

## Chapter 2

# SUPERFLUID HELIUM

### 2.1 Superfluid Helium

Helium exists in two stable isotopic forms,  $^3\text{He}$  and  $^4\text{He}$ . In this study,  $^4\text{He}$  is exclusively concerned with. The liquefaction of helium was first conducted by H. K. Onnes in 1908. The phase diagram of helium is schematically shown in Fig. 2.1. The normal boiling point is  $4.2\text{ K}$  and the critical temperature is  $5.19\text{ K}$ . Liquid helium exists in two phases, He I and He II (Superfluid Helium), separated by a phase boundary called the  $\lambda$ -line. At saturated vapor pressure this transition occurs at the temperature

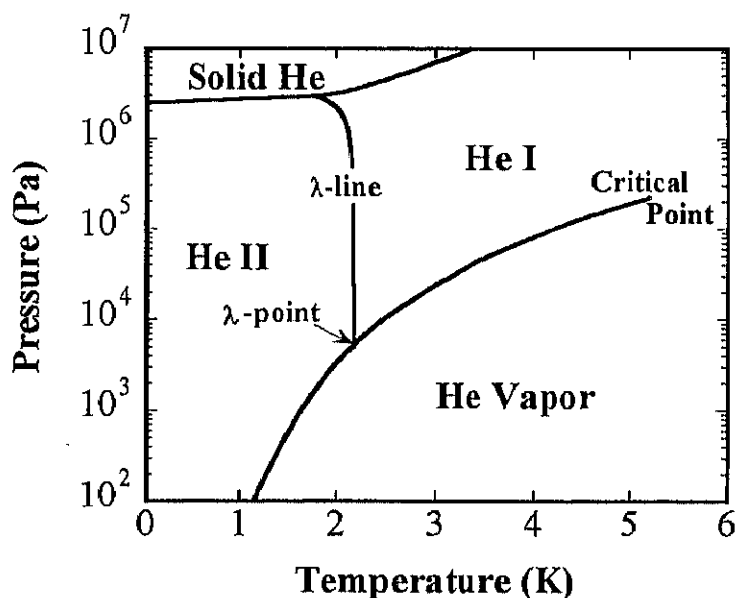


Fig. 2.1: The phase diagram of  $^4\text{He}$

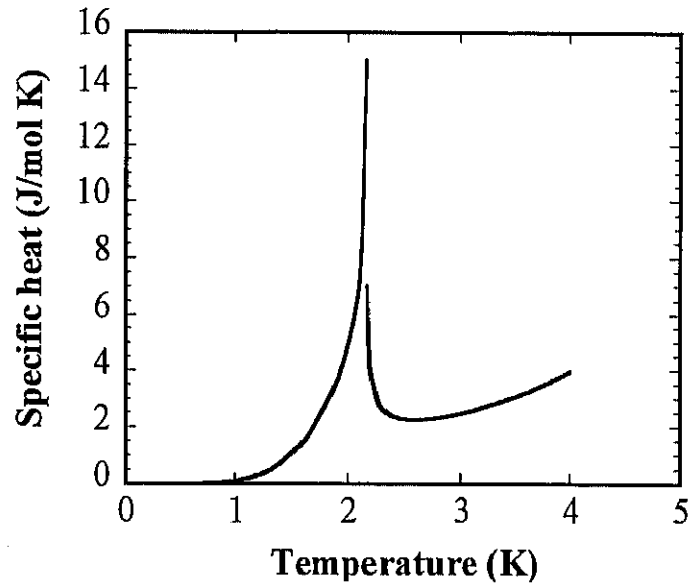


Fig. 2.2: The specific heat of liquid helium along the saturated vapor curve as a function of the temperature.

$T_0 = 2.17 \text{ K}$  founded by H. K. Onnes' research group in 1911. There is no latent heat associated with this phase transition and He II never solidifies even at the temperature, absolute zero, if the pressure is below  $2.5 \text{ MPa}$ . Moreover, the heat capacity of liquid helium increase towards infinity at  $\lambda$ -point as shown in Fig. 2.2. Below the  $\lambda$ -point liquid helium (He II) has the most remarkable character, superfluidity and super thermal conduction. Superfluidity was discovered by Kapitza in 1927 and super thermal conduction was also discovered by H. K. Onnes in 1927. Superfluidity and super thermal conduction are the typical phenomena which are regarded as the appearance of quantum effect in macroscopic level. Owing to superfluidity property can flow without any friction even through narrow capillaries. The practical thermal conductivity of He II is several hundreds times higher than that of pure copper. This property is called super thermal conduction. He II is not a classical liquid but a quantum liquid and He II obey Bose-Einstein static. Liquid helium is only substance that dose not solidify under its own vapor pressure. The reason for this is quantum mechanical in that the zero point energy is sufficient to overcome the weak binding potential between helium atoms. And He II has several attractive

properties for cryogen, and so it is expected to be the efficient coolant for superconducting magnets and space-borne infrared telescope.

## 2.2 Two-Fluid Model for He II

The fluid-dynamic behavior of He II is described on the basis of the two-fluid model introduced by Tisza[35], which was then led to the two-fluid equation presented by Landau[36]. The two-fluid model is considered that He II is composed of two components, the normal fluid and superfluid components. The normal fluid component is a regular viscous fluid being subject to classical fluid dynamics. It has finite viscosity and entropy. On the other hand, the superfluid component has zero viscosity and entropy. Both of the two components have their own densities,  $\rho_n$  and  $\rho_s$  respectively. The total density  $\rho$  is given by

$$\rho = \rho_n + \rho_s. \quad (2.1)$$

In Fig. 2.3, the proportion of the super fluid component increases from zero up to unity at absolute zero as the temperature goes down. In the view-

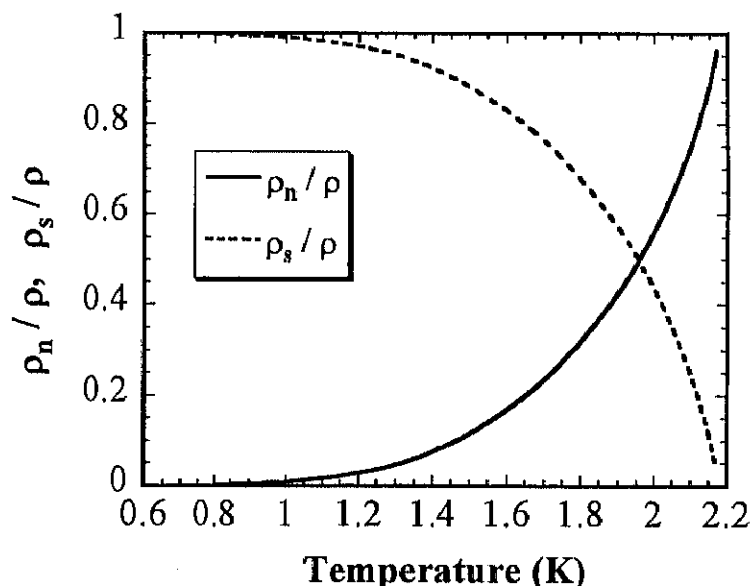


Fig. 2.3: The ratio of the normal fluid density  $\rho_n$  and the superfluid density  $\rho_s$  to the total density  $\rho$  as a function of temperature.

point of the quantum microscopic, the superfluid component corresponds to helium atoms which occupy the quantum ground state, Bose-Einstein condensation phase. On the other hand, the normal fluid component is the thermodynamically excited phase. The excitations are described by the dispersion curve of energy as a function of the momentum in Fig. 2.4. There are two regions of specific interest on this curve:

- 1) The low momentum range, where the curve is linear: Phonons
- 2) The high momentum valley in the energy spectrum: Rotons

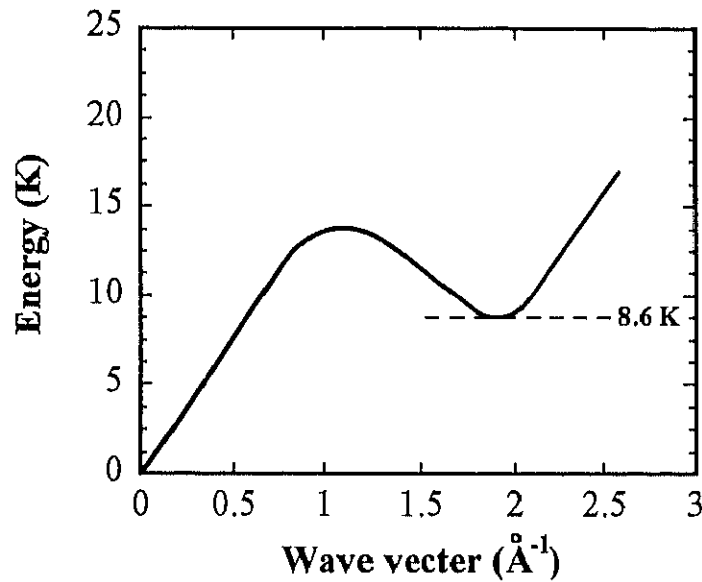


Fig. 2.4: The dispersion curve for liquid helium at the saturated vapor and  $T_0 = 1.0 K$

The energy spectra of these two groups of excitations are usually expressed by following simple analytical expression:

$$\text{Phonons : } \epsilon = cp \quad (2.2)$$

$$\text{Rotons : } \epsilon = \Delta_r + \frac{p - p_r}{2\mu_r} \quad (2.3)$$

where  $\epsilon$  and  $p$  are the energy and momentum of excitation,  $c$  is the speed of sound, and  $\Delta_r/\kappa_B = 8.6 K$ ,  $p_r/\hbar = 1.9 \text{\AA}^{-1}$  and  $\mu_r = 0.15 m_4$  are parameters.  $\hbar$ ,  $\kappa_B$  and  $m_4$  are the value of the Plank constant divided by  $2\pi$ , the Boltzmann constant and the mass of a helium atom, respectively. At fairly

low temperatures,  $T_0 < 0.6 K$ , phonons which are simply sound waves of quantized amplitude are excited almost to the exclusion of all other higher energy excitations. As the temperature increases, the population of rotons increases very rapidly due to its large density of states. So that above  $T_0 = 1.2 K$ , thermodynamics of He II is completely dominated by them. The relation among phonons, rotons and normal fluid component in density is given by

$$\rho_n = \rho_{np} + \rho_{nr}, \quad (2.4)$$

where  $\rho_{np}$  and  $\rho_{nr}$  are the density of phonons and that of rotons, respectively.

At fairly low temperature,  $T_0 < 0.6 K$ , a phonon and a roton can excite a helium atom into vapor phase. This excitation is called as a quantum evaporation phenomena. Recently, the quantum evaporation is studied by many authors[37]-[39].

### 2.3 Landau Two-Fluid Equations

The basic hydrodynamic equations for He II in the case of non-dissipative effects are first introduced by Landau. The fundamental assumptions on which Landau two-fluid equations is based are the following: The ordered motion of the excitations carries along with it only part of the liquid, characterized by the normal fluid density  $\rho_n$ . The remaining part, the superfluid is characterized by superfluid density  $\rho_s$ , and performs an independent motion. This independent motion has the important property of being irrotational. Thus in a superfluid, there can exist two simultaneous but independent motions, one normal and the other superfluid, with velocities  $\vec{v}_n$  and  $\vec{v}_s$ , with

$$\text{curl } \vec{v}_s = 0. \quad (2.5)$$

Eq. (2.5) which states that the superfluid motion is irrotational will not be violated until the flow velocities have reached certain critical values where the normal fluid and superfluid parts of liquid begin to interact. The total density of He II,  $\rho$  is defined by Eq. (2.1). The equation of conservation

mass is

$$\frac{\partial \rho}{\partial t} + \text{div } \vec{j} = 0, \quad (2.6)$$

where  $\vec{j}$  is the total mass flux shown by

$$\vec{j} = \rho_n \vec{v}_n + \rho_s \vec{v}_s. \quad (2.7)$$

Since dissipation is neglected and the entropy is transported only the normal fluid component, the equation of conservation entropy takes the form

$$\frac{\partial(\rho s)}{\partial t} + \text{div}(\rho s \vec{v}_n) = 0. \quad (2.8)$$

The flow of superfluid component is driven by the gradient of chemical potential. The equation of conservation momentum for the superfluid component is

$$\frac{D\vec{v}_s}{Dt} + \nabla \mu = 0, \quad (2.9)$$

where  $\mu$  is the chemical potential per volume of He II and its gradient is given as follows

$$\nabla \mu = \frac{\nabla P}{\rho} - s \nabla T - \frac{\rho_n}{2\rho} \nabla (\vec{v}_n - \vec{v}_s)^2, \quad (2.10)$$

Here, the conservation equation for the total momentum is introduced.

$$\frac{\partial \vec{j}_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_i} = 0, \quad (2.11)$$

where  $\Pi_{ik}$  is the momentum flux density tensor indicated by

$$\Pi_{ik} = P \delta_{ik} + \rho_n \rho_{ni} \rho_{nk} + \rho_s \rho_{si} \rho_{sk}. \quad (2.12)$$

## 2.4 Two Mode of Sound Wave Propagation in He II

The structure of two-fluid model Eqs. (2.6), (2.8), (2.9) and (2.11) suggests that there are two kinds of propagating wave modes. In sound wave the velocities  $\vec{v}_n$  and  $\vec{v}_s$  and are assumed to be small, and the thermodynamic quantities almost equal to their equilibrium values. The propagation

of sound in He II is described by the system of hydrodynamic equations, Eqs. (2.6), (2.8), (2.9) and (2.11), which in this case may be linearized, and given

$$\frac{\partial \rho}{\partial t} + \text{div} \vec{j} = 0, \quad (2.13)$$

$$\frac{\partial(\rho s)}{\partial t} + \rho s \text{div} \vec{v}_n = 0, \quad (2.14)$$

$$\frac{\partial \vec{v}_s}{\partial t} + \nabla \mu = 0, \quad (2.15)$$

$$\frac{\partial \vec{j}}{\partial t} + \nabla P = 0. \quad (2.16)$$

From Eqs. (2.13) and (2.16), the momentum  $\vec{j}$  is eliminated such as

$$\frac{\partial^2 \vec{j}}{\partial t^2} + \nabla P = 0. \quad (2.17)$$

Next, the velocities of  $\vec{v}_n$  and  $\vec{v}_s$  must be also eliminated from Eqs. (2.14), (2.15) and (2.16). For this, take the time derivative of Eq. (2.14) and the divergence of Eqs. (2.15) and (2.16). By eliminating the terms  $\frac{\partial}{\partial t} \text{div} \vec{v}_n$  and  $\frac{\partial}{\partial t} \text{div} \vec{v}_s$  from the equations thus obtained as

$$\rho_s \nabla P + \frac{\rho_n}{\rho_s} \frac{\partial^2(\rho s)}{\partial t^2} + \nabla P = 0. \quad (2.18)$$

From Eqs. (2.15) and (2.8), finally, the next equation given

$$\frac{\partial^2 s}{\partial t^2} + \frac{\rho_s}{\rho_n} s^2 \nabla T = 0. \quad (2.19)$$

Eqs. (2.17) and (2.19) determine the changes in the thermodynamic quantities in the presence of sound wave. In these equations, independent variable  $P$  and  $T$  is represented by

$$P = P_0 + P' \quad (2.20)$$

$$T = T_0 + T' \quad (2.21)$$

where the quantities with subscript zero denote the equilibrium values, and the primed quantities the deviations from equilibrium due to the sound wave. As a result, Eqs. (2.17) and (2.19) take the form

$$\frac{\partial \rho}{\partial P} \frac{\partial^2 P'}{\partial t^2} - \nabla P' + \frac{\rho}{T} \frac{\partial^2 T'}{\partial t^2} = 0, \quad (2.22)$$

$$\frac{\partial \rho}{\partial P} \frac{\partial^2 P'}{\partial t^2} + \frac{\rho}{T} \frac{\partial^2 T'}{\partial t^2} - \frac{s^2 \rho_s}{\rho_n} \nabla T' = 0 \quad (2.23)$$

Eqs. (2.17) and (2.19) can be represented by a plane wave propagating in some direction. In such a wave the quantities  $P'$  and  $T'$  vary as

$$P = P_0 + P' \exp\left(\frac{ik_0 x}{u} - i\omega t\right) \quad (2.24)$$

$$T = T_0 + T' \exp\left(\frac{ik_0 x}{u} - i\omega t\right) \quad (2.25)$$

where the x-axis can be chosen as the direction of propagation of the wave,  $k_0$ ,  $u$  and  $\omega$  are the wave number, the velocity of sound and the frequency. The system of the equations, Eqs. (2.22) and (2.23), may be written in the form

$$\left(\frac{\partial \rho}{\partial P} u^2 - 1\right) P' + \frac{\partial \rho}{\partial T} u^2 T' = 0, \quad (2.26)$$

$$\frac{\partial s}{\partial P} u^2 P' + \left(\frac{\partial s}{\partial T} u^2 - \frac{s^2 \rho_s}{\rho_n}\right) T' = 0. \quad (2.27)$$

As usual, the above equations will be compatible if the determinant of their coefficient is equal to zero. Expanding the determinant, the next equation is given as

$$u^4 \frac{\partial(s, \rho)}{\partial(T, P)} - u^2 \left(\frac{\partial s}{\partial T} + s^2 \frac{\rho_s}{\rho_n} \frac{\partial \rho}{\partial P}\right) + \frac{\rho_s}{\rho_n} s^2 = 0. \quad (2.28)$$

After some simple transformations, Eq. (2.28) is simplified as

$$u^4 - u^2 \left[ \left(\frac{\partial P}{\partial \rho}\right)_s + s^2 \frac{\rho_s}{\rho_n} \left(\frac{\partial T}{\partial s}\right)_\rho \right] + s^2 \frac{\rho_s}{\rho_n} \left(\frac{\partial T}{\partial s}\right)_\rho \left(\frac{\partial P}{\partial \rho}\right)_T = 0. \quad (2.29)$$

Furthermore, Eq. (2.29) is written as

$$\left(\frac{u^2}{a_{10}^2} - 1\right) \left(\frac{u^2}{a_{20}^2} - 1\right) = 1 - \frac{c_P}{c_V}. \quad (2.30)$$



where,

$$\begin{aligned}
 c_P &= T \left( \frac{\partial s}{\partial T} \right)_\rho, \\
 c_V &= T \left( \frac{\partial s}{\partial T} \right)_T, \\
 a_{10}^2 &= \left( \frac{\partial P}{\partial \rho} \right)_s, \\
 a_{20}^2 &= \frac{\rho_s s^2 T}{\rho_n c_P}.
 \end{aligned}
 \tag{2.31}$$

Eq. (2.31) determines two possible velocities of sound in He II. The coefficient of thermal expansion is in practice, very small for all conventional substances, for He II it is anomalously small. Therefore, according to the well-known thermodynamic relations one can consider two specific heats  $c_P$  and  $c_V$  to be practically equal in He II. From this equal condition, the right hand side of Eq. (2.30) can be neglected. Then the solutions of Eq. (2.30) are given as,

$$u_{10} = a_{10} = \sqrt{\left( \frac{\partial P}{\partial \rho} \right)_s}
 \tag{2.32}$$

$$u_{20} = a_{20} = \sqrt{\frac{\rho_s s^2 T}{\rho_n c_P}}
 \tag{2.33}$$

Eq. (2.32) determines the propagation speed of the ordinary sound called first sound wave in He II. From Eq. (2.17), the first sound wave is produced by pressure (density) oscillations. On the other hand, Eq. (2.33) determines the propagation speed of the temperature wave. From Eq. (2.19), the second sound wave is produced by pressure (entropy) oscillations. According to the two-fluid model, when the first sound wave travels in He II, the normal and super components move together in phase. However, in the second sound wave, the two components move oppositely in zero center-of-mass motion. Fig. 2.5 shows the propagation speeds of the first sound and second sound as a function of temperature. The difference between  $c_P$  and  $c_V$  is not always negligible for He II, in particularly for conditions near the  $\lambda$  point and at high pressures. In such cases the term  $1 - \frac{c_P}{c_V}$  must be retained. Keeping term to first order in  $1 - \frac{c_P}{c_V}$ , the first sound and the

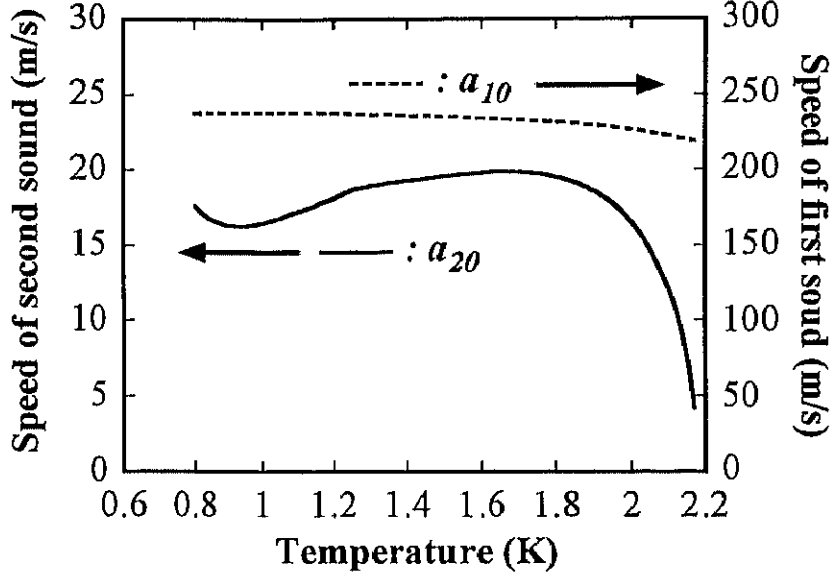


Fig. 2.5: The propagation speeds of a first sound and a second sound as a function of temperature.

second sound wave speed are given by

$$u_{10}^2 = a_{10}^2 + \frac{a_{10}^2 a_{20}^2}{a_{10}^2 - a_{20}^2} \left( 1 - \frac{c_P}{c_V} \right), \quad (2.34)$$

$$u_{10}^2 = a_{10}^2 - \frac{a_{10}^2 a_{20}^2}{a_{10}^2 - a_{20}^2} \left( 1 - \frac{c_P}{c_V} \right). \quad (2.35)$$

By recognizing the non-zero difference between  $c_P$  and  $c_V$ , additional insight into the physical nature of first and second sound wave may be achieved by Lifshitz[40] and Landau and Lifshitz[41]. From the thermodynamic identity

$$1 - \frac{c_P}{c_V} = \frac{c_P}{c_V} T \alpha_{exp}^2 \left( \frac{\partial P}{\partial \rho} \right)_s, \quad (2.36)$$

where  $\alpha_{exp} = -\frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_P$  is the thermal expansion coefficient. it is clear that keeping terms of order  $1 - \frac{c_P}{c_V}$  is equivalent to keeping terms quadratic in  $\alpha_{exp} T$ . Here, working only to first order in  $\alpha_{exp} T$ . Next, if the varying part of each quantity is denoted by prime, the next equations are given

from Eqs. (2.6), (2.8), (2.9) and (2.11).

$$\begin{aligned}
 u\rho' - j' &= 0, \\
 us\rho' + u\rho s' - s\rho v_n &= 0, \\
 uj' - P' &= 0, \\
 uv_s + sT' - \frac{1}{\rho}P' &= 0.
 \end{aligned}
 \tag{2.37}$$

Furthermore, Eq. (2.37) is simplified as follows

$$P' = u^2\rho' = uj' \tag{2.38}$$

$$T' = \frac{u^2\rho_n}{s^2\rho_s}s^2 = \frac{u\rho_n}{s\rho}(v_n - v_s). \tag{2.39}$$

From Eqs. (2.34), (2.35), (2.38) and (2.39), the coefficients of proportionality between the velocities and variable parts of the temperature and pressure in the case of first order thermal expansion coefficient  $\alpha_{exp}$  are given by Lifshitz[40] as follows;

$$\begin{aligned}
 v_s &= a v_n, \\
 P' &= b v_s, \\
 T' &= c v_s.
 \end{aligned}
 \tag{2.40}$$

where, for the first sound

$$\begin{aligned}
 a_1 &= 1 + \frac{\alpha_{exp}\rho}{\rho_s s} \frac{a_{10}^2 a_{20}^2}{a_{10}^2 - a_{20}^2}, \\
 b_1 &= \rho a_{10}, \\
 c_1 &= \frac{\alpha_{exp}\rho T}{c_P} \frac{a_{10}^2}{a_{10}^2 - a_{20}^2}.
 \end{aligned}
 \tag{2.41}$$

for the second sound

$$\begin{aligned}
 a_2 &= -\frac{\rho_n}{\rho_s} + \frac{\alpha_{exp}\rho}{\rho_n s} \frac{a_{10}^2 a_{20}^2}{a_{10}^2 - a_{20}^2}, \\
 b_2 &= \frac{\alpha_{exp}\rho}{s} \frac{a_{10}^2 a_{20}^2}{a_{10}^2 - a_{20}^2}, \\
 c_2 &= -\frac{a_{20}}{s}.
 \end{aligned}
 \tag{2.42}$$

As the thermal expansion coefficient  $\alpha_{exp}$  is negative above 1.0 K in He II, for the first sound, the pressure rise will decrease through the compression and increase through the rarefaction. On the other hand, for the second sound, the pressure and the temperature rises have the same sign.

## 2.5 Thermal Shock Wave in He II

When the amplitudes of waves becomes large, the linear theory becomes no longer valid. Moreover, each sound may develop into a shock wave. The shock wave of second sound wave origin is called a thermal shock wave. It is Khalatnikov[42] who investigated the dynamic behavior of the sounds with finite amplitude in He II. In the case of a finite amplitude disturbances the propagation speed within a wave profile differs from point to point depending on the deviation from the equilibrium value,  $\Delta T$  or  $\Delta P$ . For the first sound wave, the local propagation speed is expressed to the first order smallness by

$$u_{10}(\Delta P) = a_{10} \left[ 1 + a \left( \frac{\Delta P}{P} \right) \right], \quad (2.43)$$

where the steepening coefficient is written as

$$a = P \left( \frac{\partial}{\partial P} \right)_s \ln(\rho c_{10}), \quad (2.44)$$

and for the second sound wave,

$$u_{20}(\Delta T) = a_{20} \left[ 1 + b \left( \frac{\Delta T}{T} \right) \right], \quad (2.45)$$

where the steepening coefficient  $b$  is given by

$$b = T \left( \frac{\partial}{\partial T} \right)_P \ln \left( \frac{a_{20} c_P}{T} \right). \quad (2.46)$$

As a result of non-linear feature due to finite amplitude waves, a second sound wave develops into a thermal shock wave characterized by a discontinuity in the temperature. It is known from theoretical consideration that the shock profile in the final stage of propagation is a Buger's triangle. It is of interest that shock waves in He II appears not only in the form of a frontal shock similar to a compression shock as seen in conventional gases but also in the form of a back shock similar to an expansion shock. The distinction of these wave forms results from the sign of the steepening coefficient,  $b$ , defined by Eq. (2.46). Fig. 2.6 shows the variation of the

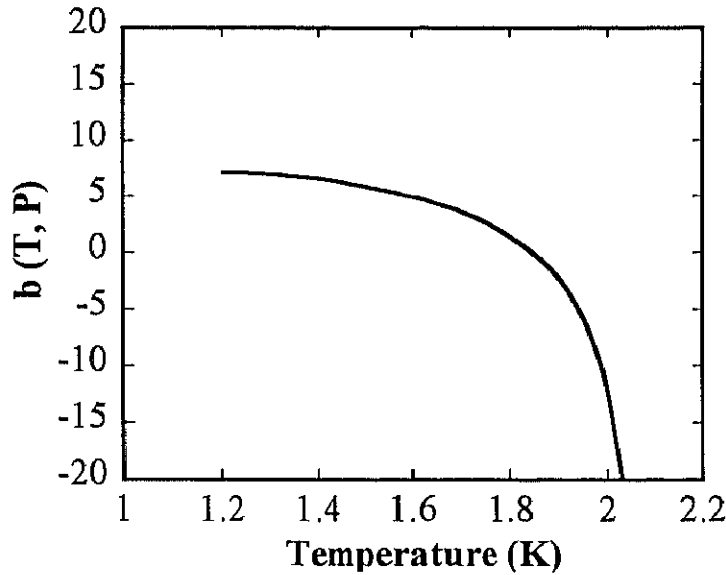


Fig. 2.6: The temperature dependence of steepening coefficient  $b$  at  $P = 1$  bar.

steepening coefficient with the temperature, calculated with the aid of the tabulated data of Maynard[43] at  $P = 1$  bar. As the steepening coefficient becomes positive below  $T = 1.884 K$ , a front-steepening shock is formed below  $1.884 K$ . In this case, points with higher in temperature within a thermal shock wave move faster than those with lower temperature. On the other hand, at temperatures above  $1.884 K$ ,  $b$  is negative, and a thermal shock wave steepens on the back side. The back shock is never seen in usual compressible fluids under regular condition. In addition to two shock waves, a double shock wave exists in He II. It has two discontinuities on both sides, front and back sides. If the temperature of undisturbed is taken a little lower than  $1.884 K$ , it can be expected that some local points stay lower than  $1.884 K$  although others might be higher than  $1.884 K$  and so the double shock is formed. In the case of a thermal pulse with large amplitude, a thermal shock wave is deformed and attenuated also by the effect of quantized vortices[44], [45], [46], [47]. Higher amplitude results in higher relative velocities between the two components. If the relative velocity  $v_{ns}$  ( $=v_s - v_n$ ) exceeds a critical value  $v_c$ , superfluid breakdown takes place. The deformation and attenuation are considered to be caused by the

interaction between a tangled mass of vortices and a shock wave, that is by the mutual friction. It gives rise to additional dissipation. The quantized vortex is almost the same as that in conventional fluids except the facts that the circulation is quantized and the radius of core is extremely small, being comparable with the inter atomic distance among helium atoms.