 0$, then the eigenvalue $1/q^2$ of $\mathbf{V}$ is simple.

(i) It is sufficient to show $|x^{(n+1)}/x^{(n)}| \to 0$ $(n \to \infty)$. This, however, is already proved in (3.2.20).

(ii) Take the contraposition. Namely, one only has to prove: 'the eigenvalue $1/q^2$ of $\mathbf{V}$ has a generalized eigenvector of rank 2 or more, or has two or more independent eigenvectors $\Rightarrow$ $\mathbf{x}^T\mathbf{x} = 0$'.

First, the second case 'the eigenvalue $1/q^2$ of $\mathbf{V}$ has two or more independent eigenvectors' is ruled out (this is already shown in the proof of Theorem 3.2.1). This means we can only assume the first case 'the eigenvalue $1/q^2$ of $\mathbf{V}$ has a generalized eigenvector of rank 2 or more'. From the definition, there exists a vector $\mathbf{0} \neq v \in \ell^2$, given an eigenvector $\mathbf{x}$, which satisfies

$$\begin{aligned}
\mathbf{x} &= (\mathbf{V} - \frac{1}{q^2}\mathbf{I})v, \\
(\mathbf{V} - \frac{1}{q^2}\mathbf{I})\mathbf{x} &= (\mathbf{V} - \frac{1}{q^2}\mathbf{I})^2 v = \mathbf{0}.
\end{aligned}$$

The computation of $\mathbf{x}^T\mathbf{x}$ yields

$$\begin{aligned}
\mathbf{x}^T\mathbf{x} &= \{(\mathbf{V} - \frac{1}{q^2}\mathbf{I})v\}^T(\mathbf{V} - \frac{1}{q^2}\mathbf{I})v \\
&= v^T(\mathbf{V} - \frac{1}{q^2}\mathbf{I})^2 v \text{ (from the symmetry of } \mathbf{V}) = 0.
\end{aligned}$$

Thus, Theorem A may apply to the problem (3.2.16). The error formula (2.1) is computed as follows:

$$\begin{aligned}
\text{LHS} &= \frac{1}{q^2} - \frac{1}{q_n^2} = \frac{-(q+q_n)\cdot(q-q_n)}{q^2\cdot q_n^2} = \frac{-2(q-q_n)}{q^3}[1+o(1)], \\
\text{RHS} &= \frac{\frac{1}{r_{4n+2}\sqrt{r_{4n}}\sqrt{r_{4n+4}}}\sqrt{r_{4n}}B_{4n}\sqrt{r_{4n+4}}B_{4n+4}}{\mathbf{x}^T\mathbf{x}}[1+o(1)] = \frac{B_{4n}B_{4n+4}}{r_{4n+2}\cdot(\mathbf{x}^T\mathbf{x})}[1+o(1)],
\end{aligned}$$

followed by

$$q - q_n = -\frac{q^3}{2}\cdot\frac{B_{4n}B_{4n+4}}{r_{4n+2}\cdot(\mathbf{x}^T\mathbf{x})}[1+o(1)] \ (n \to \infty).$$

By the direct computation, the rate of convergence of q_n to q, defined by $(q-q_{n+1})/(q-q_n)$, is derived as

$$\begin{aligned}
\frac{q-q_{n+1}}{q-q_n} &= \left(-\frac{q^3}{2}\cdot\frac{B_{4n+4}B_{4n+8}}{r_{4n+6}\cdot(\mathbf{x}^T\mathbf{x})}[1+o(1)]\right) \Big/ \left(-\frac{q^3}{2}\cdot\frac{B_{4n}B_{4n+4}}{r_{4n+2}\cdot(\mathbf{x}^T\mathbf{x})}[1+o(1)]\right) \\
&= \frac{r_{4n+2}}{r_{4n+6}}\cdot\frac{B_{4n+4}}{B_{4n}}\cdot\frac{B_{4n+8}}{B_{4n+4}}[1+o(1)] = \left(\frac{q}{16}\right)^4\frac{1}{n^8}\ [1+o(1)] \ (n\to\infty) \text{ (by (3.2.14))}.
\end{aligned}$$

Theorem 3.2.2 guarantees that the values of q_n computed by the eigenvalues of $\mathbf{V}_n$, the nth principal submatrix of $\mathbf{V}$, reach the true value q faster as n gets larger, and the errors are evaluated precisely using (3.2.22). The four uniqueness (or usefulness) mentioned in Section 2 are succeeded to this Theorem 3.2.2.

[**Corollary 3.2.1**] The problem for the computation of the inverse eigenvalue q for given λ is also equivalent to the computation of q by the eigenvalue problem of the following matrix $\mathbf{U}$:

(3.2.24) $$\mathbf{U}\tilde{\mathbf{x}} = \frac{1}{q^2}\tilde{\mathbf{x}}, \quad \text{where}$$

$$\mathrm{U} = \begin{bmatrix} \frac{1}{r_2 r_4} & \frac{1}{r_4\sqrt{r_2}\sqrt{r_6}} & & \mathbf{0} \\ \frac{1}{r_4\sqrt{r_2}\sqrt{r_6}} & \frac{1}{r_6}(\frac{1}{r_4}+\frac{1}{r_8}) & \frac{1}{r_8\sqrt{r_6}\sqrt{r_{10}}} & \\ & \frac{1}{r_8\sqrt{r_6}\sqrt{r_{10}}} & \frac{1}{r_{10}}(\frac{1}{r_8}+\frac{1}{r_{12}}) & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} : \ell^2 \to \ell^2,\ \tilde{\mathrm{x}} = \begin{bmatrix} \sqrt{r_2}B_2 \\ \sqrt{r_6}B_6 \\ \sqrt{r_{10}}B_{10} \\ \vdots \end{bmatrix} \in \ell^2.$$

Taking $\tilde{q}_n$ by $\tilde{\xi}_n = 1/\tilde{q}_n^2$ properly, where $\tilde{\xi}_n$ is the eigenvalue of U_n and U_n is the nth principal submatrix of U, one has a sequence $\{\tilde{q}_n\}$ converging to q. Furthermore, with the assumption of $\tilde{\mathrm{x}}^T\tilde{\mathrm{x}} \neq 0$, the following error estimate is valid:

$$q - \tilde{q}_n = -\frac{q^3}{2} \cdot \frac{B_{4n-2}B_{4n+2}}{r_{4n} \cdot (\tilde{\mathrm{x}}^T\tilde{\mathrm{x}})}[1+o(1)]\ (n \to \infty). \tag{3.2.25}$$

[**Proof**] In the same way as the derivation of (3.2.18), we first delete $\{B_{4n}\}(n = 1,2,\ldots)$ from (3.2.7) and express the relations in matrix form (with symmetry) so that (3.2.24) is obtained. Let us omit the details since they are basically the same proofs as in Theorem 3.2.1 and Theorem 3.2.2. ■

[**Remark 3.2.1**] $\{q_n\}$ converges faster than $\{\tilde{q}_n\}$ to q as the following asymptotic relation represents:

$$\frac{q-q_n}{q-\tilde{q}_n} = \frac{q^2}{256n^4}[1+o(1)]\ (n \to \infty). \tag{3.2.26}$$

[**Proof**] The direct computation of $(q-q_n)/(q-\tilde{q}_n)$ by (3.2.22) and (3.2.25) gives

$$\begin{aligned} \frac{q-q_n}{q-\tilde{q}_n} &= \frac{r_{4n}}{r_{4n+2}} \cdot \frac{B_{4n}B_{4n+4}}{B_{4n-2}B_{4n+2}} \cdot \frac{\tilde{\mathrm{x}}^T\tilde{\mathrm{x}}}{\mathrm{x}^T\mathrm{x}}[1+o(1)] \\ &= \frac{(4n)^2-\lambda}{(4n+2)^2-\lambda} \cdot \frac{q}{(4n)^2} \cdot \frac{q}{(4n+4)^2} \cdot \frac{\tilde{\mathrm{x}}^T\tilde{\mathrm{x}}}{\mathrm{x}^T\mathrm{x}}[1+o(1)] = \frac{q^2}{256n^4} \cdot \frac{\tilde{\mathrm{x}}^T\tilde{\mathrm{x}}}{\mathrm{x}^T\mathrm{x}}[1+o(1)]. \end{aligned}$$

In order to evaluate $(\tilde{\mathrm{x}}^T\tilde{\mathrm{x}})/(\mathrm{x}^T\mathrm{x})$, multiplying the both sides of the lth line $(l = 1,2,\ldots)$ of (3.2.7) by B_{2l} gives

$$\begin{aligned} r_2B_2^2 + qB_2B_4 &= 0 &(3.2.27)\\ qB_2B_4 + r_4B_4^2 + qB_4B_6 &= 0 &(3.2.28)\\ qB_4B_6 + r_6B_6^2 + qB_6B_8 &= 0 &(3.2.29)\\ \ddots\quad &\ \vdots \end{aligned}$$

(3.2.27) − (3.2.28) + (3.2.29) − ⋯ leads $r_2B_2^2 - r_4B_4^2 + r_6B_6^2 - r_8B_8^2 + \cdots = 0$, or

$$r_2B_2^2 + r_6B_6^2 + \cdots = r_4B_4^2 + r_8B_8^2 + \cdots. \tag{3.2.30}$$

The LHS of (3.2.30) is $\tilde{\mathrm{x}}^T\tilde{\mathrm{x}}$, while its RHS is $\mathrm{x}^T\mathrm{x}$, leading $\tilde{\mathrm{x}}^T\tilde{\mathrm{x}} = \mathrm{x}^T\mathrm{x}$. This shows (3.2.26) holds.■

[**Corollary 3.2.2**] The problem for the computation of the inverse eigenvalue q for given λ is also equivalent to the computation of q by the eigenvalue problem of the following matrix T:

$$\text{(3.2.31)} \qquad \mathbf{T}\hat{\mathrm{x}} = -\frac{1}{q}\hat{\mathrm{x}}, \text{ where}$$

$$\mathrm{T} \equiv \begin{bmatrix} 0 & \frac{1}{\sqrt{r_2}\sqrt{r_4}} & & \mathbf{0} \\ \frac{1}{\sqrt{r_2}\sqrt{r_4}} & 0 & \frac{1}{\sqrt{r_4}\sqrt{r_6}} & \\ & \frac{1}{\sqrt{r_4}\sqrt{r_6}} & 0 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} : \ell^2 \to \ell^2, \ \hat{\mathrm{x}} \equiv \begin{bmatrix} \sqrt{r_2}B_2 \\ \sqrt{r_4}B_4 \\ \sqrt{r_6}B_6 \\ \vdots \end{bmatrix} \in \ell^2.$$

Taking $\hat{q}_n$ by $\hat{\xi}_n = -1/\hat{q}_n$ properly, where $\hat{\xi}_n$ are the eigenvalues of T_n and T_n is the nth principal submatrix of T, one has a sequence $\{\hat{q}_n\}$ converging to q. Furthermore, with the assumption of $\hat{\mathrm{x}}^T\hat{\mathrm{x}} \neq 0$, the following error estimate is valid:

$$\text{(3.2.32)} \qquad q - \hat{q}_n = \frac{q^2 B_{2n}B_{2n+2}}{\hat{\mathrm{x}}^T\hat{\mathrm{x}}}[1 + o(1)] \ (n \to \infty).$$

[**Proof**] One gets (3.2.31) after changing (3.2.7) and putting into matrix form. Again, the above is derived in the same way as Theorem 3.2.1 and Theorem 3.2.2.■

[**Remark 3.2.2**] $\{q_n\}$ converges faster than $\{\hat{q}_n\}$ to q.

[**Proof**] $(q - \hat{q}_{n+1})/(q - \hat{q}_n)$ by (3.2.32) yields

$$\text{(3.2.33)} \qquad \frac{q - \hat{q}_{n+1}}{q - \hat{q}_n} = \frac{q^2}{16} \cdot \frac{1}{n^4} \ [1 + o(1)] \ (n \to \infty) \ \ \text{(the details are skipped)}.$$

From the comparison between (3.2.23) and (3.2.33), it is clear that $\{q_n\}$ converges faster than $\{\hat{q}_n\}$ to q. ■

[**Remark 3.2.3**] (3.2.16), (3.2.24) are derived by (3.2.31).

[**Proof**] $\mathrm{T}^2\hat{\mathrm{x}} = \frac{1}{q^2}\hat{\mathrm{x}}$ is easily derived from $\mathbf{T}\hat{\mathrm{x}} = -\frac{1}{q}\hat{\mathrm{x}}$. T^2 in a concrete form is

$$\mathrm{T}^2 = \begin{bmatrix} \frac{1}{r_2r_4} & 0 & \frac{1}{r_4\sqrt{r_2}\sqrt{r_6}} & & \mathbf{0} \\ 0 & \frac{1}{r_4}(\frac{1}{r_2} + \frac{1}{r_6}) & 0 & \frac{1}{r_6\sqrt{r_4}\sqrt{r_8}} & \\ \frac{1}{r_4\sqrt{r_2}\sqrt{r_6}} & 0 & \frac{1}{r_6}(\frac{1}{r_4} + \frac{1}{r_8}) & 0 & \ddots \\ & \frac{1}{r_6\sqrt{r_4}\sqrt{r_8}} & 0 & \frac{1}{r_8}(\frac{1}{r_6} + \frac{1}{r_{10}}) & \ddots \\ \mathbf{0} & & \ddots & \ddots & \ddots \end{bmatrix},$$

which is symmetric penta-diagonal with $(i, i+1), (i+1, i)$ components vanished ($i = 1, 2, ..$
Then, expanding $\mathbf{T}^2\hat{\mathbf{x}} = \frac{1}{q^2}\hat{\mathbf{x}}$, followed by picking up odd and even lines, and reformulat
them into matrix form give (3.2.16) and (3.2.24), respectively. ∎

[**Remark 3.2.4**] If the truncation size of the matrices of (3.2.16), (3.2.24) and (3.2.31) the same, the numbers of the approximate inverse eigenvalues obtained from (3.2.16) a (3.2.24) are twice as much as that of (3.2.31).

[**Proof**] One gets n approximate inverse eigenvalues from the nth principal submatrix (3.2.31). On the other hand, in the cases of (3.2.16) and (3.2.24), $2n$ of the inverse eigenvalu are obtained since it is obvious that if q_n or $\tilde{q}_n$ satisfies (3.2.16) or (3.2.24), so does $-q_n$ $-\tilde{q}_n$. ∎

This concludes the theorem for the computation of inverse eigenvalues of Mathieu d ferential equation for the (a) case (where $\lambda \neq (2k)^2$ $(k = 1, 2, \ldots)$). Next, the solution f the (b) case, where $\lambda = (2k)^2$ (for some natural number k), is given. The same meth applies to this case as (a) case, then the details shall be omitted. Instead, the process f the problem to be reformulated and the error formula given by the application of Theore A are shown, only as for the one with the fastest rate of convergence.

(b), the case of $\lambda = (2k)^2$ (for some natural number k):

First, let us take the case of $k = 1$. The first three relations of (3.2.7) are

$$\begin{array}{ccccccccc} r_2B_2 & + & qB_4 & & & & & = & 0, \\ qB_2 & + & r_4B_4 & + & qB_6 & & & = & 0, \\ & & qB_4 & + & r_6B_6 & + & qB_8 & = & 0. \end{array}$$

$B_4 = 0$ since $r_2 = 0$ and $q \neq 0$. Also, $r_6B_6 + qB_8 = 0$ from the third equation. This and t rest relations can eventually be applied by Theorem A as the (a) case.

Secondly, take the case $k \geq 2$. For convenience, define $B_0 = 0$. Picking up the k - $1, k, k+1$th equations of (3.2.7), one gets

$$qB_{2k-4} + r_{2k-2}B_{2k-2} + qB_{2k} = 0, \tag{3.2.34}$$

$$qB_{2k-2} + r_{2k}B_{2k} + qB_{2k+2} = 0, \tag{3.2.35}$$

$$qB_{2k} + r_{2k+2}B_{2k+2} + qB_{2k+4} = 0. \tag{3.2.36}$$

$B_{2k-2} = -B_{2k+2}$ since $r_{2k} = 0$ (by (3.2.35)). (3.2.34) - (3.2.36) computes

$$\begin{aligned} \text{LHS} &= qB_{2k-4} + r_{2k-2}B_{2k-2} - r_{2k+2}B_{2k+2} - qB_{2k+4} \\ &= qB_{2k-4} - (r_{2k-2} + r_{2k+2})B_{2k+2} - qB_{2k+4} = 0 = \text{RHS}, \end{aligned}$$

where $r_{2k-2} + r_{2k+2} = 8$ as $r_{2k} = 0$. Then,

$$q(-B_{2k-4}) + 8B_{2k+2} + qB_{2k+4} = 0. \tag{3.2.37}$$

Substituting $B_{2k-2} = -B_{2k+2}$ into the $k-2$th equation gives

$$q(-B_{2k-6}) + r_{2k-4}(-B_{2k-4}) + qB_{2k+2} = 0. \tag{3.2.38}$$

Likewise, the $1 \sim (k-3)$th equations of (3.2.7) are transformed as

$$\begin{aligned} r_2(-B_2) + q(-B_4) &= 0 \\ q(-B_{2m-2}) + r_{2m}(-B_{2m}) + q(-B_{2m+2}) &= 0 \quad (2 \le m \le k-3). \end{aligned} \tag{3.2.39}$$

Thus, using the $k+2$th equation of (3.2.7), along with (3.2.39),(3.2.38),(3.2.37), one only has to take the same steps as the (a) case. Refer to Appendix 1 for the results.

***The Computation of Inverse Eigenvalues of $w(z) = ce_{2m}(z,\lambda), se_{2m+1}(z,\lambda)$ and $ce_{2m+1}(z,\lambda)$ Types** The eigenvalue problem for $w(z) = ce_{2m}(z,\lambda)$ type can be reformulated as matrix eigenvalue problem by the same process as we handled $se_{2m}(z,\lambda)$ type. The matrix forms and error estimates are shown for the cases (a) $\lambda \ne (2k)^2$ $(k = 0, 1, 2, \ldots)$, and (b) $\lambda = (2k)^2$ (for some k), only as for the one with the fastest convergence. See Appendix 1.

As for the types of $w(z) = se_{2m+1}(z,\lambda)$ and $ce_{2m+1}(z,\lambda)$, one can apply Theorem A, by reformulating (3.2.8), (3.2.6) into the form of $\mathbf{Ty} = -\frac{1}{q}\mathbf{y}$. However, since the $(1,1)$ component of matrix $\mathbf{T}$ is nonzero, which is not the case of $se_{2m}(z,\lambda), ce_{2m}(z,\lambda)$ types, one is not allowed to separate the eigenvalue problems into two, in order to have a faster rate of convergence. See also Appendix 1.

3.2.4 Numerical Experiments and q–λ Graph Making

In this section, the numerical results of error estimates obtained in Section 3.2.3 are shown for the cases of $se_{2m}(z,\lambda)$ and $ce_{2m}(z,\lambda)$ types. Also, pairs of real points (q,λ) where λ are eigenvalues and q are inverse eigenvalues, will be plotted. All the computations were done in quadruple precision using Fortran77 on Fujitsu VPP-500, using comqr.f in EISPACK [28] for the computations of matrix eigenvalues.

In the experiments, actual (relative) errors $(q - q_n)/q$ and theoretical errors (the RHS of the error estimate without the term $[1 + o(1)]$, again divided by q) were computed and compared to see its validity. The true value q was given by one of the q_n's, which were obtained from the eigenvalues of truncated matrices of sufficiently large size n. For each n, the

closest q_n to q, where q_n were computed from the eigenvalues of the nth truncated matri
was regarded as the approximate value. $\{B_{2n}\}$, which appears in the error estimate ($\{A$
for ce_{2m} type), were computed by backward substitution from (3.2.7) (likewise, (3.2.5)) w
$B_{2N} = 0$ for sufficiently large N (likewise, $A_{2N} = 0$), from the fact $B_{2n} \to 0$ ($n \to$
(likewise, $A_{2n} \to 0$ ($n \to \infty$)).

In the tables, three digits are displayed after rounding, and 'Re', 'Im' represent the r and imaginary parts of the observed data, respectively.

Experiment 3.2.1 **Results of error estimate($se_{2m}(z, \lambda)$ type)**

$$\begin{aligned} \text{Given } \lambda &= 50 + 80i, \\ \text{compute } q &= (263.9649620\cdots) - (95.28516350\cdots)i. \end{aligned}$$

Table 3.2.2. Actual errors & estimates of (3.2.22)

	Actual	Error	Theoretical	Error
n	Re	Im	Re	Im
2	1.04e+00	-2.05e-01	-5.16e-02	-2.54e-01
3	8.24e-01	2.34e-01	9.19e-02	-2.91e-01
4	6.37e-01	2.11e-01	2.83e-01	-3.76e-03
5	-1.27e-01	1.60e-01	-1.33e-01	2.25e-02
6	2.74e-03	1.27e-02	2.65e-03	1.36e-02
7	2.52e-04	2.28e-05	2.54e-04	1.82e-05
8	5.07e-07	-1.24e-06	5.01e-07	-1.24e-06
9	-1.95e-09	-1.64e-09	-1.95e-09	-1.64e-09
10	-1.70e-12	1.08e-12	-1.69e-12	1.08e-12
11	1.96e-16	7.02e-16	1.96e-16	7.02e-16
12	1.31e-19	1.52e-21	1.31e-19	1.49e-21
13	3.50e-24	-1.20e-23	3.38e-24	-1.20e-23

$$\begin{cases} \text{Actual Error} : (q - q_n)/q \\ \text{Theoretical Error} : -\dfrac{q^3}{2} \cdot \dfrac{B_{4n}B_{4n+4}}{r_{4n+2} \cdot (\mathrm{x}^T\mathrm{x})} \cdot \dfrac{1}{q} \end{cases}$$

Table 3.2.3. Actual errors & estimates of (3.2.25)

	Actual	Error	Theoretical	Error
n	Re	Im	Re	Im
2	1.02e+00	-1.95e-01	9.31e-02	6.86e-01
3	8.49e-01	2.15e-01	2.38e-01	5.05e-01
4	6.60e-01	2.09e-01	-1.07e-01	2.59e-01
5	-6.45e-01	4.59e-01	-6.49e-02	-1.71e-01
6	-1.61e-02	5.39e-02	-2.83e-02	4.85e-02
7	1.61e-03	1.47e-03	1.70e-03	1.47e-03
8	1.81e-05	-1.11e-05	1.80e-05	-1.13e-05
9	-1.57e-08	-6.38e-08	-1.59e-08	-6.38e-08
10	-7.91e-11	-5.35e-12	-7.91e-11	-5.26e-12
11	-1.51e-14	3.91e-14	-1.51e-14	3.92e-14
12	8.42e-18	6.49e-18	8.42e-18	6.49e-18
13	1.11e-21	-8.21e-22	1.11e-21	-8.21e-22

$$\begin{cases} \text{Actual Error} : (q - \tilde{q}_n)/q \\ \text{Theoretical Error} : -\dfrac{q^3}{2} \cdot \dfrac{B_{4n-2}B_{4n+2}}{r_{4n} \cdot (\tilde{\mathrm{x}}^T \tilde{\mathrm{x}})} \cdot \dfrac{1}{q} \end{cases}$$

Table 3.2.4. Actual errors & estimates of (3.2.32)

	Actual	Error	Theoretical	Error
n	Re	Im	Re	Im
2	0.00e+00	0.00e+00	8.16e-03	-5.16e-02
4	9.42e-01	4.68e-01	4.44e-02	-1.08e-01
6	6.78e-01	5.58e-01	1.63e-01	-8.59e-02
8	1.03e+00	-2.07e-01	1.17e-01	1.84e-01
10	8.31e-01	2.35e-01	-2.88e-01	-1.13e-01
12	8.31e-01	2.35e-01	-2.46e-02	6.83e-02
14	1.61e-03	1.47e-03	2.04e-03	1.50e-03
16	1.81e-05	-1.11e-05	1.87e-05	-1.29e-05
18	-1.57e-08	-6.38e-08	-1.83e-08	-6.58e-08
20	-7.91e-11	-5.35e-12	-8.12e-11	-3.96e-12
22	-1.51e-14	3.91e-14	-1.48e-14	4.00e-14
24	8.42e-18	6.49e-18	8.58e-18	6.49e-18
26	1.11e-21	-8.21e-22	1.12e-21	-8.35e-22
28	-3.49e-26	-9.07e-26	-3.56e-26	-9.11e-26

$$\begin{cases} \text{Actual Error} : (q - \hat{q}_n)/q \\ \text{Theoretical Error} : \dfrac{q^2 B_{2n} B_{2n+2}}{\hat{\mathrm{x}}^T \hat{\mathrm{x}}} \cdot \dfrac{1}{q} \end{cases}$$

Table 3.2.5. Actual errors & estimates of (3.2.26)

n	Actual Error Re	Actual Error Im	Theoretical Error Re	Theoretical Error Im
2	1.02e+00	-6.60e-03	1.48e+01	-1.23e+01
3	9.77e-01	2.87e-02	2.92e+00	-2.43e+00
4	9.69e-01	1.31e-02	9.25e-01	-7.68e-01
5	2.49e-01	-7.18e-02	3.79e-01	-3.14e-01
6	2.03e-01	-1.11e-01	1.83e-01	-1.52e-01
7	9.22e-02	-7.03e-02	9.86e-02	-8.18e-02
8	5.10e-02	-3.74e-02	5.78e-02	-4.80e-02
9	3.13e-02	-2.28e-02	3.61e-02	-2.99e-02
10	2.04e-02	-1.50e-02	2.37e-02	-1.97e-02
11	1.39e-02	-1.04e-02	1.62e-02	-1.34e-02
12	9.86e-03	-7.42e-03	1.14e-02	-9.48e-03
13	7.19e-03	-5.46e-03	8.29e-03	-6.88e-03

$$\begin{cases} \text{Actual Error} : \dfrac{q - q_n}{q - q_n} \\ \text{Theoretical Error} : \dfrac{q^2}{256n^4} \end{cases}$$

As one can tell from Table 3.2.2 and Table 3.2.3, the actual and theoretical (relative) errors are in agreement for about two digits of both of real and imaginary parts at around $n = 8$. Table 3.2.4 shows the results of (3.2.32) and also shows that the matrix size needs to be approximately double in order to obtain inverse eigenvalues with the same precision in Table 3.2.2 and Table 3.2.3. Table 3.2.5 is the result of the computation of both sides of (3.2.26) in Remark 3.2.1 (as for the RHS, the term [1+o(1)] is neglected).

As for the $ce_{2m}(z, \lambda)$ type, the experiments were performed only as for the case with the fastest convergence of approximate inverse eigenvalues. (Refer to Appendix 1 for its error estimate). Table 3.2.6 is the result and shows that actual and theoretical (relative) errors are in agreement for about two digits at around $n = 6$.

Lastly, real pairs of (q, λ) computed by the algorithm proposed in this section and the previous theorem[13, Chapter 4] are shown for $se_{2m}(z, \lambda)$ type (Fig. 3.2.1).

In Fig. 3.2.1, one can find where $d\lambda/dq \simeq 0$ occurs. At those points, the problem of getting q, given λ will be unstable, then the algorithm in [13, Chapter 4] is used at such points. Except such cases, however, the algorithm in this section is adopted. Since the rate of convergence of approximate inverse eigenvalues in this paper is faster than the ones by [13, Chapter 4], one can compute such pairs of (q, λ) more quickly. The rate of convergence for each case shall be shown:

• The rate of convergence of $\{q_n\}$ obtained in this paper ((3.2.23)) :

$$\frac{q - q_{n+1}}{q - q_n} = \left(\frac{q}{16}\right)^4 \cdot \frac{1}{n^8} \, [1 + o(1)] \ (n \to \infty),$$

• The rate of convergence of $\{\lambda_n\}$ obtained by [13, Chapter 4, formula(10)]:

$$\frac{\lambda - \lambda_{n+1}}{\lambda - \lambda_n} = \frac{q^2}{(\lambda - 4(n+1)^2)^2} \, [1 + o(1)] = \frac{q^2}{16} \cdot \frac{1}{n^4} \, [1 + o(1)] \ (n \to \infty).$$

As was shown in Lemma 3.2.1, given $\lambda \leq 4$, one only has real inverse eigenvalues q, tl all (q, λ) are included in the real q–real λ graph shown in Fig. 3.2.1.

Experiment 3.2.2 **Results of error estimate ($ce_{2m}(z, \lambda)$ type)**

Given $\lambda = 60 + 30i$, compute
$q = (79.56777345 \cdots) - (50.87969961 \cdots)i.$

Table 3.2.6. Actual errors & estimates
of $ce_{2m}(z, \lambda)$ case

	Actual	Error	Theoretical	Error
n	Re	Im	Re	Im
2	7.51e-01	-6.48e-01	-3.97e-01	1.01e+00
3	1.07e+00	-6.46e-01	9.84e-01	7.45e-01
4	-3.88e-02	2.79e-01	-1.70e-01	1.23e-01
5	4.88e-03	3.57e-04	4.99e-03	2.74e-04
6	-7.55e-06	-1.19e-05	-7.57e-06	-1.18e-05
7	-3.24e-09	7.60e-09	-3.24e-09	7.60e-09
8	1.34e-12	-2.80e-13	1.34e-12	-2.80e-13
9	-5.67e-17	-5.20e-17	-5.67e-17	-5.20e-17
10	-1.80e-22	1.68e-21	-1.80e-22	1.68e-21
11	1.38e-26	-8.28e-27	1.38e-26	-8.28e-27

$$\begin{cases} \text{Actual Error : } (q - q_n)/q \\ \text{Theoretical Error : } -\dfrac{q^3}{2} \cdot \dfrac{A_{4n-2}A_{4n+2}}{r_{4n} \cdot (\mathbf{z}^T\mathbf{z})} \cdot \dfrac{1}{q} \end{cases}$$

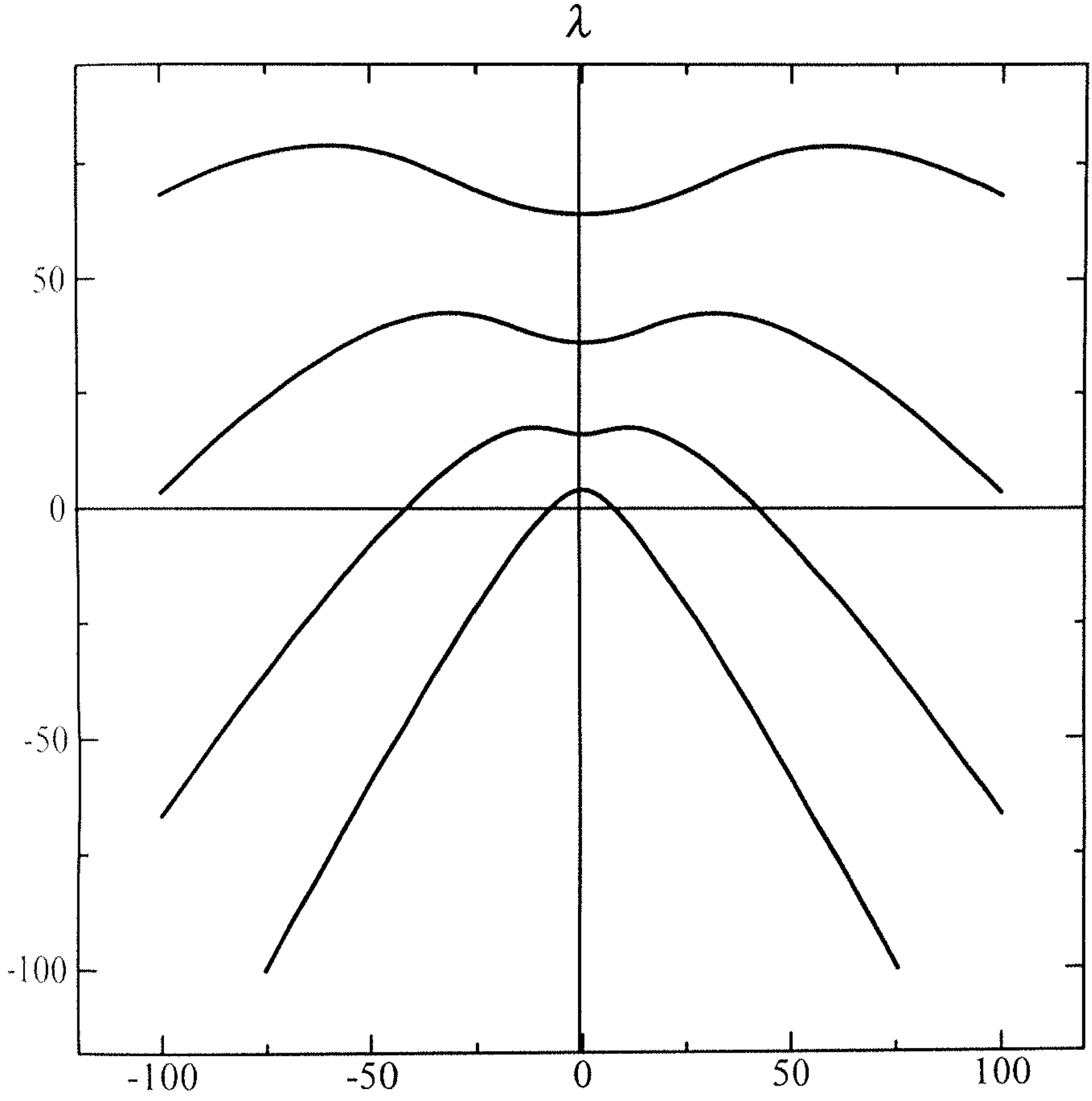

Fig. 3.2.1. Real q–real λ relation ($w = se_{2m}(z,q)$ case)

3.2.5 Theorems on Double Eigenvalues and Their Computations

In the last section, the q–λ graph of $se_{2m}(z, \lambda)$ type was shown in Fig. 3.2.1. The graph evidently suggests the existence of double eigenvalues (q, λ), where $\mathrm{d}\lambda/\mathrm{d}q = 0$. In this section, a theorem on double eigenvalues is presented such that a method for the fast computation of those double eigenvalues is obtained (the method will be explained in Section 4.4.3).

In the later discussion, three types of the rest are omitted since they, or $(i)\sim(iv)$ are essentially the same problems. Also, in this section, we limit our focus only to the case where q and λ are both real. Note that the more generalized discussion for the computation of double eigenvalues of matrices of a certain type will be provided in Section 4. Then, only briefly, and at the same time, the points specific to the Mathieu's case are stated for the rest of this section:

[**Theorem 3.2.3**] An eigenvalue λ, an inverse eigenvalue $q \neq 0$, and coefficients $\{B_{2n}\}$ $(n = 1, 2, \ldots)$ have the next relation in general:

$$\begin{aligned} \left(\frac{\mathrm{d}\lambda}{\mathrm{d}q}\right) \cdot (\mathbf{y}^T\mathbf{y}) &= -\frac{2}{q} \cdot (\mathbf{x}^T\mathbf{x}), \text{where} \\ \mathbf{y} &= [B_2, B_4, B_6, \ldots]^T \in \ell^2, \\ \mathbf{x} &= [\sqrt{r_4}B_4, \sqrt{r_8}B_8, \sqrt{r_{12}}B_{12}, \ldots]^T \in \ell^2. \end{aligned} \tag{3.2.40}$$

Note that $\mathbf{y}$ and $\mathbf{x}$ defined here are nothing but the eigenvectors of (3.2.2) and (3.2.16).

[**Proof**] It is known that (3.2.7) is replaced by the matrix equation (3.2.2) [13, Chapter 4], and differentiating the both sides of (3.2.2) with respect to q (in the sequel, let us define that " $'$ " represents the differentiation with respect to q) gives

$$\lambda'\mathbf{y} = (\mathbf{T} - \lambda\mathbf{I})\mathbf{y}' + \mathbf{T}'\mathbf{y}. \tag{3.2.41}$$

On the other hand, transformation of (3.2.2) yields

$$\begin{bmatrix} 0 & 1 & & 0 \\ 1 & 0 & 1 & \\ & 1 & 0 & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix} \mathbf{y} = -\frac{1}{q} \begin{bmatrix} r_2 & & & 0 \\ & r_4 & & \\ & & r_6 & \\ 0 & & & \ddots \end{bmatrix} \mathbf{y}. \tag{3.2.42}$$

This informs that the LHS of this equation is equal to $\mathbf{T}'\mathbf{y}$, the second term of RHS of (3.2.41). Operating $\mathbf{y}^T$ from the left on the both sides of (3.2.41) gives

$$\lambda'(\mathbf{y}^T\mathbf{y}) = \mathbf{y}^T(\mathbf{T} - \lambda\mathbf{I})\mathbf{y}' + \mathbf{y}^T\mathbf{T}'\mathbf{y} = \mathbf{y}^T\mathbf{T}'\mathbf{y} \ \ (\text{by } \mathbf{y}^T(\mathbf{T} - \lambda\mathbf{I})\mathbf{y}' = \{(\mathbf{T} - \lambda\mathbf{I})\mathbf{y}\}^T\mathbf{y}' = 0)$$

$$= y^T \cdot \left(-\frac{1}{q}\right) \begin{bmatrix} r_2 & & 0 \\ & r_4 & \\ 0 & & \ddots \end{bmatrix} y \ \ (\text{by } (3.2.42)) = -\frac{1}{q}(r_2 B_2^2 + r_4 B_4^2 + r_6 B_6^2 + \cdots)$$

$$= -\frac{2}{q}(r_4 B_4^2 + r_8 B_8^2 + r_{12} B_{12}^2 + \cdots) \ \ (\text{by } (3.2.30)) = -\frac{2}{q} \cdot (x^T x). \blacksquare$$

The fact is that Theorem 3.2.3 is applied to the general complex numbers q and λ as well. Furthermore, the next is stated when q and λ are real:

[**Lemma 3.2.2**] When $q \neq 0, \lambda$ are real, the necessary and sufficient condition for $d\lambda/dq = 0$ is $x^T x = 0$.

[**Proof**] By (3.2.40), it is sufficient to show $y^T y \neq 0$. When q and λ are real, y is a real vector. That means $y^T y = B_2^2 + B_4^2 + \cdots = 0$ leads $B_{2k} = 0$ $(k = 1, 2, \ldots)$, or $y = 0$. This contradicts the assumption that y is an eigenvector of $\mathbf{T}$ defined in (3.2.2). Then, $y^T y \neq 0.\blacksquare$

[**Lemma 3.2.3**] When $\lambda \leq 16$, no pairs of real points (q, λ) $(0 \neq q$: inverse eigenvalue) satisfying $d\lambda/dq = 0$ exist.

[**Proof**] It suffices to show $x^T x \neq 0$, from Lemma 3.2.2. When q and λ are real, so are B_{2n} $(n = 1, 2, \ldots)$. And when $\lambda \leq 16$, $r_{4n} \geq 0$ $(n = 1, 2, \ldots)$. Then,

$$x^T x = r_4 B_4^2 + r_8 B_8^2 + r_{12} B_{12}^2 + \cdots \geq 0.$$

Also, since $x \neq 0$, there are no cases where equality sign ("=") is valid in the above equation. Therefore, $x^T x > 0.\blacksquare$

A method for the computation of such double eigenvalues is then proposed based on the last theorem. Let's leave it, however, to Section 4.4.3 because Section 4 is where the computation of double eigenvalues is its main topic.

3.2.6 Summary of Section 3.2

In this section, we attempted an approach to the problem for the computation of inverse eigenvalues q of the Mathieu differential equation, from the standpoint that one applies the achievements for the eigenvalue problem for infinite matrices (in Theorem A) to some new problem. As a consequence, the author obtained a new theorem on the error estimate of q_n to q, for a given complex λ. In addition, the analysis on double eigenvalues (q, λ) was treated. To show its validity, enough numerical experiments were tested, and q–λ graph was created.

The method for the computation of the inverse eigenvalues of the Mathieu differential equation is seldom researched as of today. Then, it is thought to be a great impact that we newly gave a solution for the inverse problem, which should be treated equally as the ordinary eigenvalue problem.

What is expected to progress in the future for this research is to make a contribution to the solution for se_{2m+1} and ce_{2m+1} types with small matrix size n and with enough precision, together with the ordinary eigenvalue problem such as [13, Chapter 4], which is complementary to the research in this paper.

3.3 The Eigenvalue Problem of Spheroidal Differential Equation

3.3.1 Description of the Problem

When the Helmholtz equation $\Delta\psi(x,y,z) + K^2\psi(x,y,z) = 0$ is expressed by the prolate spheroidal coordinates

$$\begin{cases} x &= \tilde{c}\sinh u \sin v \cos\varphi, \\ y &= \tilde{c}\sinh u \sin v \sin\varphi, \\ z &= \tilde{c}\cosh u \cos v \qquad (0 < \tilde{c},\ 0 \le v, \varphi < 2\pi,\ 0 < u < \infty), \end{cases}$$

followed by the separation of variables, one of the given equations, which is the second-order linear ordinary differential equation, has the form

$$\frac{d}{dz}\{(1-z^2)\frac{dw}{dz}\} + (\lambda_{mn} - c^2 z^2 - \frac{m^2}{1-z^2})w = 0 \tag{3.3.1}$$

(there is another type "oblate", in which case c^2 is replaced by $-c^2$ in (3.3.1)),

where m is used as $\psi = X(u)Y(v)\cos m\varphi$, and $c = \tilde{c}K$. From the physical consideration, m has to be an integer parameter and c^2 is a real parameter, respectively. In this paper, we only deal with $m \ge 0$ case (even when m is a negative integer, one finds, easily from (3.3.1), that the case may turn out to be the same case as m is positive). We define the "(ordinary) eigenvalue problem" of spheroidal wave equation by finding λ_{mn} such that w, the solution of (3.3.1), is regular in $(-1,1)$, given m and c, and we let such λ_{mn} be called an "eigenvalue". As will be stated later, λ_{mn} is real when $m \ge 0$. And sorted eigenvalues in an increasing order correspond to have $n = m$, $n = m+1$, ..., each. On the other hand, we also think of the inverse problem to the ordinary eigenvalue problem. We let the problem of finding c^2 (or c) for given m and λ such that again, w, the solution of (3.3.1), is regular in (-1,1), be called the "inverse eigenvalue problem". Let's call such c^2 an "inverse eigenvalue".

The equation (3.3.1) has two independent solutions. They are denoted as $pe_n^m(c,z)$ and $qe_n^m(c,z)$. We only deal with $pe_n^m(c,z)$, or the spheroidal wave function of the first kind.

The expansion of $pe_n^m(c,z)$ by the associate Legendre function $P_{m+k}^m(z)$ gives

$$pe_n^m(c,z) = \sum_{k=0,1}^{\infty}{}' A_{n,k}^m \cdot P_{m+k}^m(z), \tag{3.3.2}$$

where $A_{n,k}^m$ represents expansion coefficients, and $\Sigma_{k=0,1}^{\infty}{}'$ is defined as

$$\begin{cases} \text{the sum of even} - \text{numbered terms, or } k = 0,2,4,\cdots \text{ (if } n-m \text{ is even)}, \\ \text{the sum of odd} - \text{numbered terms, or } k = 1,3,5,\cdots \text{ (if } n-m \text{ is odd)}. \end{cases}$$

$P_{m+k}^m(z)$ is the function of the form

$$P_n^m(z) = (1-z^2)^{m/2}\frac{\mathrm{d}^m P_n(z)}{\mathrm{d}z^m}\ (m \geq 0, -1 \leq z \leq 1),$$

where $P_n(z)$ represents Legendre polynomial

$$P_n(z) = \frac{1}{2^n n!}\cdot\frac{\mathrm{d}^n}{\mathrm{d}z^n}(z^2-1)^n\ (n = 0,1,2,\ldots),$$

with orthogonality over [-1,1]:

$$\begin{aligned} \int_{-1}^{1} P_l^m(z)P_k^m(z)dz &= 0\ (l \neq k), \\ \int_{-1}^{1}\{P_n^m(z)\}^2 dz &= \frac{2}{2n+1}\cdot\frac{(n+m)!}{(n-m)!}. \end{aligned} \tag{3.3.3}$$

3.3.2 Three-Term Recurrence Relations Regarding Expansion Coefficients

It is known that the substitution of (3.3.2) into (3.3.1) gives the following three-term recurrence relations[2, Chap 21, formula 21.7.3]. For convenience, let $A_{n,k}^m$ be simply rewritten A_k and λ_{mn} be λ in the sequel.

When $n-m$ is even:

$$\begin{aligned} \beta_0 A_0 + \alpha_0 A_2 &= \lambda A_0, \\ \gamma_{2k}A_{2k-2} + \beta_{2k}A_{2k} + \alpha_{2k}A_{2k+2} &= \lambda A_{2k}\ (k = 1,2,3,\ldots). \end{aligned} \tag{3.3.4}$$

When $n-m$ is odd:

$$\begin{aligned} \beta_1 A_1 + \alpha_1 A_3 &= \lambda A_1, \\ \gamma_{2k+1}A_{2k-1} + \beta_{2k+1}A_{2k+1} + \alpha_{2k+1}A_{2k+3} &= \lambda A_{2k+1}\ (k = 1,2,3,\ldots). \end{aligned} \tag{3.3.5}$$

In (3.3.4) and (3.3.5), each of α_k, β_k, and γ_k represents

$$\alpha_k = \frac{(2m+k+2)(2m+k+1)}{(2m+2k+3)(2m+2k+5)}(\pm c^2) \equiv a_k\cdot(\pm c^2) \sim \pm\frac{c^2}{4}\ (k \to \infty), \tag{3.3.6}$$

$$\begin{aligned} \beta_k &= (m+k)(m+k+1) + \frac{2(m+k)(m+k+1)-2m^2-1}{(2m+2k-1)(2m+2k+3)}(\pm c^2) \\ &\equiv (m+k)(m+k+1) + b_k\cdot(\pm c^2) \sim k^2\ (k \to \infty), \end{aligned} \tag{3.3.7}$$

$$\gamma_k = \frac{k(k-1)}{(2m+2k-3)(2m+2k-1)}(\pm c^2) \equiv r_k\cdot(\pm c^2) \sim \pm\frac{c^2}{4}\ (k \to \infty), \tag{3.3.8}$$

for $k = 0, 1, 2, \cdots$, Note that $\pm$ signifies '+' in the prolate case, while '−' in the oblate case.

The three-term relations of (3.3.4) and (3.3.5) type are generalized as one single type of three-term relations as follows:

$$\begin{aligned} \beta_s A_s + \alpha_s A_{s+2} &= \lambda A_s, \\ \gamma_{2k+s} A_{2(k-1)+s} + \beta_{2k+s} A_{2k+s} + \alpha_{2k+s} A_{2(k+1)+s} &= \lambda A_{2k+s} \ (k = 1, 2, 3, \ldots), \end{aligned} \tag{3.3.9}$$

where

$$s \equiv \mathrm{mod}(n-m, 2) = \begin{cases} 0 & (\text{if } n-m \text{ is even}) \\ 1 & (\text{if } n-m \text{ is odd}) \end{cases} \tag{3.3.10}$$

(Let $\mathrm{mod}(i, j)$ be the remainder when i is divided by j).

Since (3.3.9) covers both of the cases where $n-m$ is even and odd (or (3.3.4) and (3.3.5)), one can handle both by dealing with (3.3.9) only. Let the behavior of the coefficients $\{A_{2k+s}\}$ be examined then.

[**Lemma 3.3.1**] The next inequality holds regarding $\{A_{2k+s}\}$ $(k = 0, 1, 2, \ldots)$:

$$\sum_{k=0,1,2,\ldots}^{\infty} A_{2k+s}^2 \cdot K(m, 2k+s) < \infty, \text{ where } K(m, a) = \frac{2}{2m+2a+1} \cdot \frac{(2m+a)!}{a!}. \tag{3.3.11}$$

[**Proof**] Considering the integrability of the spheroidal wave function of the first kind $pe_n^m(c, z)$, one is given

$$\int_{-1}^{1} \{pe_n^m(c, z)\}^2 dz = \int_{-1}^{1} \left(\sum_{k=0,1}^{\infty} {}'A_k \cdot P_{m+k}^m(z) \right)^2 dz \text{ (by (3.3.2))} < \infty. \tag{3.3.12}$$

Now, by the orthogonality (3.3.3) of $P_{m+k}^m(z)$ over $[-1, 1]$,

$$\begin{aligned} \text{the LHS of (3.3.12)} &= \int_{-1}^{1} \sum_{k=s,2+s,\ldots}^{\infty} A_k^2 P_{m+k}^m(z)^2 dz = \sum_{k=s,2+s,\ldots}^{\infty} A_k^2 \int_{-1}^{1} P_{m+k}^m(z)^2 dz \\ &= \sum_{k=s,2+s,\ldots}^{\infty} A_k^2 \cdot \frac{2}{2m+2k+1} \cdot \frac{(2m+k)!}{k!} = \sum_{k=s,2+s,\ldots}^{\infty} A_k^2 \cdot K(m, k) < \infty. \quad \blacksquare \end{aligned}$$

Now, with (3.3.9) and (3.3.11), let the behavior of $\{A_{2k+s}\}$ be analyzed, by the application of Theorem C:

First, consider the recurrence relations with the same coefficients as (3.3.9), or

$$\begin{aligned} \beta_s h_1 + \alpha_s h_2 &= \lambda h_1, \\ \gamma_{2k+s} h_{k-1} + \beta_{2k+s} h_k + \alpha_{2k+s} h_{k+1} &= \lambda h_k \quad (k = 2, 3, 4, \ldots). \end{aligned} \tag{3.3.13}$$

Theorem C may apply since (3.3.13) satisfies the conditions imposed on the theorem. As a result of application, the existence of two independent solutions (say, $\{h_{n,1}\}, \{h_{n,2}\}$) is guaranteed with the behaviors

$$(3.3.14)\qquad \frac{h_{n+1,1}}{h_{n,1}} = -\frac{16n^2}{c^2}[1+o(1)],\ \frac{h_{n+1,2}}{h_{n,2}} = -\frac{c^2}{16n^2}[1+o(1)] \to 0\ (n\to\infty).$$

With the consideration of (3.3.11), it is obvious that $\{A_{2n+s}\}$ $(n = 0, 1, 2, \ldots)$ can only show the same behaviors as $\{h_{n,2}\}$, the minimal solution of (3.3.9). Thus,

$$(3.3.15)\qquad \frac{A_{2n+s+2}}{A_{2n+s}} = -\frac{c^2}{16n^2}[1+o(1)]\ (n\to\infty).$$

3.3.3 The Solution of (Ordinary) Eigenvalue Problem

[**Theorem 3.3.1**] Given integer $m \geq 0$ and real c, $\lambda \neq 0$ is an eigenvalue of (3.3.1) if and only if λ is an eigenvalue of an infinite real symmetric tridiagonal matrix $\mathbf{T}$ acting as a linear transformation from X into ℓ^2 defined below:

$$(3.3.16)\qquad \mathbf{T} = \begin{bmatrix} \beta_s & \sqrt{\alpha_s}\sqrt{\gamma_{2+s}} & & 0 \\ \sqrt{\alpha_s}\sqrt{\gamma_{2+s}} & \beta_{2+s} & \sqrt{\alpha_{2+s}}\sqrt{\gamma_{4+s}} & \\ & \sqrt{\alpha_{2+s}}\sqrt{\gamma_{4+s}} & \beta_{4+s} & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix}.$$

The definitions of α_k, β_k and γ_k are retained in (3.3.6), (3.3.7) and (3.3.8), respectively. X is a subspace of ℓ^2 defined as follows:

$$X = \{\mathrm{y} \in \ell^2 : \begin{bmatrix} \beta_s & & 0 \\ & \beta_{2+s} & \\ 0 & & \ddots \end{bmatrix} \mathrm{y} \in \ell^2\} \subset \ell^2.$$

Moreover, if one lets an eigenvector of $\mathbf{T}$ corresponding to λ be $\mathbf{0} \neq \mathrm{x} \equiv [x^{(1)}, x^{(2)}, \ldots]^T \in \ell^2$, or

$$(3.3.17)\qquad \mathbf{T}\mathrm{x} = \lambda \mathrm{x}$$

holds, $x^{(i)}$ $(i = 1, 2, \ldots)$ are expressed with a scalar t $(\neq 0)$:

$$(3.3.18)\qquad x^{(i)} = t \cdot \Pi_{j=1}^{i-1} \frac{\sqrt{\alpha_{2(j-1)+s}}}{\sqrt{\gamma_{2j+s}}} A_{2(i-1)+s}.$$

[**Proof**] If (3.3.9) is expressed in matrix form, that directly becomes an eigenvalue problem of the infinite tridiagonal matrix. Also, since $c \neq 0$ and $m \geq 0$, $\alpha_k \cdot \gamma_{k+2} \neq 0$ $(k = s, 2+s, \cdots)$ hold which means that the coefficient matrix can be symmetrized. Since $m \geq 0$, each component of the symmetrized matrix $\mathbf{T}$ becomes real. From the behavior of A_{2k} in (3.3.15),

$0 \neq \mathrm{x} \in \ell^2$ is also proved easily. Conversely, we shall show that letting the eigenvalues of T be λ, one obtains an eigenvector corresponding to λ being a scalar multiple of x whose components $x^{(i)}$ $(i = 1, 2, \ldots)$ are expressed as (3.3.18). However, this is likewise shown as the last part of Theorem 3.2.1 (Mathieu's case). Then, let us omit this. ∎

[Lemma 3.3.2] λ is real and simple.

[Proof] Let λ's being real be proved first, using (3.3.16). In the prolate case, one finds that $\alpha_k > 0$, $\gamma_{k+2} > 0$ $(k = s, 2+s, \ldots)$, while in oblate case, $\alpha_k < 0$, $\gamma_{k+2} < 0$ $(k = s, 2+s, \ldots)$. This means $\alpha_k \gamma_{k+2} > 0$ $(k = s, 2+s, \ldots)$, in either case. Then $\mathbf{T}$ in (3.3.16) is real symmetric tridiagonal. Since the eigenvalues of real symmetric matrices are always real, λ is guaranteed to be real.

Second, the proof that λ is simple follows. As was proved, $\mathbf{T}$ is an infinite real symmetric matrix in Hilbert space. In this case, there are no general eigenvectors of rank two or more corresponding to the eigenvalue of $\mathbf{T}$ (See standard books on functional analysis, e.g. [26]). Also, once the first component of the three-term recurrence relations is given, all the other components are uniquely determined (This is because $\alpha_k \cdot \gamma_{k+2} \neq 0$ $(k = s, 2+s, \cdots)$). This indicates that there is only one independent eigenvector. The two facts show that λ is simple. ∎

[Lemma 3.3.3] Let us first define the existence of T^{-1} when the solution of $\mathrm{Tx} = \mathbf{0}$ turns out to $\mathrm{x} = 0$. Then,

(i) In the prolate case, $\mathbf{T}^{-1}$ always exists.

(ii) In the oblate case, with the appropriate scalar δ taken, $(\mathbf{T} + \delta\mathbf{I})^{-1}$ exists.

[Proof]

(i) [In the prolate case] One only has to show that 0 is not an eigenvalue of $\mathbf{T}$.

By Lemma 3.3.4, as will be shown later in this section, when $m \geq 1$ or when $m = 0$ and $s = 1$, $\lambda \geq 2$ holds. Therefore, $\lambda \neq 0$.

When $m = 0$ and $s = 0$, $\lambda > 0$ in case of $c^2 \neq 0$. Then, there exists $\mathbf{T}^{-1}$.

(ii) [In the oblate case] Let $\hat{\mathbf{T}}$ be defined to be a similar matrix to $\mathbf{T}$,

$$\hat{\mathbf{T}} = \begin{bmatrix} \beta_0 & \alpha_0 & & 0 \\ \gamma_2 & \beta_2 & \alpha_2 & \\ & \gamma_4 & \beta_4 & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix}.$$

It is sufficient to show 0 is not an eigenvalue of $(\hat{\mathbf{T}} + \delta\mathbf{I})$, with δ taken appropriately. First, let a_k, b_k and r_k be defined by

$$\begin{aligned} a_k &= \frac{(2m+k+2)(2m+k+1)}{(2m+2k+3)(2m+2k+5)}, \\ b_k &= \frac{2(m+k)(m+k+1) - 2m^2 - 1}{(2m+2k-1)(2m+2k+3)}, \\ r_k &= \frac{k(k-1)}{(2m+2k-3)(2m+2k-1)}, \end{aligned}$$

which give the behaviors $a_k \sim 1/4, b_k \sim 1/2$ and $r_k \sim 1/4$. The computation of $\Delta_k \equiv |\beta_k| - |\alpha_k| - |\gamma_k|$ derives

$$|\beta_k| - |\alpha_k| - |\gamma_k| \;=\; |\beta_k| - a_k c^2 - r_k c^2.$$

Also, since

$$|\beta_k| = \left|(m+k)(m+k+1) - b_k c^2\right| \geq (m+k)(m+k+1) - b_k c^2,$$

then,

$$\Delta_k \geq (m+k)(m+k+1) - (a_k + b_k + r_k)c^2$$

holds. Since $\{a_k\}, \{b_k\}$ and $\{r_k\}$ are all confirmed to be convergent, they are bounded. Supposing $a_k < A, b_k < B, r_k < R$ gives, for all k,

$$0 < a_k + b_k + r_k < A + B + R \text{ (constant)} \equiv \mu.$$

Thus,

$$\begin{aligned} \tilde{\Delta}_k &\equiv \left|c^2\mu + \beta_k\right| - |\alpha_k| - |\gamma_k| \geq c^2\mu + (m+k)(m+k+1) - (a_k + b_k + r_k)c^2 \\ &\geq \{\mu - (a_k + b_k + r_k)\}c^2 = \{(A - a_k) + (B - b_k) + (R - r_k)\}c^2 > 0. \end{aligned}$$

This shows that each Gershgorin Disc of the matrix defined by $(\hat{\mathbf{T}} + \delta\mathbf{I})$ $(\delta \equiv c^2\mu)$ never includes the origin. Namely, 0 is not an eigenvalue of $(\hat{\mathbf{T}} + \delta\mathbf{I})$, or $(\mathbf{T} + \delta\mathbf{I})$. In other words, $(\mathbf{T} + \delta\mathbf{I})$ always has an inverse. ∎

[**Theorem 3.3.2**] Let $\beta_k \neq 0$ $(k = s, 2+s, \cdots)$, $c \neq 0$, and $\mathbf{T}^{-1}$ exists. Also let λ be one of the eigenvalues of $\mathbf{T}$. Now, for each n, one can take λ_n, one of the eigenvalues of $\mathbf{T}_n$, the nth principal submatrix of $\mathbf{T}$, such that $\lambda_n \to \lambda$.

Further, let $\{\lambda_n\}$ converge to λ, as stated above. Then, the following error estimate is valid:

$$(3.3.19) \qquad \lambda - \lambda_n = \left(\frac{\alpha_0\alpha_2\cdots\alpha_{2n-2}}{\gamma_2\gamma_4\cdots\gamma_{2n-2}}\right)\cdot\frac{A_{2n-2}A_{2n}}{x^Tx}[1+o(1)],$$

moreover,

$$(3.3.20) \frac{\lambda-\lambda_{n+1}}{\lambda-\lambda_n} = \frac{\alpha_{2n}}{\gamma_{2n}}\cdot\frac{A_{2n+2}}{A_{2n-2}}[1+o(1)] = \frac{\alpha_{2n}\cdot\gamma_{2n+2}}{(\lambda-\beta_{2n})^2}[1+o(1)] = \left(\frac{c^2}{16}\right)^2\cdot\frac{1}{n^4}[1+o(1)].$$

[Proof] By Lemma 3.3.2, λ is a simple eigenvalue of $\mathbf{T}$. Besides, it is easily proved that x is real (easily leading $\mathbf{x}^T\mathbf{x} \neq 0$ since $\mathbf{x} \neq \mathbf{0}$) since all the eigenvectors of a real symmetric matrix are real. By the behavior of $\{x^{(n)}\}$, $\frac{f_{n+1}x^{(n+1)}}{x^{(n)}} \to 0$ $(n \to \infty)$ is also derived, for, substituting $f_{n+1}, x^{(n)}$ and $x^{(n+1)}$ into the LHS gives

$$\begin{aligned}\left|\frac{f_{n+1}x^{(n+1)}}{x^{(n)}}\right| &= \left|\frac{\sqrt{\alpha_{2n-2}}\cdot\sqrt{\gamma_{2n}}\frac{\sqrt{\alpha_{2n-2}}}{\sqrt{\gamma_{2n}}}\frac{\sqrt{\alpha_{2n-4}}}{\sqrt{\gamma_{2n-2}}}\cdots\frac{\sqrt{\alpha_0}}{\sqrt{\gamma_2}}A_{2n}}{\frac{\sqrt{\alpha_{2n-4}}}{\sqrt{\gamma_{2n-2}}}\frac{\sqrt{\alpha_{2n-6}}}{\sqrt{\gamma_{2n-4}}}\cdots\frac{\sqrt{\alpha_0}}{\sqrt{\gamma_2}}A_{2n-2}}\right| \\ &= |\alpha_{2n-2}|\cdot\left|\frac{A_{2n}}{A_{2n-2}}\right| \to 0 \ (n\to\infty) \text{ (by the boundedness of } \{\alpha_{2n-2}\} \text{ and (3.3.15))}.\end{aligned}$$

Hence, Theorem B may apply (3.3.17) and rate of convergence (3.3.20) as well as the error estimate (3.3.19) are obtained by the direct computation. ∎

3.3.4 The Solution of Inverse Eigenvalue Problem

Letting a_k, b_k and r_k be defined by $\alpha_k = a_kc^2, \beta_k = (m+k)(m+k+1)+b_kc^2$ and $\gamma_k = r_kc^2$, as (3.3.6), (3.3.7), and (3.3.8), we have

$$\begin{aligned}(3.3.21) \qquad a_k &= \frac{(2m+k+2)(2m+k+1)}{(2m+2k+3)(2m+2k+5)} \sim \frac{1}{4}\ (k\to\infty),\\ (3.3.22) \qquad b_k &= \frac{2(m+k)(m+k+1)-2m^2-1}{(2m+2k-1)(2m+2k+3)} \sim \frac{1}{2}\ (k\to\infty),\\ (3.3.23) \qquad r_k &= \frac{k(k-1)}{(2m+2k-3)(2m+2k-1)} \sim \frac{1}{4}\ (k\to\infty).\end{aligned}$$

Also define d_k by

$$(3.3.24) \qquad d_k = \lambda - (m+k)(m+k+1).$$

(3.3.9) is rewritten as follows, using (3.3.21), (3.3.22), (3.3.23), and (3.3.24):

$$\begin{aligned} b_sA_s + a_sA_{s+2} &= \frac{d_s}{c^2}A_s,\\ (3.3.25)\ r_{2k+s}A_{2(k-1)+s} + b_{2k+s}A_{2k+s} + a_{2k+s}A_{2(k+1)+s} &= \frac{d_{2k+s}}{c^2}A_{2k+s}\ (k=1,2,3,\ldots).\end{aligned}$$

The following discussion is divided into two cases, or (a) $d_{2k+s} \neq 0$ ($k = 0, 1, 2, \ldots$), and (b) $d_{2k+s} = 0$ (for some non-negative integer k).

(a) $d_{2k+s} \neq 0$ ($k = 0, 1, 2, \ldots$) case:

[**Theorem 3.3.3**] Let m and λ be given. Then, $c^2 \neq 0$ is an inverse eigenvalue of (3.3.1) if and only if $1/c^2$ is an eigenvalue of the compact transformation $\mathbf{T}$ in ℓ^2, where

$$(3.3.26)\mathbf{T} = \begin{bmatrix} b_s/d_s & \sqrt{a_s r_{s+2}}/\sqrt{d_s d_{s+2}} & & 0 \\ \sqrt{a_s r_{s+2}}/\sqrt{d_s d_{s+2}} & b_{s+2}/d_{s+2} & \sqrt{a_{s+2} r_{s+4}}/\sqrt{d_{s+2} d_{s+4}} & \\ & \sqrt{a_{s+2} r_{s+4}}/\sqrt{d_{s+2} d_{s+4}} & b_{s+4}/d_{s+4} & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix}.$$

Note that a_k, b_k, r_k and d_k are the ones defined in (3.3.21),(3.3.22),(3.3.23), and (3.3.24). Furthermore, if we let an eigenvector of $\mathbf{T}$ corresponding to $1/c^2$ be $\mathbf{0} \neq \mathrm{x} \equiv [x^{(1)}, x^{(2)}, \ldots]^T \in \ell^2$, or

$$(3.3.27) \qquad \mathbf{T}\mathrm{x} = \frac{1}{c^2}\mathrm{x},$$

then, $x^{(i)}$ ($i = 1, 2, \ldots$) are expressed as, using a scalar $\tilde{t}$ ($\neq 0$):

$$(3.3.28) \qquad x^{(i)} = \tilde{t} \cdot \sqrt{d_{2(i-1)+s}} \Pi_{j=1}^{i-1} \frac{\sqrt{a_{2(j-1)+s}}}{\sqrt{r_{2j+s}}} A_{2(i-1)+s}.$$

[**Proof**] Suppose that $c^2 \neq 0$ is an inverse eigenvalue of (3.3.1). Then reformulating the three-term relations (3.3.9) into matrix form gives

$$\mathbf{T}\mathrm{x} = \frac{1}{c^2}\mathrm{x}.$$

What remains to show that $1/c^2$ is an eigenvalue of $\mathbf{T}$ is $\mathrm{x} \neq \mathbf{0}$ and $\mathrm{x} \in \ell^2$.

$\mathrm{x} = \mathbf{0}$ directly means $A_{2k+s} = 0$ ($k = 0, 1, 2, \ldots$). Clearly, the solution of (3.3.1) is trivial in this case, then we may assume $\mathrm{x} \neq \mathbf{0}$. In order to show $\mathrm{x} \in \ell^2$, one only has to prove $\|\mathrm{x}\|^2 < \infty$ from its definition. Since $\|\mathrm{x}\|^2 = |x^{(1)}|^2 + |x^{(2)}|^2 + \cdots$, the convergence of $x^{(n)}$ ($n = 1, 2, \ldots$) is guaranteed if

$$\bar{R} \equiv \lim_{n\to\infty} \sup \left| \frac{x^{(n+1)}}{x^{(n)}} \right| < 1$$

is satisfied, according to [6, Theorem 8.25]. By (3.3.15),

$$(3.3.29) \left| \frac{x^{(n+1)}}{x^{(n)}} \right| = \left| \frac{\sqrt{d_{2n}}}{\sqrt{d_{2n-2}}} \cdot \frac{\sqrt{a_{2n-2}}}{\sqrt{r_{2n}}} \right| \cdot \left| \frac{A_{2n}}{A_{2n-2}} \right| = \frac{c^2}{16n^2}[1 + o(1)] \ \to 0 \ (n \to \infty)$$

is easily derived. Then, $\bar{R} < 1$, meaning $\|\mathrm{x}\|^2 < \infty$. Thus, $1/c^2$ is an eigenvalue of $\mathbf{T}$.

On the contrary, the proof follows that c^2 is the inverse eigenvalue of (3.3.1) when $1/c^2$ is assumed to be the eigenvalue of $\mathbf{T}$. First, let us define $\tilde{\mathbf{T}}$ which is similar to $\mathbf{T}$:

$$\tilde{\mathbf{T}} = \begin{bmatrix} b_s/d_s & a_s/d_s & & 0 \\ r_{s+2}/d_{s+2} & b_{s+2}/d_{s+2} & a_{s+2}/d_{s+2} & \\ & r_{s+4}/d_{s+4} & b_{s+4}/d_{s+4} & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix}.$$

Since $\mathbf{T}$ and $\tilde{\mathbf{T}}$ are similar, the set of eigenvalues of $\tilde{\mathbf{T}}$ are identical to the one of $\mathbf{T}$. From now on, then, let us think of solving the eigenvalue problem for $\tilde{\mathbf{T}}$.

Before one may assume the eigenvalue of $\tilde{\mathbf{T}}$ to have the form of $1/c^2$, one needs to prove that 0 can never be an eigenvalue of $\tilde{\mathbf{T}}$. Let an eigenvector of $\tilde{\mathbf{T}}$ corresponding to 0 be $\mathbf{w} = [w_1, w_2, \ldots]^T \in \ell^2$. Then,

$$\tilde{\mathbf{T}}\mathbf{w} = 0 \cdot \mathbf{w} = \mathbf{0}.$$

The expansion of this equation gives the next three-term recurrence relations:

$$\begin{aligned} b_s w_1 + a_s w_2 &= 0, \\ r_{2(k-1)+s} w_{k-1} + b_{2(k-1)+s} w_k + a_{2(k-1)+s} w_{k+1} &= 0 \ (k = 1, 2, 3, \ldots). \end{aligned}$$

If one applies [11, Theorem 2.3] to the relations, one finds, from the behaviors of the coefficients of the recurrence relations $a_{2(k-1)+s} \sim \frac{1}{4}, b_{2(k-1)+s} \sim \frac{1}{2}, r_{2(k-1)+s} \sim \frac{1}{4}$ $(k \longrightarrow \infty)$, that this type of the relations is categorized into [11, Theorem 2.3, case (b)] case, and that the behavior of the solution $\{w_k\}$ is given by

$$\limsup_{k \longrightarrow \infty} |w_k| = 1.$$

This, however, contradicts the fact $\mathbf{w} \in \ell^2$. Then, one can say that 0 can never be an eigenvalue of $\tilde{\mathbf{T}}$.

In the above, it was guaranteed that one may let the eigenvalue of $\tilde{\mathbf{T}}$ be $1/c^2$ and further let us define the corresponding eigenvector to $1/c^2$ by $\mathbf{h} = [h_2, h_4, h_6, \ldots]^T$. Expanding $\tilde{\mathbf{T}}\mathbf{h} = \frac{1}{c^2}\mathbf{h}$ yields the same recurrence relations as (3.3.9). However, the solution $\{h_{2k}\}$ $(k = 1, 2, \ldots)$ of (3.3.9) must be a minimal one $\{h_{2n,2}\}$ since it is to converge to 0. Namely, $h_{2n} = h_{2n,2} = A_{2n}$ $(n = 1, 2, \ldots)$. Thus, (3.3.27) holds. ∎

Furthermore, the next Theorem 3.3.4 is derived by applying Theorem A to Theorem 3.3.3.

[**Theorem 3.3.4**] Given m and λ, consider the eigenvalue problem (3.3.27), where $c^2 \neq 0$ is assumed. The definitions of $\mathbf{T}, \mathbf{x}$ in (3.3.27) are retained in (3.3.26),(3.3.28), respectively.

For each natural number n, one can have $\{c_n^2\}$ converge to c^2, if each $c_n^2 = 1/\xi_n$ is taken appropriately, where ξ_n represents one of the eigenvalues of $\mathbf{T}_n$, and $\mathbf{T}_n$ the nth principal submatrix of $\mathbf{T}$. Moreover, assuming $\mathrm{x}^T\mathrm{x} \neq 0$, one finds that the following error estimate is valid:

$$c^2 - c_n^2 = \frac{-c^4 l_{m,n,s} A_{2n-2+s} A_{2n+s}}{\mathrm{x}^T\mathrm{x}}[1 + o(1)](n \to \infty), \tag{3.3.30}$$

where $l_{m,n,s} = \dfrac{(2m+1)(2m+2)(2m+2s+1)}{(2m+1)^s(2m+4n+2s-3)(2m+4n+2s-1)(2m+4n+2s+1)} \cdot {}_{2m+2n+s}C_2$

By (3.3.30), the next rate of convergence is easily derived:

$$\frac{c^2 - c_{n+1}^2}{c^2 - c_n^2} = \left(\frac{c}{4}\right)^4 \cdot \frac{1}{n^4}\,[1 + o(1)]\,(n \to \infty). \tag{3.3.31}$$

[**Proof**] Let us show that Theorem A can apply the eigenvalue problem of (3.3.27). The components of $\mathbf{T}$ obviously satisfy the conditions of Theorem A, and $\mathrm{x} \in \ell^2$ is already proven in Theorem 3.3.1. This signifies that one may only have to prove the next two:

(i) Letting $\mathrm{x} = [x^{(1)}, x^{(2)}, \dots]^T$, $x^{(n+1)}/x^{(n)}$ is bounded for all sufficiently large n.

(ii) $\mathrm{x}^T\mathrm{x} \neq 0$ leads that the eigenvalue $1/c^2$ of matrix $\mathbf{T}$ is simple.

(i) It suffices to show $|x^{(n+1)}/x^{(n)}| \to 0$ $(n \to \infty)$. Actually, this is already proved by (3.3.29).

(ii) Take the contraposition. Namely, think of proving 'the eigenvalue $1/c^2$ of $\mathbf{T}$ has an eigenvector of rank two or more, or, there exist more than one independent eigenvectors $\Rightarrow$ $\mathrm{x}^T\mathrm{x} = 0$'.

First, the fact is that there is no case where 'there exist more than one independent eigenvectors'. This is because it is shown in Lemma 3.3.2 that x is uniquely determined up to scalar multiplication. Therefore, take only the case where 'the eigenvalue $1/c^2$ of $\mathbf{T}$ has an eigenvector of rank two or more'. When this holds, there exists a vector $v \in \ell^2$, given x, such that

$$\begin{aligned} \mathbf{0} \neq \mathrm{x} &= (\mathbf{T} - \frac{1}{q^2}\mathbf{I})v, \\ (\mathbf{T} - \frac{1}{q^2}\mathbf{I})\mathrm{x} &= (\mathbf{T} - \frac{1}{q^2}\mathbf{I})^2 v = \mathbf{0}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{x}^T\mathrm{x} &= \{(\mathbf{T} - \frac{1}{q^2}\mathbf{I})v\}^T(\mathbf{T} - \frac{1}{q^2}\mathbf{I})v \\ &= v^T(\mathbf{T} - \frac{1}{q^2}\mathbf{I})^2 v \text{ (by the symmetry of } \mathbf{T}) = 0. \end{aligned}$$

Then, Theorem A may apply. The error estimate (3.3.30) is computed as follows:

$$\begin{aligned}
\text{LHS} &= \frac{1}{c^2} - \frac{1}{c_n^2} = -\frac{c^2 - c_n^2}{c^4}[1 + o(1)], \\
\text{RHS} &= \left(\tilde{t} \cdot \sqrt{d_{2n-2+s}} \Pi_{j=1}^{n-1} \frac{\sqrt{a_{2(j-1)+s}}}{\sqrt{r_{2j+s}}} A_{2n-2+s}\right) \cdot \left(\tilde{t} \cdot \sqrt{d_{2n+s}} \Pi_{j=1}^{n} \frac{\sqrt{a_{2j+s}}}{\sqrt{r_{2(j+1)+s}}} A_{2n+s}\right) \\
&\cdot \frac{\sqrt{a_{2n-2+s}}\sqrt{r_{2n+s}}}{\sqrt{d_{2n-2+s}}\sqrt{d_{2n+s}}} \cdot \frac{[1 + o(1)]}{\mathrm{x}^T \mathrm{x}} = \frac{l_{m,n,s} A_{2n-2+s} A_{2n+s}}{(\mathrm{x}^T \mathrm{x})/\tilde{t}^2}[1 + o(1)](n \to \infty).
\end{aligned}$$

This yields

$$c^2 - c_n^2 = \frac{-c^4 l_{m,n,s} A_{2n-2+s} A_{2n+s}}{\mathrm{x}^T \mathrm{x}}[1 + o(1)](n \to \infty).$$

Also, the rate of convergence $(c^2 - c_{n+1}^2)/(c^2 - c_n^2)$ is given as

$$\begin{aligned}
\frac{c^2 - c_{n+1}^2}{c^2 - c_n^2} &= \left(\frac{-c^4 l_{m,n+1,s} A_{2n+s} A_{2n+2+s}}{\mathrm{x}^T \mathrm{x}}[1 + o(1)]\right) / \left(\frac{-c^4 l_{m,n,s} A_{2n-2+s} A_{2n+s}}{\mathrm{x}^T \mathrm{x}}[1 + o(1)]\right) \\
&= \frac{l_{m,n+1,s}}{l_{m,n,s}} \cdot \frac{A_{2n+s}}{A_{2n-2+s}} \cdot \frac{A_{2n+2+s}}{A_{2n+s}}[1 + o(1)] = \left(\frac{c}{4}\right)^4 \frac{1}{n^4}\ [1 + o(1)]\ (n \to \infty)\ \text{(by (3.3.1}
\end{aligned}$$

(b) $d_{2k+s} = 0$ (for some non-negative integer k) case :

For starters, take the case of $k = 0$, or $d_s = \lambda - (m + s)(m + s + 1) = 0$. The first line of (3.3.25) thus yields

$$b_s A_s + a_s A_{s+2} = \frac{d_s}{c^2} A_s = 0, \text{ or } A_s = -\frac{a_s}{b_s} A_{2+s}.$$

Substituting this into the second line of (3.3.25) turns

$$\left(b_{2+s} - \frac{a_s r_{2+s}}{b_s}\right) A_{2+s} + a_{2+s} A_{4+s} = \frac{d_{2+s}}{c^2} A_{2+s} = 0. \tag{3.3.32}$$

Considering (3.3.32) and the equations in the sequel of (3.3.25), one can reformulate them as the matrix eigenvalue problem as was derived in the (a) case.

Next, take the case where k is a natural number. Picking up the $k, k + 1$, and $k + 2$ th lines of (3.3.25) each gives

$$r_{2(k-1)+s} A_{2(k-2)+s} + b_{2(k-1)+s} A_{2(k-1)+s} + a_{2(k-1)+s} A_{2k+s} = \frac{d_{2(k-1)+s}}{c^2} A_{2(k-1)+s}, \tag{3.3.33}$$

$$r_{2k+s} A_{2(k-1)+s} + b_{2k+s} A_{2k+s} + a_{2k+s} A_{2(k+1)+s} = \frac{d_{2k+s}}{c^2} A_{2k+s} = 0, \tag{3.3.34}$$

$$r_{2(k+1)+s} A_{2k+s} + b_{2(k+1)+s} A_{2(k+1)+s} + a_{2(k+1)+s} A_{2(k+2)+s} = \frac{d_{2(k+1)+s}}{c^2} A_{2(k+1)+s}. \tag{3.3.35}$$

By (3.3.34), $A_{2k+s} = -\frac{1}{b_{2k+s}}\left(r_{2k+s} A_{2(k-1)+s} + a_{2k+s} A_{2(k+1)+s}\right) = 0$. Substituting this into (3.3.33) and (3.3.35) yields

$$r_{2(k-1)+s} A_{2(k-2)+s} \quad + \quad \left(b_{2(k-1)+s} - \frac{a_{2(k-1)+s} r_{2k+s}}{b_{2k+s}}\right) A_{2(k-1)+s} \tag{3.3.36}$$

$$\begin{aligned}
(3.3.37)\ -\frac{r_{2k+s}r_{2(k+1)+s}}{b_{2k+s}}A_{2(k-1)+s} \quad &- \quad \frac{a_{2(k-1)+s}a_{2k+s}}{b_{2k+s}}A_{2(k+1)+s} = \frac{d_{2(k-1)+s}}{c^2}A_{2(k-1)+s}, \\
&+ \quad \left(b_{2(k+1)+s} - \frac{a_{2k+s}r_{2(k+1)+s}}{b_{2k+s}}\right)A_{2k+s} \\
&+ \quad a_{2(k+1)+s}A_{2(k+2)+s} = \frac{d_{2(k+1)+s}}{c^2}A_{2(k+1)+s}.
\end{aligned}$$

Regarding the first to $(k-1)$th, (3.3.36), (3.3.37), and later than $(k+3)$th equations of (3.3.25) as a new series of recurrence relations, one can reformulate them into the form of a matrix eigenvalue problem, likewise.

Lastly, a few of geometrical properties of λ–c^2 graph obtained shall be shown below.

[Lemma 3.3.4] In the prolate case, λ is always non-negative. More precisely, $\lambda \geq (m+s)\cdot(m+s+1)$ holds, given m.

[Proof] It suffices to show $\lambda \geq (m+s)\cdot(m+s+1)$ for given m, since the first statement is covered by the second.

Suppose the contrary, or $\lambda < (m+s)\cdot(m+s+1)$, and let the contradiction be reached. Since $\lambda < (m+s)\cdot(m+s+1)$,

$$d_k = \lambda - (m+k)\cdot(m+k+1) < 0 \ (k = s, 2+s, 4+s, \ldots).$$

Transforming (3.3.27) gives

$$(3.3.38)\mathbf{T}\cdot\begin{bmatrix} A_s \\ A_{2+s} \\ A_{4+s} \\ \vdots \end{bmatrix} = -\frac{1}{c^2}\begin{bmatrix} (-d_s)A_s \\ (-d_{2+s})A_{2+s} \\ (-d_{4+s})A_{4+s} \\ \vdots \end{bmatrix}, \text{ where } \mathbf{T} = \begin{bmatrix} b_s & a_s & & \mathbf{0} \\ r_{2+s} & b_{2+s} & a_{2+s} & \\ & r_{4+s} & b_{4+s} & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix}.$$

Also, operating $\begin{bmatrix} \frac{1}{\sqrt{-d_s}} & & & \mathbf{0} \\ & \frac{1}{\sqrt{-d_{2+s}}} & & \\ & & \frac{1}{\sqrt{-d_{4+s}}} & \\ \mathbf{0} & & & \ddots \end{bmatrix}$ from the left on the both sides of (3.3.38) gives

$$\mathbf{U}\mathbf{y} \ = \ -\frac{1}{c^2}\mathbf{y}, \text{ where}$$

$$
\begin{aligned}
\mathbf{U} &= \begin{bmatrix} \frac{b_s}{-d_s} & \frac{a_s}{\sqrt{-d_s}\sqrt{-d_{2+s}}} & & \mathbf{0} \\ \frac{r_{2+s}}{\sqrt{-d_s}\sqrt{-d_{2+s}}} & \frac{b_{2+s}}{-d_{2+s}} & \frac{a_{2+s}}{\sqrt{-d_{2+s}}\sqrt{-d_{4+s}}} & \\ & \frac{r_{4+s}}{\sqrt{-d_{2+s}}\sqrt{-d_{4+s}}} & \frac{b_{4+s}}{-d_{4+s}} & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix}, \\
\mathbf{y} &= [\sqrt{-d_s}A_s, \sqrt{-d_{2+s}}A_{2+s}, \sqrt{-d_{4+s}}A_{4+s}, \ldots]^T.
\end{aligned}
$$

Symmetrizing $\mathbf{U}$ by a similarity transformation gives

$$
\begin{aligned}
\mathbf{V}\mathbf{z} &= -\frac{1}{c^2}\mathbf{z}, \text{ where} \\
\text{(3.3.39)} \quad \mathbf{V} &= \begin{bmatrix} \frac{b_s}{-d_s} & \frac{\sqrt{a_s}\sqrt{r_{2+s}}}{\sqrt{-d_s}\sqrt{-d_{2+s}}} & & \mathbf{0} \\ \frac{\sqrt{a_s}\sqrt{r_{2+s}}}{\sqrt{-d_s}\sqrt{-d_{2+s}}} & \frac{b_{2+s}}{-d_{2+s}} & \frac{\sqrt{a_{2+s}}\sqrt{r_{4+s}}}{\sqrt{-d_{2+s}}\sqrt{-d_{4+s}}} & \\ & \frac{\sqrt{a_{2+s}}\sqrt{r_{4+s}}}{\sqrt{-d_{2+s}}\sqrt{-d_{4+s}}} & \frac{b_{4+s}}{-d_{4+s}} & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix}, \\
\mathbf{z} &= \left[\sqrt{-d_s}A_s, \frac{\sqrt{a_s}}{\sqrt{r_{2+s}}}(\sqrt{-d_{2+s}}A_{2+s}), \frac{\sqrt{a_{2+s}}}{\sqrt{r_{4+s}}}\frac{\sqrt{a_s}}{\sqrt{r_{2+s}}}(\sqrt{-d_{4+s}}A_{4+s}), \ldots\right]^T,
\end{aligned}
$$

where one finds that all of the components of $\mathbf{V}$ are real (and furthermore, positive). Now, $\mathbf{V}$ is found to be decomposed, using a real matrix $\mathbf{S}$, as $\mathbf{V} = \mathbf{S}^T\mathbf{S}$, where

$$
\text{(3.3.40)} \qquad \mathbf{S} = \begin{bmatrix} e_1 & & & \mathbf{0} \\ e_2 & e_3 & & \\ & e_4 & e_5 & \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix}, \text{ with}
$$

$$
\begin{aligned}
e_k &= \frac{1}{\sqrt{-d_{P_k}}} \cdot \sqrt{\frac{(k-1)(2m+k-1)}{(2m+2k-3)(2m+2k-1)}} \ (k = 1, 2, 3, \ldots), \\
P_k &= k - 2 + (k \bmod 2).
\end{aligned}
$$

This shows that $\mathbf{V}$ is positive definite (refer to Appendix 2 for its proof). This leads to a contradiction (since the eigenvalue of $\mathbf{V}$ was assumed to be $-1/c^2 < 0$). Thus, $\lambda \geq (m+s)\cdot(m+s+1)$. ∎

[**Lemma 3.3.5**] In the prolate case, if one assumes $c^2 \neq 0$, then $\lambda > 0$ always holds.

[**Proof**] Lemma 3.3.4 shows $\lambda > 0$ if $m \geq 1$ or if $m = 0$ and $s = 1$. Then one only has to show $\lambda \neq 0$ when $m = 0$ and $s = 0$.

By taking the similar measures as the proof for Lemma 3.3.4, one may assume $\lambda = 0$ a $c^2 \neq 0$ for leading a contradiction. Since $d_k = -k(k+1)$ when $m = s = \lambda = 0$,

$$d_0 = 0, \text{ and } d_k < 0 \ (k = 1, 2, \ldots).$$

Obtaining $A_0 = -\frac{a_0}{b_0} A_2$ from the first line of (3.3.25), and substituting this into t subsequent lines yield

$$\begin{aligned}
\left(b_2 - \frac{a_0 r_2}{b_0}\right) A_2 + a_2 A_4 &= \frac{d_2}{c^2} A_2, \\
r_{2k} A_{2(k-1)} + b_{2k} A_{2k} + a_{2k} A_{2(k+1)} &= \frac{d_{2k}}{c^2} A_{2k} \ (k = 2, 3, 4, \ldots).
\end{aligned}$$

After reformulating them into matrix form and transforming for symmetry, one is given

$$\begin{aligned}
\hat{\mathbf{V}} \hat{\mathbf{z}} &= -\frac{1}{c^2} \hat{\mathbf{z}}, \text{ where} \\
\hat{\mathbf{V}} &= \begin{bmatrix}
(b_2 - \frac{a_0 r_2}{b_0})/(-d_s) & \frac{\sqrt{a_2}\sqrt{r_4}}{\sqrt{-d_2}\sqrt{-d_4}} & & \mathbf{0} \\
\frac{\sqrt{a_2}\sqrt{r_4}}{\sqrt{-d_2}\sqrt{-d_4}} & \frac{b_4}{-d_4} & \frac{\sqrt{a_4}\sqrt{r_6}}{\sqrt{-d_4}\sqrt{-d_6}} & \\
& \frac{\sqrt{a_4}\sqrt{r_6}}{\sqrt{-d_4}\sqrt{-d_6}} & \frac{b_6}{-d_6} & \ddots \\
\mathbf{0} & & \ddots & \ddots
\end{bmatrix}, \\
\hat{\mathbf{z}} &= \left[\sqrt{-d_2} A_2, \frac{\sqrt{a_2}}{\sqrt{r_4}} (\sqrt{-d_4} A_4), \frac{\sqrt{a_4}}{\sqrt{r_6}} \frac{\sqrt{a_2}}{\sqrt{r_4}} (\sqrt{-d_6} A_6), \ldots \right]^T.
\end{aligned}$$

Also, one finds that all of the components of $\hat{\mathbf{V}}$ are real (moreover, positive). $\hat{\mathbf{V}}$ is aga found to be decomposed as $\hat{\mathbf{V}} = \hat{\mathbf{S}}^T \hat{\mathbf{S}}$, where

$$\hat{\mathbf{S}} = \begin{bmatrix}
e'_1 & & & \mathbf{0} \\
e'_2 & e'_3 & & \\
& e'_4 & e'_5 & \\
\mathbf{0} & & \ddots & \ddots
\end{bmatrix}, \text{ with}$$

$$\begin{aligned}
e'_k &= \frac{1}{\sqrt{-d_{P_{k+2}}}} \cdot \frac{(k+1)}{\sqrt{(2k+1)(2k+3)}} \ (k = 2, 3, 4, \ldots), \\
P_k &= k - 2 + (k \bmod 2).
\end{aligned}$$

This means that $\hat{\mathbf{V}}$ is positive definite. Arriving at a contradiction, we have $\lambda > 0$. ∎

3.3.5 Numerical Experiments and λ–c^2 Graph Making

In this section, some experiments are executed to show the validity of the error estir The results are shown in the table below.

Given m and c, the author let one of the eigenvalues λ be computed. And error estim was done to the approximate eigenvalues. By putting those computed values into the estimate, we compared the results between the LHS and RHS of the estimate formula (a the RHS, the term $[1 + o(1)]$ is neglected). The computation was done on FUJITSU V 500 with quadruple precision, and subroutine COMQR in EISPACK[28] was used fo computation of eigenvalues. We continued until the absolute value of the error rea 10e-25.

In the table, one can confirm that they agree to the first three digits for n large eno

Experiment 3.3.1

Results of error estimate

$$\text{given } m = 0,\ c^2 = 5,$$

$$\text{compute}$$

$$\begin{aligned}\lambda^{(1)} &= 44.46262\ 13744\ 59746\ 44443\ 47485 \cdots,\\ \lambda^{(2)} &= 74.51992\ 09530\ 20538\ 80224\ 85287 \cdots,\\ \lambda^{(3)} &= 112.51296\ 13927\ 08998\ 16536\ 96080 \cdots.\end{aligned}$$

Table 3.3.1 Actual errors and estimate of (3.3.19)

n	$\lambda^{(1)} - \lambda_n^{(1)}$	$E^{(1)}$	$\lambda^{(2)} - \lambda_n^{(2)}$	$E^{(2)}$	$\lambda^{(3)} - \lambda_n^{(3)}$	$E^{(3)}$
1	-5.252e-02	-5.242e-02	7.452e+01	5.240e-02	1.125e+02	2.521e-05
2	-4.030e-05	-4.033e-05	-4.131e-02	-4.127e-02	1.125e+02	4.129e-02
3	-8.152e-09	-8.154e-09	-2.029e-05	-2.030e-05	-3.408e-02	-3.405e-02
4	-6.667e-13	-6.668e-13	-2.745e-09	-2.745e-09	-1.160e-05	-1.161e-05
5	-2.701e-17	-2.701e-17	-1.558e-13	-1.558e-13	-1.122e-09	-1.122e-09
6	-6.124e-22	-6.125e-22	-4.516e-18	-4.516e-18	-4.674e-14	-4.674e-14
7			-7.521e-23	-7.521e-23	-1.017e-18	-1.017e-18
8					-1.296e-23	-1.296e-23

, where

$\lambda^{(k)}$ denotes λ_k^m, the exact eigenvalue of (3.3.17),

$\lambda_n^{(k)}$ denotes an approximation of $\lambda^{(k)}$ computed from $n \times n$ matrix,

$E^{(k)} = (\frac{\alpha_0\alpha_2\cdots\alpha_{2n-2}}{\gamma_2\gamma_4\cdots\gamma_{2n-2}}) \cdot \frac{A_{2n-2}A_{2n}}{x^T x}$,

and $x = \left[A_s, \frac{\sqrt{\alpha_s}}{\sqrt{\gamma_{2+s}}}A_{2+s}, \frac{\sqrt{\alpha_{2+s}}}{\sqrt{\gamma_{4+s}}}\frac{\sqrt{\alpha_s}}{\sqrt{\gamma_{2+s}}}A_{4+s}, \cdots\right]^T$.

Also, the graph showing the relation of λ and c^2 is shown in Fig. 3.3.1 (the up (above λ-axis) of the figure). This was made by computing eigenvalues λ, given c^2, t Theorem 3.3.2. This figure visually shows that λ is simple. And for comparison, the graph in the oblate case is also plotted (the lower half (below λ-axis) of the figure).

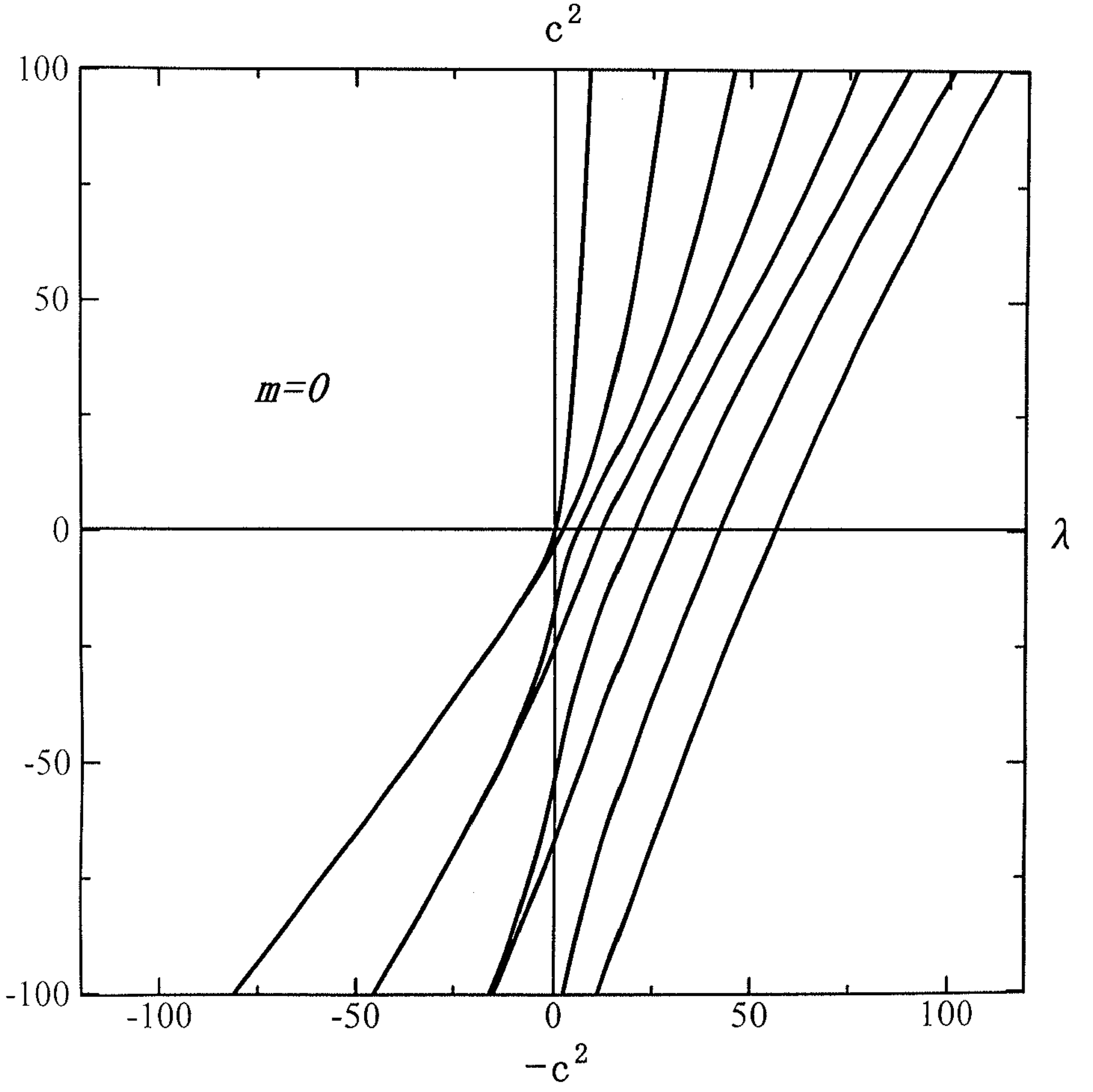

Fig. 3.3.1 λ-c^2 graph

3.3.6 Summary of Section 3.3

The methods for computing approximate eigenvalues as wel as inverse eigenvalu the spheroidal wave equations were proposed both by matrix method. Also, some c geometrical properties were also showed.

3.4 Summary of Section 3

Throughout Section 3, the author proved that three more problems may be applied e by Theorem A or Theorem B. With previous applications included, up to present, it has proved that Theorem A and Theorem B each may apply to the problems of

<Theorem A>:

- the computation of the zeros z of $J_\nu(z)$,
- the computation of the zeros z of $zJ'_\nu(z) + HJ_\nu(z)$,
- the inverse EVP of the Mathieu differential equation,
- the inverse EVP of the spheroidal wave equation,
- the computation of the zeros ρ of Coulomb wave function $F_L(\eta, \rho)$ and its first de tive.

<Theorem B>:

- the computation of the zeros ν of $J_\nu(z)$,
- the computation of the zeros ν of $zJ'_\nu(z) + HJ_\nu(z)$,
- the ordinary EVP of the Mathieu differential equation,
- the ordinary EVP of the spheroidal wave equation.

Very lately, the author has found that the computation of the zeros of Whittaker func ($M_{\kappa,\mu}(z)$, for further explanations on this function, refer to [9]) is likely to be applied by orem A. This research will appear somewhere else to the details although the investiga is immature yet.