Chapter 3

NUMERICAL SOLUTION OF SHOCK JUMP CONDITION

Although the Khalatnikov approximations provide valuable insight into what types of shocks may be observed in *HeII*, they are limited to weak shocks (§ 2.1.3, p 20). Furthermore, it is not clear, without examining thermodynamic data for *HeII*, that neglecting the coefficient of thermal expansion is consistent with this second order approximation. Owing to the complexity of the jump condition for any more exact solutions, one must resort to numerical methods. The numerical solutions are limited only by the quality of available thermodynamic data. We have used the data of *HEPAK*, which is the thermophysical properties software (by CRYODATA INC.), and have approximated the thermodynamic dependence on counter-flow velocity to leading order in w.

3.1 Shock Jump Condition in Superfluid Helium

Considering the shock-fixed frame in one-dimensional steady flow, the equations of mass, momentum, energy conservation and the equation of the superfluid motion were rewritten in the following set of shock jump conditions (§ 2.49, p 18).

$$\rho_1 v_1 = \tilde{\rho} v \tag{3.1}$$

$$p_1 + \rho_1 {v_1}^2 = p + \tilde{\rho} v^2 + \frac{\rho_s \rho_n}{\tilde{\rho}} w^2 \tag{3.2}$$

$$\rho_1 s_1 T_1 v_1 = \tilde{\rho}\tilde{s}T\left(v - \frac{\rho_s}{\tilde{\rho}}w\right) - \rho_n w\left[v - \frac{\rho_s}{\tilde{\rho}}w\right]^2 \tag{3.3}$$

$$\mu_1 + \frac{1}{2}{v_1}^2 = \tilde{\mu} + \frac{1}{2}\left(v + \frac{\rho_n}{\tilde{\rho}}w\right)^2 \tag{3.4}$$

where the subscripted variables $(\ )_1$ denote the undisturbed, equilibrium conditions ahead of the shock, and the notation $\tilde{\rho}$, $\tilde{s}$, $\tilde{\mu}$ over a quantity indicates that it is a function of p, T, and w.

The above four equations are the superfluid shock jump conditions, and one can note they reduce to their counterflow parts for a classical fluid when w =0. It is helpful to rearrange the set in non-dimensional form. By using Eqs.(2.1) and (2.2) to replace v_n and v_s in favor of v and w, and then eliminating v by Eq.(3.1),we may write

$$f_1 = \frac{p - p_1}{\rho_1 a_1^2} + \left(\frac{\rho_1}{\tilde{\rho}} - 1\right)M_{SL}^2 + \frac{\rho_s \rho_n w^2}{\rho_1 \tilde{\rho} a_1^2} = 0 \tag{3.5}$$

$$f_2 = \frac{\tilde{\mu} - \mu_1}{a_1^2} + \frac{1}{2a_1^2}\left(\frac{\rho_1}{\tilde{\rho}}a_1 M_{SL} - \frac{\rho_n}{\tilde{\rho}}w\right) - \frac{1}{2}M_{SL}^2 = 0 \tag{3.6}$$

$$f_3 = \left(\frac{\tilde{s}T}{s_1 T_1} - 1\right)M_{SL} + \frac{\rho_s \tilde{s} T w}{\rho_1 s_1 T_1 a_1} + \frac{\rho_n w}{\rho_1 s_1 T_1 a_1}\left(\frac{\rho_1}{\tilde{\rho}}a_1 M_{SL} + \frac{\rho_s}{\tilde{\rho}}w\right)^2 = 0 \tag{3.7}$$

The wave speed $U_I = v_1$ has been expressed as the product of a shock Mach number M_{SL} and sound speed of initial state a_1. It is generic because shock waves in $HeII$ are two type which correspond to the linear wave modes of first and second sound in the limit of vanishingly small wave strengths. Thus the Mach number for a compression (or first sound) shock which primary jumps for the pressure, the density and total flow velocity v is based on first sound speed a_I, while the Mach number for a thermal (or second sound) shock which primary jumps the temperature, entropy and counterflow velocity w is based on the second sound speed a_{II}. For a given initial thermodynamic state (specified p_1 and T_1) and given wave speed $U_I = M_{SL}$ a_1, the three jump conditions, Eqs.(3.5) $\sim$ (3.7), supplemented by state equations for $\tilde{\mu}$, $\tilde{s}$, $\tilde{\rho}$ and $\rho_n/\tilde{\rho}$, specify the state behind shocks p, T, and w.

3.2 Khalatnikov Approximation

The validity of experimental data requires comparison to solutions of the jump conditions which follow from those equations. The solutions are compared with the experimental data. Furthermore, it is compared with an approximations made by Khalatnikov to the exact solutions, too. These solutions are approximate in the sense that weak shock waves are considered, in which case the jumps in all quantities across the shocks are small. Jumps in pressure $\Delta p = p - p_1$, temperature $\Delta T = T - T_1$ and counterflow velocity w are chosen as the independent variables and each one is considered $o(\epsilon)$. Then all quantities which appear in the jump conditions are expanded as Taylor's series in these variables, and only terms through $o(\epsilon^2)$ are retained (§2.1.3, p 20). As a further approximation, the coefficient of thermal expansion α is neglected. The results for compression shocks are

$$\Delta p = 2(M_{SL} - 1)\left(\frac{\partial}{\partial p}\ln(\rho a_I)\right)_1^{-1} \tag{3.8}$$

$$\Delta T = 0 \tag{3.9}$$

$$w = 0 \tag{3.10}$$

while for thermal shocks Khalatnikov finds

$$\Delta T = 2(M_{SL} - 1)\left[\frac{\partial}{\partial T}\ln\left(\frac{a_{II}^3 c_p}{T}\right)\right]_1^{-1} \tag{3.11}$$

$$w = -\left(\frac{\rho s}{\rho_n a_{II}}\right)_1 \Delta T \tag{3.12}$$

$$\Delta p = -\left[\frac{\rho_s \rho_n}{\rho} - \frac{\rho^2 a_{II}^2}{2}\frac{\partial}{\partial p}\left(\frac{\rho_n}{\rho}\right)\right]_1 w^2 \tag{3.13}$$

Here, as in Eqs.(3.5) through (3.7), the coordinate system is shock-fixed, and the subscript 1 refers to the unshocked fluid. The denominator on the right-hand side of Eq.(3.8) is always positive which means that only compression shocks are to bc expected in $HeII$. High-density regions in a propagating

pressure disturbance will therefore travel faster than those of low density. However, the denominator on the right-hand side of Eq.(3.11) can be either positive or negative depending on the location on the p-T diagram. On the saturated vapor pressure curve it is positive for $T < 0.5K$ and for $0.95\ K < T < 1.88K$ which means that temperature raising shocks are formed within these regions. Outside of these regions temperature lowering shocks occur (§ 2.1.3, p 21).

3.3 Solution Method

Newton's method in three variables may be used to numerically solve the nonlinear jump conditions *R-H-HeII* as follows. For a fixed initial pressure p_1 and temperature T_1 ($w = 0$ always), we seek solution vectors

$$\mathbf{x} = (p, T, w) \tag{3.14}$$

for the system

$$\mathbf{f}(\mathbf{x}, M_{SL}) = 0 \tag{3.15}$$

where

$$\mathbf{f} = (f_1, f_2, f_3) \tag{3.16}$$

is given by Eqs.(3.5) through (3.7). The shock Mach number M_{SL} is taken as the independent parameter (In this section, the shock Mach number M_{SL} is replaced with M.) and is referenced to either the first or second sound speed for compression or thermal shocks, respectively. For M slightly greater than unity, the Khalatnikov approximations would be expected to give a good first guess x_0. A better solution is then found by Newton's method,

$$\mathbf{x}_1 = \mathbf{x}_0 - J_0^{-1}\mathbf{f}_0 \tag{3.17}$$

J_0^{-1} represents the inverse of the Jacobian at the zeroth value.

$$J_0 = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_0 \tag{3.18}$$

Iterations continue,

$$\mathbf{x}_{\nu+1} = \mathbf{x}_\nu - J_\nu^{-1}\mathbf{f}_\nu \tag{3.19}$$

until each f_i approaches zero to within a specified to tolerance. At higher mach numbers, the Khalatnikov approximation may be such a poor guess that at best a large number of iterations are required for convergence or at worst no convergence at all is obtained. In such cases, the previous solution $\mathbf{x}^\alpha$ at Mach number M^α may be extrapolated in M and used as the initial guess $x_0^{\alpha+1}$ for thc current Mach number $M^{\alpha+1}$. This is effected by letting

$$\mathbf{x}_\mathbf{0}^{\alpha+1} = \mathbf{x}^\alpha + \left(\frac{\partial \mathbf{x}}{\partial M}\right)^\alpha \left(M^{\alpha+1} - M^\alpha\right) \tag{3.20}$$

where

$$\left(\frac{\partial \mathbf{x}}{\partial M}\right)^\alpha = \left[\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^\alpha\right]^{-1}\left(\frac{\partial \mathbf{f}}{\partial M}\right)^\alpha = [J^\alpha]^{-1}\left(\frac{\partial \mathbf{f}}{\partial M}\right)^\alpha \tag{3.21}$$

If the solution vector $\mathbf{x}$ changes rapidly with M, the Mach number increment on thc right side of Eq. (3.20) may be made very small to improve the quality of $\mathbf{x}_\mathbf{0}^{\alpha+1}$.

Although it has long been known that linear "mode coupling" between first and second sound is very weak, the finite amplitude results also show surprisingly weak nonlinear "mode coupling". The two solutions remain very distinct to high Mach numbers.

3.4 Process of Interpolation for Thermodynamic Quantities

In the case of an ordinary fluid, two thermodynamic quantities are calculated from Euler or Navier-Stokes equation system, and the others are given with the aid of the equation of state. On the other hand, in the case of *HeII*, reliable equation of state doesn't exist. Therefore, such thermophysical quantities as the total density ρ, the total entropy s, the density ratio ρ_n/ρ and the chemical potential μ are calculate with the aid of a thermophysical properties software called *HEPAK*. Since there quantities are

directly shown as a function of temperature and pressure, the data is used for numerical solution after which was linearly interpolated.

The interpolation procedure for the other thermodynamic quantities from the data in a table is as follows. When $T(T_1 \leq T < T_2)$ and $p(p_1 \leq p < p_2)$ are known, another thermodynamic quantity $f(T,p)$ is given in terms of four data $f(T_1,p_1)$, $f(T_1,p_2)$, $f(T_2,p_1)$ and $f(T_2,p_2)$, denoted by f_{11}, f_{12}, f_{21}, f_{22}, respectively, for brevity as shown in Figure ??. The Taylor's series expansion of f with respect to p is written to the first order of accuracy as

$$f(T,p+\Delta p) = f(T,p) + \left(\frac{\partial f}{\partial p}\right)_T \Delta p + o\left(\Delta p^2\right) \tag{3.22}$$

Thus, f_1 and f_2 representing $f(T_1,p)$ and $f(T_2,p)$ are written in the forms

$$f_1 = f_{11} + \frac{f_{12}-f_{11}}{p_2-p_1}(p-p_1) \tag{3.23}$$

$$f_2 = f_{21} + \frac{f_{22}-f_{21}}{p_2-p_1}(p-p_1) \tag{3.24}$$

Similarly, the Taylor's series of f with respect to T is written as

$$f(T+\Delta T,p) = f(T,p) + \left(\frac{\partial f}{\partial T}\right)_p \Delta T + o\left(\Delta T^2\right) \tag{3.25}$$

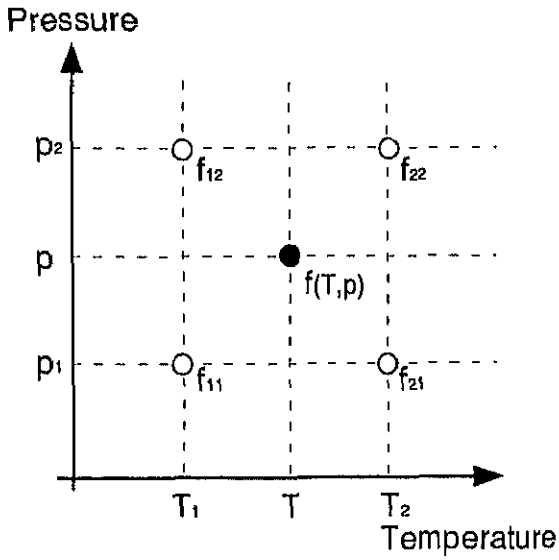

Figure 3.1: Schematic illustration of interpolation for thermodynamic quantity.

Therefore the quantity $f(T,p)$ is calculated from the expansion in terms of f_1, f_2, T_1 and T_2 as

$$f(T,p) = f_1 + \frac{f_2 - f_1}{T_2 - T_1}(T - T_1) \tag{3.26}$$

Thermodynamic quantities ρ and s are numerically calculated at each numerical time step in Newton's method. To obtain new quantities at the next time step by Newton's method, μ, p, ρ_n and ρ_s must be calculated in advance by the use of thermophysical table. The temperature T and the pressure p are calculated in terms of ρ and s in the following manner. First, the assumed temperature T^{n-1} equal to one on the previous time step yields an approximately value of p^n from the inverse process.

$$p^n = f^{-1}\left(T^{n-1}, \rho\right) \tag{3.27}$$

Similarly, new approximate temperature T^n is obtained from the pressure p^n derived from Eq.(3.27) and the finite difference result of entropy s at the present time step.

$$T^n = f^{-1}\left(p^n, s\right) \tag{3.28}$$

This series of procedure is repeated until the temperature converges as follows:

$$\frac{\left|T^n - T^{n-1}\right|}{T^{n-1}} < \delta \tag{3.29}$$

where δ is a quantity as small as 10^{-6}. The temperature and the pressure are now obtained. Consequently, s, ρ, ρ_n and μ are readily computed by using the interpolation.