Chapter 2

THEORETICAL FORMULATION

2.1 Superfluid Hydrodynamics

2.1.1 Landau two-fluid equation

Helium exists in two stable isotopic states, ^3He and ^4He. We primarily treat ^4He. Helium is the only element that can exist as a liquid at absolute zero (0 K).

The structural symmetry, both nuclear and electronic, of a ^4He atom results in extremely weak interaction of the helium attractive forces. Consequently, the condensation of helium into liquid or solid state is possible only at very low temperatures where the disruptive thermal energy is significantly reduced. For temperatures approaching absolute zero, the thermal energy is vanishingly small and the total energy of a system of helium atoms is the sum of the potential energy and the quantum mechanical ground state, or zero-point, energy which is no longer negligible. As a result of the contribution of the zero-point energy, the liquid is the stable phase of ^4He for temperatures approaching absolute zero (at pressure not exceeding 0.25 MPa). The low temperatures associated with this liquid permit the manifestation of certain quantum effects which have a profound influence on the hydrodynamics.

The phase diagram for ^4He is shown in Figure 2.1. The critical point is at $T_{cr} = 5.20\ K$ and $P_{cr} = 0.2274\ MPa$, and normal boiling point (for $P = 0.1013\ MPa$.) is at 4.21 K. There is no triple point at which gas, liquid, and solid are in equilibrium. The solid can be produced even at the lowest temperatures only by application of pressures in excess of 0.25 MPa. At lower pressures, the liquid state is preferred. As indicated by the phase dia-

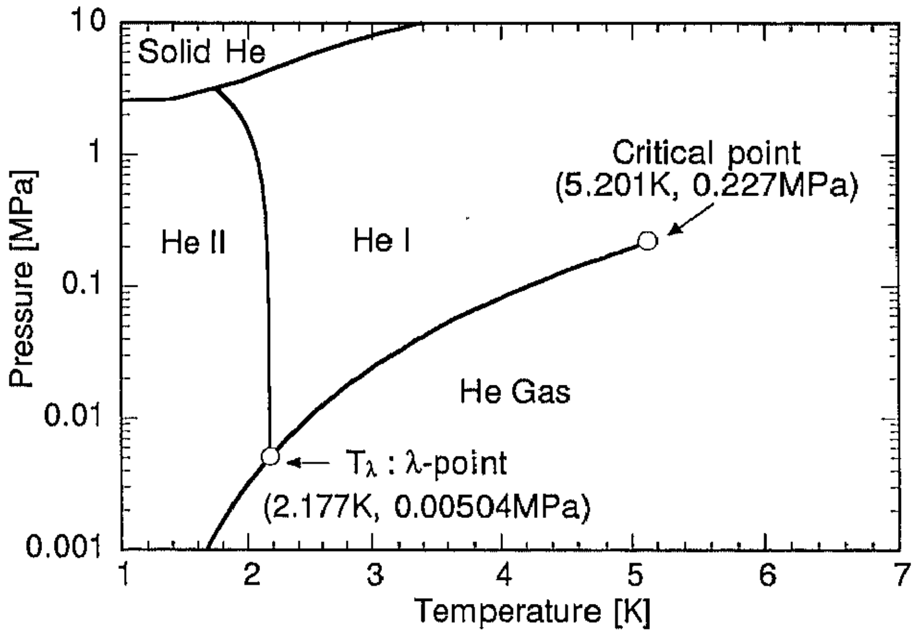

Figure 2.1: Phase diagram for ^{4}helium

gram, there are two liquid phases – referred to as liquid helium I (HeI) and liquid helium II, which is called a superfluid helium ($HeII$). The transition from HeI to $HeII$ is called the "λ-transition" and involves no latent heat and no discontinuous change in the density. HeI is a classical fluid which exhibits some properties which are liquid-like and others such as the viscosity and the thermal conductivity, which are more gas-like. On the other hand, $HeII$ exhibits behavior which in fact is dominated by macroscopic quantum effects. Table 1 shows the feature of liquid helium.

Table 1:Feature of liquid helium (^4He)

HeI	• Ordinary Fluid
$HeII$ *(Superfluid)*	• Superfluidity • Super Heat Transfer • Second Sound Wave exist

To account for the unique behavior of $HeII$, a theory known as the two-fluid model has been developed. This theory postulates that $HeII$ may be viewed as a mixture of two interpenetrating liquids — one being referred to as "superfluid" and the other as "normal fluid" seen in Table 2. The ^4He atom has no net spin, either nuclear or electronic; therefore, Bose-Einstein statistics governs an assembly of ^4He atoms, permitting unlimited occupation of any energy level. The superfluid, then corresponds to that fraction of the liquid which has actually dropped into the ground state, having undergone a Bose-Einstein "condensation" in momentum space. The normal fluid corresponds to particles in the excited energy levels. The superfluid has no entropy and its motion is inviscid; the normal fluid has non-vanishing entropy and exhibits viscous effects.

Table 2:Two-fluid model

	Superfluid component	Normal fluid component
Density	ρ_s	ρ_n
Viscosity	0	η_n
Entropy	0	s
Bose-Einstein statics	Ground state	Excited state

The hydrodynamic behavior of $HeII$ on a macroscopic scale is described by the two-fluid equations. In this model, the total density of the liquid, ρ, is taken to be the sum of the normal fluid density, ρ_n, and the superfluid density, ρ_s:

$$\rho = \rho_s + \rho_n \tag{2.1}$$

For temperatures approaching absolute zero, $\rho_s/\rho \to 1$; conversely, $\rho_s/\rho \to 0$ at the λ-point where the liquid reverts completely to normal fluid seen in Figure 2.2. The superfluid and normal fluid have separate velocity fields, $\mathbf{v_s}$ and $\mathbf{v_n}$ respectively, and their relative velocity is denoted by

$$\mathbf{w} = \mathbf{v_n} - \mathbf{v_s} \tag{2.2}$$

The two fluids can move relative to one another without exchange of the energy or momentum as long as the magnitude of the relative velocity does not exceed a critical value, $\mathbf{w_c}$. The net mass flux (or momentum density) of the liquid is given by

$$\mathbf{J} = \rho\mathbf{v} = \rho_s\mathbf{v_s} + \rho_n\mathbf{v_n}, \tag{2.3}$$

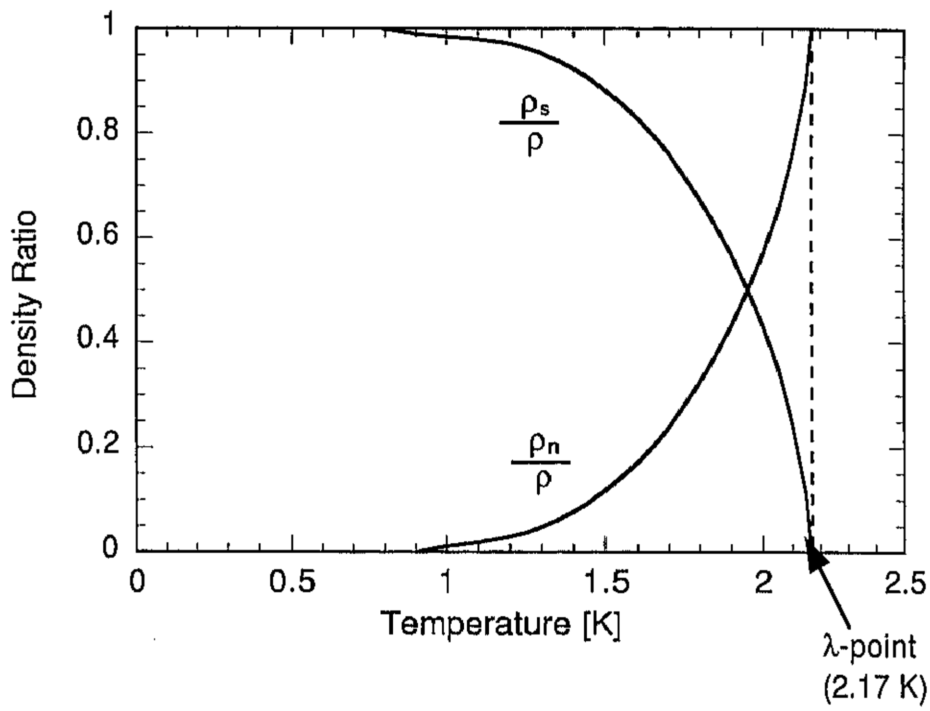

Figure 2.2: Temperature dependence of the normal fluid density ratio of total density ρ_n/ρ, the superfluid density ratio of total density ρ_s/ρ at the saturated vapor pressure.

where $\mathbf{v}$ is the effective bulk fluid, or center-of-mass, velocity.

On the basis of the above definitions, and considering only non-dissipative processes, the equations of mass, momentum, and entropy conservation are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \tag{2.4}$$

$$\frac{\partial J_i}{\partial t} + \frac{\partial P_{ij}}{\partial x_j} = 0 \tag{2.5}$$

$$\frac{\partial \rho s}{\partial t} + \nabla \cdot (\rho s \mathbf{v_n}) = 0 \tag{2.6}$$

where $P_{ij} = p\delta_{ij} + \rho_s \mathbf{v_{sj}} + \rho_n \mathbf{v_{nj}}$ = momentum flux
p=pressure
=entropy/unit mass

This system of equations is closed by an equation of motion for the superfluid. Landau postulates that the superfluid flow is irrotational and that it is driven by gradients in the chemical potential [29]. The resulting equation of motion for the superfluid is

$$\frac{D\mathbf{v_s}}{Dt} = \frac{\partial \mathbf{v_s}}{\partial t} + (\mathbf{v_s} \cdot \nabla)\,\mathbf{v_s} = -\nabla \mu, \tag{2.7}$$

or equivalently,

$$\frac{\partial \mathbf{v_s}}{\partial t} + \nabla \left(\frac{v_s^2}{2} + \mu \right) = 0 \tag{2.8}$$

where μ is the chemical potential per unit mass. The chemical potential μ is considered as the driving force for the superfluid to obey the following formula using $\mathbf{w}$.

$$d\mu = -sdT + \left(\frac{1}{\rho} \right) dp - \frac{\rho_n}{\rho} \mathbf{w} \cdot d\mathbf{w} \tag{2.9}$$

where the first two terms are contributed by the fluid at rest, and the last term is due to the relative motion of the two fluids. Thus superfluid is accelerated not only by the pressure gradient and the temperature gradient but also by the gradient of the square of the relative velocity $\mathbf{w}$. The above equations are the Landau two-fluid equations. These eight equations in eight independent variables (two thermodynamic variables and six velocity components) yield a complete description of the hydrodynamics for non-dissipative processes when supplemented by equations of state for the remaining thermodynamic variables. The state equations are complicated by the fact that quantities such as ρ, s, and μ are functions of two independent thermodynamic variables and relative velocity, $\mathbf{w}$.

The law of energy conservation is a consequence of the two-fluid equations and is given by

$$\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{Q} = 0 \tag{2.10}$$

where

$$E = E_0 + \mathbf{v_s} \cdot (\mathbf{J} - \rho \mathbf{v_s}) + \frac{1}{2} \rho v_s^2 \tag{2.11}$$

$$E_0 = \text{Galilean invariant energy density}$$

$$Q = \left(\mu + \frac{1}{2}v_s^2\right)\mathbf{J} + \rho s T \mathbf{v_n} = \rho_n \mathbf{v_n}\left(\mathbf{v_n}\cdot(\mathbf{v_n}-\mathbf{v_s})\right) \qquad (2.12)$$
$$= \text{energy flux}$$

The boundary conditions for the above hydrodynamic equations require that the normal component of $\mathbf{J}$ and the tangential component of $\mathbf{v_n}$ must vanish at a solid surface. In addition, the normal component of the heat flux, $\mathbf{q} = \rho s T \mathbf{v_n}$ must be continuous across the interface.

2.1.2 Propagation of sound in superfluid helium

Weak disturbances propagate at two characteristic velocities in $HeII$. Pressure disturbances, known as "first" sound, travel at the classical isentropic sound speed, which is on the order of 220 m/s. In addition, weak temperature disturbances, referred to as "*second*" sound, travel at a separate characteristic velocity which is typically on the order of 0-20 m/s.

Landau has theoretically investigated the propagation of sound in $HeII$ by linearizing the hydrodynamic equations. The equations are simplified by (i) assuming that $\mathbf{v}$ and $\mathbf{w}$ are small for sound waves, and (ii) considering small perturbations of p, ρ, T, and s about their equilibrium values. By neglecting terms that are quadratic in small quantities, the system of equations (2.4), (2.5), (2.15), (2.8) for sound waves becomes

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\mathbf{J} = 0 \qquad (2.13)$$

$$\frac{\partial \mathbf{J}}{\partial t} + \nabla p = 0 \qquad (2.14)$$

$$\frac{\partial \rho s}{\partial t} + \rho s(\nabla\cdot\mathbf{v_n}) = 0 \qquad (2.15)$$

$$\frac{\partial \mathbf{v_s}}{\partial t} + \nabla\mu = 0 \qquad (2.16)$$

Differntiating Eq.(2.13) with respect to time, and eliminating $\partial\mathbf{J}/\partial t$ with the aid of Eq.(2.14) one obtains,

$$\frac{\partial^2 \rho}{\partial t^2} = \nabla^2 p \qquad (2.17)$$

A combination of Eqs.(2.16) and (2.14) supplemented with the thermodynamic identity $d\mu = -sdT + \frac{1}{\rho}dp$ yeilds

$$\rho_n \frac{\partial \mathbf{w}}{\partial t} + \rho s \nabla T = 0 \tag{2.18}$$

Taking the divergence of Eq.(2.18) and using the relation

$$\frac{\partial s}{\partial t} = \left(\frac{1}{\rho}\frac{\partial \rho s}{\partial t} - \frac{s}{\rho}\frac{\partial \rho}{\partial t} = -s\nabla \cdot \mathbf{v_n} + \frac{s}{\rho}\nabla \cdot \mathbf{J}) = \right) - \frac{s\rho_s}{\rho}\nabla \cdot \mathbf{w} \tag{2.19}$$

We obtain

$$\frac{\partial^2 s}{\partial t^2} = \frac{\rho_s}{\rho_n} s^2 \nabla^2 T \tag{2.20}$$

This pair of Eqs.(2.17), (2.20) governs the propagation of sound in $HeII$, and it follows that there will be two characteristic wave speeds. Expanding p, ρ, T, and s in terms of equilibrium $(\ )_1$ and perturbation $(\ )'$ quantities yields,

$$p = p_1 + p' \qquad T = T_1 + T' \tag{2.21}$$

$$\rho = \rho_1 + \rho' \qquad s = s_1 + s' \tag{2.22}$$

and, for small perturbations,

$$s' = \left(\frac{\partial s}{\partial p}\right)_T p' + \left(\frac{\partial s}{\partial T}\right)_p T' \tag{2.23}$$

$$\rho' = \left(\frac{\partial \rho}{\partial p}\right)_T p' + \left(\frac{\partial \rho}{\partial T}\right)_p T' \tag{2.24}$$

Using the above relations, the wave equations may be rewritten as

$$\left(\frac{\partial \rho}{\partial p}\right)_T \frac{\partial^2 p'}{\partial t^2} + \left(\frac{\partial \rho}{\partial T}\right)_p \frac{\partial^2 T'}{\partial t^2} = \nabla^2 p' \tag{2.25}$$

$$\left(\frac{\partial s}{\partial p}\right)_T \frac{\partial^2 p'}{\partial t^2} + \left(\frac{\partial s}{\partial T}\right)_p \frac{\partial^2 T'}{\partial t^2} = \frac{\rho_s}{\rho_n} s^2 \nabla^2 T' \tag{2.26}$$

Assuming a plane wave solution in which p' and T' are proportional to $e^{-i\omega\left(t-\frac{x}{u}\right)}$ where u is the wave speed of disturbance, we have

$$\left[\frac{1}{u^2}-\left(\frac{\partial\rho}{\partial p}\right)_T\right]p'-\left(\frac{\partial\rho}{\partial T}\right)_p T'=0 \tag{2.27}$$

$$\left(\frac{\partial s}{\partial p}\right)_T p'-\left[\frac{\rho_s s^2}{\rho_n u^2}-\left(\frac{\partial s}{\partial T}\right)_p\right]T'=0 \tag{2.28}$$

The above two equations are rewritten in a matrix form as

$$\begin{bmatrix}\left[\frac{1}{u^2}-\left(\frac{\partial\rho}{\partial p}\right)_T\right] & -\left(\frac{\partial\rho}{\partial T}\right)_p \\ \left(\frac{\partial s}{\partial p}\right)_T & -\left[\frac{\rho_s s^2}{\rho_n u^2}-\left(\frac{\partial s}{\partial T}\right)_p\right]\end{bmatrix}\begin{bmatrix}p' \\ T'\end{bmatrix}=O \tag{2.29}$$

The condition for compatibility of Eq.(2.29) demanding that the determinant of the coefficient matrix is equal to 0 results in

$$\left[\frac{1}{u^2}-\left(\frac{\partial\rho}{\partial p}\right)_T\right]\left[\frac{\rho_s s^2}{\rho_n u^2}-\left(\frac{\partial s}{\partial T}\right)_p\right]-\left(\frac{\partial\rho}{\partial T}\right)_p\left(\frac{\partial s}{\partial p}\right)_T=0 \tag{2.30}$$

The Eq.(2.30) can be reduced to an equation of u as

$$\frac{\partial(s,\rho)}{\partial(T,p)}u^4-\left[\frac{\rho_s s^2}{\rho_n}\left(\frac{\partial s}{\partial p}\right)_T-\left(\frac{\partial s}{\partial T}\right)_p\right]u^2+\frac{\rho_s s^2}{\rho_n}=0 \tag{2.31}$$

where $\partial(s,\rho)/\partial(T,p)$ is the *Jacobian* of the transformation of s and ρ with respect to T and p. From the thermodynamic relations, one has

$$\frac{\partial(s,\rho)}{\partial(T,p)}=\frac{c_v}{T}\left(\frac{\partial\rho}{\partial p}\right)_T \tag{2.32}$$

$$\left(\frac{\partial s}{\partial T}\right)_p\frac{T}{c_v}\left(\frac{\partial p}{\partial\rho}\right)_T=\left(\frac{\partial p}{\partial\rho}\right)_s \tag{2.33}$$

Finally, we obtain

$$u^4-u^2\left[\left(\frac{\partial p}{\partial\rho}\right)_s+\frac{\rho_s}{\rho_n}\frac{s^2T}{c_v}\right]+\frac{\rho_s}{\rho_n}\frac{s^2T}{c_v}\left(\frac{\partial p}{\partial\rho}\right)_T=0 \tag{2.34}$$

where c_v is the specific heat per unit mass at constant volume. The condition should be satisfied for the existence of nontrivial solutions for p' and T'.

In general, $\left(\frac{\partial p}{\partial \rho}\right)_s$ and $\left(\frac{\partial p}{\partial \rho}\right)_T$ are related by the thermodynamic identity

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \frac{c_p}{c_v}\left(\frac{\partial p}{\partial \rho}\right)_T = \gamma\left(\frac{\partial p}{\partial \rho}\right)_T \tag{2.35}$$

where c_p is the specific heat per unit mass at constant pressure, and $\gamma = c_p/c_v$. By defining the quantities a_I and a_{II} as

$$a_I = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} \tag{2.36}$$

$$a_{II} = \sqrt{\frac{\rho_s}{\rho_n}\frac{s^2 T}{c_v}}, \tag{2.37}$$

the condition (2.34) is rewritten in terms of a_I and a_{II}.

$$\left(\frac{u^2}{a_I^2} - 1\right)\left(\frac{u^2}{a_{II}^2} - 1\right) = 1 - \frac{c_v}{c_p} \tag{2.38}$$

For $HeII$ at low pressures and low temperatures, the specific heats at constant volume and constant pressure are nearly equal, $c_p \doteq c_v$. Under such circumstance the term (1 - c_v/c_p) is negligible, and the two characteristic wave speeds are given by

$$u_I = a_I = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} \tag{2.39}$$

$$u_{II} = a_{II} = \sqrt{\frac{\rho_s}{\rho_n}\frac{s^2 T}{c_v}} \tag{2.40}$$

Disturbances propagating with the speed a_I are referred to as "*first*" sound which are quite analogous to ordinary acoustic waves. Disturbances which travel at speed a_{II} are referred to as "*second*" sound, which is unique to $HeII$. The expression for a_{II} indicates that $a_{II} \rightarrow 0$ as the λ-point is approached since $\rho_s \rightarrow 0$. In Figure 2.3 and Figure 2.4, a_I and a_{II} are plotted, respectively, as functions of temperature under the saturated vapor pressure and the other four pressure condition. As pointed out by Putterman the difference between c_p and c_v is not always negligible for $HeII$, particularly under the conditions near the λ-transition and at high pressures. For example, at $T = 1.80\ K$ and $p = 0.20\ MPa$, $\gamma = 1.0935$; at $T = 1.85\ K$ and $p = 0.25\ MPa$, $\gamma = 1.397$ (Maynard 1976). In such cases the term $(1 - c_v/c_p)$ must be retained. Keeping terms to the first order in 2.38, the *first-* and *second-*sound wave speeds are given by

$$u_I^2 = a_I^2 + \frac{a_I^2 a_{II}^2}{a_I^2 - a_{II}^2}\left(1 - \frac{c_v}{c_p}\right) \tag{2.41}$$

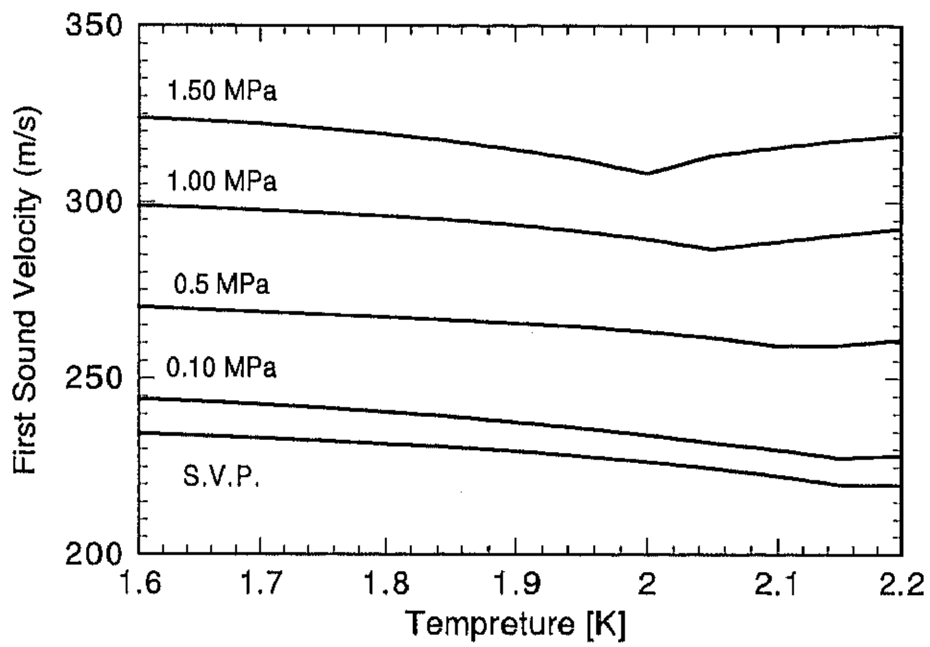

Figure 2.3: Dependence of first-sound velocity on temperature and pressure.

By recognizing the non-zero difference between c_p and c_v, additional insight into the physical nature of first and second sounds may be achieved. From the thermodynamic identity, one has

$$1 - \frac{c_v}{c_p} = \frac{c_v T}{c_p^2}\alpha^2 \left(\frac{\partial p}{\partial \rho}\right)_s \tag{2.42}$$

where

$$\alpha = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \quad = \text{Thermal expansion coefficient}$$

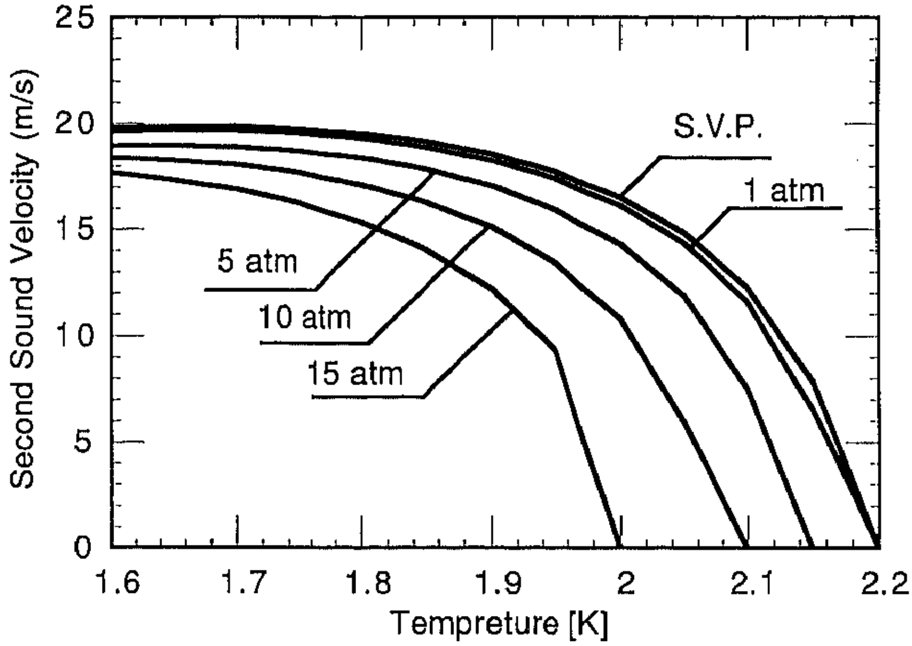

Figure 2.4: Dependence of second-sound velocity on temperature and pressure.

It is clear that keeping terms of (1 - c_v/c_p) is equivalent to keeping terms quadratic in αT. Working only to first order in αT, the following relations are found to be valid to a first-sound wave:

$$\mathbf{v_n} = \mathbf{v_s}\left[1 + \frac{\alpha T s}{c}\frac{\rho}{\rho_n}\frac{a_I^2}{(a_I^2 - a_{II}^2)}\right] \tag{2.43}$$

$$\Delta p = p' = v\,(\rho a_I)\left[1 + o\,(\alpha T)^2\right] \tag{2.44}$$

$$\Delta T = T' = v\left[\frac{\alpha T}{c}\frac{a_I^3}{a_I^2 - a_{II}^2}\right] \tag{2.45}$$

where $c \doteq c_p \doteq c_v$ since $c_p = c_v\left[1 + o\,(\alpha T)^2\right]$. The equivalent relations for a second-sound wave are:

$$v_n = -v_s\left(\frac{\rho_s}{\rho_n}\right)\left[1 - \frac{\alpha T s}{c}\frac{\rho}{\rho_n}\frac{a_I^2}{(a_I^2 - a_{II}^2)}\right] \tag{2.46}$$

$$\Delta p = p' = -w\,(\rho_n a_{II})\left[\frac{\alpha T s}{c}\frac{a_I^2}{(a_I^2 - a_{II}^2)}\right] \tag{2.47}$$

$$\Delta T = T' = w\left(\frac{\rho_s a_{II}}{\rho s}\right)\left[1 + \frac{\alpha T s}{c_h}\frac{\rho_s}{\rho_n}\frac{a_I^2}{(a_I^2 - a_{II}^2)}\right] \tag{2.48}$$

Since αT is small, those quantities which have factors of αT are small compared to those which do not. For a first-sound wave, $v_n \doteq v_s \doteq v$ and so the superfluid and normal fluid move together in phase. For a second-sound wave, $\mathbf{v_n} \doteq -(\rho_s/\rho_n)$ Thus, the superfluid and normal fluid move in anti-phase and to the lowest order the net mass flux is zero ($\mathbf{J} = \rho_s \mathbf{v_s} + \rho_n \mathbf{v_n} \doteq 0$). The first-sound wave is basically a pressure disturbance which produces only higher-order temperature changes. On the other hand, the second-sound wave is basically a temperature disturbance which produces only higher-order pressure changes.

The examination of thermodynamic data reveals that the thermal expansion coefficient, α, of $HeII$ is negative as seen in Figure 2.5. As a result, for a first-sound wave the preceding analysis predicts that the temperature decreases through a compression and increases through a rarefaction; in other words, the pressure and temperature fluctuations have "*opposite*" signs for first sound. In the case of a second-sound wave, the pressure and temperature fluctuations have the "*same*" sign.

2.1.3 Shock waves in superfluid helium (Khalatnikov theory)

The theoretical analysis of sound propagation in $HeII$ has been achieved by considering disturbances for which the induced velocities and perturbations in thermodynamic quantities are small enough to permit linearization of the hydrodynamic equations. For finite-amplitude disturbances, the velocities and changes in thermodynamic quantities are no longer negligible, and the full two-fluid equations must be used. The inherent nonlinearity of these equations results in a variation of wave velocity with amplitude and the subsequent development of shock waves for large amplitudes. Corresponding to the two possible types of weak wave propagation in $HeII$, first- and second-sound shock waves may be generated by sufficiently strong disturbances. First-sound shocks produce large changes in pressure and only higher-order temperature changes; second-sound shocks produce large

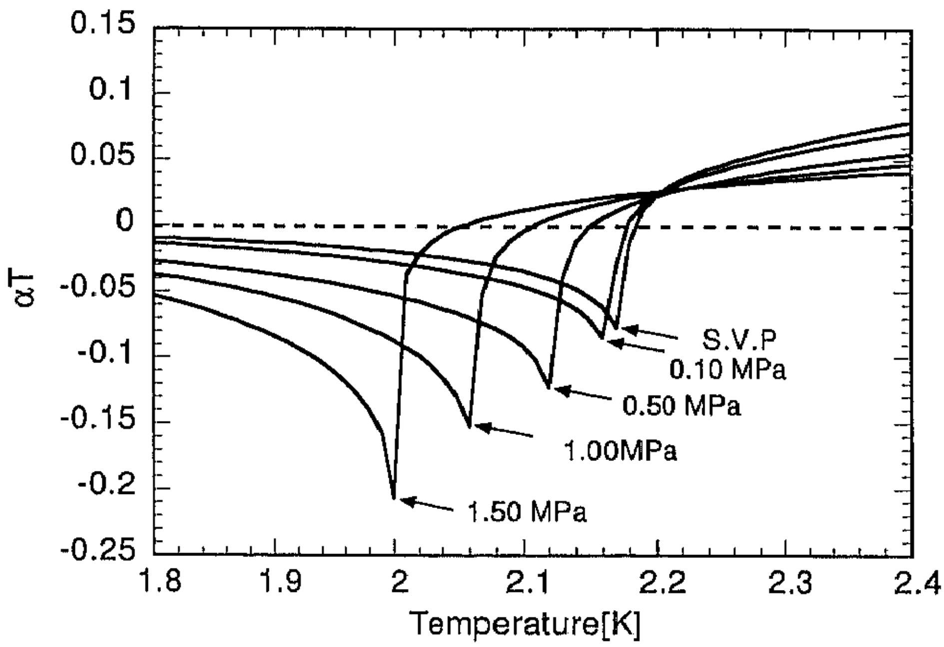

Figure 2.5: Thermal expansion coefficient α of liquid helium at saturated vapor pressure. α:Thermal expansion coefficient.

changes in temperature and only higher-order pressure changes. The propagation of a one-dimensional discontinuity, or shock wave, in $HeII$ has been examined theoretically by Khalatnikov. If the velocity, U_I, of the discontinuity in the laboratory reference frame is constant, then the analysis may be simplified by transforming to a frame of reference moving with the discontinuity (at velocity U_I in the lab frame). In the shock-fixed, or "steady", coordinates, the equations of mass, momentum, and energy conservation and the equation of motion for the superfluid result in the following set of jump conditions which relate the fluid properties on the two sides (1 and 2) of the shock:

$$[\rho v]_1^2 = 0 \tag{2.49}$$

$$\left[p + \rho_s v_s^2 + \rho_n v_n^2 w\right]_1^2 = 0 \tag{2.50}$$

$$\left[\rho s T v_n + \rho_n v_n^2 w\right]_1^2 = 0 \tag{2.51}$$

$$\left[\mu + \frac{1}{2} v_s^2\right]_1^2 = 0 \tag{2.52}$$

where $[f]_1^2$ denotes the difference f_2 - f_1.

Assuming that the fluid ahead of the shock wave is at rest ($\mathbf{v_n} = \mathbf{v_s} = 0$) in the laboratory reference frame, transformation of the jump conditions from the shock-fixed back to the lab-fixed reference frame yields

$$\rho_1 v_1 = \tilde{\rho}\left(U_I - v\right) \tag{2.53}$$

$$p_1 + \rho_1 {U_I}^2 = p + \tilde{\rho}\left(U_I - v\right)^2 + \frac{\rho_s \rho_n}{\tilde{\rho}} w^2 \tag{2.54}$$

$$\rho_1 s_1 T_1 v_1 = \tilde{\rho}\tilde{s}T\left[U_I - v - \frac{\rho_s}{\tilde{\rho}}w\right] - \rho_n w\left[U_I - v - \frac{\rho_s}{\tilde{\rho}}w\right]^2 \tag{2.55}$$

$$\mu_1 + \frac{1}{2}{U_I}^2 = \tilde{\mu} + \frac{1}{2}\left[U_I - v + \frac{\rho_n}{\tilde{\rho}}w\right]^2 \tag{2.56}$$

where the subscripted variables $(\ )_1$ denote the undisturbed, equilibrium conditions ahead of the shock, and the notation $\tilde{\rho}$, $\tilde{s}$, $\tilde{\mu}$ over a quantity indicates that it is a function of p, T, and w.

In principle, the above system of equations may be solved to find the shock velocity, U_I, and the magnitude of the jumps in the velocities and the thermodynamic variables across the shock. Such a solution is extremely difficult due to the complicated dependence of the thermodynamic variables on the relative velocity, w. However, by considering small discontinuities. Khalatnikov [53] has solved the system of jump conditions by expanding the variables $\tilde{\mu}$, $\tilde{s}$, and $\tilde{\rho}$ to the second order in w:

$$\tilde{\mu}\left(p, T, w\right) = \mu\left(p, T\right) - \frac{\rho_n}{\rho}\frac{w^2}{2} \tag{2.57}$$

$$\tilde{s}\left(p, T, w\right) = s\left(p, T\right) + \frac{w^2}{2}\frac{\partial}{\partial T}\left(\frac{\rho_n}{\rho}\right) \tag{2.58}$$

$$\tilde{\rho}\left(p, T, w\right) = \rho\left(p, T\right) + \frac{1}{2}\rho^2 w^2 \frac{\partial}{\partial p}\left(\frac{\rho_n}{\rho}\right) \tag{2.59}$$

Defining $\Delta p = p - p_1$ and $\Delta T = T - T_1$, the analysis continues by expanding ρ, ρ_n, ρ_s, μ, and s in Taylor's series in Δp and ΔT (up to quadratic terms). Since the coefficient of thermal expansion, α, is small, additional simplification is achieved by neglecting the dependence of the density ρ on the temperature (i.e., assume α=0).

For a first-sound shock wave of strength Δp, Khalatnikov's results show

$$U_I = a_I \left[1 + \frac{\Delta p}{2}\frac{\partial}{\partial p}\ln(\rho a_I)\right] \tag{2.60}$$

$$v = \frac{\Delta p}{\rho u_I} \tag{2.61}$$

$$\frac{c\Delta T}{a_I^2} = o\left[\frac{\Delta p}{\rho a_I^2}\right]^3 \tag{2.62}$$

$$\frac{w}{a_I} = o\left[\frac{\Delta p}{\rho a_I^2}\right]^3 \tag{2.63}$$

where U_I is the first-sound shock velocity and a_I is the first-sound wave speed ahead of the shock. Since the factor $\frac{\partial}{\partial p}\ln(\rho a_I)$ is on the order of $5 \times 10^{-2} atm.^{-1}$ for $HeII$, pressure jumps ΔP of only several atmospheres can lead to shock Mach numbers, U_I/a_I, appreciably greater than unity.

For a second-sound shock wave of strength ΔT, Khalatnikov's results are

$$u_2 = a_{II}\left[1 + \frac{\Delta T}{2}\frac{\partial}{\partial T}\ln\left(a_{II}^3\frac{\partial s}{\partial T}\right)\right] \tag{2.64}$$

$$w = \Delta T\left[\frac{\rho}{\rho_n}\frac{s}{a_{II}}\right] \tag{2.65}$$

$$\frac{\Delta p}{\rho a_{II}^2} = -\frac{w^2}{a_{II}^2}\left[\frac{\rho_s\rho_n}{\rho^2} - \frac{1}{2}\rho a_{II}^2\frac{\partial}{\partial p}\left(\frac{\rho_n}{\rho}\right)\right] \tag{2.66}$$

and, using the acoustic approximation $\Delta p = \rho a_I v$,

$$\frac{v a_I}{a_{II}^2} = o\left[\frac{w}{a_{II}}\right]^2 \tag{2.67}$$

where U_{II} is the second-sound shock velocity and a_{II} is the second-sound wave speed ahead of the shock. The expression for the second-sound shock velocity may be rewritten as

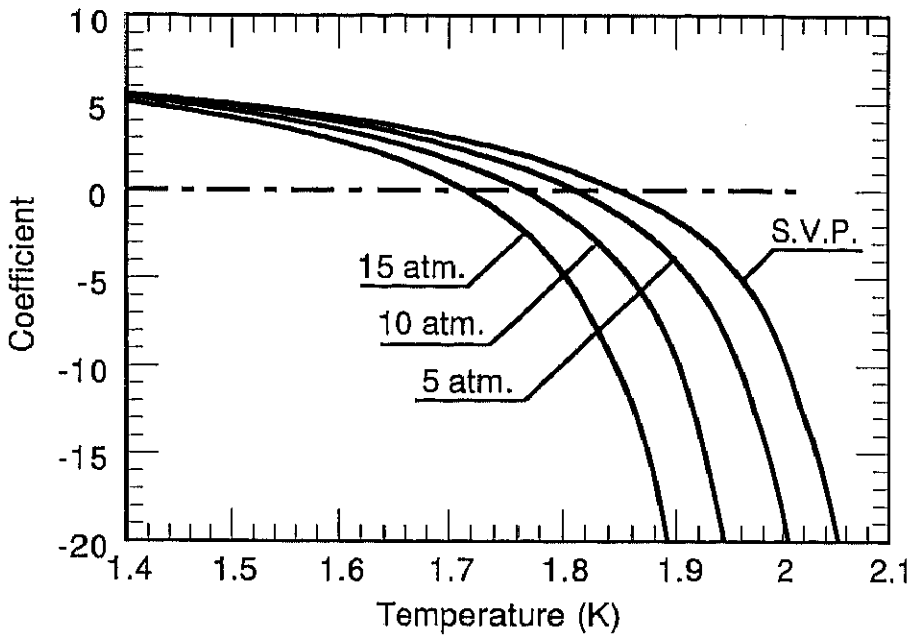

Figure 2.6: Variation of the coefficient $\frac{T}{2}\frac{\partial}{\partial T}\ln\left(a_{II}^3\frac{\partial s}{\partial T}\right)$ with temperature at saturated vapor and other four pressures.

$$\frac{U_{II}}{a_{II}} = 1 + \frac{\Delta T}{T}\left[\frac{T}{2}\frac{\partial}{\partial T}\ln\left(a_{II}^3\frac{\partial s}{\partial T}\right)\right] \tag{2.68}$$

Using the thermodynamic identity $\left(\frac{\partial s}{\partial T}\right)_p = {}^c\!/_T$ and the definition $a_{II}^2 = \frac{\rho_s}{\rho_n}\frac{s^2 T}{c}$, the coefficient of $\Delta T/\mathrm{T}$ in the above equation may be evaluated as a function of temperature at the saturated vapor pressure using data given by Maynard (1976). The result is shown in Figure 2.6, and it is immediately apparent that the coefficient $\frac{T}{2}\frac{\partial}{\partial T}\ln\left(a_{II}^3\frac{\partial s}{\partial T}\right)$ changes sign in the vicinity of 1.88 K. For regions where the coefficient is positive (e.g., $T < 0.50K$ and $0.95 < T < 1.88K$), those parts of a second-sound wave in which ΔT is positive will steepen. However, for regions where the coefficient is negative (e.g., $0.5 < T < 0.95K$ and $1.88 < T < 2.172K$) , those parts of a second-sound wave in which ΔT is negative will steepen, and portions with positive ΔT will disperse.

2.2 General Properties of Gasdynamic Shock Wave

A small pressure disturbance in a compressible fluid propagates as a sound wave in every direction, while a large amplitude pressure change that is generated as a compression wave in explosion propagates with a speed exceeding speed of sound, which is called a shock wave. A shock wave is a surface in a flow field across which the flow variables change discontinuously.

A shock wave is a relatively thin region of rapid state variation across which there is a flow of matter. Because the region of variation is quite thin, it can almost always be idealized as a surface of discontinuity in space. The surface may propagate through the fluid and is not necessarily stationary. In general, all fluid properties — the pressure p, velocity v, density ρ, etc. — are discontinuous across the surface. As a shock wave can be regarded as a surface with zero thickness in the absence of such dissipative effects as viscosity and heat conductivity, it is usually treated in the framework of an inviscid gasdynamics.

2.2.1 Shock conditions in a gas

To investigate the relation among the thermodynamic properties before and after a shock wave, one selects a control volume bounded by surfaces just before and after the wave as in Figure 2.7. It is assumed that the gas is an inviscid perfect gas with zero thermal conductivity. The following one-dimensional steady-flow relations are considered by letting the region 1 be upstream and 2 be downstream sides of the shock wave to compute every variation in physical property as well as the wave speed. It is preferable to transform the lab-fixed coordinate system as shown in Figure 2.7 (a) into the system in which the shock is fixed as shown in Figure 2.7 (b). For the control volume in the shock-fixed system, the conservation law of mass and momentum are applied.

$$\rho_1 (U_S - v_1) = \rho_2 (U_S - v_2) \tag{2.69}$$

$$p_1 + \rho_1 (U_S - v_1)^2 = p_2 + \rho_2 (U_S - v_2)^2 \tag{2.70}$$

Here v is the particle velocity, p is the pressure and U_S is the shock wave

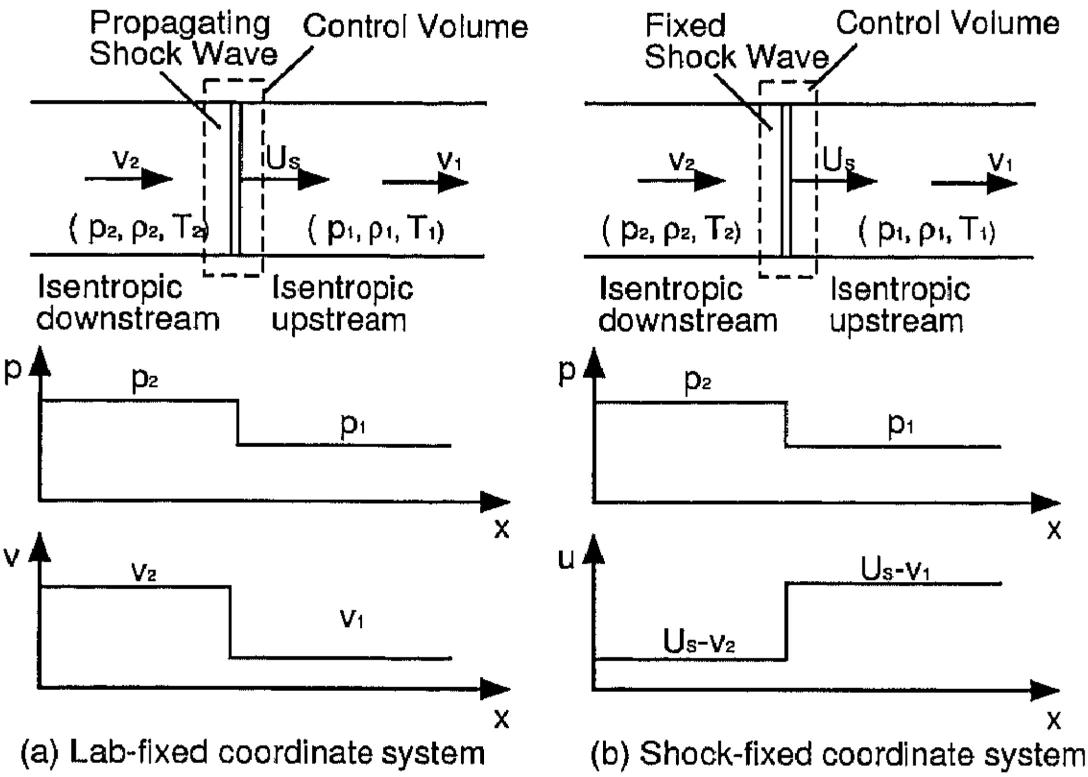

Figure 2.7: One-dimensional steady flow for shock wave.

speed in the gas. (The shock wave speed U_S is equal to U_I for the compression shock, which mean first sound shock, in § 2.1.3.)

The velocity and the temperature gradient are extremely large in inside of a shock wave and, as a result, the entropy of the gas passing through the shock wave increases owing to such dissipative effects as viscosity and heat conductivity. Nevertheless, one may consider an adiabatic flow because the heat exchange between the control volume and the outside of it can be ignored. Accordingly, it is reasonable to consider that the stagnation point enthalpy of a flow is kept constant across a shock wave. Consequently, the conservation of energy is represented as follows,

$$\frac{\gamma_1}{\gamma_1 - 1}\frac{p_1}{\rho_1} + \frac{1}{2}\left(U_S - v_1\right)^2 = \frac{\gamma_2}{\gamma_2 - 1}\frac{p_2}{\rho_2} + \frac{1}{2}\left(U_S - v_2\right)^2 \qquad (2.71)$$

where γ is the ratio of the specific heats of the gas. These three equations compose the basic equations. In the case of superfluid helium, the equation of momentum for superfluid component is necessary in addition to the above equations as described in § 2.52.

The thermodynamic states of before and after a shock wave are derived from equations, (2.69) $\sim$ (2.70) supplemented by the expressions, $a_{12} = \gamma \frac{p_1}{\rho}$, $M_S = \frac{(U_S - v_1)}{a_1}$, as follows,

$$\frac{p_2}{p_1} = \frac{2\gamma_1 M_s^2 - (\gamma_1 - 1)}{\gamma_1 + 1} \tag{2.72}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma_1 + 1) M_S^2}{(\gamma_1 - 1) M_S^2 + 2} \tag{2.73}$$

$$\frac{T_2}{T_1} = \frac{\left[2\gamma_1 M_S^2 - (\gamma_1 - 1)\right]\left[(\gamma_1 - 1) M_S^2 + 2\right]}{(\gamma_1 + 1)^2 M_S^2} = \left(\frac{a_2}{a_1}\right)^2 \tag{2.74}$$

where a is the speed of general sound and M_S is the shock Mach number defined by U_S/a_1. These equations are called the Rankine-Hugoniot relation. In this thesis, the Rankine-Hugoniot relation for gas is abbreviated to the *R-H-Gas* and that for superfluid helium to the *R-H-HeII*.

Let consider the entropy change passing through a shock wave. The entropy change is calculated from the second law of thermodynamics as follows,

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} \tag{2.75}$$

Substituting, eqs.(2.72) and (2.74) into this equation, one obtains

$$\frac{s_2 - s_1}{R} = \frac{\gamma_1}{\gamma_1 - 1} \ln \left[\frac{(\gamma_1 - 1) M_s^2 + 2}{(\gamma_1 + 1) M_s^2}\right] + \frac{1}{\gamma_1 - 1} \ln \left[\frac{2\gamma_1 M_s^2 - (\gamma_1 - 1)}{\gamma_1 + 1}\right] \tag{2.76}$$

where s is the specific entropy, R is the gas constant and c_p is the specific heat at constant pressure. For a weak shock wave in which the discontinuity in every quantity is small, the entropy change is further simplified as follows,

$$s_2 - s_1 = \frac{1}{12T_1} \left(\frac{\partial^2 v}{\partial p_1^2}\right) (p_2 - p_1)^3 \tag{2.77}$$

The entropy change in a weak shock wave is of the third order of smallness relative to the discontinuity of pressure. Accordingly, when the discontinuity of pressure the $(p_2 - p_1)$ is small, there is not a large error even if an isentropic process is assumed for a shock wave. That is the entropy change by a shock wave can be considered as isentropic.

2.3 Simple Theory of Shock Tube Operation

A shock tube is a versatile tool for the studies of a shock wave and a high-speed gas flow, etc. A shock tube was first designed by Vieille in 1899, in which a long tube was partitioned into a high-pressure gas section and a low-pressure driven gas section by cellophane diaphragm. He observed that the shock wave propagated into the low-pressure section when it was broken. This very simple device is called a shock tube. Some improved types of shock tubes are used for studies of aerodynamics, physics and chemistry. In this section, the operation principle and the theory of a shock tube in the simplest standard form is described.

2.3.1 Operation principle and the feature

In a simple model of a shock tube, a fixed long tube is divided into a high-pressure chamber and a low-pressure tube with a diaphragm as shown in Figure 2.8 (a). The extremely simple device is filled with gas of low-pressure in the low-pressure tube and with high pressure gas in the high-pressure chamber. The high-pressure gas expands suddenly into the low-pressure section when the diaphragm is broken by large pressure difference after piercing it with a pin, and the low-pressure gas is compressed. By the piston effect of expanding high pressure-gas, a shock wave is formed in the low-pressure tube, propagating rightward into the tube. A high-speed gas flow develops behind the shock wave propagating.

Shock tube has the following characteristic in the device which is been most suitable for to study various phenomena with propagating shock wave.

(1) A device is simple, and production is easy, and is economical compared with other wind tunnel.

(2) The initial pressure ratio between the high-pressure chamber and the low-pressure tube let to be only decided, from subsonic speed to super sonic speed flow, Mach number can be changed in one range voluntarily.

(3) It is suitable for a study of transient flow and non- steady flow.

(4) Because a change of pressure and temperature induced by shock wave are in a moment, it is employed by proofreading of various detector element of high response of the small damping time constant.

(5) As for heating by shock wave, it is done a stepwise and it can be get an extremely high temperature homogeneous gas. Accordingly, shock tube is suited for a flow of high speed and high enthalpy and a study of a chemical reaction in a high temperature.

On the other hand, the continuance time of a flow to be provided with shock tube is extremely short and is generally from the several $msec$ to 10 several $msec$. On this account, advanced technology is demanded from the measurement of power and moments to act on a test model. However, one of a good point is been because, now, shock tube become widely used by the progress of the measurement technique in high-speed phenomenon, and it does consideration such as an insulation or cooling of a device with uselessness that the continuance time is short.

2.3.2 Simple theory of shock tube

The flow in a shock tube is treaded by assuming the followings:

(1) The flow is one-dimensional and isentropic (adiabatic flow).

(2) The gas is an inviscid perfect gas with zero thermal conductivity.

(3) As soon as the diaphragm is broken, a shock wave is formed and propagates with a constant strength specified by initial condition.

A flow in shock tube on the basis of an above-mentioned assumption is shown in Figure 2.8. Figure 2.8 (a) shows an initial state in the high-pressure chamber and low-pressure tube, each state of which is denoted by the regions 4 and 1, respectively. And Figure 2.8 (b) shows a wave motion diagram. Upon breaking a diaphragm, a shock wave propagates into the rightward of the low-pressure tube with a constant velocity U_S, and an expansion fan (rarefaction wave) propagates into the leftward of the high-pressure chamber. Finally after repeated reflections of the shock wave, the whole system will reach a uniform pressure state. A speed of wave front and tail for an expansion fan are shown in -a_4 and v_3 - a_3, respectively. This figure shows a case of v_3 ¿ a_3, that is a wave tail propagates into the rightward. Since we assume that a diaphragm is broken in a moment, an expansion fan formed is a centered rarefaction wave.

The gas in the low-pressure tube designated by the region 2 is heated up and accelerated by the shock wave compression and called as hot gas. The gas remained in the high-pressure chamber designated by the region 3 is cooled down in the action of an expansion fan and called a cold gas. The dashed line in Figure 2.8 (b) is a contact surface between regions 2 and 3, or between gases originally in the high-pressure chamber and low-pressure tube. The temperature and density differ on both sides of the contact surface, but the fluid velocity and pressure must be equal on the both sides. Accordingly, the relations of $p_2 = p_3$ and $v_2 = v_3$ hold. Figure 2.8 (c) $\sim$ (f) show spatial distributions of the fluid velocity, the pressure, the temperature, and the density at the time $t = t_1$ denoted in Figure 2.8 (b). In a centered rarefaction wave, the distribution of fluid velocity in the x-direction is indicated as a straight line at arbitrary time. Accordingly, the variation from the v_2 to v_3 is a straight line.

The Rankine-Hugoniot relations (*R-H-Gas*) and other relations among physical quantities before and after a shock wave yield the following relations.

$$\frac{p_2}{p_1} = \frac{2\gamma_1 M_s^2 - (\gamma_1 - 1)}{\gamma_1 + 1} \tag{2.78}$$

$$\frac{T_2}{T_1} = \frac{\left[2\gamma_1 M_S^2 - (\gamma_1 - 1)\right]\left[(\gamma_1 - 1) M_S^2 + 2\right]}{(\gamma_1 + 1)^2 M_S^2} = \left(\frac{a_2}{a_1}\right)^2 \tag{2.79}$$

$$\frac{v_2}{a_1} = \frac{2}{\gamma_1 + 1}\left(M_S - \frac{1}{M_S}\right) \tag{2.80}$$

Here γ_1 is the ratio of the specific heat of driven gas, M_S is the shock Mach number in a gas, v is the fluid velocity and a is the sound velocity.

Since the Riemann invariant, $P = v + \frac{2a}{\gamma_1+1}$, must be conserved, the following relation holds across the expansion fan in the high-pressure chamber.

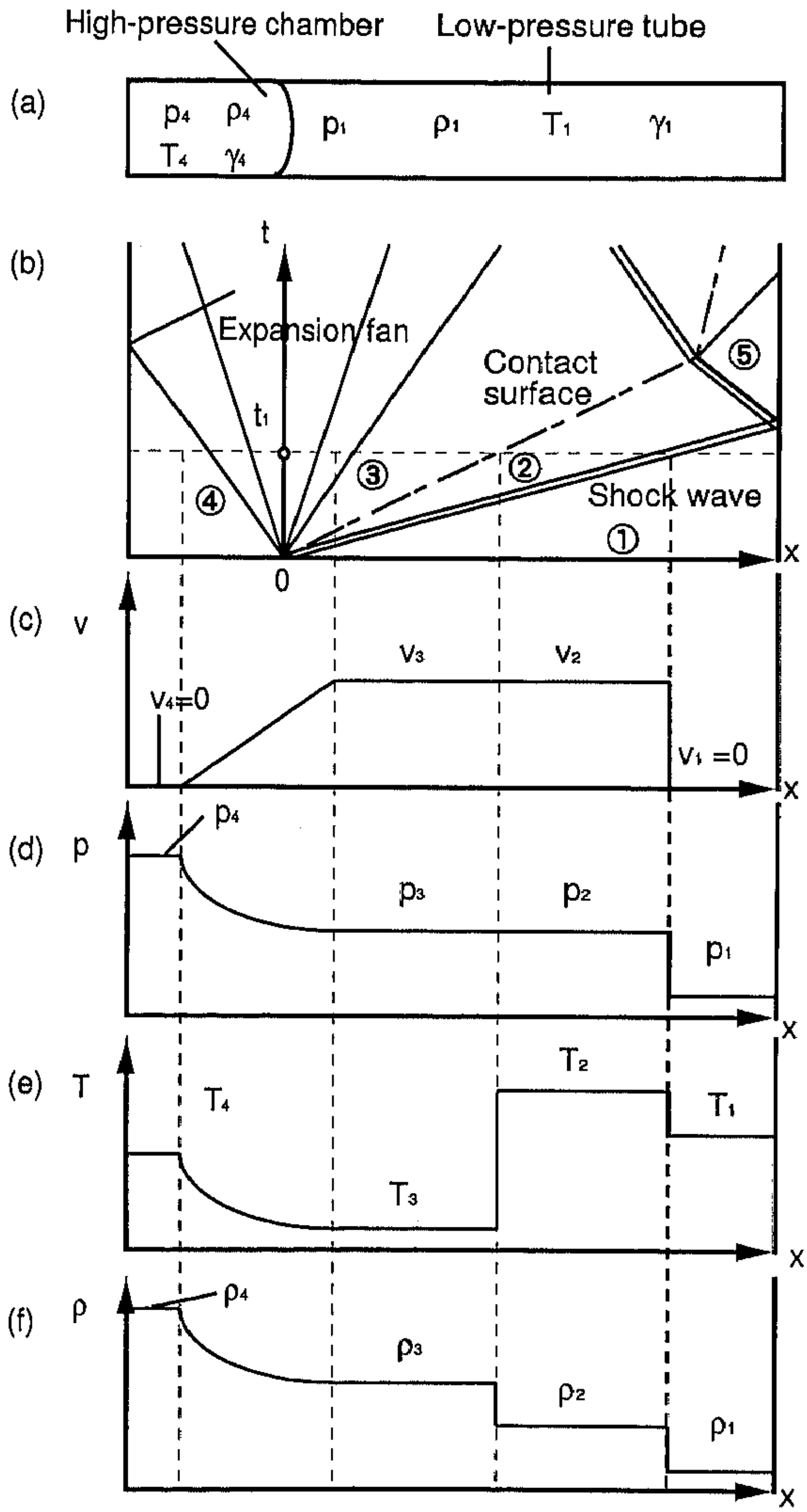

Figure 2.8: One-dimensional steady flow in shock tube.

$$\frac{2a_4}{\gamma_4 - 1} = v_3 + \frac{2a_3}{\gamma_4 - 1} \tag{2.81}$$

As an isentropic relation is held between the regions 3 and 4, one obtains

$$\frac{p_4}{p_3} = \left(\frac{a_4}{a_3}\right)^{\frac{2\gamma_4}{\gamma_4-1}} = \left(1 - \frac{\gamma_4 - 1}{2}\frac{v_3}{a_4}\right)^{-\frac{2\gamma_4}{\gamma_4-1}} \tag{2.82}$$

Moreover, since the relations $p_2 = p_3$, $v_2 = v_3$ hold on both sides of the contact surface, one obtains

$$\frac{p_4}{p_2} = \left(\frac{a_4}{a_3}\right)^{\frac{2\gamma_4}{\gamma_4-1}} = \left[1 - \frac{\gamma_4 - 1}{\gamma_1 + 1}\frac{a_1}{a_4}\left(M_S - \frac{1}{M_S}\right)\right]^{-\frac{2\gamma_4}{\gamma_4-1}} \tag{2.83}$$

Multiplication of Eqs. (2.78) and (2.83) yields

$$\frac{p_4}{p_1} = \frac{2\gamma_1 M_S^2 - (\gamma_1 - 1)}{\gamma_1 + 1}\left[1 - \frac{\gamma_4 - 1}{\gamma_1 + 1}\frac{a_1}{a_4}\left(M_S - \frac{1}{M_S}\right)\right]^{-\frac{2\gamma_4}{\gamma_4-1}} \tag{2.84}$$

Accordingly, the ratio γ_4/γ_1, the ratio of speeds of sound a_1/a_4 and the pressure ratio p_1/p_4 in the initial state is found to give a shock Mach number M_s.

Next let consider a hot and a cold gases flow following after a shock wave. The Mach number M_2 in the region 2 is written as

$$M_2 = \frac{v_2}{a_2} = \frac{v_2}{a_1}\frac{a_1}{a_2} = \left(M_S^2 - 1\right)\left[\left(\gamma_1 M_S^2 - \frac{\gamma_1 - 1}{2}\right)\left(\frac{\gamma_1 - 1}{2}M_S^2 + 1\right)\right]^{-\frac{1}{2}} \tag{2.85}$$

And the Mach number M_3 in the region 3 is written as

$$M_3 = \frac{v_3}{a_3} = \frac{v_2}{a_1}\frac{a_1}{a_4}\frac{a_4}{a_3} = \frac{2}{\gamma_1 + 1}\frac{a_1}{a_4}\left(M_S - \frac{1}{M_S}\right)\left[1 - \frac{\gamma_4 - 1}{\gamma_1 + 1}\frac{a_1}{a_4}\left(M_S - \frac{1}{M_S}\right)\right]^{-1} \tag{2.86}$$

Generally a hot gas flow is used for an experiment in shock tube.

Furthermore let consider a shock wave reflected from the bottom of the shock tube. The reflected shock Mach number M_R are described that,

$$M_R = \frac{U_R + v_2}{a_2} \tag{2.87}$$

Applying this Mach number to the Eq.(2.78), the pressure ratio p_5/p_2 is obtained.

$$\frac{p_5}{p_2} = \frac{2\gamma_1 M_R^2 - (\gamma_1 - 1)}{\gamma_1 + 1} \tag{2.88}$$

And applying the Mach number M_S to Eq.(2.88), the reflected shock Mach number M_R is eliminated.

$$\frac{p_5}{p_2} = \frac{(3\gamma_1 - 1) M_S^2 - 2(\gamma_1 - 1)}{(\gamma_1 - 1) M_S^2 + 2} \tag{2.89}$$

Multiplication of Eqs. (2.78) and (2.89) yields,

$$\frac{p_5}{p_1} = \left[\frac{2\gamma_1 M_S^2 - (\gamma_1 - 1)}{\gamma_1 + 1}\right] \left[\frac{(3\gamma_1 - 1) M_S^2 - 2(\gamma_1 - 1)}{(\gamma_1 - 1) M_S^2 + 2}\right] \tag{2.90}$$

In the same way, the temperature ratio T_5/T_1 is represented as follows.

$$\frac{T_5}{T_1} = \frac{\left[2(\gamma_1 - 1) M_S^2 + (3 - \gamma_1)\right] \left[(3\gamma_1 - 1) M_S^2 - 2(\gamma_1 - 1)\right]}{(\gamma_1 + 1)^2 M_S^2} \tag{2.91}$$

In addition, the speed ratio between the incident and reflected shock wave and the reflected shock Mach number M_R are represented as

$$\frac{U_R}{U_S} = \frac{2(\gamma_1 - 1) M_S^2 + (3 - \gamma_1)}{(\gamma_1 + 1) M_S^2} \tag{2.92}$$