1 Introduction

This paper is strongly based on two powerful general theorems, proved by Ikebe, et.al in 1993[15] and 1996[13], which will be referred to as Theorem A and Theorem B in this paper for convenience. In particular, they justify the approximate computation of simple eigenvalues of infinite matrices of certain types by truncation, giving an extremely accurate error estimates. The matrices (let's say, $\mathbf{A}$) to which these two theorems apply are the ones which satisfy the following three conditions:

- $D(\mathbf{A}) \subset \ell^2$ and $R(\mathbf{A}) \subset \ell^2$ (the range as well as the domain of $\mathbf{A}$ are subspaces of the Hilbert space ℓ^2).
- $\mathbf{A}$ is complex symmetric tridiagonal.
- $\mathbf{A}$ or $\mathbf{A}^{-1}$ is compact.

Applications of these two theorems include:

(A) the computation of the zeros of $J_\nu(z)$ (where $J_\nu(z)$ denotes the Bessel function of the first kind of order ν) [15],[13],

(B) the computation of the zeros of $zJ'_\nu(z) + HJ_\nu(z)$ [7],

(C) the (ordinary) EVP (EigenValue Problem) of the Mathieu differential equation [13].

The main author of Theorem A and Theorem B, Ikebe, had earlier proposed the matrix method for the computation of the zeros of Coulomb wave function $F_L(\eta, \rho)$ and its first derivative[12]. The method was justified, but no error estimates are given.

Expansion of the said type of eigenvalue problem yields a set of linear three-term recurrence relations involving a few parameters, one of which or a simple function of it represents the said eigenvalue. Since the special functions of mathematical physics, e.g., Bessel functions, or Mathieu functions cited above, are often solutions of the recurrence relations of the said type, applicability of Theorem A and Theorem B may be wider than to those already mentioned. It is precisely the major purpose of this paper to demonstrate that this is in fact the case. Indeed, further applications of Theorem A and Theorem B studied in this paper include:

ⓐ the computation of the zeros of Coulomb wave function $F_L(\eta, \rho)$ and its first derivative (with an explicit and a closed error formula) [23],[25],

ⓑ the inverse EVP of the Mathieu differential equation [21],

ⓒ the ordinary and inverse EVP of the spheroidal wave equation [22],

each of which is discussed in Section 3.1, Section 3.2, and Section 3.3, respectively. In each section, the superiority of the method presented in this paper is emphasized, in comparison with the previous research. Also, not only the method for the solutions, but more, such as geometrical properties and some other remarks are given, along with the proof by matrix theory. Specifically, in the case of ⓐ, or the zeros of Coulomb wave function $F_L(\eta, \rho)$, a new relation useful for the simplification of the given error estimate was found as well as the proof that one of the theorems may apply to the problem. This finding might mean a lot in the sense that the error estimate will be more simplified into a closed form, if the relation holds to more general problems. The details are discussed in the later sections.

Another objective of this article is to give supplementary theorems to Theorem A and Theorem B for their reinforcements. As will be shown in Section 2, Theorem A and Theorem B guarantee the asymptotic error estimates for eigenvalues of truncated matrices to the true eigenvalue of its original infinite matrix, but only when the eigenvalue is simple. This means that it will be difficult to obtain double eigenvalues by the method in the theorems, and further that it will be safer and more secure if one knows where such eigenvalues exist in advance. The supplementary theorems cover the problem although the class of infinite matrices is more limited to the one Theorem A or Theorem B may apply. In the theorem, the author proposes an algorithm to compute such double eigenvalues with a good rate of convergence and precision, by showing a new relation involving the first derivative of the eigenvalue. In fact, computing those is easily realized by the combination of Newton-Raphson method and Theorem B. This will be discussed in Section 4.