2 Three Key Theorems

In this section, three theorems are introduced for the later use. Each of these theorems is strongly related to the present paper, therefore, one section is allocated for their introductions to the details. First, two previous theorems mentioned in the last section, or Theorem A and Theorem B, shall be introduced.

[Theorem A] [15, Theorem 1.1 & 1.4] Given a complex symmetric tridiagonal matrix

$$\mathbf{A} = \begin{bmatrix} d_1 & f_2 & & \mathbf{0} \\ f_2 & d_2 & f_3 & \\ & f_3 & d_3 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix},$$

where $d_k \to 0, f_k \to 0\ (k \to \infty), f_k \neq 0\ (k = 2, 3, \ldots)$, representing a compact operator in the Hilbert space ℓ^2. Let $\mathbf{A}$ have a simple eigenvalue $\lambda \neq 0$, and $\mathbf{0} \neq \chi = [\chi^{(1)}, \chi^{(2)}, \ldots]^T \in \ell^2$ be an eigenvector corresponding to λ. Then

(*i*) Letting $\mathbf{A}_n\ (n = 1, 2, \ldots)$ denote the nth order principal submatrix of $\mathbf{A}$, there is a sequence $\{\lambda_n\}$ of eigenvalues of $\mathbf{A}_n$ which converges to λ.

(*ii*) Letting $\chi^T\chi \neq 0$, and $\chi^{(n+1)}/\chi^{(n)}$ be bounded for all sufficiently large n, we have the following error estimate:

$$\lambda - \lambda_n = \frac{f_{n+1}\chi^{(n)}\chi^{(n+1)}}{\chi^T\chi}[1 + o(1)] \quad (n \to \infty). \tag{2.1}$$

[Theorem B] [13, Theorem 1] Given a non-compact complex symmetric tridiagonal matrix

$$\mathbf{T} = \begin{bmatrix} d_1 & f_2 & & 0 \\ f_2 & d_2 & f_3 & \\ & f_3 & d_3 & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix} : D(\mathbf{T}) \to \ell^2,$$

where $0 < |d_k| \to \infty\ (k \to \infty), 0 < |f_k| < \text{const}\ (k = 2, 3, \ldots), D(\mathbf{T}) = \{[u^{(1)}, u^{(2)}, \ldots]^T : [d_1 u^{(1)}, d_2 u^{(2)}, \ldots]^T \in \ell^2\}$. Let $\mathbf{T}$ have a simple eigenvalue $\lambda \neq 0$, and $\mathbf{0} \neq \chi = [\chi^{(1)}, \chi^{(2)}, \ldots]^T$ be an eigenvector corresponding to λ, and assume the existence of $\mathbf{T}^{-1}$. Then

(*i*) Letting $\mathbf{T}_n\ (n = 1, 2, \ldots)$ denote the nth order principal submatrix of $\mathbf{T}$, there is a sequence $\{\lambda_n\}$ of eigenvalues of $\mathbf{T}_n$ which converges to λ.

(*ii*) Letting $\chi^T\chi \neq 0$ and $f_{n+1}\chi^{(n+1)}/\chi^{(n)} \to 0$ $(n \to \infty)$, we have the following error estimate:

$$\lambda - \lambda_n = \frac{f_{n+1}\chi^{(n)}\chi^{(n+1)}}{\chi^T\chi}[1 + o(1)] \quad (n \to \infty). \tag{2.2}$$

In Theorem A & B, we define ℓ^2 as the complex Hilbert space $\ell^2 \equiv \{[c_1, c_2, \ldots]^T : c_1, c_2, \ldots \in \mathbf{C}, \sum_{n=1}^{\infty} |c_n|^2 < \infty\}$; $o(1)$ as the quantity converging to zero as $n \to \infty$; and the *simple* eigenvalue λ as having the unique corresponding eigenvector (up to scalar multiplication) and also no corresponding generalized eigenvectors of rank 2, namely, no vectors $\mathbf{0} \neq \mathbf{v}_1 \in \ell^2$ (or $\mathbf{0} \neq \mathbf{v}_2 \in D(\mathbf{T})$, in the Theorem B's case) satisfying $(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_1 = \mathbf{0}$ (or $(\mathbf{T} - \lambda\mathbf{I})^2\mathbf{v}_2 = \mathbf{0}$, likewise). Also, in Theorem B, the definition of the existence of $\mathbf{T}^{-1}$ shall be that there is only one solution $\mathbf{z} = \mathbf{0}$ which satisfies $\mathbf{Tz} = \mathbf{0}$. We use these definitions throughout this paper.

Theorem A & B are complementarily related (refer to [7],[13], and [15]), and have the next 4 favorable properties in common:

1. The computed eigenvalues shall be approximated with any precision.
2. An asymptotic error estimate for approximated eigenvalues to the true one is obtained.
3. There is no need to use different methods separately for getting real or complex eigenvalues (you only have to use one method consistently).
4. The simplicity of the algorithm makes the implementation to computers easy.

One more attention to be paid to the error estimate formulas in the two theorems is that they are identical in their forms.

Another theorem which plays an important role in this article is introduced. This theorem guarantees the behavior of the solutions for three-term recurrence relations of a certain type:
[**Theorem C**] [11, Theorem 2.3, Case(a)] Consider three-term recurrence relations of the form

$$h_{n+1} + p_n h_n + q_n h_{n-1} = 0 \ (n = 1, 2, \ldots). \tag{2.3}$$

If the following conditions

$$\begin{aligned} &p_n = p \cdot n^P[1 + o(1)], q_n = q \cdot n^Q[1 + o(1)] \ (n \to \infty), q_n \neq 0 \ (n = 1, 2, \ldots), \\ &p, q \text{ are non} - \text{zero, and } P, Q \text{ are both real satisfying } 2P > Q \end{aligned} \tag{2.4}$$

hold, then (2.3) has two linearly independent solutions, $\{h_{n,1}\}$ and $\{h_{n,2}$ behaves as

$$\frac{h_{n+1,1}}{h_{n,1}} = -pn^P[1+o(1)], \quad \frac{h_{n+1,2}}{h_{n,2}} = -\frac{q}{p}n^{Q-P}[1+o(1)] \ (n \to \infty$$

In [11], a solution $\{g_n\}$ of (2.3) is defined as "minimal solution of (2.3)" if

$$\lim_{n\to\infty} g_n/h_n = 0$$

holds for all the solutions $\{h_n\}$ but $\{g_n\}$ and any constant multiple of $\{g_n$ case, $\{h_{n,2}\}$ is the minimal solution of (2.3).

This theorem is considered one of the key theorems in that we are a concrete behavior of the components of eigenvectors of matrix **A** (in Theore T (in Theorem B), which is indispensable for obtaining error estimates (2.1