# Penrose Limits and $\theta$-Expansion of the Green-Schwarz Action

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**Abstract**

We study the Green-Schwarz action of the Heterotic and the Type IIB string in a plane-wave background obtained by the Penrose limit. We show that the action is quadratic with respect to the fermionic coordinates $\theta$ in such a background, in the light-cone gauge. Because of this property, superstring theory may be relatively easy to be quantized in such a background. We use the normal coordinate expansion in superspace to show this property of the action. Especially we present the $\theta$-expansion of the Heterotic string action in general background explicitly in some approximation and show that a claim previously made about it in the literature is not correct.

# Contents

# Chapter 1

# Introduction

String theory is considered to be the most promising candidate of a unified theory including gravity. In string theory, we assume that the "elementary particles" are actually states of a string. Since a string can oscillate and there are infinite oscillation modes, one can express infinitely many particles as states of such a string. A massless particle with spin 2 is included in the spectrum and it can be considered as the graviton. One can see that string theory includes gravity more explicitly by considering this theory in a curved space-time. Demanding the consistency of such a theory, one can show that the metric tensor should satisfy the Einstein equation with stringy corrections.

Superstring theory is the supersymmetric version of string theory and is considered to be more promising phenomenologically. It includes supergravity and the equations of motion of supergravity can be derived by considering superstring theory in a supergravity background. What we would like to study in this paper is the superstring action in such a background.

There are two different formalisms of superstring theory: the Ramond-Neveu-Schwarz(R-NS) formalism and the Green-Schwarz formalism [1][2]. The R-NS superstring action is given by generalizing the worldsheet coordinates $(\tau, \sigma)$ in the bosonic string action to the superspace coordinates $(\tau, \sigma, \theta)$. On the contrary, the Green-Schwarz superstring action is given by generalizing the space-time coordinates $X^m$ to the superspace coordinates $(X^m, \theta^\alpha)$. These two formalisms are physically equivalent, and each formalism has its own advantages and disadvantages. In the R-NS formalism, it is easy to quantize the action covariantly. However the space-time supersymmetry is not manifest and it is difficult to incorporate the so-called Ramond-Ramond background field into the action. On the other hand, in the Green-Schwarz formalism, it is easy to consider the theory in a general background, but it is difficult to quantize the theory unless one takes the light-cone gauge.

In this paper, since we would like to deal with superstring theories in general supergravity backgrounds, we use the Green-Schwarz formalism. The Green-Schwarz action in general supergravity background is uniquely given in terms of superfields by demanding that supersymmetry and a fermionic local gauge symmetry, the so-called $\kappa$-symmetry are preserved by the action. For the Type IIB superstring, the Green-Schwarz action is given as

$$I = \frac{1}{2}\int d^2\sigma \left[\sqrt{-g}g^{ij}\Phi E_i^a E_j^b \eta^{ab} + \epsilon^{ij} E_i^B E_k^A \mathcal{B}_{AB}\right]. \tag{1.1}$$

To analyze the dynamics of superstring theory, we need to expand superfields into component fields. However this expansion itself is a problem. Most generally the Green-Schwarz action become a very complex polynomial up to order 16 with respect to the fermionic space-time coordinates $\theta$ (or up to order 8 after gauge fixing the $\kappa$-symmetry). We never know how to quantize and analyze such an action correctly.

In the case of flat background, there is no problem since the Green-Schwarz action is known to become quadratic when the $\kappa$-symmetry is fixed by taking the light-cone gauge. It is trivial to quantize this action. If the Green-Schwarz action becomes as simple as the flat case in some of non-trivial backgrounds, it will be very helpful for quantizing the theory. In particular, superstring theory in non-trivial Ramond-Ramond backgrounds is very important. According to the AdS/CFT correspondence [3][4], one can get non-perturbative information on super Yang-Mills theories, if one can quantize superstring theory in such backgrounds.

The first example of such an analysis was about the Type IIB string in $AdS_5 \times S^5$ with non-trivial Ramond-Ramond background fields [5]. This $AdS_5 \times S^5$ background is maximally supersymmetric, i.e. it preserves 32 supersymmetries which are as many as those in the flat background. Since this background is highly symmetric, a method called supercoset method was applied to determine the explicit expression of the Green-Schwarz action. The supercoset method makes the best use of the supersymmetry to construct the action, and it does not require too much knowledge about the background fields of supergravity. The light-cone gauge Green-Schwarz action in this $AdS_5 \times S^5$ background is discovered to be order $\theta^4$ by the supercoset method, which is far simpler than the general case but not simple enough for direct quantization.

A few years later, a new maximally supersymmetric plane-wave background of the Type IIB superstring theory was found [6]. Since this new background is maximally supersymmetric as the $AdS_5 \times S^5$ background is, the supercoset method can be applied [7][8]. The light-cone gauge Green-Schwarz action in this plane-wave background is quadratic with respect to both the bosonic coordinates and the fermionic coordinates, as in the flat background case. This new background belongs to a class of backgrounds called plane-wave, and can be derived from the $AdS_5 \times S^5$ background by taking the special limit called the Penrose limit. The Penrose limit is originally defined in general relativity, and it is known that the Penrose limit of any solution of general relativity leads to a plane-wave solution

[9]. Plane-wave is a special case of pp-wave or "Plane-fronted wave with Parallel rays" which form a bigger class of solutions.

After it was discovered that the Green-Schwarz action in this particular background is very simple, people examined various type of pp-wave and plane-wave, and showed that some of them have this property. In particular, various plane-wave backgrounds obtained by the Penrose limit were discovered [11][12][13] to have similar property; the light-cone gauge Green-Schwarz action was quadratic with respect to the fermionic coordinates $\theta$. Therefore one expects that the light-cone gauge Green-Schwarz action in some more general plane-wave or pp-wave may be relatively simple too. In [14], we showed that the light-cone gauge Green-Schwarz action of the Type IIB superstring in a plane-wave background given as the Penrose limit of an arbitrary background is always quadratic with respect to $\theta$. In showing this, we used the method called the normal coordinate expansion in superspace, which was used for the Heterotic string [15] and for the Type IIB string [16][17].

The normal coordinate expansion method is a generalization of the ordinary Taylor expansion, in which coefficients are covariant tensors in the superspace. Since we would like to consider superstring theory in a background satisfying the equations of motion of supergravity and these equations are given in terms of the tensors in the superspace, the normal coordinate expansion is very efficient to get the explicit form of the action. The calculation in the normal coordinate expansion method is straightforward, but generally very lengthy and complicated, and requires full knowledge about the on-shell superfields of supergravity. This is why the normal coordinate expansion method was not fully applied for all string theories.

In this paper we would like to explain the method and the results of [14]. We also apply the normal coordinate expansion method to the Heterotic string action, and present the explicit form of the second and the fourth order terms of the $\theta$-expansion of the action in some approximation, which will be explained later. Since the Heterotic string theory is a theory with $\mathcal{N} = 1$ supersymmetry, the calculation is easier than the Type IIB case, but still lengthy.

This paper is organized as follows.

In chapter 2, we present the definition of the Green-Schwarz action of superstring theory in flat space-time, and its symmetries: the space-time supersymmetry and the so-called $\kappa$-symmetry.

In chapter 3 we present the definition of the Penrose limit [18][9], and review the example of the action in the maximally supersymmetric plane-wave background [6][7][8], in which the action is found to be quadratic by using supercoset method [5][7]. Then we explain why we pay attention to the Penrose limit.

In chapter 4, we present the solutions of the Bianchi identities of the supergravity theories. With an appropriate set of constraints, these identities automatically imply the equations of motion.

In chapter 5, we determine the superstring action in a general supergravity background by using the results of chapter 4. The action is written in terms of the superfields, and it is the starting point of the normal coordinate expansion method.

In chapter 6, we describe the techniques of the normal coordinate expansion in superspace.

In chapter 7, we explicitly apply the normal coordinate expansion method to obtain the second and the fourth order terms of the $\theta$-expansion of the Green-Schwarz string action of the Heterotic string up to some approximation. The purpose of this calculation is twofold. Firstly, we explicitly check whether and how the light-cone gauge Green-Schwarz action becomes quadratic with respect to $\theta$ in the plane-wave backgrounds obtained by the Penrose limit. Secondly, we show that the claim made in [15], which is that the light-cone gauge action is quadratic with respect to $\theta$ when the background fields depend only on the transverse coordinates, is wrong.

In chapter 8, we study the $\theta$-expansion of the light-cone gauge Green-Schwarz action of the Type IIB string in the plane-wave backgrounds obtained by the Penrose limit. We use the dimensional analysis which was developed in [15] to show that it is quadratic with respect to $\theta$.

Chapter 9 is devoted to the conclusions and discussions.

# Chapter 2

# The Green-Schwarz formalism

In this chapter, we review the definition of the Green-Schwarz action of superstring theory in flat space-time [2][19], and its symmetries.

## 2.1 The Green-Schwarz action

The Polyakov action of the bosonic string theory is given as

$$S = -\frac{1}{2}\int d^2\sigma\sqrt{-g}g^{ij}\partial_i X^m \partial_j X^n \eta_{mn}. \tag{2.1}$$

The superstring action is given as a generalization of this action. In the Green-Schwarz formalism, one generalizes the action so that the space-time coordinates $X^m$ become $(X^m, \theta^{1\mu}, \theta^{2\mu})$, where $\theta^{I\mu}(I = 1, 2)$ are space-time spinors playing the role of the fermionic coordinates. The standard Green-Schwarz action $S$ of $\mathcal{N} = 2$ superstring theory in flat space-time is given [2] as

$$\begin{aligned} S &= S_1 + S_2, \qquad\qquad (2.2)\\ S_1 &= -\frac{1}{2}\int d^2\sigma\sqrt{-g}g^{ij}\Pi_i^m\Pi_j^n\eta_{mn}, \quad \Pi_i^m = \partial_i X^m - i\bar{\theta}^I\Gamma^m\partial_i\theta^I \\ S_2 &= \int d^2\sigma\Big[-i\epsilon^{ij}\partial_i X^m(\bar{\theta}^1\Gamma_m\partial_j\theta^1 - \bar{\theta}^2\Gamma_m\partial_j\theta^2) + \epsilon^{ij}\bar{\theta}^1\Gamma^m\partial_i\theta^1\bar{\theta}^2\Gamma_m\partial_j\theta^2\Big], \end{aligned}$$

where $\Gamma^m$ is the gamma matrix, $\eta_{mn}$ is the flat space-time metric, $g^{ij}$ is the worldsheet metric, and $\epsilon^{ij}$ is the totally anti-symmetric tensor density. $i, j = 0, 1$ denote the worldsheet indices, and $m, n = 0, 1, \cdots, D-1$ denote the space-time indices. This action is invariant under the space-time supersymmetry transformation. In addition to the supersymmetry, there exists a fermionic local gauge symmetry the so-called the $\kappa$-symmetry.

## 2.2 Supersymmetry transformation

The supersymmetry transformation is given as

$$\begin{aligned} &\delta_\epsilon \theta^I = \epsilon^I \qquad \delta_\epsilon X^m = i\bar{\epsilon}^I \Gamma^m \theta^I \\ &\delta_\epsilon \bar{\theta}^I = \bar{\epsilon}^I \qquad \delta_\epsilon g^{ij} = 0, \end{aligned} \tag{2.3}$$

where the transformation parameters $\epsilon^I (I = 1, 2)$ are constant spinors. $S_1$ is obviously invariant under (2.3) because $\Pi_i^m$ is invariant. One can check that the second term $S_2$ is also invariant if $D = 10$ and $\theta^I$ are chosen to be Majorana-Weyl spinors.

## 2.3 The $\kappa$-symmetry transformation

As an important feature of the Green-Schwarz formalism, the action is invariant under a local fermionic gauge transformation, i.e. the $\kappa$-symmetry transformation. The $\kappa$-symmetry transformation is given as

$$\begin{aligned} \delta_\kappa \theta^I &= 2i\Gamma^m \Pi_{im} \kappa^{Ii} \\ \delta_\kappa X^m &= i\bar{\theta}^I \Gamma^m \delta\theta^I, \\ \delta_\kappa(\sqrt{g} g^{ij}) &= -16\sqrt{g}(P_-^{ik} \bar{\kappa}^{1j} \partial_k \theta^1 + P_+^{ik} \bar{\kappa}^{2j} \partial_k \theta^2), \end{aligned} \tag{2.4}$$

where

$$P_\pm^{ij} = \frac{1}{2}(g^{ij} \pm \epsilon^{ij}/\sqrt{g}). \tag{2.5}$$

The transformation parameters $\kappa^{Ii\mu}(I = 1, 2)$ have both the worldsheet vector index $i$ and the space-time spinor index $\mu$, which is suppressed in (2.4). $\kappa^{1i}$ is restricted to be an anti-self-dual worldsheet vector, and $\kappa^{2i}$ is restricted to be self-dual. Although the two terms $S_1$ and $S_2$ in the action are not separately invariant under the transformation (2.4), the total action $S = S_1 + S_2$ is invariant. As a consequence of this local $\kappa$-symmetry, half of the degrees of freedom of $\theta^I$ are decoupled from the theory.

It is very important to keep the $\kappa$-symmetry when one generalizes the action in flat space-time to that in curved superspace. Without the $\kappa$-symmetry, the physical degrees of freedom in $\theta^I$ would be doubled and the theory would correspond to something different from the one defined by using the R-NS formalism.

## 2.4 The light-cone gauge fixing

Since the action (2.2) is in a rather complicated form, covariantly quantizing this theory is very difficult. But the action will be simplified in the light-cone gauge, and quantization becomes very easy in this case. The light-cone gauge conditions are

$$\begin{aligned} X^{+}(\sigma,\tau) &= x^{+} + p^{+}\tau, \\ \Gamma^{+}\theta^{I} &= 0, \end{aligned} \tag{2.6}$$

where $X^{\pm}(\sigma,\tau) \equiv \frac{X^0 \pm X^9}{\sqrt{2}}$. The second condition $\Gamma^{+}\theta^{I} = 0$ fixes the $\kappa$-symmetry. Thus the unphysical degrees of freedom in $\theta^I$ are fixed to be zero. An important consequence of this gauge condition is that $\bar{\theta}\Gamma^m \partial_i \theta = 0$ unless $m = -$. One can check it using the anti-commutation relations of the gamma matrices:

$$\begin{aligned} \bar{\theta}\Gamma^m \partial_i \theta &= \frac{1}{2}\bar{\theta}\left(\Gamma^{+}\Gamma^{-} + \Gamma^{-}\Gamma^{+}\right)\Gamma^m \partial_i \theta \\ &= \frac{1}{2}(\bar{\theta}\Gamma^{+})\Gamma^{-}\Gamma^m \partial_i \theta + \frac{1}{2}\bar{\theta}\Gamma^{-}\Gamma^m \partial_i(\Gamma^{+}\theta) \\ &= 0. \quad (\text{unless } m = -) \end{aligned} \tag{2.7}$$

In this gauge, the action becomes

$$\begin{aligned} S &= -\frac{1}{2}\int d^2\sigma \eta^{ij}\partial_i X^{\tilde{m}} \partial_j X^{\tilde{n}} \eta_{\tilde{m}\tilde{n}} \\ &\quad + \int d^2\sigma i p^{+}(\bar{\theta}^1\Gamma^{-}\partial_0\theta^1 + \bar{\theta}^2\Gamma^{-}\partial_0\theta^2 - \bar{\theta}^1\Gamma^{-}\partial_1\theta^1 + \bar{\theta}^2\Gamma^{-}\partial_1\theta^2), \end{aligned} \tag{2.8}$$

or is rewritten as

$$S = -\frac{1}{2}\int d^2\sigma \left[\partial_i X^{\tilde{m}} \partial^i X_{\tilde{m}} - i\overline{S^{\mu}}\rho^i \partial_i S^{\nu} \delta_{\mu\nu}\right], \tag{2.9}$$

where we have combined the two space-time spinors $\theta^I$ into a single two-component Majorana worldsheet spinor as

$$S^{\mu} = \begin{pmatrix} \sqrt{p^{+}}\theta^1 \\ \sqrt{p^{+}}\theta^2 \end{pmatrix}, \tag{2.10}$$

and $\rho^i$ are $2\times 2$ gamma matrices. The action (2.9) becomes the one for free scalars $X^{\tilde{m}}$ and free spinors $S^{\mu}$ and one can quantize the theory very easily. Thus we have seen that the complicated action (2.2) is simplified vastly after taking the gauge (2.6) in flat space-time. What we would like to pursue in this paper is if there are any other space-time backgrounds in which the Green-Schwarz action becomes simple by taking the light-cone gauge, similarly as the one in this chapter.

# Chapter 3

# The plane-wave backgrounds and the Penrose limit

As mentioned in the introduction, the light-cone gauge Green-Schwarz action in the maximally supersymmetric plane-wave [6] is known to be quadratic [7]. Since this background is obtained as the Penrose limit of the $AdS_5 \times S^5$ background, we expect that any background obtained by the Penrose limit may be simple too. In this chapter, we present the definition of the Penrose limit [18][9]. Then we briefly review the features of the maximally supersymmetric plane-wave.

## 3.1 The definition of the Penrose limit

Let us consider a supergravity theory in ten dimensions with a set of $p$-form potentials $A^{(p)}$, and $(p+1)$-form field strengths $F^{(p+1)}$ defined as

$$F^{(p+1)} = dA^{(p)} + A^{(p-q)}F^{(q)} + \cdots . \tag{3.1}$$

We would like to consider the so-called Penrose limit in this theory. We will show that any classical background in this theory becomes a plane-wave background in such a limit. A plane-wave background is a background which can be expressed as

$$\begin{aligned} ds^2 &= dx^+dx^- - h_{\tilde{m}\tilde{n}}(x^+)x^{\tilde{m}}x^{\tilde{n}}(dx^+)^2 - \delta_{\tilde{m}\tilde{n}}dx^{\tilde{m}}dx^{\tilde{n}}, & (3.2)\\ A^{(p)} &= \frac{1}{p!}A^{(p)}_{m_p\cdots m_2 m_1}(x^+)dx^{m_1}dx^{m_2}\cdots dx^{m_p}, & (3.3)\\ F^{(p+1)} &= \frac{1}{p!}F^{(p+1)}_{+\tilde{m}_p\cdots\tilde{m}_2\tilde{m}_1}(x^+)dx^{\tilde{m}_1}dx^{\tilde{m}_2}\cdots dx^{\tilde{m}_p}dx^+, & (3.4)\end{aligned}$$

where $x^{\pm} = \frac{x^0 \pm x^9}{\sqrt{2}}$ and $\tilde{m}, \tilde{n} = 1, 2, \cdots, 8$.

To consider the Penrose limit, it is convenient to choose a gauge for the metric and the $p$-form potentials as follows. For the metric, we introduce a coordinate system $\{Y^+, Y^-, Y^{\tilde{m}}\}$ so that the string frame metric takes the form

$$ds^2 = 2dY^-\{dY^+ + \alpha dY^- + \beta_{\tilde{m}} dY^{\tilde{m}}\} - C_{\tilde{m}\tilde{n}} dY^{\tilde{m}} dY^{\tilde{n}} \tag{3.5}$$

where $\alpha$, $\beta$, and $C_{\tilde{m}\tilde{n}}$ are functions of $Y^{\pm}$ and $Y^{\tilde{m}}$. Such a coordinate system can always be introduced in the neighborhood of a null geodesic unless there are conjugate points in the neighborhood; in other words, this coordinate system can be introduced unless any two points in the neighborhood are connected by multiple different null geodesics. From the form of the metric (3.5), one can easily see that the null geodesics are given by $Y^- =$ const., $Y^{\tilde{m}} =$ const., and $Y^+$ is an affine parameter along the null geodesics. For the $p$-form potential, we need to choose a gauge so that

$$A^{(p)}_{+\tilde{m}_{p-1}\cdots\tilde{m}_2\tilde{m}_1} = 0. \tag{3.6}$$

This gauge choice is always possible locally in the chosen neighborhood of the null geodesic using the gauge transformation, $A^{(p)} \to A^{(p)} + d\lambda^{(p-1)}$.

Taking the Penrose limit is essentially to concentrate on what is happening in the small neighborhood of a null geodesic $Y^- = 0$, $Y^{\tilde{m}} = 0$. In order to do so, we introduce a new rescaled coordinate system $\{X^+, X^-, X^{\tilde{m}}\}$ such that

$$Y^+ = X^+, \quad Y^- = \Omega^2 X^-, \quad Y^{\tilde{m}} = \Omega X^{\tilde{m}}, \tag{3.7}$$

and rewrite the basis one-forms of the metric, the $p$-form potentials, and the $(p+1)$-form field strengths by using these new coordinates.

$$\begin{aligned}
ds^2 &= \Omega^2 \left[2dX^- dX^+ - C_{\tilde{m}\tilde{n}}(Y) dX^{\tilde{m}} dX^{\tilde{n}}\right] \\
&\quad + 2\Omega^4 dX^- \alpha(Y) dX^- + 2\Omega^3 dX^- \beta_{\tilde{m}}(Y) dX^{\tilde{m}}, \\
A^{(p)} &= \frac{\Omega^p}{p!}\Big[A^{(p)}_{\tilde{m}_p\cdots\tilde{m}_2\tilde{m}_1}(Y) dX^{\tilde{m}_1} dX^{\tilde{m}_2}\cdots dX^{\tilde{m}_p} \\
&\qquad + A^{(p)}_{+-\tilde{m}_{p-2}\cdots\tilde{m}_2\tilde{m}_1}(Y) dX^{\tilde{m}_1} dX^{\tilde{m}_2}\cdots dX^{\tilde{m}_{p-2}} dX^- dX^+\Big] \\
&\quad + \frac{\Omega^{p+1}}{p!} A^{(p)}_{-\tilde{m}_{p-1}\cdots\tilde{m}_2\tilde{m}_1}(Y) dX^{\tilde{m}_1} dX^{\tilde{m}_2}\cdots dX^{\tilde{m}_{p-1}} dX^-, \\
F^{(p+1)} &= \frac{\Omega^p}{p!}\frac{\partial}{\partial X^n} A^{(p)}_{\tilde{m}_p\cdots\tilde{m}_2\tilde{m}_1}(Y) dX^{\tilde{m}_1} dX^{\tilde{m}_2}\cdots dX^{\tilde{m}_p} dX^n, \\
&\quad + \frac{\Omega^{p+1}}{(p-q)!q!} A^{(p-q)}_{\tilde{m}_{p-q}\cdots\tilde{m}_2\tilde{m}_1} A^{(q)}_{\tilde{n}_q\cdots\tilde{n}_2\tilde{n}_1}(Y) dX^{\tilde{m}_1}\cdots dX^{\tilde{m}_{p-q}} dX^{\tilde{n}_1}\cdots dX^{\tilde{n}_q} \\
&\quad + \cdots.
\end{aligned} \tag{3.8}$$

We have omitted various other terms of $F^{(p+1)}$ which are constructed from various combinations of the potentials and the field strengths without derivatives; all these combinations are of order $\mathcal{O}(\Omega^{p+1})$ or higher, and will vanish after the limiting process in the following.

We rescale each field as

$$\begin{aligned} g_{mn} &\rightarrow \overline{g}_{mn} = \Omega^{-2} g_{mn}, \\ A^{(p)} &\rightarrow \overline{A}^{(p)} = \Omega^{-p} A^{(p)}, \\ F^{(p+1)} &\rightarrow \overline{F}^{(p+1)} = \Omega^{-p} F^{(p+1)}, \end{aligned} \tag{3.9}$$

and take the limit $\Omega \rightarrow 0$ to obtain

$$ds^2 = 2dX^- dX^+ - C_{\tilde{m}\tilde{n}}(X^+) dX^{\tilde{m}} dX^{\tilde{n}}, \tag{3.10}$$

$$A^{(p)} = \frac{1}{p!}\Big[A^{(p)}_{\tilde{m}_p \cdots \tilde{m}_2 \tilde{m}_1}(X^+) dX^{\tilde{m}_1} dX^{\tilde{m}_2} \cdots dX^{\tilde{m}_p} \tag{3.11}$$

$$+ A^{(p)}_{+-\tilde{m}_{p-2} \cdots \tilde{m}_2 \tilde{m}_1}(X^+) dX^{\tilde{m}_1} dX^{\tilde{m}_2} \cdots dX^{\tilde{m}_{p-2}} dX^- dX^+\Big], \tag{3.12}$$

$$\begin{aligned} F^{(p+1)} &= \frac{1}{p!}\frac{\partial}{\partial X^+} A^{(p)}_{\tilde{m}_p \cdots \tilde{m}_2 \tilde{m}_1}(X^+) dX^{\tilde{m}_1} dX^{\tilde{m}_2} \cdots dX^{\tilde{m}_p} dX^+ \\ &= \frac{1}{p!} F^{(p+1)}_{+\tilde{m}_p \cdots \tilde{m}_2 \tilde{m}_1}(X^+) dX^{\tilde{m}_1} dX^{\tilde{m}_2} \cdots dX^{\tilde{m}_p} dX^+, \end{aligned} \tag{3.13}$$

where we have used

$$\begin{aligned} \frac{\partial}{\partial X^{\tilde{n}}} A^{(p)}_{\tilde{m}_p \cdots \tilde{m}_2 \tilde{m}_1}(Y) &= \mathcal{O}(\Omega) \rightarrow 0, \\ \frac{\partial}{\partial X^-} A^{(p)}_{\tilde{m}_p \cdots \tilde{m}_2 \tilde{m}_1}(Y) &= \mathcal{O}(\Omega^2) \rightarrow 0. \end{aligned} \tag{3.14}$$

Since the metric (3.10) becomes singular when $\det(C_{\tilde{m}\tilde{n}}) = 0$, we should define yet another coordinate frame $\{x^+, x^-, x^{\tilde{m}}\}$ as

$$\begin{aligned} X^- &= x^- + \tfrac{1}{4}\dot{C}_{\tilde{m}\tilde{n}}(X^+) Q^{\tilde{m}}{}_{\tilde{k}}(X^+) Q^{\tilde{n}}{}_{\tilde{l}}(X^+) x^{\tilde{k}} x^{\tilde{l}}, \\ X^+ &= x^+, \quad X^{\tilde{m}} = Q^{\tilde{m}}{}_{\tilde{n}} x^{\tilde{n}}, \end{aligned} \tag{3.15}$$

with

$$C_{\tilde{m}\tilde{n}} Q^{\tilde{m}}{}_{\tilde{k}} Q^{\tilde{n}}{}_{\tilde{l}} = \delta_{\tilde{k}\tilde{l}}, \quad C_{\tilde{m}\tilde{n}}\left(\dot{Q}^{\tilde{m}}{}_{\tilde{k}} Q^{\tilde{n}}{}_{\tilde{k}} - Q^{\tilde{m}}{}_{\tilde{k}} \dot{Q}^{\tilde{n}}{}_{\tilde{k}}\right) = 0. \tag{3.16}$$

In this coordinate frame the metric takes the form:

$$ds^2 = dx^+ dx^- - h_{\tilde{m}\tilde{n}}(x^+) x^{\tilde{m}} x^{\tilde{n}} (dx^+)^2 - \delta_{\tilde{m}\tilde{n}} dx^{\tilde{m}} dx^{\tilde{n}}, \tag{3.17}$$

where

$$h_{\tilde{m}\tilde{n}} = -\left[\dot{C}_{\tilde{k}\tilde{l}} \dot{Q}^{\tilde{l}}{}_{\tilde{m}} + C_{\tilde{k}\tilde{l}} \ddot{Q}^{\tilde{l}}{}_{\tilde{m}}\right] Q^{\tilde{k}}{}_{\tilde{n}}. \tag{3.18}$$

The form of the $(p+1)$-form field strength is not changed from (3.13):

$$F^{(p+1)} = \frac{1}{p!} F^{(p+1)}_{+\tilde{m}_p \cdots \tilde{m}_2 \tilde{m}_1}(x^+) dx^{\tilde{m}_1} dx^{\tilde{m}_2} \cdots dx^{\tilde{m}_p} dx^+. \tag{3.19}$$

which means that each component of field strength tensors $F^{(p+1)}_{m_{p+1} \cdots m_2 m_1}(x^+)$ is non-vanishing only when it has a single '+' index and $p$ transverse indices. On the other hand, the $p$-form potential does not retain its form in (3.12), and all components of the $p$-form potentials may be non-vanishing:

$$A^{(p)} = \frac{1}{p!} A^{(p)}_{m_p \cdots m_2 m_1}(x^+) dx^{m_1} dx^{m_2} \cdots dx^{m_p}. \tag{3.20}$$

Thus we obtain a plane-wave solution in the universal form (3.2)-(3.4) as a result of the Penrose limit. From the metric (3.2). we can see that the only non-zero components of the curvature tensor and the Ricci tensor are

$$R_{+\tilde{m}+\tilde{n}}, \quad R_{++}, \tag{3.21}$$

up to symmetry of the tensors. Also the covariant derivatives are non-vanishing only when its index is lower '+', since all the above fields depend only on $x^+$. As we will see in the following chapters, the Green-Schwarz action in a plane-wave background is very simple because of these properties.

## 3.2 The maximally supersymmetric plane-wave

As an example of the plane-wave background obtained by the Penrose limit, let us consider the maximally supersymmetric plane-wave in the Type IIB theory, in which the metric and the Ramond-Ramond 5-form field strength are given as

$$\begin{aligned} ds^2 &= dx^+ dx^- - f^2 x^{\tilde{m}} x_{\tilde{m}}\, dx^+ dx^+ + \delta_{\tilde{m}\tilde{n}} dx^{\tilde{m}} dx^{\tilde{n}}, \\ & F_{+1234} = F_{+5678} = 2f. \end{aligned} \tag{3.22}$$

It preserves 32 supersymmetries [6]. This property allows one to construct the Green-Schwarz action by using the supercoset method. In the supercoset method, one construct the action from the Cartan 1-forms, $L^m$, $L^\mu$, $L^{\mu\nu}$, which are defined by using the supersymmetry algebra. The action is uniquely determined by the following four conditions:

- It has global supersymmetry corresponding to the supersymmetry algebra preserved by the background (3.22).
- Its bosonic part is given by the action of the bosonic string theory in the background (3.22)

- It is invariant under the $\kappa$-symmetry transformation.
- It is reduced to the known Green-Schwarz action of the Type IIB superstring theory in the flat space limit ($m \to 0$).

As a result of this method, the explicit action is given [7] in the light-cone gauge as

$$\begin{aligned} L &= -\frac{1}{2}\eta^{ij}\partial_i X^{\tilde{m}}\partial_j X_{\tilde{m}} - \frac{f^2}{2}X^{\tilde{m}}X_{\tilde{m}} \\ &\quad + i\left(\bar{\theta}\Gamma^-\partial_0\theta + \theta\Gamma^-\partial_0\bar{\theta} + \theta\Gamma^-\partial_1\theta + \bar{\theta}\Gamma^-\partial_1\theta\right) \\ &\quad - 2f^2\bar{\theta}\Gamma^-\Gamma^1\Gamma^2\Gamma^3\Gamma^4\theta. \end{aligned} \tag{3.23}$$

or

$$\begin{aligned} L &= -\frac{1}{2}\eta^{ij}\partial_i X^{\tilde{m}}\partial_j X_{\tilde{m}} - \frac{f^2}{2}X^{\tilde{m}}X_{\tilde{m}} \\ &\quad + i\bar{S}\Gamma^-\rho^i\partial_i S + if^2\bar{S}\Gamma^-\Gamma^1\Gamma^2\Gamma^3\Gamma^4 S, \end{aligned} \tag{3.24}$$

where

$$S = \begin{pmatrix} \sqrt{p^+}\theta^1 \\ \sqrt{p^+}\theta^2 \end{pmatrix} \quad \theta^1 = \frac{\theta + \bar{\theta}}{2}, \quad \theta^2 = \frac{\theta - \bar{\theta}}{2i}, \tag{3.25}$$

and $\rho^i$ are $2 \times 2$ gamma matrices. This action is quadratic in both bosonic and fermionic fields, and one can quantize the theory easily as in the flat background case. This is the first non-trivial background in which the Green-Schwarz action is discovered to be quadratic.

This background is also an example of the plane-wave obtained by the Penrose limit; actually (3.22) is obtained [20][21] as the Penrose limit of the $AdS_5 \times S^5$ background given as

$$ds^2 = R^2\left[-dt^2\cosh^2\rho + d\rho^2 + \sinh^2\rho d\Omega_3^2 + d\Psi^2\cos^2\vartheta + d\vartheta^2 + \sin^2\vartheta d\Omega_3'^2\right].$$

In this case, the Penrose limit is written as

$$x^+ = \bar{x}^+, \quad x^- = R^2\bar{x}^-, \quad \rho = \frac{r}{R}, \quad \vartheta = \frac{y}{R}, \quad R \to \infty. \tag{3.26}$$

After it was discovered that the Green-Schwarz action in this particular background is very simple, people examined various other plane-wave backgrounds which are known to be obtained by the Penrose limit [11][12][13], and showed that the Green-Schwarz action become quadratic in $\theta$ in many cases. However, thorough investigation had not been done. What we would like to do is to study the Green-Schwarz action in general plane-wave backgrounds obtained by the Penrose limit.

# Chapter 4

# Solutions of supergravity theories

In this chapter, we present the solutions of Bianchi identities in supergravity theories. Since we would like to study general plane-wave backgrounds satisfying the equations of motion of supergravity theories, we should have a convenient way to express such backgrounds. The most convenient way to do so is to use the superfield formalism. In the superfield formalism, we need a consistent set of constraints which removes the unphysical degrees of freedom. This is necessary because in the superfields formalism, there are generally far too large number of component fields included in the superfields, compared to the number of physical fields. In 10 dimensions, one can actually derive the equations of motion from these constraints using the Bianchi identities. There are a number of equations extracted from the component form of the Bianchi identities; some of the equations relate higher order component fields to the lowest component fields, and some of the equations become the equations of motion of this supergravity theory.

In section 4.1, we present the general solutions of $\mathcal{N} = 1$ supergravity coupled with super Yang-Mills theory, which is used as the background of the Heterotic superstring theory, up to some approximation about the Lorentz anomaly. In section 4.2, we present the general solutions of $\mathcal{N} = 2$ Type IIB supergravity which is used as the background of the Type IIB superstring theory. Since we would like to present the explicit calculations about the higher order terms only for the Heterotic string case, we explain this case in section 4.1 in full detail, and only briefly review the Type IIB case in section 4.2.

## 4.1 $\mathcal{N} = 1$ supergravity coupled with super Yang-Mills theory

### 4.1.1 The definition of $\mathcal{N} = 1$ curved superspace

First, let us review the superspace formulation of the $\mathcal{N} = 1, D = 10$ supergravity theories.

A curved superspace is parametrized by $Z^M = (x^m, \theta^\mu)$ where $x^m$ $(m = 0, 1, 2, \ldots, 9)$ are ordinary bosonic space-time coordinates, and $\theta^\mu$, $(\mu = 1, 2, \ldots, 16)$ are the anti-commuting fermionic coordinates, and each superfield is a function of $Z^M$. If we expand a superfield in terms of the fermionic coordinates $\theta^\mu$, we can obtain a finite number of component fields, which are functions of $x^m$.

To construct a curved superspace, we introduce a set of basis one-forms $E^A$

$$E^A = dZ^M E_M{}^A, \tag{4.1}$$

where $E_M{}^A$ is the supervielbein, which has the ordinary vielbein $e_M{}^A$ as its lowest order component. The tangent-space indices $A = (a, \alpha)$ take either bosonic values $a = 0, 1, \ldots, 9$ or fermionic values $\alpha = 1, 2, \ldots, 16$. The supervielbein $E_M{}^A$ and its inverse $E_A{}^M$ satisfies,

$$E_M{}^A E_A{}^N = \delta_M{}^N, \quad E_A{}^M E_M{}^B = \delta_A{}^B, \tag{4.2}$$

and $E_M{}^A E_N{}^B \eta_{AB} = G_{MN}$ is the metric of the curved superspace.

A basis for any p-forms is constructed from a set of one-forms $E^A$ by forming exterior products. We define the components of a $p$-form $\Omega^{(p)}$ as

$$\begin{aligned}\Omega^{(p)} &= \frac{1}{p!} dZ^{M_p} \cdots dZ^{M_2} dZ^{M_1} \Omega^{(p)}_{M_1 M_2 \cdots M_p}, \\ &= \frac{1}{p!} E^{A_p} \cdots E^{A_2} E^{A_1} \Omega^{(p)}_{A_1 A_2 \cdots A_p}.\end{aligned} \tag{4.3}$$

The exterior derivative is defined as

$$d\Omega^{(p)} = \frac{1}{p!} dZ^{M_p} \cdots dZ^{M_2} dZ^{M_1} dZ^N \partial_N \Omega^{(p)}_{M_1 M_2 \cdots M_p}. \tag{4.4}$$

It satisfies $d^2 = 0$ and obeys the Leibniz rule:

$$d(\Omega^{(p)} \Lambda^{(p')}) = \Omega^{(p)} (d\Lambda^{(p')}) + (-)^p (d\Omega^{(p)}) \Lambda^{(p')}. \tag{4.5}$$

Tangent space vectors $V^A$ and $V_A$ transform under the tangent-space group as

$$\delta V^A = V^B L_B{}^A, \quad \delta V_A = -L_A{}^B V_B. \tag{4.6}$$

The tangent space group is SO(1, 9) with ordinary bosonic vectors transforming as **10** and spinors transforming as **16** or $\overline{\mathbf{16}}$. This implies that the Lie algebra valued matrices $L_A{}^B$ must satisfy

$$L_\alpha{}^b = L_a{}^\beta = 0, \quad L_\alpha{}^\beta = \frac{1}{4} L_{ab} (\Gamma^{ab})_\alpha{}^\beta, \tag{4.7}$$

where $\Gamma^{ab}$ is a totally anti-symmetric product of gamma matrices. We use two sets of $16 \times 16$ gamma matrices, $\Gamma^a \equiv (\Gamma^a)^{\alpha\beta}$, and $\bar{\Gamma}^a \equiv (\Gamma^a)_{\alpha\beta}$ satisfying

$$(\Gamma^a)_{\alpha\delta}(\Gamma^b)^{\delta\beta} + (\Gamma^b)_{\alpha\delta}(\Gamma^a)^{\delta\beta} = 2\eta^{ab}\delta_\alpha{}^\beta. \tag{4.8}$$

Strictly speaking, $\Gamma^a$ and $\bar{\Gamma}^a$ are two different sub-matrices of $32 \times 32$ gamma matrices, and the fermionic indices cannot be raised or lowered and its contractions must be taken between an upper index and a lower index. Also we use $\Gamma^{abc\cdots}$ to denote totally anti-symmetric products of $\Gamma^a$ and $\bar{\Gamma}^a$ matrices normalized to be unit weight, for example,

$$\begin{aligned} (\Gamma^{ab})^\alpha{}_\beta &\equiv (\Gamma^{[a}\bar{\Gamma}^{b]})^\alpha{}_\beta \\ &\equiv \frac{1}{2}\left\{(\Gamma^a)^{\alpha\delta}(\Gamma^b)_{\delta\beta} - (\Gamma^b)^{\alpha\delta}(\Gamma^a)_{\delta\beta}\right\}, \end{aligned} \tag{4.9}$$

$$\begin{aligned} \Gamma^{abc} &\equiv \Gamma^{[a}\bar{\Gamma}^b\Gamma^{c]} \\ &\equiv \frac{1}{3!}\{\Gamma^a\bar{\Gamma}^b\Gamma^c + \Gamma^b\bar{\Gamma}^c\Gamma^a + \Gamma^c\bar{\Gamma}^a\Gamma^b \\ &\quad - \Gamma^a\bar{\Gamma}^c\Gamma^b - \Gamma^b\bar{\Gamma}^a\Gamma^c - \Gamma^c\bar{\Gamma}^b\Gamma^a\}, \end{aligned} \tag{4.10}$$

where [ ] represents anti-symmetrization with respect to the indices normalized to unit weight. Also ( ) and [ ) will be used to represent symmetrization and graded anti-symmetrization with respect to the indices normalized to unit weight. We may omit the bar on $\bar{\Gamma}^a$ when doing so does not cause any confusion.

The covariant exterior derivative, $D \equiv dZ^M D_M = E^A D_A$, acting on a vector valued $p$-form is defined as,

$$DV^A = dV^A + V^B\Omega_B{}^A, DV_A = dV_A - (-)^p\Omega_A{}^B V_B, \tag{4.11}$$

where $\Omega_A{}^B = dZ^M\Omega_{MA}{}^B = E^C\Omega_{CA}{}^B$ is the superconnection one-form which has the ordinary connection of the bosonic space-time as its lowest order component field. The superconnection one-form $\Omega_A{}^B$ must be Lie algebra valued to make (4.11) covariant under the SO(1, 9) transformations, thus they satisfy relations:

$$\Omega_\alpha{}^b = \Omega_a{}^\beta = 0, \quad \Omega_\alpha{}^\beta = \frac{1}{4}\Omega_{ab}(\Gamma^{ab})_\alpha{}^\beta. \tag{4.12}$$

From the the connection and the vielbein we can define the torsion two-form $T^A$ and the curvature two-form $R_A{}^B$ as

$$T^A = dE^A, \quad R_A{}^B = d\Omega_A{}^B + \Omega_A{}^C \Omega_C{}^B. \tag{4.13}$$

These two are written in terms of its components as

$$\begin{aligned} T^A &= \frac{1}{2} dZ^N dZ^M T_{MN}{}^A = \frac{1}{2} E^C E^B T_{BC}{}^A, & (4.14) \\ R_A{}^B &= \frac{1}{2} dZ^N dZ^M R_{MNA}{}^B = \frac{1}{2} E^D E^C R_{CDA}{}^B. & (4.15) \end{aligned}$$

From (4.12), the curvature two-form $R_A{}^B$ also satisfies the same equations,

$$R_\alpha{}^b = R_a{}^\beta = 0, \quad R_\alpha{}^\beta = \frac{1}{4} R_{ab} (\Gamma^{ab})_\alpha{}^\beta. \tag{4.16}$$

By using $d^2 = 0$, we can obtain the Bianchi identities for $T^A$ and $R_A{}^B$.

$$\begin{aligned} DT^A &= d(dT^A + T^B \Omega_B{}^A) + (dT^B + T^C \Omega_C{}^B) \Omega_B{}^A \\ &= T^B d\Omega_B{}^A + T^C \Omega_C{}^B \Omega_B{}^A \\ &= T^B R_B{}^A, & (4.17) \\ DR_A{}^B &= d(d\Omega_A{}^B + \Omega_A{}^C \Omega_C{}^B) \\ &\quad + (d\Omega_A{}^C + \Omega_A{}^D \Omega_D{}^C) \Omega_C{}^B - \Omega_A{}^C (d\Omega_C{}^B + \Omega_C{}^D \Omega_D{}^B) \\ &= 0. & (4.18) \end{aligned}$$

In the component form, these are

$$\begin{aligned} D_{[A} T_{BC)}{}^D + T_{[AB}{}^E T_{\hat{E}C)}{}^D - R_{[ABC)}{}^D = 0, \\ D_{[A} R_{BC)D}{}^E + T_{[AB}{}^F R_{\hat{F}C)D}{}^E = 0. & (4.19) \end{aligned}$$

In order to describe supergravity, we need an anti-symmetric tensor field in addition to the torsion and curvature tensors . It is accommodated by introducing a two-form superfield $B$,

$$B = \frac{1}{2} e^A e^B B_{AB}, \tag{4.20}$$

and the three-form field strength $H$,

$$\begin{aligned} H &= \frac{1}{3!} e^C e^B e^A H_{ABC} \\ &= dB. & (4.21) \end{aligned}$$

The Bianchi identity for $H$ is obtained as

$$dH = 0, \tag{4.22}$$

or

$$D_{[A} H_{BCD)} + \frac{3}{2} T_{[AB}{}^F H_{\hat{F}CD)} = 0, \tag{4.23}$$

in the component form.

### 4.1.2 Coupling with super Yang-Mills theory

In the Heterotic string theory, there are super Yang-Mills fields in addition to the $\mathcal{N}=1$ supergravity. An important feature of this $\mathcal{N}=1$ supergravity theory coupled to super Yang-Mills fields is that we need to require that the two-form potential $B$ transforms anomalously under the gauge transformations and the Lorentz transformations to keep the gauge/Lorentz invariance of the action. The gauge field strength is defined as

$$\begin{aligned} F &= \frac{1}{2}e^B e^A F_{AB} \\ &= dA + A^2, \end{aligned} \tag{4.24}$$

where $A$ is the one-form potential $A = dZ^M A_M = e^B A_B$. Both $A$ and $F$ take their values in the Lie algebra of the gauge group ($E_8 \times E_8$ or SO(32)), and we suppressed the gauge indices. The gauge- and superspace-covariant derivative is defined to act on a Lie-algebra valued scalar superfield $\Lambda$ as

$$\mathcal{D}\Lambda = d\Lambda - [A, \Lambda], \tag{4.25}$$

and the action on any Lie-algebra valued superfields can be defined similarly. Using this covariant derivative, the Bianchi identity for $F$ is given as

$$\mathcal{D}F = 0, \tag{4.26}$$

or

$$\mathcal{D}_{[A}F_{BC)} + T_{[AB}{}^{D}F_{\hat{D}C)} = 0, \tag{4.27}$$

in the component form.

Since the two-form potential $B$ is transformed anomalously, we need to modify the definition of the gauge invariant three-form field strength $H$ as

$$H = dB + c_1\omega_{3\mathrm{YM}} + c_2\omega_{3\mathrm{L}}, \tag{4.28}$$

where

$$\omega_{3\mathrm{YM}} \equiv \mathrm{tr}(AF - \tfrac{1}{3}A^3), \quad \omega_{3\mathrm{L}} \equiv \mathrm{tr}(\Omega R - \tfrac{1}{3}\Omega) \tag{4.29}$$

and $c_1$ and $c_2$ are some constants. The Bianchi identity for $H$ is obtained as

$$dH = c_1\mathrm{tr}F^2 + c_2\mathrm{tr}R^2, \tag{4.30}$$

which is clearly gauge invariant.

From (4.28), the anomalous gauge and Lorentz transformation of $B$ is read as

$$\delta B = -c_1\mathrm{tr}(\Lambda dA) - c_2\mathrm{tr}(Ld\Omega), \tag{4.31}$$

where $\Lambda$ and $L$ are the parameters of the gauge transformation and the local Lorentz transformation. The constant $c_1$ and $c_2$ can be determined by requiring the anomaly cancellation of the superstring action in quantum level, since the supergravity theory is a low energy effective theory of the superstring theory. However, in this paper, we will first solve the Bianchi identities assuming $c_2 = 0$ and take the effect of $c_2 \neq 0$ into account later.

### 4.1.3 The constraints

In the above two subsections, we have represented the definition of supergravity in the superspace formulation and the Bianchi identities are given as (4.19) (4.19) (4.27). The equations of motion of supergravity is derived by solving the Bianchi identities. So we need to find solutions of the Bianchi identities.

To solve the Bianchi identities, we first choose a basic set of constraints. The choices are not unique and any consistent choices are physically equivalent. The specific set of constraints used in [22] is,

$$\begin{aligned} & T_{\alpha\beta}{}^{a} = 2(\Gamma^{a})_{\alpha\beta}, \quad T^{b}_{\alpha a} = -T^{b}_{a\alpha} = 0, \\ & T_{a\alpha}{}^{\beta} = -T_{\alpha a}{}^{\beta} = (\Gamma_{a}\Psi)_{\alpha}{}^{\beta}, \quad T_{\alpha\beta}{}^{\gamma} = 0, \\ & H_{\alpha\beta\gamma} = 0, \quad F_{\alpha\beta} = 0, \end{aligned} \tag{4.32}$$

where the superfield $\Psi^{\alpha\beta}$, $T_{ab}{}^{c}$, $T_{ab}{}^{\alpha}$ and other components of field strengths are unconstrained (at least initially). We use this set of constraints to obtain the solutions of the Bianchi identities following the calculations in [22].

### 4.1.4 The Bianchi identities for $T$

Substituting the constraints (4.32) into the Bianchi identities (4.19), we obtain a number of equations,

$$D_{[a}T_{bc]}{}^{d} - T_{[ab}{}^{e}T_{c]e}{}^{d} - R_{[abc]}{}^{d} = 0, \tag{4.33}$$

$$D_{[a}T_{bc]}{}^{\delta} - T_{[ab}{}^{e}T_{c]e}{}^{\delta} - T_{[ab}{}^{\gamma}\Gamma_{c]\gamma\epsilon}\Psi^{\epsilon\delta} = 0, \tag{4.34}$$

$$R_{(\alpha\beta\gamma)}{}^{\delta} = 0, \tag{4.35}$$

$$D_{\beta}T_{bc}{}^{d} + 2T_{bc}{}^{\gamma}\Gamma^{d}_{\gamma\beta} - 2R_{\beta[bc]}{}^{d} = 0, \tag{4.36}$$

$$\begin{aligned} & D_{\beta}T_{bc}{}^{\delta} + 2D_{[b}\Psi^{\epsilon\delta}\Gamma_{c]\beta\epsilon} + T_{bc}{}^{e}\Psi^{\epsilon\delta}\Gamma_{e\beta\epsilon} \\ & + 2\Psi^{\epsilon\gamma}\Psi^{\alpha\delta}\Gamma_{[b\hat{\beta}}\hat{\epsilon}\Gamma_{c]\alpha\gamma} - R_{bc\beta}{}^{\delta} = 0, \end{aligned} \tag{4.37}$$

$$2\Psi^{\epsilon\delta}\Gamma_{a\epsilon(\beta}\Gamma^{d}_{\gamma)\delta} - \frac{1}{2}R_{\gamma\beta a}{}^{d} + \Gamma^{b}_{\gamma\beta}T_{ba}{}^{d} = 0, \tag{4.38}$$

$$R_{a(\beta\gamma)}{}^{\delta} + D_{(\gamma}\Psi^{\epsilon\delta}\Gamma_{a\beta)\epsilon} - \Gamma^{c}_{\gamma\beta}T_{ca}{}^{\delta} = 0. \tag{4.39}$$

First, we will extract an equation

$$T_{a(bc)} = 0, \tag{4.40}$$

from (4.35) and (4.38) as follows. Multiplying (4.38) with $(\Gamma^a\Gamma_d)_{\zeta\gamma}$, and taking the symmetric part with respect to $\zeta$ and $\beta$, we get

$$\begin{aligned}
0 &= 2\Psi^{\epsilon\delta}\Gamma_{a\epsilon(\beta}(\Gamma^a\Gamma_{\hat{d}}\Gamma^d)_{\zeta)\delta} - 2\Psi^{\epsilon\delta}\Gamma^d_{\delta(\beta}(\Gamma^a\Gamma_{\hat{d}}\Gamma_a)_{\zeta)\epsilon} \\
&\quad -\frac{1}{2}(\Gamma^a\Gamma_d)_{(\zeta}{}^{\gamma}R_{\hat{\gamma}\beta)a}{}^d + (\Gamma^a\Gamma_d\Gamma^b)_{(\zeta\beta)}T_{ba}{}^d \\
&= -\frac{1}{2}R_{\gamma(\beta\zeta)}{}^{\gamma} + 2\Gamma^b_{\zeta\beta}T_{bd}{}^d \\
&= -\frac{1}{2}\left[\frac{3}{2}R_{(\gamma\beta\zeta)}{}^{\gamma} - \frac{1}{2}R_{\beta\zeta\gamma}{}^{\gamma}\right] + 2\Gamma^b_{\zeta\beta}T_{bd}{}^d \\
&= 2\Gamma^b_{\zeta\beta}T_{bd}{}^d, \qquad (4.41)
\end{aligned}$$

where we used $R_{(\gamma\beta\zeta)}{}^{\gamma} = 0$ which is obtained from (4.35) and $R_{\beta\zeta\gamma}{}^{\gamma} \equiv \frac{1}{4}R_{\beta\zeta ab}\mathrm{Tr}(\Gamma^{ab}) = 0$. Thus, we get

$$T_{bd}{}^d = 0. \qquad (4.42)$$

Substituting (4.42) into (4.38) after contracting $a$ and $d$, we get

$$2\Psi^{\epsilon\delta}\Gamma_{d\epsilon(\beta}\Gamma^d_{\gamma)\delta} = 0. \qquad (4.43)$$

Then multiplying (4.43) with $\Gamma^{c\beta\gamma}$, we get

$$\begin{aligned}
0 &= 2\Psi^{\epsilon\delta}\Gamma_{d\epsilon(\beta}\Gamma^d_{\gamma)\delta}\Gamma^{c\beta\gamma} \\
&= 2\Psi^{\epsilon\delta}\left(\Gamma_a\Gamma^c\Gamma^a\right)_{(\epsilon\delta)} \\
&= -20\Psi^{\epsilon\delta}\Gamma^c_{\epsilon\delta}. \qquad (4.44)
\end{aligned}$$

Multiplying (4.38) with $\Gamma_c^{\gamma\beta}$ again, contracting fermionic indices, and taking the symmetric part with respect to $a$ and $d$, we obtain,

$$\begin{aligned}
0 &= 2(\Psi\Gamma_{(d}\Gamma_{\hat{c}}\Gamma_{a)})^{\epsilon}{}_{\epsilon} - (\Gamma_c\Gamma^b)^{\gamma}{}_{\gamma}T_{b(ad)} \\
&= 32\Psi^{\epsilon\delta}\Gamma_{c\epsilon\delta}\eta_{ad} - 16T_{c(ad)}. \qquad (4.45)
\end{aligned}$$

Together with (4.44), we get (4.40). Since $T_{abc}$ is components of two-form $T^A$, it is anti-symmetric with respect to $a$ and $b$, and (4.40) implies that $T_{abc}$ is a totally anti-symmetric tensor. This tensor will be related to $H_{abc}$ which is a totally anti-symmetric tensor by definition.

Secondly, we will extract the relation between $\Psi^{\alpha\beta}$ and $T_{abc}$ using the above results. $\Psi^{\alpha\beta}$ is solved to be

$$\Psi^{\alpha\beta} = -\frac{1}{24}T_{abc}(\Gamma^{abc})^{\alpha\beta}, \qquad (4.46)$$

as follows. We first expand $\Psi^{\epsilon\delta}$ in terms of the $\Gamma$ matrices as

$$\Psi^{\epsilon\delta} = G_a(\Gamma^a)^{\epsilon\delta} + G_{abc}(\Gamma^{abc})^{\epsilon\delta} + G_{abcde}(\Gamma^{abcde})^{\epsilon\delta}, \qquad (4.47)$$

where $G_{abc}$ is a totally anti-symmetric tensor and $G_{abcde}$ is a totally anti-symmetric self-dual tensor. This expansion is always possible since any tensor with two fermionic indices can be expanded using the complete set made from the products of $\Gamma$ matrices. Substituting (4.47) into (4.44), we get $G_a = 0$. Substituting (4.47) into (4.38) and symmetrizing with respect to $a$ and $d$, we get

$$\begin{aligned}
0 &= 2\left(\Gamma_{(a}\Psi\Gamma_{d)}\right)_{(\beta\gamma)} \\
&= 2\Big(6G_{efg}\delta^{e}_{(a}\Gamma^{f}\delta^{g}_{d)} \\
&\quad - G_{efghi}\Gamma^{efghi}\eta_{ad} + 5G_{efghi}\delta^{e}_{(a}\Gamma^{fghi}\Gamma_{d)} + 5G_{efghi}\Gamma_{(a}\Gamma^{efgh}\delta^{i}_{d)}\Big)_{(\beta\gamma)} \\
&= 2\Big( - G_{efghi}\Gamma^{efghi}\eta_{ad} + 10G_{efgh(a}\Gamma^{efgh}{}_{d)}\Big)_{(\beta\gamma)}.
\end{aligned} \tag{4.48}$$

Thus we see that $G_{abcde}$ is also vanishing, and $\Psi^{\epsilon\delta} = G_{abc}(\Gamma^{abc})^{\epsilon\delta}$. (4.35), (4.38) and above results allow us to write $G_{abc}$ in terms of $T_{abc}$. Multiplying (4.35), (4.38) with $\Gamma$ matrices and contracting some of indices, we get four equations written below:

Multiplying (4.35) with $\Gamma^{a\beta\gamma}$ and contracting $\alpha$ and $\delta$, we get

$$\begin{aligned}
0 &= R_{\alpha\beta\gamma}{}^{\alpha}\Gamma^{a\beta\gamma} \\
&= \frac{1}{4}R_{\alpha\beta bc}(\Gamma^{a}\Gamma^{bc})^{\beta\alpha} \\
&= \frac{1}{2}R_{\alpha\beta bc}\Gamma^{c\beta\alpha}.
\end{aligned} \tag{4.49}$$

Multiplying (4.35) with $\Gamma^{a\beta\gamma}(\Gamma^{ef})_{\delta}{}^{\alpha}$, we get

$$\begin{aligned}
0 &= R_{(\alpha\beta\gamma)}{}^{\delta}\Gamma^{a\beta\gamma}(\Gamma^{ef})_{\delta}{}^{\alpha} \\
&= \frac{1}{4}R_{\alpha\beta bc}(\Gamma^{bc})_{\gamma}{}^{\delta}\Gamma^{a\beta\gamma}(\Gamma^{ef})_{\delta}{}^{\alpha} + \frac{1}{4}R_{\beta\gamma bc}(\Gamma^{bc})_{\alpha}{}^{\delta}\Gamma^{a\beta\gamma}(\Gamma^{ef})_{\delta}{}^{\alpha} \\
&= \frac{1}{4}R_{\alpha\beta bc}(\Gamma^{a}\Gamma^{bc}\Gamma^{ef})^{\beta\alpha} + \frac{1}{4}R_{\beta\gamma bc}\Gamma^{a\beta\gamma}\mathrm{Tr}(\Gamma^{bc}\Gamma^{ef}) \\
&= \frac{1}{4}R_{\alpha\beta bc}\left(\Gamma^{abcef} - 18\Gamma^{a}\eta^{be}\eta^{cf} + 2\Gamma^{f}\eta^{ab}\eta^{ce} - 2\Gamma^{e}\eta^{ab}\eta^{cf}\right)^{\beta\alpha}.
\end{aligned} \tag{4.50}$$

Multiplying (4.38) with $(\Gamma^{abcef})^{\gamma\beta}$, we get

$$\begin{aligned}
0 &= 2G_{ghi}\mathrm{Tr}(\Gamma_{c}\Gamma^{ghi}\Gamma_{d}\Gamma^{abcef}) - \frac{1}{2}(\Gamma^{abcef})^{\gamma\beta}R_{\gamma\beta cd} \\
&= 2G_{ghi}(\sum_{c\neq a,e,f}\ \sum_{d\neq c,a,e,f}\delta^{c}_{c}\delta^{d}_{d})\mathrm{Tr}(-6\eta^{ga}\eta^{he}\eta^{if}\delta_{\beta}{}^{\beta}) - \frac{1}{2}(\Gamma^{acdef})^{\gamma\beta}R_{\gamma\beta cd} \\
&= -2\times 42\times 6\times 16G^{aef} - \frac{1}{2}(\Gamma^{acdef})^{\gamma\beta}R_{\gamma\beta cd}.
\end{aligned} \tag{4.51}$$

Multiplying (4.38) with ${\Gamma_e}^{\beta\gamma}$, we get

$$\begin{aligned} 0 &= 2G_{ghi}\mathrm{Tr}(\Gamma_a\Gamma^{ghi}\Gamma_d\Gamma_e) - \frac{1}{2}R_{\gamma\beta ad}{\Gamma_e}^{\beta\gamma} + \mathrm{Tr}(\Gamma_e\Gamma^b)T_{bad} \\ &= 2\times 6\times 16G_{aed} - \frac{1}{2}R_{\gamma\beta ad}{\Gamma_e}^{\beta\gamma} + 16T_{ead}. \end{aligned} \tag{4.52}$$

Using (4.50), (4.51) and (4.52), we can obtain $\Psi^{\alpha\beta}$ as (4.46). Additionally, we can determine $R_{\alpha\beta ab}$ by substituting (4.46) into (4.38) as

$$R_{\alpha\beta ad} = \frac{1}{6}T_{cde}({\Gamma_{ab}}^{cde})_{\alpha\beta} + 3T_{abc}\Gamma^c_{\alpha\beta}. \tag{4.53}$$

In the above, we have written $\Psi^{\alpha\beta}$(i.e. ${T_{a\alpha}}^\beta$) and $R_{\alpha\beta ad}$ in terms of $T_{abc}$. In the rest of this subsection, we will study the remaining equations to get the expressions for other superfields ${T_{ab}}^\alpha$, $R_{\alpha abc}$ in terms of $T_{abc}$ and $D_\alpha T_{abc}$.

Transforming (4.36) by using the fact that $R_{ABcd}$ is anti-symmetric with respect to $c$ and $d$, we get

$$R_{\beta cbd} = \frac{1}{2}D_\beta T_{cbd} + {T_{cb}}^\gamma\Gamma_{d\gamma\beta} - {T_{bd}}^\gamma\Gamma_{c\gamma\beta} + {T_{dc}}^\gamma\Gamma_{b\gamma\beta}. \tag{4.54}$$

From this and (4.39), we get the following two equations:

Multiplying (4.54) with $\frac{1}{4}(\Gamma_e\Gamma^{bd})^{\beta\delta}$, we get

$$\begin{aligned} \frac{1}{4}R_{\beta cbd}(\Gamma_e\Gamma^{bd})^{\beta\delta} &= \frac{1}{8}D_\beta T_{cbd}(\Gamma_e\Gamma^{bd})^{\beta\delta} + \frac{1}{4}{T_{cb}}^\gamma{(\Gamma_d\Gamma_e\Gamma^{bd})_\gamma}^\delta \\ &\quad - \frac{1}{4}{T_{bd}}^\gamma{(\Gamma_c\Gamma_e\Gamma^{bd})_\gamma}^\delta + \frac{1}{4}{T_{dc}}^\gamma{(\Gamma_b\Gamma_e\Gamma^{bd})_\gamma}^\delta. \end{aligned} \tag{4.55}$$

Multiplying (4.39) with $\Gamma^{\beta\gamma}_e$, we get

$$\begin{aligned} 0 &= {R_{a(\beta\gamma)}}^\delta\Gamma^{\beta\gamma}_e + D_{(\gamma}\Psi^{\epsilon\delta}\Gamma_{a\beta)\epsilon}\Gamma^{\beta\gamma}_e - \Gamma^c_{\gamma\beta}{T_{ca}}^\delta\Gamma^{\beta\gamma}_e \\ &= -\frac{1}{4}R_{\beta abd}(\Gamma_e\Gamma^{bd})^{\beta\delta} + D_\gamma(\Gamma_e\Gamma_a\Psi)^{\gamma\delta} - {T_{ca}}^\delta\mathrm{Tr}(\Gamma^c\Gamma_e). \end{aligned} \tag{4.56}$$

Substituting (4.55) into (4.56), we get

$$\begin{aligned} &6{T_{eb}}^\delta - D_\gamma\psi^{\epsilon\delta}{(\Gamma_c\Gamma_e)_\epsilon}^\delta \\ &\quad = \frac{1}{8}D_\beta T_{bcd}(\Gamma_e\Gamma^{bd})^{\beta\delta} + \frac{9}{2}{T_{cb}}^\gamma{(\Gamma^b\Gamma_e)_\gamma}^\beta \\ &\qquad + \frac{1}{4}{T_{bd}}^\gamma{(\Gamma^{bd}\Gamma_c\Gamma_e)_\gamma}^\delta - {T_{ed}}^\delta{(\Gamma^d\Gamma_c)_\gamma}^\delta. \end{aligned} \tag{4.57}$$

To analyze ${T_{ab}}^\alpha$, we decompose it into $\mathrm{SO}(1,9)$ irreducibles **560**, **144** and **16** as

$$ {T_{ab}}^\alpha = {J_{ab}}^\alpha + 2J_{\beta[a}\Gamma^{\beta\alpha}_{b]} + J^\beta{(\Gamma_{ab})_\beta}^\alpha, \tag{4.58}$$

where $J_{ab}{}^{\alpha}$, $J_{\beta a}$, $J^{\beta}$ are in **540**, **144**, **16** representations of SO$(1,9)$ respectively. They satisfy

$$J_{ab}{}^{\alpha}\Gamma^{b}_{\alpha\beta} = 0, \tag{4.59}$$

$$J_{\beta a}\Gamma^{a\beta\alpha} = 0. \tag{4.60}$$

Using this expansion, (4.57) becomes,

$$\begin{aligned} & 6T_{eb}{}^{\delta} - D_{\gamma}\psi^{\epsilon\delta}(\Gamma_{c}\Gamma_{e})_{\epsilon}{}^{\delta} - \frac{1}{8}D_{\beta}T_{bcd}(\Gamma_{e}\Gamma^{bd})^{\beta\delta} \\ = \; & 36J_{ac}\Gamma^{\alpha\delta}_{e} - 8J_{\alpha e}\Gamma^{\alpha\delta}_{c} + 27J^{\beta}(\Gamma_{c}\Gamma_{e})_{\beta}{}^{\delta} - 18J^{\delta}\eta_{ec}. \end{aligned} \tag{4.61}$$

We can determine $J^{\beta}$ by contracting $c$ and $e$:

$$J^{\beta} = -\frac{13}{90}D_{\alpha}\psi^{\alpha\beta}. \tag{4.62}$$

$J_{\beta a}$ can be determined by multiplying (4.61) with $\Gamma^{c}_{\delta\gamma}$:

$$J_{\beta a} = -\frac{1}{56}[D_{\alpha}(\Gamma_{a}\Psi)^{\alpha}{}_{\beta} + 288J^{\alpha}\Gamma_{a\alpha\beta}]. \tag{4.63}$$

Using the equations (4.60), $J_{\beta a}\Gamma^{a\beta\alpha} = 0$, we get

$$\begin{aligned} J_{\beta a}\Gamma^{a\beta\gamma} &= -\frac{1}{56}[D_{\alpha}(\Gamma_{a}\Psi\Gamma^{a})^{\alpha\gamma} + 288\times 10J^{\gamma}] \\ &= -\frac{1}{56}(10+6+288\times 10)J^{\gamma} = 0, \end{aligned} \tag{4.64}$$

and we can see $J^{\beta} = 0$. This is one of the equations of motion which is really obtained as a part of solutions of the Bianchi identities. Substituting (4.63) and $J^{\beta} = 0$ into (4.61) we can obtain an expression about $J_{ab}{}^{\alpha}$:

$$J_{ec}{}^{\delta} = \frac{1}{16}D_{\beta}T_{jk[e}(\Gamma^{jk}_{c]})^{\beta\delta} + \frac{1}{8}D_{\beta}T_{eck}\Gamma^{k\beta\delta}, \tag{4.65}$$

and thus, the full superfield $T_{ec}{}^{\delta}$ becomes

$$T_{ec}{}^{\delta} = \frac{1}{14}D_{\beta}T_{jk[e}(\Gamma^{jk}_{c]})^{\beta\delta} + \frac{3}{28}D_{\beta}T_{eck}\Gamma^{k\beta\delta}. \tag{4.66}$$

Combining it with (4.54), we can also get an equation which relate $R_{\beta cbd}$ to fermionic derivatives of $T_{abc}$.

In this subsection, we have written $\Psi^{\alpha\beta}$, $R_{\alpha\beta ad}$ $T_{ab}{}^{\alpha}$, and $R_{\alpha abc}$ in terms of $T_{abc}$ and $D_{\alpha}T_{abc}$. Actually, one can write all the unconstrained components of the torsion tensor and all the components of the curvature tensor in terms of $T_{abc}$ and its derivatives, but we have presented only important equations which will be used in the following chapters.

### 4.1.5 The Bianchi identities for $F$

Substituting the constraints (4.32) into the Bianchi identities (4.27), we obtain another set of equations,

$$\mathcal{D}_{[c}F_{ba]} - T_{[cd}{}^{d}F_{a]d} - T_{[cd}{}^{\delta}F_{a]\delta} = 0, \tag{4.67}$$
$$\Gamma^{d}_{(\gamma\beta}F_{\alpha)d} = 0, \tag{4.68}$$
$$2\mathcal{D}_{[c}F_{b]\alpha} + \mathcal{D}_{\alpha}F_{cb} + T_{cb}{}^{d}F_{d\alpha} - 2T_{\alpha[c}{}^{\delta}F_{b]\delta} = 0, \tag{4.69}$$
$$\mathcal{D}_{(\gamma}F_{\beta)a} + \Gamma^{d}_{\gamma\beta}F_{da} = 0. \tag{4.70}$$

We decompose $F_{a\alpha}$ into SO$(1,9)$ irreducibles, **144** and **16**, in a similar way as $T_{ab}{}^{\alpha}$ in the last subsection.

$$F_{a\alpha} = \chi_{a\alpha} + \Gamma_{a\alpha\beta}\chi^{\beta}, \tag{4.71}$$

where $\chi_{a\alpha}$, $\chi^{\beta}$ are in **144**, **16** representations of SO$(1,9)$ respectively. $\chi_{a\alpha}$ satisfy

$$\chi_{a\alpha}\Gamma^{a\alpha\beta} = 0. \tag{4.72}$$

Substituting this decomposition to (4.68), we can see $\chi_{a\alpha} = 0$. This $\chi^{\delta}$ is identified as the gluino, the super partner of the gauge field. Other equations (4.67), (4.69) and (4.70), allow us to obtain the equation of motion for gluino, and the Yang-Mills equation, but we do not need these equations in the calculations in the following chapters.

### 4.1.6 The Bianchi identities for $H$

The Bianchi identities for $H$ (4.30) are very hard to solve because of the presence of curvature squared term. To solve it generally, we may try to obtain the solutions as a power series of $c_2$, and the solutions will be very complicated. Therefore we consider the zero-th order approximation first. Namely, we assume $c_2 = 0$, and the component form of the Bianchi identities (4.30) becomes

$$D_{[e}H_{abd]} + \frac{3}{2}T_{[ea}{}^{f}H_{\hat{f}bd]} - \frac{3c_1}{2}\mathrm{tr}(F_{[ea}F_{bd]}) = 0, \tag{4.73}$$
$$\Gamma^{f}{}_{(\epsilon\alpha}H_{\hat{f}\beta\delta)} = 0, \tag{4.74}$$
$$D_{(\epsilon}H_{\alpha\beta)d} + 2\Gamma^{f}{}_{(\epsilon\alpha}H_{\hat{f}\beta)d} = 0, \tag{4.75}$$
$$\begin{aligned}&3D_{[e}H_{ab]\delta} - D_{\delta}H_{eab} + 3T_{[ea}{}^{F}H_{\hat{F}b]\delta}\\ &+ 3\psi^{\epsilon\gamma}\Gamma_{[e\hat{\epsilon}\hat{\delta}}H_{ab]\gamma} - 6c_1\mathrm{tr}(F_{[ea}\Gamma_{b]\alpha\delta}\chi^{\alpha}) = 0,\end{aligned} \tag{4.76}$$
$$\begin{aligned}&D_{[e}H_{a]\beta\delta} + D_{(\beta}H_{\delta)ea} + \frac{1}{2}T_{ea}{}^{f}H_{[f\beta\delta} + \Gamma^{f}_{\beta\delta}H_{fea} - \psi^{\epsilon\gamma}\Gamma_{[e\hat{\epsilon}\hat{\beta}}H_{\hat{\gamma}a]\delta}\\ &- \psi^{\epsilon\gamma}\Gamma_{[e\hat{\epsilon}\hat{\delta}}H_{\hat{\gamma}a]\beta} + 2c_1\Gamma_{[e\hat{\beta}\hat{\alpha}}\Gamma_{a]\delta\gamma}\mathrm{tr}(\chi^{\alpha}\chi^{\gamma}) = 0.\end{aligned} \tag{4.77}$$

To deal with $H_{a\alpha\beta}$, we decompose it as

$$H_{a\alpha\beta} \quad = \quad \phi\Gamma_{a\alpha\beta} + \phi^{bc}(\Gamma_{abc})_{\alpha\beta} + \phi^{bcde}(\Gamma_{abcde})_{\alpha\beta}. \tag{4.78}$$

Substituting it into (4.74), we see that $\phi^{bc} = \phi^{bcde} = 0$ and

$$H_{a\alpha\beta} \quad = \quad \phi\Gamma_{a\alpha\beta}. \tag{4.79}$$

$\phi$ is the dilaton superfield, which includes the ordinary dilaton field as its lowest order component. We define the superfield $\lambda$ by $\lambda_\alpha \equiv D_\alpha\phi$; which includes superpartner of dilaton as its lowest order component. Using the results for $H_{a\alpha\beta}$, we can solve (4.75) for $H_{ab\alpha}$ as

$$H_{ab\alpha} \quad = \quad -\frac{1}{2}\left(\Gamma_{ab}\right)_\alpha{}^\beta D_\beta\phi = \frac{1}{2}\left(\Gamma_{ab}\right)_\alpha{}^\beta\lambda_\beta. \tag{4.80}$$

Multiplying (4.77) with $(\Gamma^{bcdea})^{\beta\epsilon}$, we obtain the expression of $D_\beta\lambda$ written by $\phi$, $T_{abc}$, $\chi^\alpha$ as

$$D_\beta\lambda_\epsilon = -(\Gamma^b)_{\beta\epsilon}D_b\phi - \frac{\phi}{6}T_{abc}(\Gamma^{abc})_{\beta\epsilon}, \tag{4.81}$$

while multiplying (4.77) with $\Gamma^{d\beta\epsilon}$, we get

$$H_{dea} = -\phi T_{dea} + \frac{1}{32}(\Gamma_{dea})^{\beta\epsilon}D_\beta\lambda_\epsilon + \frac{c_1}{8}(\Gamma_{dea})_{\alpha\beta}\mathrm{tr}(\chi^\alpha\beta). \tag{4.82}$$

Then substituting (4.81) into this equation, we obtain the expression of $H_{dea}$ written by $\phi$, $T_{abc}$, $\chi^\alpha$ as

$$H_{abc} = -\frac{3}{2}\phi T_{abc} + \frac{c_1}{4}\left(\Gamma_{abc}\right)_{\alpha\beta}\mathrm{tr}(\chi^\alpha\chi^\beta). \tag{4.83}$$

In the above, we have solved the Bianchi identities for $H$ when $c_2 = 0$. We will use these solutions in the following chapters as the zero-th order approximation when we calculate the $\theta$-expansion of the Green-Schwarz action. The purposes of these calculations are to examine the claim made in [15] is right or wrong, and to examine the special backgrounds obtained by the Penrose limit. For these purposes, the results in the zero-th order approximation presented here are actually enough. Since the claim in [15] was made in the background with this zero-th order approximation, we will use the same approximation to examine this claim. For the latter purpose, one can argue as follows. To get the exact expressions for the Green-Schwarz action of the Heterotic string, we need to obtain the solutions which satisfies the Bianchi identities with $c_2 \neq 0$. The component form of the

Bianchi identities with $c_2 \neq 0$ becomes

$$D_{[e}H_{abd]} + \frac{3}{2}T_{[ea}{}^{f}H_{\hat{f}bd]} - \frac{3c_1}{2}\mathrm{tr}(F_{[ea}F_{bd]}) - \frac{9c_2}{2}R_{[ea\hat{f}}{}^{g}R_{bd]g}{}^{f} = 0, \quad (4.84)$$

$$\Gamma^{f}{}_{(\epsilon\alpha}H_{\hat{f}\beta\delta)} - \frac{3c_2}{2}R_{(\epsilon\alpha\hat{f}}{}^{g}R_{\beta\delta)g}{}^{f} = 0, \quad (4.85)$$

$$D_{(\epsilon}H_{\alpha\beta)d} + 2\Gamma^{f}{}_{(\epsilon\alpha}H_{\hat{f}\beta)d} - 6c_2R_{(\epsilon\alpha\hat{f}}{}^{g}R_{\beta)dg}{}^{f} = 0, \quad (4.86)$$

$$\begin{aligned} &3D_{[e}H_{ab]\delta} - D_{\delta}H_{eab} + 3T_{[ea}{}^{F}H_{\hat{F}b]\delta} \\ &+ 3\psi^{\epsilon\gamma}\Gamma_{[e\hat{\epsilon}\hat{\delta}}H_{ab]\gamma} - 6c_1\mathrm{tr}(F_{[ea}\Gamma_{b]\alpha\delta}\chi^{\alpha}) - 18c_2R_{[ea\hat{f}}{}^{g}R_{b)\delta g}{}^{f} = 0, \end{aligned} \quad (4.87)$$

$$\begin{aligned} &D_{[e}H_{a]\beta\delta} + D_{(\beta}H_{\delta)ea} + \frac{1}{2}T_{ea}{}^{f}H_{[f\beta\delta} + \Gamma^{f}_{\beta\delta}H_{fea} - \psi^{\epsilon\gamma}\Gamma_{[e\hat{\epsilon}\hat{\beta}}H_{\hat{\gamma}a]\delta} \\ &- \psi^{\epsilon\gamma}\Gamma_{[e\hat{\epsilon}\hat{\delta}}H_{\hat{\gamma}a]\beta} + 2c_1\Gamma_{[e\hat{\beta}\hat{\alpha}}\Gamma_{a]\delta\gamma}\mathrm{tr}(\chi^{\alpha}\chi^{\gamma}) \\ &- 3c_2R_{eaf}{}^{g}R_{\beta\delta g}{}^{f} + 3c_2R_{[e\hat{\beta}\hat{f}}{}^{g}R_{a]\delta g}{}^{f} + 3c_2R_{[e\hat{\delta}\hat{f}}{}^{g}R_{a]\beta g}{}^{f} = 0. \end{aligned} \quad (4.88)$$

However, in the special background which we are interested in, i.e. the backgrounds obtained by the Penrose limit, the solutions of the Bianchi identities are not changed by the parameter $c_2$ because the curvature squared term always vanish as shown as follows.

In this special background, each component of $R_{abcd}$ is non-vanishing only when it has two lower '+' and two transverse indices. as explained in section 3.1. Thus $R_{eaf}{}^{g}R_{bdg}{}^{f}$ in (4.84) is always zero. From (4.16), each component of $R_{\alpha\beta cd}$ is non-vanishing only when it has a bosonic index '+' and a bosonic transverse index. Thus $R_{\epsilon\alpha\hat{f}}{}^{g}R_{\beta\delta g}{}^{f}$ in (4.85) and $R_{eaf}{}^{g}R_{\beta\delta g}{}^{f}$ in (4.88) are always zero. Since we are considering a classical background, $R_{\beta dg}{}^{f}$ is always zero. Since all the curvature squared terms in (4.86) - (4.87) are always zero, the solutions of equations (4.73) - (4.77) are also satisfies equations (4.84) - (4.88). We will use this result to prove that the light-cone gauge Green-Schwarz action of the Heterotic string in this background is quadratic in $\theta$ without any approximation.

### 4.1.7 The summary of the solutions

Summarizing the above results, we list the relations which are important for the calculations of the Heterotic string action in the following chapters.

The constraints for $T_{AB}{}^{C}$:

$$\begin{aligned} &T_{\alpha\beta}{}^{a} = 2(\Gamma^{a})_{\alpha\beta}, \quad T^{b}_{\alpha a} = -T^{b}_{a\alpha} = 0, \\ &T_{a\alpha}{}^{\beta} = -T_{\alpha a}{}^{\beta} = (\Gamma_{a}\Psi)_{\alpha}{}^{\beta}, \quad \Psi^{\alpha\beta} = -\frac{1}{24}T_{abc}(\Gamma^{abc})^{\alpha\beta}, \\ &T_{\alpha\beta}{}^{\gamma} = 0. \end{aligned} \quad (4.89)$$

The constraints for $H_{ABC}$:

$$H_{a\alpha\beta} = \phi\Gamma_{a\alpha\beta}, \quad H_{ab\alpha} = \frac{1}{2}\left(\Gamma_{ab}\right)_{\alpha}{}^{\beta}\lambda_{\beta},$$

$$H_{abc} = -\frac{3}{2}\phi T_{abc} + \frac{c_1}{4}\,(\Gamma_{abc})_{\alpha\beta}\,\mathrm{tr}(\chi^\alpha \chi^\beta). \qquad (4.90)$$

The relation between $T_{ab}{}^{\gamma}$ and fermionic derivatives of $T_{abc}$:

$$T_{ec}{}^{\delta} = \frac{1}{14} D_\beta T_{jk[e}(\Gamma_{c]}{}^{jk})^{\beta\delta} + \frac{3}{28} D_\beta T_{eck}(\Gamma^k)^{\beta\delta}. \qquad (4.91)$$

The relation between the curvature tensor and $T_{abc}$:

$$R_{\alpha\beta ab} = \frac{1}{6} T_{cde}(\Gamma_{ab}{}^{cde})_{\alpha\beta} + 3T_{abc}(\Gamma^c)_{\alpha\beta}. \qquad (4.92)$$

The relation between fermionic element of the curvature tensor and the torsion:

$$R_{\beta cbd} = \frac{1}{2} D_\beta T_{cbd} + T_{cb}{}^{\gamma}\Gamma_{d\gamma\beta} + T_{db}{}^{\gamma}\Gamma_{c\gamma\beta} + T_{dc}{}^{\gamma}\Gamma_{b\gamma\beta}. \qquad (4.93)$$

Fermionic derivatives of $\lambda$:

$$D_\beta \lambda_\epsilon = -(\Gamma^b)_{\beta\epsilon} D_b\phi - \frac{\phi}{6} T_{abc}(\Gamma^{abc})_{\beta\epsilon}. \qquad (4.94)$$

Fermionic derivatives of $T_{abc}$:

$$\begin{aligned} D_\gamma T_{abc} \;&=\; 2T_{[ab}{}^{\alpha}\Gamma_{c]\alpha\gamma} + \phi^{-1} D_{[a}\lambda_{\hat\beta}(\Gamma_{bc]})_\gamma{}^{\beta} - \phi^{-1}T_{abc}\lambda_\gamma \\ &\quad - \phi^{-1}T_{[ab}{}^{d}(\Gamma_{c]d})_\gamma{}^{\beta}\lambda_\beta - \phi^{-1}(\Gamma_{[a}\Psi\Gamma_{bc]})_\gamma{}^{\beta}\lambda_\beta + \mathcal{O}(F\chi). \qquad (4.95) \end{aligned}$$

Fermionic derivatives of $T^{\alpha}_{ab}$:

$$\begin{aligned} &D_\beta T_{bc}{}^{\delta} + 2D_{[b}\Psi^{\epsilon\delta}(\Gamma_{c]})_{\beta\epsilon} + T_{bc}{}^{e}\Psi^{\epsilon\delta}(\Gamma_e)_{\beta\epsilon} \\ &+ 2\Psi^{\epsilon\gamma}\Psi^{\alpha\delta}(\Gamma_{[b})_{\hat\beta\hat\epsilon}(\Gamma_{c]})_{\alpha\gamma} - R_{bc\beta}{}^{\delta} = 0, \qquad (4.96) \end{aligned}$$

$$\begin{aligned} D_\beta T_{ab}{}^{\alpha} \;&=\; -2D_{[a}(\Gamma_{b]}\Psi)_\beta{}^{\alpha} - T_{ab}{}^{e}(\Gamma_e\Psi)_\beta{}^{\alpha} \\ &\quad - 2(\Gamma_{[a}\Psi\Gamma_{b]}\Psi)_\beta{}^{\alpha} + \frac{1}{4}R_{abcd}(\Gamma^{cd})_\beta{}^{\alpha}. \qquad (4.97) \end{aligned}$$

These results will be used when we explicitly calculate the expansion of the Green-Schwarz action in chapter 7.

## 4.2 The Type IIB supergravity

The formulation of the $\mathcal{N} = 2, D = 10$ Type IIB supergravity theory [23] is a direct generalization of the $\mathcal{N} = 1, D = 10$ supergravity theory. We use the convention that a curved superspace is parametrized by $Z^M = (x^m, \theta^\mu, \theta^{\bar\mu})$ where $\theta^{\bar\mu}$ is the complex conjugate of $\theta^\mu$, $\theta^{\bar\mu} = \bar\theta^\mu$. The tangent space group is now

SO(1, 9) × U(1), where the U(1) subgroup is the automorphism group of the $\mathcal{N} = 2$ supersymmetry algebra. Because of this additional U(1), the field contents of the Type IIB supergravity is complicated.

In this section, we only present formulas necessary for the following chapters. Following the notation and conventions in [23], the part of the field contents of the IIB supergravity theory which is necessary for us is as follows: the basis 1-form $E^A = (E^a, E^\alpha, E^{\bar\alpha} \equiv \bar{E}^\alpha)$, the SO(1, 9) × U(1) connection 1-form $\hat\Omega_A{}^B$, the complex 2-form potential $\mathcal{A}$ and the real 4-form potential $B$. From these fields, the torsion 2-form $T^A$, the curvature 2-form $R_A{}^B$, the complex 3-form $\mathcal{F}$, and the 5-form field strength $Z$ are constructed. In addition, there appears a scalar superfield, which is an element of SU(1, 1),

$$\mathcal{V} = \begin{pmatrix} u & v \\ \bar v & \bar u \end{pmatrix}, \qquad u\bar u - v\bar v = 1. \tag{4.98}$$

From this, one obtains the 1-form,

$$\mathcal{V}^{-1} d\mathcal{V} = \begin{pmatrix} 2iQ & P \\ \bar P & -2iQ \end{pmatrix}, \qquad Q = \bar Q. \tag{4.99}$$

The real 1-form $Q$ is identified with the U(1) part of $\hat\Omega_A{}^B$, whereas the complex 1-form $P$ is expanded as $P = E^a P_a + E^\alpha P_\alpha - E^{\bar\alpha} P_{\bar\alpha}$ with

$$P_\alpha = -2\Lambda_\alpha, \quad P_{\bar\alpha} = 0. \tag{4.100}$$

where $\Lambda_\alpha$ is a superfield which contains the physical spin 1/2 field. Also, the scalar field $\mathcal{V}$ and the complex 3-form $\mathcal{F}$ are combined to form an SU(1, 1) invariant 3-form field strength $F$,

$$(\bar{\mathcal{F}}, \mathcal{F})\mathcal{V} = (\bar F, F). \tag{4.101}$$

The real 2-form potential $\mathcal{B}$ and real 3-form field strength $\mathcal{H}$, which are directly corresponding to the $B$ and the $H$ in the Heterotic string, are given from the $\mathcal{F}$ as

$$\mathcal{H} \equiv \mathcal{F} + \bar{\mathcal{F}} = d\mathcal{B}, \tag{4.102}$$

and the dilaton superfield $\Phi$ is given by the components in $\mathcal{V}$ as

$$\Phi = w = \bar w, \qquad w = u - \bar v, \tag{4.103}$$

where we have taken a specific local U(1) gauge so that $w$ becomes real. In this gauge, one can derive

$$\begin{aligned} D_A\Phi &= -\frac{1}{2}(P_A + \bar P_A), \\ Q_A &= \frac{i}{4}(P_A - \bar P_A), \\ \mathcal{H} &= \Phi(F + \bar F). \end{aligned} \tag{4.104}$$

These fields satisfy the constraints:

$$
\begin{aligned}
&T_{ab}{}^{c}=T_{\alpha\beta}{}^{c}=T_{\alpha b}{}^{c}=T_{\alpha\beta}{}^{\gamma}=T_{\alpha\bar\beta}{}^{\bar\gamma}=0\,,\\
&T_{\alpha\bar\beta}{}^{c}=-i(\Gamma^{c})_{\alpha\beta}\,,\quad T_{a\beta}{}^{\bar\gamma}=-\frac{3}{16}\bar F_{abc}(\Gamma^{bc})_{\beta}{}^{\gamma}-\frac{1}{48}\bar F^{bcd}(\Gamma_{abcd})_{\beta}{}^{\gamma}\,,\\
&T_{a\beta}{}^{\gamma}=\frac{21}{2}iX_{a}\delta_{\beta}{}^{\gamma}+\frac{3}{2}iX^{b}(\Gamma_{ab})_{\beta}{}^{\gamma}+\frac{5}{4}iX_{abc}(\Gamma^{bc})_{\beta}{}^{\gamma}\\
&\qquad+\frac{1}{4}iX^{bcd}(\Gamma_{abcd})_{\beta}{}^{\gamma}+iZ_{abcde}(\Gamma^{bcde})_{\beta}{}^{\gamma},\\
&T_{a\bar\beta}{}^{\bar\gamma}=-(\overline{T_{a\beta}{}^{\gamma}})\,,\quad T_{a\bar\beta}{}^{\gamma}=-(\overline{T_{a\beta}{}^{\bar\gamma}})\,,
\end{aligned}
\tag{4.105}
$$

$$
\begin{aligned}
&\mathcal{H}_{\alpha\beta\gamma}=\mathcal{H}_{\alpha\beta\bar\gamma}=\mathcal{H}_{\alpha\bar\beta\bar\gamma}=\mathcal{H}_{\bar\alpha\bar\beta\bar\gamma}=\mathcal{H}_{a\beta\bar\gamma}=0\,,\\
&\mathcal{H}_{a\beta\gamma}=\mathcal{H}_{a\bar\beta\bar\gamma}=-i\Phi(\Gamma_{a})_{\beta\gamma}\,,\\
&\mathcal{H}_{ab\gamma}=-\Phi(\Gamma_{ab})_{\gamma}{}^{\delta}\bar\Lambda_{\delta}\,,\quad \mathcal{H}_{ab\bar\gamma}=-\Phi(\Gamma_{ab})_{\gamma}{}^{\delta}\Lambda_{\delta},
\end{aligned}
\tag{4.106}
$$

$$
D_{\alpha}\Lambda_{\beta}=-\frac{i}{24}F_{abc}(\Gamma^{abc})_{\alpha\beta}\,,\quad \bar D_{\alpha}\Lambda_{\beta}=-\frac{i}{2}P_{a}(\Gamma^{a})_{\alpha\beta},
\tag{4.107}
$$

where $X^{[\cdots]}=\frac{1}{16}\bar\Lambda_{\alpha}(\Gamma_{[\cdots]})^{\alpha\beta}\Lambda_{\beta}$.

The results of this section will be used when we analyze the Green-Schwarz action of the Type IIB string in chapter 8.

# Chapter 5

# Superstring action in supergravity backgrounds

In this chapter, we briefly review how the superstring actions in the general supergravity background are obtained. These actions are written by using superfields, and will be the starting point of $\theta$-expansion in chapter 7 and chapter 8.

## 5.1 The Heterotic string action

The standard Green-Schwarz action (2.2) in flat space-time can be rewritten as

$$I = -\frac{1}{2}\int d^2\sigma \left[\sqrt{-g}g^{ij}\partial_i Z^M \partial_j Z^N ({e_N}^a {e_M}^b)\eta_{ab} + \epsilon^{ij}\partial_i Z^M \partial_j Z^N a_{NM}\right], \quad (5.1)$$

where ${e_M}^A$ is the vielbein of the flat superspace, and $a_{NM}$ is the 2-form:

$$\begin{aligned}
{e_M}^A &= ({e_M}^a, {e_M}^\alpha), \\
e^a &\equiv dZ^M {e_M}^a = dX^a - i\theta^\alpha(\Gamma^a)_{\alpha\beta}d\theta^\beta, \\
e^\alpha &\equiv dZ^M {e_M}^\alpha = d\theta^\alpha, \\
a &= i\theta^\alpha(\Gamma_a)_{\alpha\beta}dX^a d\theta^\beta. \qquad (5.2)
\end{aligned}$$

One may generalize (5.1) to the action in general supergravity background by replacing the vielbein ${e_M}^A$ and the 2-form by the general vielbein ${E_M}^A$ of the superspace and the 2-form potential of the supergravity theory respectively. However, as we mentioned in chapter 2, we need to preserve the invariance under the $\kappa$-symmetry when we generalize the action. The $\kappa$-symmetry transformation of the vielbein is given as

$$\begin{aligned}
\delta E^a &= \delta Z^M {E_M}^a = 0, \\
\delta E^\alpha &= \delta Z^M E_M^\alpha = 2E_i^a(\Gamma_a)^{\alpha\beta}g^{ij}\kappa_{j\beta}. \qquad (5.3)
\end{aligned}$$

The action (5.1) can not be made invariant no matter how we choose the $\kappa$-symmetry transformation of $g^{ij}$. To make the action invariant, we need to add one more field $\phi$, which turns out to be the dilaton superfield. In addition to the dilaton, we need the gauge superfield in the Heterotic string theory. Therefore the Heterotic string action in general supergravity background is obtained [24] as

$$\begin{aligned} I &= \int d^2\sigma \left[ \frac{1}{2}\eta^{ij}\phi(Z)E_i^a(Z)E_{ja}(Z) \right. \\ &\left. \quad + \epsilon^{ij}\partial_i Z^N \partial_j Z^M B_{MN}(Z) + \psi^T \mathcal{D}_- \psi \right]. \end{aligned} \tag{5.4}$$

This action is invariant under the $\kappa$-symmetry transformation:

$$\begin{aligned} \delta E^a &= \delta Z^M E_M{}^a \;=\; 0, \\ \delta E^\alpha &= \delta Z^M E_M^\alpha \;=\; 2E_i^a(\Gamma_a)^{\alpha\beta} g^{ij}\kappa_{j\beta}. \\ \Psi^s &= (\delta Z^M A_M)^{st}\Psi^t, \\ \delta\phi &= \phi\delta E^\alpha \Lambda_\alpha, \\ \delta(\sqrt{-g}g^{ij}) &= 4i(g^{ik}\epsilon^{jl} + \epsilon^{ik}g^{jl}) \left[ E_k^\alpha - \frac{1}{4}\phi^{-1}E_k^c(\Gamma_c)^{\alpha\beta} \right] \kappa_{l\alpha} \\ &\quad + \phi^{-1}(4\sqrt{-g}\bar{\Psi}^s \rho^j \kappa_\alpha^i \chi_{st}^\alpha \Psi^t). \end{aligned} \tag{5.5}$$

Here, we need to substitute the solutions of the Bianchi identities into superfields $E_M^A$, $\Psi^s$, $\phi$, and $B_{MN}$, which were given in chapter 4. Unless we do so, the $\kappa$-symmetry is not preserved.

## 5.2 The Type IIB string action

The action of the Type IIB string in general supergravity background is obtained [25] in a similar way. The action becomes

$$I = \frac{1}{2}\int d^2\sigma \left[ \sqrt{-g}g^{ij}\Phi E_i^a E_j^b \eta^{ab} + \epsilon^{ij}E_i^B E_k^A \mathcal{B}_{AB} \right], \tag{5.6}$$

and the $\kappa$-symmetry transformation of this action is

$$\begin{aligned} \delta E^a &= \delta Z^M E_M{}^a \;=\; 0, \\ \delta E^\alpha &= \delta Z^M E_M^\alpha \;=\; 2E_i^a(\Gamma_a)^{\alpha\beta} g^{ij}\kappa_{j\beta}. \\ \delta\bar{E}^\alpha &= (\delta\bar{E}^\alpha), \\ \delta\phi &= \phi(\delta E^\alpha \Lambda_\alpha - \delta\bar{E}^\alpha \bar{\Lambda}_a lpha), \\ \delta(\sqrt{-g}g^{ij}) &= 4i(g^{ik}\epsilon^{jl} + \epsilon^{ik}g^{jl})(E_k^\alpha \kappa_{l\alpha} + \bar{E}_k^\alpha \bar{\kappa}_{l\alpha}) \\ &\quad + 2(g^{ij}\epsilon^{kl} - 2\epsilon^{kj}g^{il})E_k^c(\Gamma_c)^{\alpha\beta}(\bar{\kappa}_{l\alpha}\Lambda_\beta - \kappa_{l\alpha}\bar{\Lambda}_\beta). \end{aligned} \tag{5.7}$$

As in the previous section, we also need to substitute the solutions of the Bianchi identities into superfields $E_M^A$, $\phi$, and $\mathcal{B}_{MN}$, which were given in chapter 4.

# Chapter 6

# The normal coordinate expansion

In this chapter, we describe the techniques of the normal coordinate expansion in superspace, which we will use in the following. Before doing so, let us explain the reason why we need the normal coordinate expansion instead of the ordinary Taylor expansion.

## 6.1 The $\theta$-expansion of the action in the superfield formalism

To express the Green-Schwarz action in the superfield formalism in terms of its component fields, we need to expand the action in terms of $\theta$. This expansion may be obtained by the Taylor expansion in superspace around the point ${Z_0}^M = (X^m, 0)$, in terms of $\theta^\mu$. This expansion is not an approximation of the action, since the power series expansion in the fermionic variables terminates at finite order. The maximum order is 16 in the Heterotic theory, and 32 in the Type IIA and Type IIB theory. We will see that the expansion are dramatically simplified by fixing the $\kappa$-symmetry with the light-cone gauge and giving some restrictions on background fields, in the following chapters.

However in the ordinary Taylor expansion such as

$$\begin{aligned} I[Z] &\equiv I[Z_0 + \epsilon] \\ &= I[Z_0] + \int d^2\sigma\ \epsilon^M(\sigma) \frac{\delta}{\delta Z_0^M(\sigma)} I[Z_0] \\ &\quad + \frac{1}{2!} \int d^2\sigma' \int d^2\sigma\ \epsilon^N(\sigma')\epsilon^M(\sigma) \frac{\delta}{\delta Z_0^N(\sigma')} \frac{\delta}{\delta Z_0^M(\sigma)} I[Z_0] \\ &\quad + \cdots, \end{aligned} \tag{6.1}$$

the coefficients are not covariant. In this paper, we need a covariant expansion because the equations satisfied by the space-time background fields are given in terms of tensors in superspace, as we saw in the previous chapters. To get a covariant expansion, we need to use the method of the normal coordinate expansion.

## 6.2 Definition of the normal coordinate expansion

Let us expand the superspace coordinate $Z^M$ of a point in the neighborhood of $Z_0$ as

$$Z^M = Z_0^M + \epsilon^M(Z_0, y) \tag{6.2}$$

where $\epsilon^M(Z_0, y)$ is a function of $Z_0$ and the tangent space vector $y^A = y^M E_M{}^A$ at $Z_0$. $\epsilon^M(Z_0, y)$ is given as follows. Let us consider the geodesic $Z^M(t)$ in superspace satisfying the geodesic equation

$$\frac{dZ^M(t)}{dt} D_M V^A(Z(t)) = 0, \tag{6.3}$$

where $V^A = \frac{dZ^M(t)}{dt} E_M{}^A(Z(t))$ and $D_M$ is the covariant derivative. Here we take the initial condition to be

$$\left\{ \begin{array}{rcl} Z^M(t=0) & = & Z^M \\ \left[\frac{dZ^M}{dt}\right]_{t=0} & = & y^M. \end{array} \right. \tag{6.4}$$

Then we define $\epsilon^M(Z_0, y)$ to be

$$\epsilon^M(Z_0, y) \quad = \quad Z^M(t=1) - Z_0^M \tag{6.5}$$

as depicted in Figure 6.1. When $Z^M$ is in a sufficiently small neighborhood of $Z_0^M$, one can take a coordinate frame $\bar{Z}^M$ in which

$$\bar{Z}^M = \bar{Z}_0^M + \bar{y}^M, \tag{6.6}$$

such a coordinate frame $\bar{Z}^M$ is called the normal coordinate frame. Notice that $y^A(Z_0)$ satisfies an equation

$$y^M D_M y^A(Z_0) = 0, \tag{6.7}$$

which follows from the geodesic equation and means that $y^A$ is covariantly constant along the geodesic line.

We use these definitions to obtain the normal coordinate expansion from the ordinary Taylor expansion (6.1). In the normal coordinate frame $\bar{Z}^M$, all geodesic

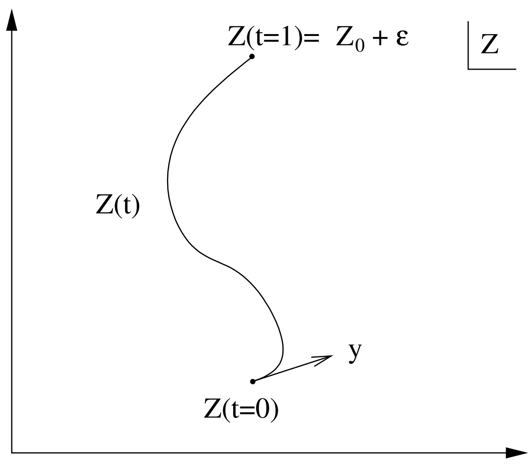


Figure 6.1: Geodesic line $Z^M(t)$

lines are straight lines and $\bar{\epsilon}^M(\bar{Z}_0, \bar{y}) = \bar{y}^M$. Thus the expansion (6.1) can be rewritten in the normal coordinate frame as

$$\begin{aligned}
I[\bar{Z}] &= I[\bar{Z}_0 + \bar{y}] \\
&= I[\bar{Z}_0] + \int d^2\sigma \ \bar{y}^M(\sigma) \frac{\delta}{\delta \bar{Z}_0^M(\sigma)} I[\bar{Z}_0] \\
&\quad + \frac{1}{2!} \int d^2\sigma' \int d^2\sigma \ \bar{y}^N(\sigma') \bar{y}^M(\sigma) \frac{\delta}{\delta \bar{Z}_0^N(\sigma')} \frac{\delta}{\delta \bar{Z}_0^M(\sigma)} I[\bar{Z}_0] \\
&\quad + \cdots \\
&= I[\bar{Z}_0] + \int d^2\sigma \ \bar{y}^M(\sigma) \frac{\delta}{\delta \bar{Z}_0^M(\sigma)} I[\bar{Z}_0] \\
&\quad + \frac{1}{2!} \int d^2\sigma' \ \bar{y}^N(\sigma') \frac{\delta}{\delta \bar{Z}_0^N(\sigma')} \int d^2\sigma \ \bar{y}^M(\sigma) \frac{\delta}{\delta \bar{Z}_0^M(\sigma)} I[\bar{Z}_0] \\
&\quad + \cdots,
\end{aligned} \tag{6.8}$$

because $\bar{y}^M(\sigma)$ commutes with the functional derivatives.

In this form all the functional derivatives are always acting on a scaler, and can be trivially replaced by the functional covariant derivatives. Once all derivatives are written by covariant derivatives $D_M(\sigma)$, we now know that this same expression is correct not only in the normal frame but also in arbitrary frames. Thus we obtain the so-called normal coordinate expansion in arbitrary frames,

$$\begin{aligned}
I[Z] &\equiv I[Z_0 + \epsilon(Z_0, y)] \\
&= I[Z_0] + \int d^2\sigma \ \epsilon^M(\sigma) \frac{\delta}{\delta Z_0^M(\sigma)} I[Z_0] \\
&\quad + \frac{1}{2!} \int d^2\sigma' \ \epsilon^N(\sigma') D_N(\sigma') \int d^2\sigma \ \epsilon^M(\sigma) D_M(\sigma) I[Z_0]
\end{aligned}$$

$$
\begin{aligned}
&+\cdots \\
=\;& I[Z_0] + \Delta[Z_0, y] I[Z_0] \\
&+ \frac{1}{2!}\Delta[Z_0, y]\Delta[Z_0, y] I[Z_0] \\
&+\cdots \\
\equiv\;& e^{\Delta[Z_0, y]} I[Z_0],
\end{aligned}
\tag{6.9}
$$

where

$$
\begin{aligned}
\Delta[Z_0, y] &= \int d^2\sigma \; y^A(\sigma) D_A(\sigma), \\
D_A(\sigma) &= E_A{}^N(Z_0(\sigma)) D_N(\sigma).
\end{aligned}
\tag{6.10}
$$

In an explicit form, the functional covariant derivative $D_A(\sigma)$ acting on a vector $T^B(\sigma')$ is given as

$$
D_A(\sigma) T^B(\sigma') = E_A{}^M(Z_0(\sigma)) \frac{\delta T^B(\sigma')}{\delta Z_0^M(\sigma)} + \delta^{(2)}(\sigma, \sigma')(-)^{AC} T^C(\sigma) \Omega_{AC}{}^B(\sigma). \tag{6.11}
$$

To obtain a covariant $\theta$-expansion, we choose

$$
Z_0^M = (X^m, 0), \quad y^M = (0, \theta^\mu). \tag{6.12}
$$

and go to the Wess-Zumino gauge in which the $\theta = 0$ components of the super-vielbein takes the following form:

$$
E_M{}^A(X) = \begin{pmatrix} e_m{}^a(X) & e_m{}^\alpha(X) \\ e_\mu{}^a(X) = 0 & e_\mu{}^\alpha(X) = \delta_\mu{}^\alpha \end{pmatrix}. \tag{6.13}
$$

The tangent space vectors $y^A$ will then be

$$
\begin{aligned}
y^a &= y^M e_M{}^a(Z_0) \\
&= y^\mu e_\mu{}^a(X) \\
&= 0,
\end{aligned}
\tag{6.14}
$$

$$
\begin{aligned}
y^\alpha &= y^M e_M{}^\alpha(Z_0) \\
&= y^\mu e_\mu{}^\alpha(X) \\
&= \theta^\mu \delta_\mu{}^\alpha.
\end{aligned}
\tag{6.15}
$$

Therefore each term of the expansion is independent of whether it is written by the superspace vectors $y^\mu$, or by the tangent space vectors $y^\alpha$.

Using all these results, we can obtain a manifestly covariant $\theta$-expansion by substituting $y^a = 0$, $y^\alpha = \theta^\alpha$ and substituting $Z_0 = (X, 0)$. In other words, we simply pick up $\theta = 0$ components of all the superfields in the coefficients of the expansion (6.9).

For later convenience, we write down some of the useful identities:

$$
\begin{aligned}
\Delta X^{BC\cdots}_{FG\cdots} &= \int d^2\sigma' y^A(\sigma')(D_A X^{BC\cdots}_{FG\cdots})\delta^{(2)}(\sigma-\sigma') \\
&= y^A D_A X^{BC\cdots}_{FG\cdots}, &(6.16)\\
\Delta y^A &= y^B D_B y^A \\
&= \left[\frac{dZ^M(t)}{dt} D_M V^A(Z(t))\right]_{t=0} \equiv 0, &(6.17)\\
\Delta E_i^A &= \int d^2\sigma' y^M(\sigma') D_M(\sigma') \left(\partial_i Z^N(\sigma) E_N{}^A(\sigma)\right) \\
&= \int d^2\sigma' y^M(\sigma') \Big[\partial_i \left(\delta^{(2)}(\sigma,\sigma')\delta_M{}^N\right) E_N{}^A(\sigma) \\
&\quad + (-)^{MN}\partial_i Z^N(\sigma)\delta^{(2)}(\sigma,\sigma')\partial_M E_N{}^A(\sigma) + (-)^{MB}\delta^{(2)}(\sigma,\sigma')E_i^B(\sigma)\Omega_{MB}{}^A(\sigma)\Big] \\
&= \partial_i y^M E_M{}^A + (-)^{MN} y^M \partial_i Z^N \partial_M E_N{}^A + (-)^{MB} y^M E_i^B \Omega_{MB}{}^A \\
&= \partial_i (y^M E_M{}^A) + (-)^{BC} E_i^B y^C \Omega_{BC}{}^A \\
&\quad + \partial_i Z^N y^M \Big[\partial_M E_N{}^A - (-)^{MN}\partial_N E_M{}^A \\
&\quad + (-)^{N(N+C)} E_N{}^C \Omega_{MC}{}^A - (-)^{NC} E_M{}^C \Omega_{NC}{}^A\Big] \\
&= D_i y^A + \partial_i Z^N y^M T_{MN}{}^A \\
&= D_i y^A + E_i^B y^C T_{CB}{}^A, &(6.18)\\
\Delta(D_i y^A) &= y^B E_i^D y^C R_{CDB}{}^A, &(6.19)
\end{aligned}
$$

where $E_i^A \equiv \partial_i X^M E_M{}^A$ and $D_i \equiv E_i^A D_A$.

Using this normal coordinate expansion method, one can straightforwardly calculate each term of the action. To get the $n$-th order term of the action, we must act the $\Delta$ operator $n$ times on the original action by using the above identities, and then pick up the lowest components of superfields of the given background, for example, the plane-wave with vanishing background fermions, after all the $\Delta$ operations are done. Especially, one can not set the background fermions to zero before all the $\Delta$ operations are done. On the contrary, we can see that any factors including $y^a$ will always vanish when we substitute $y^a = 0$ after all the $\Delta$ operations are done, because of (6.17) and (6.19). Thus one can set $y^a = 0$ anytime, which considerably simplifies the algebra.

# Chapter 7

# The $\theta$-expansion of the Heterotic string action

Now with all the necessary techniques and identities at hand, we start studying the Green-Schwarz action for various cases. In this chapter, we apply the normal coordinate expansion method to obtain the $\theta$-expansion of the heterotic string action. We explicitly prove that the light-cone gauge Green-Schwarz action becomes quadratic in the fermionic coordinates in the backgrounds obtained by the Penrose limit. In addition, we explicitly examine that the claim made in [15] which is that the light-cone gauge Green-Schwarz action in the zero-th order approximation with respect to $c_2$ is quadratic in $\theta$ when the background fields depend only on the transverse coordinates, and prove that it is wrong. Although this claim is made for the action in very restricted backgrounds, essentially the same logic was used in a more recent paper [17] for the Type IIB string, thus our disproof is meaningful as a caution for such mistakes.

In this chapter, we use the notation,

$$Q_{abc} \equiv \theta\Gamma_{abc}\theta, \tag{7.1}$$

$$\begin{aligned} Q_{abcdefg} &\equiv \theta\Gamma_{abcdefg}\theta \\ &= \epsilon_{abcdefghij}\left(\theta\Gamma^{hij}\theta\right), \end{aligned} \tag{7.2}$$

$$Q'_{iabc\cdots} \equiv D_i\theta\Gamma_{abc\cdots}\theta, \tag{7.3}$$

for bilinears of $\theta$.

## 7.1 Generic form of expansion

Before explicitly calculating each term, we present the generic form of each term.

The heterotic string action is written using superfields as

$$I[Z] = \int d^2\sigma \left[L_{\text{kin}} + L_{\text{WZ}} + L_{\text{gauge}}\right],$$

$$
\begin{aligned}
L_{\text{kin}} &= \frac{1}{2}\eta^{ij}\phi(Z)E_i^a(Z)E_{ja}(Z), \\
L_{\text{WZ}} &= \epsilon^{ij}\partial_i Z^N \partial_j Z^M B_{MN}(Z), \\
L_{\text{gauge}} &= \psi^T \mathcal{D}_- \psi.
\end{aligned} \tag{7.4}
$$

We consider the expansion of this action with vanishing gauge field backgrounds for simplicity. Therefore we ignore the third term, $L_{\text{gauge}}$, in (7.4).

From now on, we give the general expression for the $n$-th order terms which is obtained by acting the $\Delta$ operator $n$ times on the action. The terms first order in $\Delta$ are,

$$
\begin{aligned}
\Delta I &= \int d^2\sigma \left[\Delta L_{\text{kin}} + \Delta L_{\text{WZ}}\right], \\
\Delta L_{\text{kin}} &= \eta^{ij}\left(\phi(Z)E_i^a(Z)\Delta E_{ja}(Z) + \frac{1}{2}\Delta\phi(Z)E_i^a(Z)E_{ja}(Z)\right), \\
\Delta L_{\text{WZ}} &= \epsilon^{ij}\Delta\left(\partial_i Z^N \partial_j Z^M B_{MN}(Z)\right) \\
&= \epsilon^{ij}\int d^2\sigma' y^L \frac{\delta}{\delta Z^L(\sigma')}\left(\partial_i Z^N \partial_j Z^M B_{MN}\right) \\
&\sim \epsilon^{ij}\partial_i Z^N \partial_j Z^M y^L \left(\partial_L B_{MN} + (-)^{L(M+N)}\partial_M B_{NL} + (-)^{N(L+M)}\partial_N B_{LM}\right) \\
&= \epsilon^{ij}\partial_i Z^N \partial_j Z^M y^L H_{LMN} \\
&= \epsilon^{ij}E_i^C E_j^B y^A H_{ABC} \\
&= \epsilon^{ij}\left(2E_i^c E_j^\beta y^\alpha \phi(\Gamma_c)_{\alpha\beta} - \frac{1}{2}E_i^b E_j^a y^\alpha (\Gamma_{ab})_\alpha{}^\beta \lambda_\beta\right),
\end{aligned} \tag{7.5}
$$

where the $\sim$ denotes that the expressions are equal up to total derivative terms. Here we have used the definition of $H_{ABC}$ with vanishing gauge field. (7.5) can be written in a more compact form using the light-cone worldsheet indices, $\sigma^\pm = \frac{1}{\sqrt{2}}(\sigma^0 \pm \sigma^1)$:

$$
\begin{aligned}
\Delta L &= \eta^{+-}\phi\left\{E_+^a \Delta E_{-a} + E_-^a \Delta E_{+a}\right\} + \eta^{+-}(y\lambda)E_+^a E_{-a} \\
&\quad + \epsilon^{+-}\phi\left\{E_{+c}\Delta E_-^c - E_{-c}\Delta E_+^c\right\} - \epsilon^{+-}E_+^b E_-^a y\Gamma_{ab}\lambda \\
&= 2\phi E_-^a \Delta E_{+a} + E_+^b E_-^a y\Gamma_a\Gamma_b\lambda,
\end{aligned} \tag{7.6}
$$

where we have used $\Delta\phi = y^\alpha\lambda_\alpha$, $\Delta E_i^a = 2E_i^\beta y^\alpha(\Gamma^a)_{\alpha\beta}$, and $\eta^{+-} = 1$, $\epsilon^{+-} = -1$. The $n$-th order in $\Delta$ will be obtained by further acting $\Delta$'s on (7.5) or (7.6). The general form of the $n$-th order terms are

$$
\begin{aligned}
I^{(n)} &= \frac{1}{n!}\int d^2\sigma \sum_{q=0}^{n-1-q}\sum_{p=0}^{n-1}\frac{(n-1)!}{p!q!(n-1-p-q)!} \\
&\quad \times \Big[(\eta^{ij} + \epsilon^{ij})(\Delta^{n-1-p-q}\phi)(\Delta^p E_i^a)(\Delta^{q+1}E_{ja})
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\eta^{ij}(\Delta^{n-p-q}\phi)(\Delta^p E_i^a)(\Delta^q E_{ja}) \\
&-\frac{1}{2}\epsilon^{ij}(\Delta^p E_i^b)(\Delta^q E_j^a)\{\Delta^{n-1-p-q}(y\Gamma_{ab}\lambda)\}\Bigg],
\end{aligned} \tag{7.7}
$$

or

$$
\begin{aligned}
I^{(n)} &= \frac{1}{n!}\int d^2\sigma \sum_{q=0}^{n-1-q}\sum_{p=0}^{n-1}\frac{(n-1)!}{p!q!(n-1-p-q)!} \\
&\quad \times \eta^{+-}\Big[2(\Delta^{n-1-p-q}\phi)(\Delta^{p+1}E_{-a})(\Delta^q E_+^a) \\
&\quad + \{\Delta^{n-1-p-q}(y\Gamma_a\Gamma_b\lambda)\}(\Delta^p E_-^a)(\Delta^q E_+^b)\Big].
\end{aligned} \tag{7.8}
$$

Then, the $n$-th order terms in the $\theta$-expansion will be obtained from these expressions after performing all the $\Delta$ operations and substituting the background field. Since we set $Z_0^M = (X^m, 0)$, we only need the background superfields at $\theta = 0$.

## 7.2 The zero-th and 2nd order contributions

The zero-th order term of the action is simply given as the original action (7.4) at $\theta = 0$:

$$
I^{(0)} = \int d^2\sigma \left[\frac{1}{2}\eta^{ij}\phi(X)E_i^a(X)E_{ja}(X) + \epsilon^{ij}E_i^d(X)E_j^c(X)B_{cd}(X) + \psi^T\mathcal{D}_-\psi\right]. \tag{7.9}
$$

We often use the same letters as superfields themselves for the lowest components of these.

For the second order term, (7.8) becomes

$$
I^{(2)} = \int d^2\sigma\eta^{+-}\left[\phi(\Delta E_{-a})E_+^a + \frac{1}{2}\Delta(y\Gamma_a\Gamma_b\lambda)E_-^a E_+^b\right]. \tag{7.10}
$$

Here, we have omitted all Grassmann-odd factors after performing $\Delta$ operations. Since we consider vanishing fermionic backgrounds, Grassmann-odd factors such as $\Delta\phi$ and $(y\Gamma_a\Gamma_b\lambda)$ will always vanish after substituting the background fields. We need to evaluate $\Delta E_{ia}$ and $\Delta(y\Gamma_a\Gamma_b\lambda)$. These are easily obtained using (6.16), (6.17), and (6.18) as

$$
\begin{aligned}
\Delta E_i^a &\to E_i^\gamma(X)y^\beta T_{\beta\gamma}{}^a(X) \\
&= 2E_i^\gamma(X)(y\Gamma^a)_\gamma, \\
\Delta(y\Gamma_a\Gamma_b\lambda) &= y\Gamma_a\Gamma_b\Delta\lambda
\end{aligned} \tag{7.11}
$$

$$
\begin{aligned}
&= (y\Gamma_a\Gamma_b)^\alpha \left\{ -(\Gamma^c)_{\alpha\beta} D_c\phi + \frac{\phi}{6} T_{efg} (\Gamma^{efg})_{\alpha\beta} \right\} y^\beta \\
&= -y\Gamma_{ab}{}^c y D_c\phi + \frac{\phi}{2} T_{bfg} y \Gamma_a{}^{fg} y \\
&\quad - \frac{\phi}{2} T_{afg} y \Gamma_b{}^{fg} y + \frac{\phi}{6} \eta_{ab} T_{efg} y \Gamma_{efg} y. \qquad (7.12)
\end{aligned}
$$

By the arrows, we indicate that the substitutions $Z^M \to (X^m, 0)$, and $y^A \to (0, \theta^\alpha)$ have been done. Note that these substitutions should be done after all the $\Delta$ operations are performed. Substituting these two results into (7.10) we obtain

$$
I^{(2)} = -\int d^2\sigma \phi(X) E_{-a}(X) \theta \Gamma^a \tilde{D}_+ \theta, \qquad (7.13)
$$

where

$$
\begin{aligned}
(\tilde{D}_+\theta)^\alpha &= \partial_+\theta^\alpha + \frac{1}{4}\theta^\beta (\Gamma^{bc})_\beta{}^\alpha E_+^a(X) \\
&\quad \times \Big( \omega_{abc}(X) + T_{abc}(X) - (\phi(X))^{-1} \eta_{a[b} D_{c]} \phi(X) \Big).
\end{aligned}
$$

If we take the gauge field background into account, the final result for the second order term of the Green-Schwarz action becomes [15]

$$
\begin{aligned}
I^{(2)} &= \int d^2\sigma \Big[ -\phi E_{-a}(X) \theta \Gamma^a \tilde{D}_+ \theta \\
&\quad - \frac{1}{4} E_{-a}(X) \theta \Gamma^a \Gamma^{bc} \theta \psi^T F_{bc}(X) \psi \\
&\quad - \frac{1}{2} c_1 \epsilon^{ij} E_{-a}(X) \theta \Gamma^a \Gamma^{bc} \theta \mathrm{Tr}\{ A_j(X) F_{bc}(X) \} \Big]. \qquad (7.14)
\end{aligned}
$$

The sum of the zero-th and the second order terms $I^{(0)} + I^{(2)}$ turns out to be the only non-vanishing contributions in the plane-wave background.

## 7.3 The 4th order contributions

In this section, we explicitly calculate the 4th order terms. This calculation is very lengthy, and we need several subsections to present it.

### 7.3.1 A preliminary remark

Before the explicit calculation, we remark on some point which is quite confusing. In constructing the fourth order term, it appears that most terms are trivially vanishing in the light-cone gauge. Indeed it was claimed in [15] that any products of the bilinears of the fermionic coordinates are always vanishing without any explicit proof. But actually their claim is wrong and there are many non-zero contributions, which will be shown below. We will explicitly show that these remaining higher order contributions will vanish only in special backgrounds including the ones obtained by the Penrose limit. Here, let us explain what was wrong about the claim in [15].

Using the characteristic of $\theta$ being a Majorana-Weyl spinor, we can get the relation

$$\theta^\alpha\theta^\beta = \frac{1}{96}(\Gamma_{abc})^{\alpha\beta}(\theta\Gamma^{abc}\theta). \tag{7.15}$$

Thus all the worldsheet scalars made from $\theta$ are written using the bilinear

$$Q^{abc} = \theta\Gamma^{abc}\theta. \tag{7.16}$$

In the light-cone gauge only non-vanishing elements of $Q^{abc}$ are $Q^{-\tilde{a}\tilde{b}}$ where $\tilde{a}$ and $\tilde{b}$ take the transverse values only. The higher order contributions may include product of $Q^{-\tilde{a}\tilde{b}}$'s. We must note that there are relations among $Q^{-\tilde{a}\tilde{b}}Q^{-\tilde{c}\tilde{d}}$ such as

$$Q^{-\tilde{a}\tilde{b}}Q^{-\tilde{c}\tilde{d}} = -\frac{1}{16}\frac{1}{32}Q^{-\tilde{e}\tilde{f}}Q^{-\tilde{g}\tilde{h}}\mathrm{Tr}\left[\Gamma^-\Gamma^+\Gamma^{\tilde{a}\tilde{b}}\Gamma_{\tilde{e}\tilde{f}}\Gamma^{\tilde{c}\tilde{d}}\Gamma_{\tilde{g}\tilde{h}}\right]. \tag{7.17}$$

At first sight, it seems to be possible to show that all such products of two $Q^{-\tilde{a}\tilde{b}}$'s are vanishing because

$$\begin{aligned} &\mathrm{Tr}\left[\Gamma^-\Gamma^+\Gamma^{\tilde{a}\tilde{b}}\Gamma_{\tilde{e}\tilde{f}}\Gamma^{\tilde{c}\tilde{d}}\Gamma_{\tilde{g}\tilde{h}}\right] \\ \simeq\ &\mathrm{Tr}\left[\Gamma^+\Gamma^-\Gamma^{\tilde{a}\tilde{b}}\Gamma_{\tilde{g}\tilde{h}}\Gamma^{\tilde{c}\tilde{d}}\Gamma_{\tilde{e}\tilde{f}}\right] \end{aligned} \tag{7.18}$$

which may be shown by transposing the matrices in the trace. *If* the equation (7.18) is correct, we obtain the relations

$$\begin{aligned} &Q^{-\tilde{e}\tilde{f}}Q^{-\tilde{g}\tilde{h}}\mathrm{Tr}\left[\Gamma^-\Gamma^+\Gamma^{\tilde{a}\tilde{b}}\Gamma_{\tilde{e}\tilde{f}}\Gamma^{\tilde{c}\tilde{d}}\Gamma_{\tilde{g}\tilde{h}}\right] \\ =\ &Q^{-\tilde{e}\tilde{f}}Q^{-\tilde{g}\tilde{h}}\mathrm{Tr}\left[\Gamma^{\tilde{a}\tilde{b}}\Gamma_{\tilde{e}\tilde{f}}\Gamma^{\tilde{c}\tilde{d}}\Gamma_{\tilde{g}\tilde{h}}\right], \end{aligned} \tag{7.19}$$

$$\begin{aligned} Q^{-\tilde{a}\tilde{b}}Q^{-\tilde{c}\tilde{d}} &= -\frac{1}{16}\frac{1}{32}Q^{-\tilde{e}\tilde{f}}Q^{-\tilde{g}\tilde{h}}\mathrm{Tr}\left[\Gamma^{\tilde{a}\tilde{b}}\Gamma_{\tilde{e}\tilde{f}}\Gamma^{\tilde{c}\tilde{d}}\Gamma_{\tilde{g}\tilde{h}}\right] \\ &= \frac{1}{4}\left(-Q^{-\tilde{a}\tilde{b}}Q^{-\tilde{c}\tilde{d}} - Q^{-\tilde{a}\tilde{c}}Q^{-\tilde{b}\tilde{d}} + Q^{-\tilde{a}\tilde{d}}Q^{-\tilde{b}\tilde{c}}\right), \end{aligned} \tag{7.20}$$

and (7.20) do not have any solutions except $Q^{-\tilde{a}\tilde{b}}Q^{-\tilde{c}\tilde{d}} = 0$ when we consider it as simultaneous equations of $Q^{-\tilde{a}\tilde{b}}Q^{-\tilde{c}\tilde{d}}$ .

Actually the equation (7.18) is not correct. The two sets of $\Gamma$ matrices with two lower/upper spinor indices , $\bar{\Gamma}^a \equiv (\Gamma^a)_{\alpha\beta}$ and $\Gamma^a \equiv (\Gamma^a)^{\alpha\beta}$, are different sets of matrices. We can omit the bar on $\bar{\Gamma}$ only when it causes no confusion, but this is not the case here. The equation (7.18) should be corrected as

$$\begin{aligned} & \mathrm{Tr}\left[\Gamma^- \bar{\Gamma}^+ \Gamma^{\tilde{a}\tilde{b}} \Gamma_{\tilde{e}\tilde{f}} \Gamma^{\tilde{c}\tilde{d}} \Gamma_{\tilde{g}\tilde{h}}\right] \\ = \; & \mathrm{Tr}\left[\bar{\Gamma}^+ \Gamma^- \bar{\Gamma}^{\tilde{c}\tilde{d}} \bar{\Gamma}_{\tilde{e}\tilde{f}} \bar{\Gamma}^{\tilde{a}\tilde{b}} \bar{\Gamma}_{\tilde{g}\tilde{h}}\right]. \end{aligned} \tag{7.21}$$

Therefore the product of two $Q^{-\tilde{a}\tilde{b}}$'s are not zero.

In addition to the $QQ$ terms, there are terms with covariant derivative of $\theta^\alpha$, $D_i\theta^\alpha$, but in a following analysis we can easily see that all the contributions of order $\theta^4$ and higher with some factors of $D_i\theta^\alpha$ do always vanish in the light-cone gauge.

### 7.3.2 Various factors in $I^{(4)}$

Let us begin to calculate the fourth order term explicitly. From (7.7), we can extract the fourth order terms,

$$\begin{aligned} I^{(4)} = \; & \int d^2\sigma \Big[\frac{1}{24}\phi(\eta^{ij} + \epsilon^{ij}) E_{ia}\Delta^4 E^a_j + \frac{1}{8}\phi\eta^{ij}\Delta^2 E_{ia}\Delta^2 E^a_j \\ & + \frac{1}{8}\Delta^2\phi(2\eta^{ij} + \epsilon^{ij}) E_{ia}\Delta^2 E^a_j + \frac{1}{32}\Delta^4\phi\eta^{ij} E_{ia} E^a_j \\ & + \frac{1}{8}\Delta(y\Gamma_{ab}\lambda)\epsilon^{ij}\Delta^2 E^a_i E^b_j \\ & + \frac{1}{48}\Delta^3(y\Gamma_{ab}\lambda)\epsilon^{ij} E^a_i E^b_j\Big]. \end{aligned} \tag{7.22}$$

Here, we have used $\Delta(y\lambda) \equiv \Delta^2\phi$. Now we need various factors given by the action of $\Delta$ operators:

$$\Delta^4 E^\alpha_i, \quad \Delta^4\phi, \quad \Delta^3(y\Gamma_{ab}\lambda). \tag{7.23}$$

$\Delta^4\phi$ is the easiest one to get. Repeatedly substituting (4.94), we obtain

$$\begin{aligned} \Delta^2\phi &= y^\alpha y^\beta D_\beta\lambda_\alpha \\ &= \frac{\phi}{6} T_{abc} y^\beta (\Gamma^{abc})_{\beta\alpha} y^\alpha, \\ \Delta^4\phi &\to \frac{1}{6}\Delta^2\phi T_{abc} y^\beta (\Gamma^{abc})_{\beta\alpha} y^\alpha + \frac{1}{6}\phi\Delta^2 T_{abc} y^\beta (\Gamma^{abc})_{\beta\alpha} y^\alpha \\ &= \frac{1}{6\cdot 6}\phi T_{abc} T_{def} (y\Gamma^{abc} y)(y\Gamma^{def} y) + \frac{1}{6}\phi\Delta^2 T_{abc} y\Gamma^{abc} y, \end{aligned} \tag{7.24}$$

where the arrow indicates that we have substituted $Z^M \to (X^m, 0)$, and $y^A \to (0, \theta^\alpha)$ in the same way as in section 7.2. We can do this substitution now since this factor is already of the fourth order in $\theta^\alpha$, and all the $\Delta$ operations have already been done. Therefore $\Delta^4\phi$ is now written by using $\Delta^2 T_{abc}$. The calculation of $\Delta^2 T_{abc}$ is very complicated and will be done later.

$\Delta^3(y\Gamma_{ab}\lambda)$ is also written by using $\Delta^2 T_{abc}$:

$$\begin{aligned}\Delta(y\Gamma_{ab}\lambda) &= (y\Gamma_{ab})^\alpha \left\{ -(\Gamma^c)_{\alpha\beta} D_c\phi + \frac{\phi}{6} T_{efg} (\Gamma^{efg})_{\alpha\beta} \right\} y^\beta \\ &= -y\Gamma_{ab}{}^c y D_c\phi + \frac{\phi}{2} T_{bfg} y\Gamma_a{}^{fg} y - \frac{\phi}{2} T_{afg} y\Gamma_b{}^{fg} y. \end{aligned} \tag{7.25}$$

$$\begin{aligned}\Delta^3(y\Gamma_{ab}\lambda) &= -y\Gamma_{ab}{}^c y D_c \left( \frac{\phi}{6} T_{efg} \right) y\Gamma^{efg} y \\ &\quad + \Delta^2 \left( \frac{\phi}{2} T_{bfg} \right) y\Gamma_a{}^{fg} y - \Delta^2 \left( \frac{\phi}{2} T_{afg} \right) y\Gamma_b{}^{fg} y. \end{aligned} \tag{7.26}$$

The calculation of $\Delta^4 E_i^a$ is much more complicated. Using (6.18) and (6.19), we act the $\Delta$ operators on $E_i^a$ one by one:

$$\begin{aligned}\Delta^2 E_i^a &= 2\Delta E_i^\gamma (y\Gamma^a)_\gamma \\ &= 2D_i y\Gamma^a y + \frac{1}{12} E_i^c T_{def} y\Gamma_c \Gamma^{def} \Gamma^a y \\ &\to 2D_i y\Gamma^a y + \frac{1}{4} E_i^c T_{cef} y\Gamma^{efa} y \\ &\quad - \frac{1}{12} E_i^a T_{def} y\Gamma^{def} y + \frac{1}{4} E_i^c T_{de}{}^a y\Gamma_c{}^{de} y, \end{aligned} \tag{7.27}$$

$$\begin{aligned}\Delta^4 E_i^a &= 2\Delta^3 E_i^\gamma (y\Gamma^a)_\gamma \\ &= \frac{1}{12} y\Gamma_c \Gamma^{efg} \Gamma^a y \Delta^2 E_i^c T_{efg} \\ &\quad - \frac{1}{2} y\Gamma^{cd} \Gamma^a y \cdot y^\gamma R_{\gamma\delta cd} \Delta E_i^\delta \\ &\quad + \frac{1}{12} y\Gamma_c \Gamma^{efg} \Gamma^a y E_i^c \Delta^2 T_{efg} \\ &\quad - \frac{1}{2} y\Gamma^{cd} \Gamma^a y \cdot y^\gamma \Delta R_{\gamma ecd} E_i^e. \end{aligned} \tag{7.28}$$

The result given in (7.28) includes four terms. The first term is easily calculated by substituting $\Delta^2 E_i^a$. For the second term, we need $R_{\alpha\beta cd}$ which is related to $T_{abc}$ by (4.92). For the third term, we again need $\Delta^2 T_{abc}$. For the fourth term, we need $\Delta R_{\alpha bcd}$ which is related to $\Delta T_{ab}{}^\gamma$ and $\Delta D_\alpha T_{bcd}$ by (4.93). $T_{ab}{}^\gamma$ is again related to $D_\alpha T_{bcd}$ by (4.91). Each calculation of the four terms in $\Delta^4 E_i^a$ is written below. We postpone the calculations of two complicated factors $\Delta^2 T_{abc}$ and $\theta^\gamma\theta^\delta D_\delta T_{[ec}{}^\beta\Gamma_{d]\beta\gamma}$ until the next subsection.

The first term of $\Delta^4 E_i^a$ is

$$\begin{aligned}
&\frac{1}{12}y\Gamma_c\Gamma^{efg}\Gamma^a y\Delta^2 E_i^c T_{efg} \\
= \;& \frac{1}{4}y\Gamma^{fga}y\Delta^2 E_i^c T_{cfg} - \frac{1}{12}y\Gamma^{efg}y\Delta^2 E_i^a T_{efg} + \frac{1}{4}y\Gamma^{cef}y\Delta^2 E_{ic}T_{ef}{}^a \\
= \;& \frac{1}{4}Q^{afg}T_{fgc}\Big[2Q_i'^c + \frac{1}{4}Q^{ce'f'}T_{e'f'c'}E_i^{c'} \\
& - \frac{1}{12}E_i^c T_{d'e'f'}Q^{d'e'f'} + \frac{1}{4}T^c{}_{d'e'}Q^{d'e'}{}_{c'}E_i^{c'}\Big] \\
& - \frac{1}{12}Q^{efg}T_{efg}\Big[2Q_i'^a + \frac{1}{4}Q^{ae'f'}T_{e'f'c'}E_i^{c'} \\
& - \frac{1}{12}E_i^a T_{d'e'f'}Q^{d'e'f'} + \frac{1}{4}T^a{}_{d'e'}Q^{d'e'}{}_{c'}E_i^{c'}\Big] \\
& + \frac{1}{4}T^a{}_{ef}Q^{ef}{}_c\Big[2Q_i'^c + \frac{1}{4}Q^{ce'f'}T_{e'f'c'}E_i^{c'} \\
& - \frac{1}{12}E_i^c T_{d'e'f'}Q^{d'e'f'} + \frac{1}{4}T^c{}_{d'e'}Q^{d'e'}{}_{c'}E_i^{c'}\Big].
\end{aligned} \tag{7.29}$$

The second term of $\Delta^4 E_i^a$ is

$$\begin{aligned}
& -\frac{1}{2}y\Gamma^{cd}\Gamma^a y\cdot y^\gamma R_{\gamma\delta cd}\Delta E_i^\delta \\
= \;& -\frac{1}{2}y\Gamma^{cda}y\cdot y^\gamma\Delta E_i^\delta\left[\frac{1}{6}T_{efg}(\Gamma_{cd}{}^{efg})_{\gamma\delta} + 3T_{cde}\Gamma^e{}_{\gamma\delta}\right] \\
= \;& +\frac{1}{2}y\Gamma^{cda}y\cdot D_i y\left[\frac{1}{6}T_{efg}\Gamma_{cd}{}^{efg} + 3T_{cde}\Gamma^e\right]y \\
& + \frac{1}{48}y\Gamma^{cda}y\cdot E_i^{c'}T_{d'e'f'}\cdot y\left[\frac{1}{6}T_{efg}\Gamma_{c'}\Gamma^{d'e'f'}\Gamma_{cd}{}^{efg} + 3T_{cde}\Gamma_{c'}\Gamma^{d'e'f'}\Gamma^e\right]y \\
= \;& +\frac{1}{2}Q^{cda}\cdot\left[\frac{1}{6}T_{efg}Q'_{icd}{}^{efg} + 3T_{cde}Q_i'^e\right] \\
& + \frac{1}{48}Q^{cda}\cdot E_i^{c'}\Big[-T_{efg}T^{efg}Q_{c'cd} + 12T_{dfg}T^{fge}Q_{c'ce} \\
& - 6T_{cd}{}^e T_{efg}Q_{c'}{}^{fg} + 9T_{cde}T^e{}_{fg}Q_{c'}{}^{fg}\Big] \\
& + \frac{1}{48}Q^{cda}\cdot E_i^b\Big[-12T_b{}^{ef}T_{efg}Q_{cd}{}^g - 2T_{bcd}T_{efg}Q^{efg} \\
& - 6T_{bd}{}^e T_{efg}Q_c{}^{fg} + 9T_{bfg}T_{cde}Q^{fge} - 3T_{efg}T_{cdb}Q^{efg}\Big] \\
& + \frac{1}{48}Q^{cda}\cdot E_{id}\Big[+6T_{cb}{}^e T_{efg}Q^{bfg}\Big] \\
& + \frac{3\cdot 5}{48\cdot 6}E_i^{c'}T^{d'e'}{}_c Q^{a[cd}T^{efg]}y\Gamma_{c'd'e'defg}y
\end{aligned}$$

$$+\frac{3}{48\cdot 6}E_i^{c'}T_{c'}{}^{e'f'}Q^{cda}T^{efg}y\Gamma_{e'f'cdefg}y$$
$$+\frac{5}{48\cdot 6}E_{ic}T^{d'e'f'}Q^{a[cd}T^{efg]}y\Gamma_{d'e'f'defg}y. \quad (7.30)$$

Here, we have used the algebra of gamma matrices to decompose various bilinear factors and rewritten it using $Q$'s.

The third term of $\Delta^4 E_i^a$ is

$$\frac{1}{12}y\Gamma_c\Gamma^{efg}\Gamma^a y E_i^c \Delta^2 T_{efg}$$
$$\to \frac{1}{4}Q^{afg}\Delta^2 T_{fge}E_i^e + \frac{1}{4}\Delta^2 T^a{}_{fg}Q^{fg}{}_c E_i^c - \frac{1}{12}E_i^a Q^{efg}\Delta^2 T_{efg}. \quad (7.31)$$

And the fourth term of $\Delta^4 E_i^a$ is

$$-\frac{1}{2}y\Gamma^{cd}\Gamma^a y\cdot y^\gamma \Delta R_{\gamma ecd}E_i^e$$
$$\to -\frac{1}{2}Q^{cda}\cdot E_i^e\left[\frac{1}{2}\Delta^2 T_{ecd} + 3\theta^\gamma\theta^\delta D_\delta T_{[ec}{}^\beta\Gamma_{d]\beta\gamma}\right]. \quad (7.32)$$

Using all these results on $\Delta^4 E_i^\alpha$, $\Delta^4\phi$ and $\Delta^3(y\Gamma_{ab}\lambda)$, each term in (7.22) is given as follows.

$$\frac{1}{24}\phi(\eta^{ij}+\epsilon^{ij})E_{ia}\Delta^4 E_j^a$$
$$= \frac{1}{24}\phi(\eta^{ij}+\epsilon^{ij})\Bigg[+\frac{1}{8}E_{ia}Q^{ade}T_{def}T^{fgh}Q_{ghb}E_j^b + \frac{1}{8}E_{ia}T^{ade}Q_{def}Q^{fgh}T_{ghb}E_j^b$$
$$+\frac{1}{4}E_{ia}Q^{ade}T_{def}Q^{fgh}T_{ghb}E_j^b - \frac{1}{4}E_{ia}Q^{ade}Q_{def}T^{fgh}T_{ghb}E_j^b$$
$$-\frac{1}{4}E_{ia}Q^{ahd}T_{def}T^{efg}Q_{ghb}E_j^b - \frac{1}{8}E_{ia}Q^{ahd}Q_{def}T^{efg}T_{ghb}E_j^b$$
$$+\frac{1}{8}E_{ia}Q^{ahb}E_{jb}T_{hd}{}^e T_{efg}Q^{fgd}$$
$$+2E_{ia}Q^{ade}T_{deb}Q'^{\,b}_j + \frac{1}{2}E_{ia}T^{ade}Q_{deb}Q'^{\,b}_j$$
$$+\frac{1}{12}E_{ia}Q^{ade}Q'_{jdefgh}T^{fgh} + \frac{1}{48}E_{ia}E_j^{c'}T^{d'e'}{}_c Q^{acd}T^{efg}Q_{c'd'e'defg}$$
$$+\frac{3}{48\cdot 2}E_{ia}E_j^{c'}T^{d'e'}{}_c Q^{afg}T^{cde}Q_{c'd'e'defg} + \frac{1}{48\cdot 3}E_{ia}E_{jc}T^{d'e'f'}Q^{acd}T^{efg}Q_{d'e'f'defg}$$
$$+\frac{1}{48}E_{ia}Q^{afg}T^{d'e'f'}Q_{fgd'e'f'de}T^{dec}E_{jc}$$
$$+\frac{1}{4}E_{ia}\Delta^2 T^a{}_{fg}Q^{fg}{}_c E_j^c - \frac{1}{12}E_{ia}E_j^a Q^{efg}\Delta^2 T_{efg} - \frac{3}{2}E_{ia}Q^{acd}\theta^\gamma\theta^\delta D_\delta T_{[ec}{}^\beta\Gamma_{d]\beta\gamma}E_j^e$$
$$-\frac{7}{48}E_{ia}Q^{ade}T_{deb}E_j^b(T\cdot Q) - \frac{1}{24}E_{ia}T^{ade}Q_{deb}E_j^b(T\cdot Q)$$

$$
-\frac{2}{12}E_{ia}Q'^{\,a}_{j}(T\cdot Q)+\frac{1}{12\cdot 12}E_{ia}E^a_j(T\cdot Q)(T\cdot Q)
$$
$$
-\frac{1}{48}E_{ia}Q^{ade}Q_{deb}E^b_j(T\cdot T)\Bigg], \qquad (7.33)
$$

where $(T\cdot Q)\equiv T_{abc}Q^{abc}$, $(T\cdot T)\equiv T_{abc}T^{abc}$.

$$
\begin{aligned}
&\frac{1}{32}\Delta^4\phi\eta^{ij}E_{ia}E^a_j\\
&= \frac{1}{24}\phi(\eta^{ij}+\epsilon^{ij})\left[\frac{1}{48}E_{ia}E^a_j(T\cdot Q)(T\cdot Q)+\frac{1}{8}E_{ia}E^a_j\Delta^2T_{def}Q^{def}\right]. \qquad (7.34)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{48}\Delta^3(y\Gamma_{ab}\lambda)\epsilon^{ij}E^a_iE^b_j\\
&= \frac{1}{48}\epsilon^{ij}\Bigg[E_{ia}Q^{acb}E_{jb}D_c\left(\frac{\phi}{6}T_{efg}\right)Q^{efg}\\
&\quad+\frac{\phi}{2}E^a_iQ_a{}^{fg}\Delta^2T_{fgb}E^b_j-\frac{\phi}{2}E^a_i\Delta^2T_{afg}Q_b{}^{fg}E^b_j\\
&\quad+\frac{\phi}{12}E^a_iQ_a{}^{fg}T_{fgb}E^b_j(T\cdot Q)-\frac{\phi}{12}E^a_iT_{afg}Q_b{}^{fg}E^b_j(T\cdot Q)\Bigg]. \qquad (7.35)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{8}\phi\eta^{ij}\Delta^2E_{ia}\Delta^2E^a_j\\
&= \frac{1}{8}\phi\eta^{ij}\\
&\quad\times\left[2Q'^{a}_i+\frac{1}{4}E^{c'}_iT_{c'e'f'}Q^{e'f'a}-\frac{1}{12}E^a_i(T\cdot Q)+\frac{1}{4}E^{c'}_iQ_{c'}{}^{d'e'}T_{d'e'}{}^a\right]\\
&\quad\times\left[2Q'_{ja}+\frac{1}{4}Q_a{}^{ef}T_{efc}E^c_j-\frac{1}{12}E_{ja}(T\cdot Q)+\frac{1}{4}T_{ade}Q^{de}{}_cE^c_j\right]\\
&= \frac{1}{24}\phi(\eta^{ij}+\epsilon^{ij})\\
&\quad\times\Bigg[\frac{3}{2}Q'^{a}_iQ_a{}^{ef}T_{efc}E^c_j+\frac{3}{2}E^{c'}_iT_{c'e'f'}Q^{e'f'a}Q'_{ja}\\
&\quad-\frac{1}{2}Q'^{a}_iE_{ja}(T\cdot Q)-\frac{1}{2}E^a_i(T\cdot Q)Q'_{ja}\\
&\quad+\frac{3}{2}Q'^{a}_iT_{ade}Q^{de}{}_cE^c_j+\frac{3}{2}E^{c'}_iQ_{c'}{}^{d'e'}T_{d'e'}{}^aQ'_{ja}\\
&\quad+\frac{3}{16}E^{c'}_iT_{c'e'f'}Q^{e'f'a}Q_a{}^{ef}T_{efc}E^c_j+\frac{3}{16}E^{c'}_iQ_{c'}{}^{d'e'}T_{d'e'}{}^aT_{ade}Q^{de}{}_cE^c_j\\
&\quad+\frac{3}{16}E^{c'}_iT_{c'e'f'}Q^{e'f'a}T_{ade}Q^{de}{}_cE^c_j+\frac{3}{16}E^{c'}_iQ_{c'}{}^{d'e'}T_{d'e'}{}^aQ_a{}^{ef}T_{efc}E^c_j
\end{aligned}
$$

$$
- \frac{1}{8} E_i^a Q_a{}^{ef} T_{efc} E_j^c (T \cdot Q) - \frac{1}{8} E_i^a T_{ade} Q^{de}{}_c E_j^c (T \cdot Q)
+ \frac{1}{48} E_i^a E_{ja} (T \cdot Q)(T \cdot Q) \Big]. \tag{7.36}
$$

$$
\begin{aligned}
&\frac{1}{8} \Delta^2 \phi (2\eta^{ij} + \epsilon^{ij}) E_{ia} \Delta^2 E_j^a \\
&= \frac{1}{48} \phi (T \cdot Q)(2\eta^{ij} + \epsilon^{ij}) \\
&\quad \times \Big[ 2 E_i{}^a Q'_{ja} + \frac{1}{4} E_i{}^a Q_a{}^{ef} T_{efc} E_j^c - \frac{1}{12} E_i{}^a E_{ja} (T \cdot Q) + \frac{1}{4} E_i{}^a T_{ade} Q^{de}{}_c E_j^c \Big] \\
&= \frac{1}{24} \phi (T \cdot Q)(\eta^{ij} + \epsilon^{ij}) \\
&\quad \times \Big[ \frac{3}{2} E_i{}^a Q'_{ja} + \frac{1}{2} Q'_{ia} E_j{}^a - \frac{1}{12} E_i{}^a E_{ja} (T \cdot Q) \\
&\quad + \frac{1}{4} E_i{}^a Q_a{}^{ef} T_{efc} E_j^c + \frac{1}{4} E_i{}^a T_{ade} Q^{de}{}_c E_j^c \Big].
\end{aligned} \tag{7.37}
$$

$$
\begin{aligned}
&\frac{1}{48} \Delta (y \Gamma_{ab} \lambda) \epsilon^{ij} \Delta^2 E_i^a E_j^b \\
&= \frac{1}{48} \Big[ -Q_{ab}{}^c D_c \phi + \frac{\phi}{2} Q_a{}^{cd} T_{cdb} - \frac{\phi}{2} T_{acd} Q^{cd}{}_b \Big] \epsilon^{ij} \\
&\quad \times \Big[ 2 Q_i'^a + \frac{1}{4} E_i^e T_{efg} Q^{fga} - \frac{1}{12} E_i^a (T \cdot Q) + \frac{1}{4} E_i^e Q_{efg} T^{fga} \Big] E_j^b \\
&= \frac{\phi}{24} (\eta^{ij} + \epsilon^{ij}) \Big[ - \phi^{-1} Q_i'^a Q_{ab}{}^c D_c \phi E_j^b \\
&\quad + \frac{1}{4} Q_i'^a Q_{acd} T^{cdb} E_{jb} - \frac{1}{4} E_{ib} T^{bcd} Q_{cda} Q_j'{}^a \\
&\quad - \frac{1}{4} Q_i'^a T_{acd} Q^{cdb} E_{jb} + \frac{1}{4} E_{ib} Q^{bcd} T_{cda} Q_j'{}^a \\
&\quad - \frac{1}{16} E_i^e T_{efg} Q^{fga} T_{acd} Q^{cdb} E_{jb} + \frac{1}{16} E_i^e Q_{efg} T^{fga} Q_{acd} T^{cdb} E_{jb} \\
&\quad - \frac{\phi^{-1}}{16} E_i^e T_{efg} Q^{fga} Q_{ab}{}^c D_c \phi E_j^b - \frac{\phi^{-1}}{16} E_i^b Q_{ba}{}^c D_c \phi Q^{afg} T_{fge} E_j^e \\
&\quad - \frac{\phi^{-1}}{16} E_i^e Q_{efg} T^{fga} Q_{ab}{}^c D_c \phi E_j^b - \frac{\phi^{-1}}{16} E_i^b Q_{ba}{}^c D_c \phi T^{afg} Q_{fge} E_j^e \\
&\quad + \frac{\phi^{-1}}{24} E_i^a Q_{ab}{}^c D_c \phi E_j^b (T \cdot Q) \\
&\quad - \frac{1}{48} E_i^a Q_{acd} T^{cdb} E_{jb} (T \cdot Q) + \frac{1}{48} E_i^a T_{acd} Q^{cdb} E_{jb} (T \cdot Q) \Big].
\end{aligned} \tag{7.38}
$$

Summing up these six terms (and substituting the two remaining factors), we will get the fourth order contributions, $I^{(4)}$, of the $\theta$-expansion.

### 7.3.3 The $\Delta^2 T_{abc}$

We will calculate the two remaining factors

$$\Delta^2 T_{abc} \quad \equiv \quad \theta^\gamma \theta^\delta [D_\delta D_\gamma T_{abc}] \tag{7.39}$$

and

$$\theta^\gamma \theta^\delta D_\delta T_{[ec}{}^\beta \Gamma_{d]\beta\gamma} \tag{7.40}$$

in this subsection. Actually the latter one will be obtained in the course of the calculation of $\Delta^2 T_{abc}$.

For the first fermionic derivative of $T_{abc}$, we can use (4.95),

$$\begin{aligned} D_\gamma T_{abc} \quad = \quad & 2T_{[ab}{}^\alpha \Gamma_{c]\alpha\gamma} + \phi^{-1} D_{[a}\lambda_{\hat{\beta}}(\Gamma_{bc]})_\gamma{}^\beta - \phi^{-1} T_{abc}\lambda_\gamma \\ & - \phi^{-1} T_{[ab}{}^d (\Gamma_{c]d})_\gamma{}^\beta \lambda_\beta - \phi^{-1}(\Gamma_{[a}\Psi\Gamma_{bc]})_\gamma{}^\beta \lambda_\beta + \mathcal{O}(F\chi). \end{aligned} \tag{7.41}$$

$D_\beta \lambda_\alpha$ and $D_\beta T_{bc}{}^\alpha$ in this equation are related to the lowest component fields by (4.94) and (4.97). An expression of $D_\delta D_\gamma T_{abc}$ can be obtained by acting one more fermionic derivative $D_\delta$ on (4.95) as

$$\begin{aligned} D_\delta D_\gamma T_{abc} \quad \rightarrow \quad & 2D_\delta T_{[ab}{}^\alpha \Gamma_{c]\alpha\gamma} \\ & - \phi^{-1} D_{[a} D_{\hat{\delta}} \lambda_{\hat{\beta}} (\Gamma_{bc]})^\beta{}_\gamma - \phi^{-1} T_{[a\hat{\delta}}{}^\alpha D_{\hat{\alpha}} \lambda_{\hat{\beta}} (\Gamma_{bc]})^\beta{}_\gamma \\ & - \phi^{-1} T_{abc} D_\delta \lambda_\gamma \\ & + \phi^{-1} T_{[ab}{}^d D_{\hat{\delta}} \lambda_{\hat{\beta}} (\Gamma_{c]d})^\beta{}_\gamma - \phi^{-1} D_\delta \lambda_\beta (\Gamma_{[bc}\Psi\Gamma_{a]})^\beta{}_\gamma \\ = \quad & \Big[ - 4D_{[a}(\Gamma_b \Psi \Gamma_{c]})_{\delta\gamma} - 2T_{[ab}{}^e (\Gamma_e \Psi \Gamma_{c]})_{\delta\gamma} \\ & - 4(\Gamma_{[a}\Psi\Gamma_b\Psi\Gamma_{c]})_{\delta\gamma} + \frac{1}{2} R_{[ab\hat{d}\hat{e}} (\Gamma^{de}\Gamma_{c]})_{\delta\gamma} \Big] \\ & - \phi^{-1} \left[ -D_{[a} D_{\hat{d}} \phi \Gamma^d \Gamma_{bc]} - \frac{1}{6} D_{[a}\phi T_{\hat{d}\hat{e}\hat{f}} \Gamma^{def} \Gamma_{bc]} - \frac{\phi}{6} D_{[a} T_{\hat{d}\hat{e}\hat{f}} \Gamma^{def} \Gamma_{bc]} \right]_{\delta\gamma} \\ & - \phi^{-1} \left[ -D_d \phi \Gamma_{[a} \Psi \Gamma^d \Gamma_{bc]} - \frac{1}{6} T_{def} \Gamma_{[a} \Psi \Gamma^{def} \Gamma_{bc]} \right]_{\delta\gamma} \\ & - \phi^{-1} T_{abc} \left[ -\Gamma^d D_d \phi - \frac{\phi}{6} T_{def} \Gamma^{def} \right]_{\delta\gamma} \\ & + \phi^{-1} \left[ -T_{[ab}{}^d D_{\hat{e}} \phi \Gamma^e \Gamma_{c]d} - \frac{\phi}{6} T_{[ab}{}^d T_{\hat{e}\hat{f}\hat{g}} \Gamma^{efg} \Gamma_{c]d} \right]_{\delta\gamma} \\ & - \phi^{-1} \left[ -D_d \phi \Gamma^d \Gamma_{[bc} \Psi \Gamma_{a]} - \frac{\phi}{6} T_{def} \Gamma^{def} \Gamma_{[bc} \Psi \Gamma_{a]} \right]_{\delta\gamma} . \end{aligned} \tag{7.42}$$

We will calculate each terms in parenthesis.
The the first term in $D_\delta D_\gamma T_{abc}$ is :

$$
\begin{aligned}
& \theta^\gamma\theta^\delta\Big[-4D_{[a}(\Gamma_b\Psi\Gamma_{c]})_{\delta\gamma}-2T_{[ab}{}^e(\Gamma_e\Psi\Gamma_{c]})_{\delta\gamma} \\
& -4(\Gamma_{[a}\Psi\Gamma_b\Psi\Gamma_{c]})_{\delta\gamma}+\frac{1}{2}R_{[ab\hat{d}\hat{e}}(\Gamma^{de}\Gamma_{c]})_{\delta\gamma}\Big] \\
= & -\frac{4}{24}D_{[a}T_{\hat{d}\hat{e}\hat{f}}\theta\Gamma_b\Gamma^{def}\Gamma_{c]}\theta \\
& -\frac{2}{24}T^{fgh}T_{[ab}{}^e\theta\Gamma_{\hat{e}}\Gamma_{\hat{f}\hat{g}\hat{h}}\Gamma_{c]}\theta \\
& +\frac{4}{24\cdot 24}T_{def}T_{ghi}\theta\Gamma_{[a}\Gamma^{def}\Gamma_b\Gamma^{ghi}\Gamma_{c]}\theta \\
& -\frac{1}{2}R_{[ab\hat{d}\hat{e}}\theta\Gamma^{de}\Gamma_{c]}\theta \\
= & \Big[-\frac{3}{12}T_e{}^{gh}T_{[ab}{}^eQ_{\hat{g}\hat{h}c]}-\frac{3}{12}T^{fg}{}_{[c}T_{ab]}{}^eQ_{\hat{e}\hat{f}\hat{g}}+\frac{1}{12}T^{fgh}T_{abc}Q_{\hat{f}\hat{g}\hat{h}}\Big] \\
& +\Big[\frac{1}{12}T_{abc}T_{def}Q^{def}-\frac{1}{4}T_{[ab}{}^fT_{c]}{}^{hi}Q_{fhi}-\frac{3}{4}T_{[ab}{}^fT_{\hat{f}}{}^{hi}Q_{\hat{h}\hat{i}c]}-\frac{3}{4}T_{[a}{}^{ef}T_{b\hat{e}}{}^iQ_{\hat{f}\hat{i}c]} \\
& -\frac{1}{4}T_{[a}{}^{ef}T_{\hat{e}\hat{f}}{}^iQ_{bc]i}+\frac{1}{24}T_{[a}{}^{ef}T^{ghi}Q_{bc]efghi}-\frac{1}{16}T^{def}T_f{}^{hi}Q_{abcdehi}\Big] \\
& -\frac{1}{2}R_{[ab\hat{d}\hat{e}}Q^{de}{}_{c]} \\
= & -T_{[ab}{}^dT_{\hat{d}\hat{f}\hat{g}}Q^{fg}{}_{c]}-\frac{1}{2}T_{[ab}{}^eQ_{\hat{e}\hat{f}\hat{g}}T^{fg}{}_{c]}+\frac{1}{6}T_{abc}T_{def}Q^{def} \\
& -\frac{3}{4}T_{[a\hat{d}}{}^eT_{b\hat{e}}{}^fQ_{c]f}{}^d-\frac{1}{4}T_{[a}{}^{de}T_{\hat{d}\hat{e}\hat{f}}Q^f{}_{bc]} \\
& +\frac{1}{24}T_{[a}{}^{ef}T^{ghi}Q_{\hat{e}\hat{f}\hat{g}\hat{h}\hat{i}bc]}-\frac{1}{16}T^{def}T_f{}^{gh}Q_{deghabc} \\
& -\frac{1}{2}R_{[ab\hat{d}\hat{e}}Q^{de}{}_{c]}. \qquad (7.43)
\end{aligned}
$$

Here, note that this contribution is in the same form as $2\theta^\gamma\theta^\delta D_\delta T_{[ab}{}^\alpha\Gamma_{c]\alpha\gamma}$ which is just the other one of the two unknown factors in $I^{(4)}$. We will use it later.

The second term in $D_\delta D_\gamma T_{abc}$ is

$$
\begin{aligned}
& -\theta^\gamma\theta^\delta\phi^{-1}\left[-D_{[a}D_{\hat{d}}\phi\Gamma^d\Gamma_{bc]}-\frac{1}{6}D_{[a}\phi T_{\hat{d}\hat{e}\hat{f}}\Gamma^{def}\Gamma_{bc]}-\frac{\phi}{6}D_{[a}T_{\hat{d}\hat{e}\hat{f}}\Gamma^{def}\Gamma_{bc]}\right]_{\delta\gamma} \\
= & -\phi^{-1}Q_{[ab}{}^dD_{c]}D_d\phi+\phi^{-1}D_{[a}\phi Q_b{}^{ef}T_{\hat{e}\hat{f}c]}-Q_{[a}{}^{ef}D_bT_{c]ef}. \qquad (7.44)
\end{aligned}
$$

The third term in $D_\delta D_\gamma T_{abc}$ is

$$
-\theta^\gamma\theta^\delta\phi^{-1}\left[-D_d\phi\Gamma_{[a}\Psi\Gamma^d\Gamma_{bc]}-\frac{\phi}{6}T_{def}\Gamma_{[a}\Psi\Gamma^{def}\Gamma_{bc]}\right]_{\delta\gamma}
$$

$$
\begin{aligned}
&= -\phi^{-1}D_d\phi\theta\Gamma_{[a}\Psi\Gamma^d\Gamma_{bc]}\theta + \frac{1}{6\cdot 24}T_{ghi}T_{def}\theta\Gamma_{[a}\Gamma^{ghi}\Gamma^{def}\Gamma_{bc]}\theta \\
&= \phi^{-1}D_d\phi\theta(\Gamma^d\Gamma_{[bc} + 4\Gamma_{[b}\delta^d_{c}))\Psi\Gamma_{a]}\theta \\
&\quad + \frac{6}{6\cdot 24}T_{def}T^{def}Q_{abc} + \frac{-72+72+72}{6\cdot 24}T_{[a\hat{h}}{}^dT_{b\hat{d}}{}^eQ_{c]e}{}^h \\
&\quad + \frac{36+36-18-18}{6\cdot 24}T_{[ab}{}^dT_{\hat{d}\hat{e}\hat{f}}Q^{ef}{}_{c]} \\
&\quad + \frac{9}{6\cdot 24}T^{de}{}_fT^{fgh}Q_{deghabc}.
\end{aligned} \tag{7.45}
$$

The fourth term in $D_\delta D_\gamma T_{abc}$ is

$$
\begin{aligned}
&-\theta^\gamma\theta^\delta\phi^{-1}T_{abc}\left[-\Gamma^dD_d\phi - \frac{\phi}{6}T_{def}\Gamma^{def}\right]_{\delta\gamma} \\
&= -\frac{1}{6}T_{abc}T_{def}Q^{def}.
\end{aligned} \tag{7.46}
$$

The fifth term in $D_\delta D_\gamma T_{abc}$ is

$$
\begin{aligned}
&\theta^\gamma\theta^\delta\phi^{-1}\left[-T_{[ab}{}^dD_{\hat{e}}\phi\Gamma^e\Gamma_{c]d} - \frac{\phi}{6}T_{[ab}{}^dT_{\hat{e}\hat{f}\hat{g}}\Gamma^{efg}\Gamma_{c]d}\right]_{\delta\gamma} \\
&= \phi^{-1}T_{[ab}{}^dQ_{c]d}{}^eD_e\phi + \frac{1}{2}T_{[ab}{}^dQ_{\hat{d}\hat{e}\hat{f}}T^{ef}{}_{c]} - \frac{1}{2}T_{[ab}{}^dT_{\hat{d}\hat{f}\hat{g}}Q^{fg}{}_{c]}.
\end{aligned} \tag{7.47}
$$

The sixth term in $D_\delta D_\gamma T_{abc}$ is

$$
\begin{aligned}
&-\theta^\gamma\theta^\delta\phi^{-1}\left[-D_d\phi\Gamma^d\Gamma_{[bc}\Psi\Gamma_{a]} - \frac{\phi}{6}T_{def}\Gamma^{def}\Gamma_{[bc}\Psi\Gamma_{a]}\right]_{\delta\gamma}. \\
&= -\phi^{-1}D_d\phi\theta\Gamma^d\Gamma_{[bc}\Psi\Gamma_{a]}\theta + \frac{1}{6\cdot 24}T_{ghi}T_{def}\theta\Gamma^{def}\Gamma_{[bc}\Gamma^{ghi}\Gamma_{a]}\theta \\
&= -\phi^{-1}D_d\phi\theta\Gamma^d\Gamma_{[bc}\Psi\Gamma_{a]}\theta \\
&\quad - \frac{6}{6\cdot 24}T_{def}T^{def}Q_{abc} + \frac{18-18+36+36}{6\cdot 24}T_{[a}{}^{de}T_{\hat{d}\hat{e}\hat{f}}Q^f{}_{bc]} \\
&\quad + \frac{-72-72+72}{6\cdot 24}T_{[a\hat{h}}{}^dT_{b\hat{d}}{}^eQ_{c]e}{}^h + \frac{-18-18+36-36}{6\cdot 24}T_{[ab}{}^dT_{\hat{d}\hat{e}\hat{f}}Q^{ef}{}_{c]} \\
&\quad + \frac{9}{6\cdot 24}T^{de}{}_fT^{fgh}Q_{deghabc} + \frac{3-3-6-6}{6\cdot 24}T_{[a}{}^{de}T^{fgh}Q_{\hat{d}\hat{e}\hat{f}\hat{g}\hat{h}bc]}.
\end{aligned} \tag{7.48}
$$

Summing all these six terms we get an expression of $\Delta^2 T_{abc}$ written by using the bilinear $Q$'s as

$$
\begin{aligned}
\Delta^2 T_{abc} \;\to\; &+\frac{1}{4}T_{[a}{}^{de}T_{\hat{d}\hat{e}\hat{f}}Q^f{}_{bc]} - \frac{3}{2}T_{[ab}{}^dT_{\hat{d}\hat{f}\hat{g}}Q^{fg}{}_{c]} \\
&- \frac{3}{4}T_{[a\hat{d}}{}^eT_{b\hat{e}}{}^fQ_{c]f}{}^d
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2} R_{[ab\hat{d}\hat{e}} Q^{de}{}_{c]} \\
&- \phi^{-1} Q_{[ab}{}^{d} D_{c]} D_d \phi + \phi^{-1} D_{[a} \phi Q_b{}^{ef} T_{\hat{e}\hat{f}c]} - Q_{[a}{}^{ef} D_b T_{c]ef} \\
&+ \phi^{-1} T_{[ab}{}^{d} Q_{c]d}{}^{e} D_e \phi \\
&- \frac{1}{24} T_{[a}{}^{de} T^{fgh} Q_{\hat{d}\hat{e}\hat{f}\hat{g}\hat{h}bc]} + \frac{1}{16} T^{de}{}_f T^{fgh} Q_{deghabc}. \qquad (7.49)
\end{aligned}
$$

We have done all the explicit calculations. The last thing to do is simply substituting the results for two factors in this subsection into $I^{(4)}$ in the last subsection.

### 7.3.4 The full expression of $I^{(4)}$

Using all the results of the above subsections, we can finally obtain an expression of the fourth order terms of the Green-Schwarz action written by using only the bilinear $Q$'s and the lowest component fields.

$$
\begin{aligned}
L^{(4)} &= \frac{1}{24}\phi(\eta^{ij} + \epsilon^{ij}) E_{ia} \Delta^4 E^a_j + \frac{1}{32}\Delta^4 \phi \eta^{ij} E_{ia} E^a_j + \frac{1}{48}\Delta^3 (y\Gamma_{ab}\lambda)\epsilon^{ij} E^a_i E^b_j \\
&+ \frac{1}{8}\phi\eta^{ij}\Delta^2 E_{ia}\Delta^2 E^a_j + \frac{1}{8}\Delta^2\phi(2\eta^{ij} + \epsilon^{ij}) E_{ia}\Delta^2 E^a_j + \frac{1}{48}\Delta(y\Gamma_{ab}\lambda)\epsilon^{ij}\Delta^2 E^a_i E^b_j \\
&= \frac{1}{24}\phi(\eta^{ij} + \epsilon^{ij}) \Bigg[ + \frac{5}{16} E_{ia} Q^{ade} T_{def} T^{fgh} Q_{ghb} E^b_j + \frac{5}{16} E_{ia} T^{ade} Q_{def} Q^{fgh} T_{ghb} E^b_j \\
&+ \frac{1}{8} E^e_i T_{efg} Q^{fga} T_{acd} Q^{cdb} E_{jb} + \frac{1}{2} E_{ia} Q^{ade} T_{def} Q^{fgh} T_{ghb} E^b_j \\
&- \frac{1}{4} E_{ia} Q^{ade} Q_{def} T^{fgh} T_{ghb} E^b_j \\
&- \frac{1}{4} E_{ia} Q^{ahd} T_{def} T^{efg} Q_{ghb} E^b_j - \frac{1}{8} E_{ia} Q^{ahd} Q_{def} T^{efg} T_{ghb} E^b_j \\
&+ \frac{1}{8} E_{ia} Q^{ahb} E_{jb} T_{hd}{}^{e} T_{efg} Q^{fgd} \\
&+ \frac{15}{4} E^c_i Q_c{}^{de} T_{de}{}^{a} Q'_{ja} + \frac{7}{4} E_{ia} T^{ade} Q_{deb} Q'^{b}_j \\
&+ \frac{7}{4} Q'^{a}_i Q_{acd} T^{cdb} E_{jb} + \frac{5}{4} Q'^{a}_i T_{ade} Q^{de}{}_c E^c_j - \phi^{-1} Q'^{a}_i Q_{ab}{}^{c} D_c \phi E^b_j \\
&+ \frac{1}{12} E_{ia} Q^{acd} R_{cdfg} Q^{fg}{}_b E^b_j + \frac{1}{6} E_{ia} Q^{acd} R_{bcfg} Q^{fg}{}_d E^b_j \\
&- \frac{\phi^{-1}}{12} E_{ia} Q^{acd} Q_{cd}{}^{f} D_b D_f \phi E^b_j + \frac{\phi^{-1}}{6} E_{ia} Q^{ahd} D_d D_e \phi Q^{e}{}_{hb} E^b_j \\
&- \frac{\phi^{-1}}{16} E^e_i T_{efg} Q^{fga} Q_{ab}{}^{c} D_c \phi E^b_j - \frac{7}{48}\phi^{-1} E^b_i Q_{ba}{}^{c} D_c \phi Q^{afg} T_{fge} E^e_j \\
&- \frac{\phi^{-1}}{16} E^e_i Q_{efg} T^{fga} Q_{ab}{}^{c} D_c \phi E^b_j + \frac{\phi^{-1}}{48} E^b_i Q_{ba}{}^{c} D_c \phi T^{afg} Q_{fge} E^e_j
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\phi^{-1}}{12} E_{ia} Q^{acd} T_{dfg} Q^{fg}{}_{c} E_j^b D_b \phi - \frac{1}{12} E_{ia} Q^{acd} Q_c{}^{fg} D_d T_{fgb} E_j^b \\
&- \frac{1}{12} E_{ia} Q^{acd} Q_d{}^{fg} E_j^b D_b T_{cfg} - \frac{1}{12} E_{ia} Q^{acd} D_c T_{dfg} Q^{fg}{}_{b} E_j^b \\
&+ \frac{\phi^{-1}}{12} E_{ia} Q^{acd} T_{cd}{}^{f} Q_{bf}{}^{g} D_g \phi E_j^b + \frac{\phi^{-1}}{6} E_{ia} Q^{acd} T_{bc}{}^{f} Q_{df}{}^{g} D_g \phi E_j^b \\
&+ \frac{1}{12} E_{ia} Q^{ade} Q'_{jdefgh} T^{fgh} + \frac{1}{48} E_{ia} E_j^{c'} T^{d'e'}{}_{c} Q^{acd} T^{efg} Q_{c'd'e'defg} \\
&+ \frac{1}{32} E_{ia} E_j^{c'} T^{d'e'}{}_{c} Q^{afg} T^{cde} Q_{c'd'e'defg} + \frac{1}{144} E_{ia} E_{jc} T^{d'e'f'} Q^{acd} T^{efg} Q_{d'e'f'defg} \\
&+ \frac{1}{48} E_{ia} Q^{afg} T^{d'e'f'} Q_{fgd'e'f'de} T^{dec} E_{jc} \\
&- \frac{1}{36} E_{ia} Q^{acd} T_c{}^{fg} T^{f'g'h'} Q_{fgf'g'h'db} E_j^b - \frac{1}{72} E_{ia} Q^{acd} T^{f'g'h'} Q_{cdf'g'h'fg} T^{fg}{}_{b} E_j^b \\
&+ \frac{1}{16} E_{ia} Q^{acd} T^{fg}{}_{h} T^{hf'g'} Q_{cdfgf'g'b} E_j^b \\
&- \frac{11}{48} E_{ia} Q^{ade} T_{deb} E_j^b (T \cdot Q) + \frac{5}{48} E_i{}^{a} Q_a{}^{ef} T_{efc} E_j^c (T \cdot Q) \\
&- \frac{2}{24} E_{ia} T^{ade} Q_{deb} E_j^b (T \cdot Q) + \frac{7}{48} E_i{}^{a} T_{ade} Q^{de}{}_{c} E_j^c (T \cdot Q) \\
&- \frac{5}{144} E_i{}^{a} E_{ja} (T \cdot Q)(T \cdot Q) \\
&- \frac{5}{6} E_{ia} Q'_j{}^{a} (T \cdot Q) + \frac{\phi^{-1}}{24} E_i^a Q_{ab}{}^{c} D_c \phi E_j^b (T \cdot Q) - \frac{1}{48} E_{ia} Q^{ade} Q_{deb} E_j^b (T \cdot T) \\
&+ \frac{\phi^{-1}}{12} E_{ia} Q^{acb} E_{jb} D_c \left(\phi T_{efg}\right) Q^{efg} \\
&+ \frac{1}{24} E_{ia} E_j^a \left( \frac{1}{4} T_{ade} T^{def} Q_{fbc} Q^{bca} - \frac{3}{2} T_{abd} T^{def} Q_{efc} Q^{cab} \right. \\
&- \frac{3}{4} T_{ad}{}^{e} T_{be}{}^{f} Q_{cf}{}^{d} Q^{abc} - \frac{1}{2} R_{abde} Q^{de}{}_{c} Q^{cab} \\
&- \phi^{-1} Q^{cab} Q_{ab}{}^{d} D_c D_d \phi + \phi^{-1} D_a \phi Q_b{}^{ef} T_{efc} Q^{cab} - Q_a{}^{ef} D_b T_{cef} Q^{cab} \\
&+ \phi^{-1} Q^{cab} T_{ab}{}^{d} Q_{cd}{}^{e} D_e \phi \\
&\left. - \frac{1}{24} T_a{}^{de} T^{fgh} Q_{defghbc} Q^{bca} + \frac{1}{16} T^{de}{}_{f} T^{fgh} Q_{deghabc} Q^{bca} \right) \Bigg]. \qquad (7.50)
\end{aligned}
$$

We will use this result in next section. Although the result is very lengthy, we only need to check whether each term in the expression survives or not in the two special background, for our purpose.

## 7.4 Simplification in special backgrounds

Until now, the only assumption on the supergravity background was that all the fermionic backgrounds are vanishing, which is a reasonable condition classically.

One may hope that this lengthy expression will become very simple in some restricted background fields. Actually, all the fourth order terms vanish when we set the background to the plane-wave background obtained by the Penrose limit in the light-cone gauge. We can easily check this by using the known properties of tensor fields, $T_{abc}$, $R_{abcd}$, $D_a\phi$, and the bilinear $Q^{abc}$. we know that only non-vanishing components of these tensors in the plane-wave background obtained by the Penrose limit in the light-cone gauge are $T_{+\tilde{b}\tilde{c}}$, $R_{+\tilde{b}+\tilde{d}}$, $D_+\phi$, and the bilinear $Q^{-\tilde{b}\tilde{c}}$, up to symmetries of the tensors. This property of $T_{abc}$ can be derived by using the fact that $T_{abc}$ is related to $H_{abc}$ by the constraint (4.83). Since $H_{abc}$ is field strength, only non-vanishing components are the ones in the form of $H_{+\tilde{b}\tilde{c}}$, and thus only $T_{+\tilde{b}\tilde{c}}$ are non-vanishing.

Since the lower '$-$' indices can only be supplied from the two $E_{ia}$'s, each tensor, $T_{abc}$, $R_{abcd}$, $D_a\phi$, and the bilinear $Q^{abc}$ must be contracted with these $E_{ia}$, which leaves a single possibly non-vanishing combination; a product of two $E_{ia}$ and two $T_{abc}$, for the fourth order term. However there is no such term in (7.50). Thus all the terms in (7.50) vanish.

As we noted in section 4.1.6, we have used the solutions of the Bianchi identities with $c_2 = 0$. However in this background which is obtained by the Penrose limit, the solutions of the Bianchi identities with $c_2 = 0$ also satisfies the real Bianchi identities with $c_2 \neq 0$, and the above statement is correct irrespective of $c_2$.

As a final note, we point out that we can easily give a disproof of the wrong claim in [15] by using the general fourth order terms (7.50). In [15], they used the solutions of the Bianchi identities with $c_2 = 0$, in the special background with restrictions such as

- Fermionic background fields are vanishing,
- Background fields depend only on transverse coordinates,
- Background field tensors have only transverse components.

and claimed that all the fourth order terms vanish. However, for example, $E_{ia}Q^{ade}Q_{deb}E^b_j(T\cdot T)$ and $E_{ia}Q^{ahd}T_{def}T^{efg}Q_{ghb}E^b_j$ are non-vanishing in this background.

# Chapter 8

# The $\theta$-expansion of the Type IIB string action

The Type IIB string action is given in (5.6),

$$I = \frac{1}{2}\int d^2\sigma \left[\sqrt{-g}g^{ij}\Phi E_i^a E_j^b \eta^{ab} + \epsilon^{ij} E_i^B E_k^A \mathcal{B}_{AB}\right]$$

In the last chapter, we studied the $\theta$-expansion of the Heterotic string action in a general background, and had to perform very lengthy calculation. Here in this chapter we study the $\theta$-expansion of the Type IIB string action only in the plane-wave backgrounds obtained by the Penrose limit. We use the dimensional analysis to show that the $\theta$-expansion always terminates at the quadratic order in $\theta$. This method was first developed for the Heterotic strings in [15], and applied for the Type IIB strings in our paper [14].

## 8.1 Generic form of the expansion

We need to slightly generalize the manipulations in chapter 6 for the Type IIB case.

Since the superspace vectors have two sets of fermionic components,

$$Z^M = Z_0^M \equiv (X^m, 0, 0), \quad y^M = y_0^M \equiv (0, y_0^\mu, y_0^{\bar{\mu}}), \tag{8.1}$$

and we take the Wess-Zumino gauge of the $\mathcal{N} = 2$ case so that the lowest components of the super-vielbein take the following form:

$$E_M{}^A(X) = \begin{pmatrix} e_m{}^a(X) & e_m{}^\alpha(X) & e_m{}^{\bar{\alpha}}(X) \\ e_\mu{}^a(X) = 0 & e_\mu{}^\alpha(X) = \delta_\mu{}^\alpha & e_\mu{}^{\bar{\alpha}}(X) = 0 \\ e_{\bar{\mu}}{}^a(X) = 0 & e_{\bar{\mu}}{}^\alpha(X) = 0 & e_{\bar{\mu}}{}^{\bar{\alpha}}(X) = -\delta_{\bar{\mu}}{}^{\bar{\alpha}} \end{pmatrix}. \tag{8.2}$$

Thus the tangent vector at the origin of the expansion, $y^A = y^M e_M^A(Z_0)$, will become

$$\begin{aligned}
y^a &= y^M e_M^a(X) \\
&= 0, &(8.3) \\
y^\alpha &= y^\mu e_\mu^\alpha \\
&= y^\mu \delta_\mu^\alpha \equiv \theta^\alpha, \\
y^{\bar\alpha} &= -y^{\bar\mu} e_{\bar\mu}^{\bar\alpha} \\
&= y^{\bar\mu} \delta_{\bar\mu}^{\bar\alpha} \equiv \bar\theta^\alpha, &(8.4)
\end{aligned}$$

where our definition of the contraction is

$$x^A y_A = x^a y_a + x^\mu y_\mu - x^{\bar\mu} y_{\bar\mu}. \tag{8.5}$$

To calculate $\Delta I$, we need the results of the $\Delta$ operations on $\Phi$ and $E_i^a$. Using the expressions of the solutions of Bianchi identities, these become as follows:

$$\begin{aligned}
\Delta\Phi &= \Phi(y^\alpha \Lambda_\alpha - \bar y^\alpha \bar\lambda_\alpha), &(8.6) \\
\Delta E_i^a &= E_i^\beta y^\gamma T_{\gamma\beta}{}^a, \\
&= -i \left[ y^\alpha \Gamma_{a\alpha\beta} \bar E_i^\beta + \bar y^\alpha \Gamma_{a\alpha\beta} E_i^\beta \right]. &(8.7)
\end{aligned}$$

Then we can calculate $I^{(1)} \equiv \Delta I$,

$$\begin{aligned}
I^{(1)} &= \int d^2\sigma L^{(1)}, \\
L^{(1)} &= \frac{1}{2}\left[\sqrt{-g} g^{ij} \left(\Delta\Phi E_i^a E_j^b + 2\Phi E_i^a \Delta E_j^b\right)\eta_{ab} + \epsilon^{ij} y^C V_i^B V_j^A \mathcal{H}_{ABC}\right] \\
&= \frac{1}{2}\Phi E_i^a (\sqrt{-g} g^{ij} y^\alpha - \epsilon^{ij} \bar y^\alpha) \Gamma_{a\alpha\beta}\left[\Gamma_b^{\beta\gamma}\Lambda_\gamma E_j^b - 2i \bar E_j^\beta\right] + \text{h.c.}\ , &(8.8)
\end{aligned}$$

where h.c. denotes the Hermitian conjugates. The second and the higher order terms can be explicitly obtained by further acting $\Delta$'s on (8.8). In this and the next section, we concentrate on studying the general form of the possible terms in the expansion to obtain the form of the terms in the highest possible order. Ignoring coefficients, any terms induced by $\Delta$ operations on (8.8) are in the form as

$$\begin{aligned}
&(\Delta^{2k}\Phi)(\Delta^{2l} E_i^a)(\Delta^{2m} E_j^b)(\sqrt{-g} g^{ij}\theta - \epsilon^{ij}\bar\theta)(\Gamma_a\Gamma_b)(\Delta^{2n+1}\Lambda), \\
&(\Delta^{2k}\Phi)(\Delta^{2l} E_i^a)(\sqrt{-g} g^{ij}\theta^\alpha - \epsilon^{ij}\bar\theta^\alpha)(\Gamma_a)_{\alpha\beta}(\Delta^{2m+1}\bar E_j^\beta), &(8.9)
\end{aligned}$$

or the Hermitian conjugates of these.

## 8.2 Highest possible order in plane-wave

Applying the results of chapter 3.1 to the fields in the Type IIB string theory, we will find that the following properties are satisfied by any tensor fields with all the indices lowered:

(I) The non-vanishing components of the tensors (tensor densities) never have lower '−' indices, except $\eta_{mn}, \epsilon_{m_1,\ldots,m_{10}}, \eta_{ab}, \epsilon_{a_1,\ldots,a_{10}}$.

(II) The tensors have lower '+' indices at least as many as their mass dimension.

When we raise, lower or contract some of lower indices of these tensor fields, the sum of the number of the lower '+' and the upper '−' is preserved because $\eta_{+-}$, $\epsilon_{+-a_3,\ldots,a_{10}}$, etc. always have both '+' and '−' indices in a pair.

Now, we will consider the $\theta$-expansion of the light-cone gauge Green-Schwarz action in the plane-wave backgrounds with vanishing background fermions $\Gamma^+_{\alpha\beta}\theta^\beta = \Gamma^+_{\alpha\beta}\bar{\theta}^\beta = 0$. Thus the supergravity background fields with odd number of spinor indices are vanishing. Background fields with even number of spinor indices are all expanded by the bilinears of the fermionic coordinates,

$$\hat{\theta}\Gamma^-\hat{\theta}, \quad \hat{\theta}\Gamma^{-\tilde{a}\tilde{b}}\hat{\theta}, \quad \hat{\theta}\Gamma^{-\tilde{a}\tilde{b}\tilde{c}\tilde{d}}\hat{\theta}, \tag{8.10}$$

where $\hat{\theta}$ denotes both $\theta$ and $\bar{\theta}$.

For bilinears without worldsheet derivatives, the combination of $\theta$ and $\bar{\theta}$ is more restricted because of their chirality. The possible combinations are:

$$\bar{\theta}\Gamma^-\theta, \quad \theta\Gamma^{-\tilde{a}\tilde{b}}\theta, \quad \bar{\theta}\Gamma^{-\tilde{a}\tilde{b}}\bar{\theta}, \quad \bar{\theta}\Gamma^{-\tilde{a}\tilde{b}}\theta, \quad \bar{\theta}\Gamma^{-\tilde{a}\tilde{b}\tilde{c}\tilde{d}}\theta. \tag{8.11}$$

Now, we list and examine all the fields that appear in the $\theta$-expansion:

- The constant tensors; $\eta^{ab}$, $\epsilon_{a_1,a_2,\cdots,a_{10}}$.
- The tensor fields explicitly in the original action and its derivatives; $\Phi$, $H_{ABC}$, $T_{AB}{}^C$, $R_{ABC}{}^D$, $\mathcal{B}_{AB}$ and its derivatives.
- The fields with worldsheet index; $E_i^A$, $D_i y^\alpha$, $D_i y^{\bar{\alpha}}$.
- The parameters $y^A$ of the expansion.

No other fields are included in the initial action nor introduced by the $\Delta$ operation. In this list, the two-form potential $\mathcal{B}_{AB}$ only appears in the zero-th order term because $\mathcal{B}_{AB}$ is completely converted into $H_{ABC}$ and neither one of the four basic $\Delta$ operations (6.16) - (6.19) nor covariant derivatives of other tensor fields introduce $\mathcal{B}_{AB}$. It is important that $\mathcal{B}_{AB}$ is the only factor which has more '+' indices than their mass dimension in the sense of the property (II). Thus we now see that the property (II) can be made more precise for the second order term and higher:

(II') The tensors with mass dimension $m$ always have $m$ more lower '+' indices than lower '−' indices (when all the indices are lowered).

Using these facts, let us examine the factors of the form $W^{(2n;p)}_{a_1,\cdots,a_k}$, where $W^{(2n;p)}_{a_1,\cdots,a_k}$ is made from a combination of $2n$ spinors and the background fields in the above list, and has total mass dimension $p \geq 0$.

In the light-cone gauge, from the properties (I), (II') combined with the properties of the bilinears (8.10), we can prove that the following statement holds:

*Components of $W^{(2n;p)}_{a_1,\cdots,a_k}$ are non-vanishing only when*

$$m_+ = m_- + 2n + p, \tag{8.12}$$

*where $m_\pm$ are the number of $\pm$ indices in $\{a_1, \cdots, a_k\}$.*

For $n = 0$, $W^{(0;p)}_{a_1,\cdots,a_k}$ is made only from bosonic fields. From the property (I), we see that only $\eta_{-+}$ and $\epsilon_{-+a_3,\cdots,a_10}$ have lower '−' indices, and each lower '−' index always appears as one of a pair of lower '−' and '+' indices in these tensors. In addition to these pairs, there can be lower '+' indices as many as the total mass dimension $p$. So the equation $m_+ = m_- + p$ is satisfied.

For $n \geq 1$, we can show the statement by the induction on $n$; Assume that the statement holds for $W^{(2(n-1);p)}_{a_1,\cdots,a_k}$. Any factor with $2n$ spinors can be represented as

$$W^{(2n;p)}_{a_1,\cdots,a_k} \;\; = \;\; \sum_{\{\tilde{a}\tilde{b}\cdots\}} O^{-[\tilde{a}\tilde{b}\cdots]} W^{(2(n-1);p+1)}_{-[\tilde{a}\tilde{b}\cdots]a_1,\cdots,a_k}, \tag{8.13}$$

where we expressed the bilinears (8.10) by $O^{-[\tilde{a}\tilde{b}\cdots]}$, and the sum over the set of indices $\{\tilde{a}\tilde{b}\cdots\}$ really means the sum over $O^-$, $O^{-[\tilde{a}\tilde{b}]}$, and $O^{-[\tilde{a}\tilde{b}\tilde{c}\tilde{d}]}$ . From the assumption, the right hand side is non-vanishing only when

$$\begin{aligned} m_+ \;\; &= \;\; (m_- + 1) + 2(n-1) + (p+1) \\ &= \;\; m_- + 2n + p. \end{aligned}$$

Therefore the left hand side, $W^{(2n;p)}_{a_1,\cdots,a_k}$, is non-vanishing only when the equation (8.12) is satisfied. Thus the statement holds for any $n$.

We use this statement to sift out non-vanishing terms on the plane-wave background from (8.9) . The possible terms in (8.9) are constructed from factors

$$\Delta^{2n}\Phi, \quad \hat{\theta}\Gamma_a\Gamma_b(\Delta^{2n+1}\Lambda), \quad \Delta^{2n}E_{ia}, \quad \hat{\theta}^\alpha(\Gamma_a)_{\alpha\beta}(\Delta^{2n+1}E^\beta_i), \tag{8.14}$$

and the Hermitian conjugate of these.

$\Delta^{2n}\Phi$ has dimension 0 and is made from $2n$ spinors, and with no indices. Thus from the statement (8.12), it is non-vanishing only when $n = 0$ in the second or higher order term of the $\theta$-expansion.

$\hat{\theta}\Gamma_a\Gamma_b(\Delta^{2n+1}\Lambda)$ has dimension 0 and is made from $(2n+2)$ spinors and with two indices. Thus it can be non-vanishing when $n=0$, and both indices are '+'. However $\Gamma_+\Gamma_+$ identically vanishes because $\Gamma_a\Gamma_b = \Gamma_{ab} + \eta_{ab}$. So $\hat{\theta}\Gamma_a\Gamma_b(\Delta^{2n+1}\Lambda)$ is always vanishing.

$\Delta^{2n}E_{ia}$ has dimension $-1$ and is made from $2n$ spinors, with a tangent space vector index, and a worldsheet vector index. Note that the *'indices'* in the statement (8.12) is about the tangent space vector indices in this case. To apply the statement to the factors with worldsheet vector indices, we need to decompose it into worldsheet derivatives of basic fields and the tangent space tensors,

$$\Delta^{2n}E_{ia} \quad \to \quad W_{ab}E_i^b + \sum_{\{\tilde{b}\tilde{c}\cdots\}} S_{a-[\tilde{b}\tilde{c}\cdots]}\left(\hat{\theta}\Gamma^{-[\tilde{b}\tilde{c}\cdots]}D_i\hat{\theta}\right). \tag{8.15}$$

In this form, $W_{ab}$ has dimension 0 and is made from $2n$ spinors. $S_{a-[\tilde{b}\tilde{c}\cdots]}$ has dimension 0 and is made from $(2n-2)$ spinors. Thus only $W_{++}$ is non-vanishing when $n=1$, $W_{+-}$, $W_{-+}$ and $S_{+-[\tilde{b}\tilde{c}\cdots]}$ are non-vanishing when $n=0$, and both terms vanish when $n>1$. So $\Delta^{2n}E_{i+}$ is at most of order two in $\theta$, and $\Delta^{2n}E_{i-}$ is of order zero in $\theta$.

Similarly, $\hat{\theta}^\alpha(\Gamma_a)_{\alpha\beta}(\Delta^{2n+1}E_i^\beta)$ has dimension $-1$ and is made from $(2n+2)$ spinors, with a tangent space vector index, and a worldsheet vector index. Using the same expansion as $\Delta^{2n}E_{ia}$, $\hat{\theta}^\alpha(\Gamma_+)_{\alpha\beta}(\Delta^{2n+1}E_i^\beta)$ is also at most of order two in $\theta$.

Substituting the highest order factors into (8.9), we get

$$\Phi E_i^+(\sqrt{-g}g^{ij}\theta^\alpha - \epsilon^{ij}\bar{\theta}^\alpha)(\Gamma_+)_{\alpha\beta}(\Delta\bar{E}_j^\beta), \tag{8.16}$$

and its Hermitian conjugate as the only non-vanishing terms. These terms are of order two in $\theta$.

Thus we have shown that the $\theta$-expansion of the Type IIB Green-Schwarz action in the plane-wave background are at most of order two in $\theta$ in the light-cone gauge.

## 8.3 The explicit form of the $\theta$-expansion

In the last section, we have shown that the Green-Schwarz action of Type IIB superstring theory becomes quadratic in the plane-wave background obtained by the Penrose limit. Now we would like to see the explicit form of the quadratic action. Since the expressions become clearer in the general background, we do not assume the plane-wave background nor the light-cone gauge here.

Operating $\Delta$ on (8.8) and substituting $y^A = (0, \theta^\alpha, \bar{\theta}^\alpha)$ and $Z_0^M = (X^m, 0, 0)$, we explicitly get the second order term of the Green-Schwarz action as

$$L^{(2)} \quad \to \quad \frac{1}{4}\Phi E_i^a\left(\sqrt{-g}g^{ij}\theta - \epsilon^{ij}\bar{\theta}\right)\gamma_a\left[\gamma_b(\Delta\Lambda)E_j^b - 2i\Delta\bar{E}_j\right] + \text{h.c.} \ . \tag{8.17}$$

$\Delta E_j^\alpha$, and $\Delta\Lambda^\alpha$ are given by substituting solutions of Bianchi identities (4.105):

$$\begin{aligned} \Delta E_j^\alpha &\to E_i^c y^\beta T_{\beta c}{}^\alpha \\ &= E_j^a (\mathcal{D}_a\theta)^\alpha \,, \\ \Delta\Lambda_\alpha &\to \theta^B D_B \Lambda_\alpha \\ &= (\mathcal{D}\theta)_\alpha \,, \end{aligned} \tag{8.18}$$

where

$$\begin{aligned} (\mathcal{D}_a\theta)^\alpha &\equiv D_a\theta^\alpha - iZ_{abcde}(\gamma^{bcde})^\alpha{}_\beta\theta^\beta - \frac{3}{16}F_{abc}(\gamma^{bc})^\alpha{}_\beta\bar{\theta}^\beta + \frac{1}{48}F^{bcd}(\gamma_{abcd})^\alpha{}_\beta\bar{\theta}^\beta \,, \\ (\mathcal{D}\theta)_\alpha &\equiv \frac{i}{2}P_a(\gamma^a)^\alpha{}_\beta\bar{\theta}^\beta + \frac{i}{24}F_{abc}(\gamma^{abc})^\alpha{}_\beta\theta^\beta \,. \end{aligned} \tag{8.19}$$

Here notice that $(\mathcal{D}_a\theta)^\alpha$ and $(\mathcal{D}\theta)_\alpha$ coincides with the form of the supersymmetry transformation law of the gravitino field, and the dilatino field respectively [23], with the parameter $\theta$.

# Chapter 9

# Conclusions

In this paper, we have studied the $\theta$-expansion of the Green-Schwarz action of the Heterotic and Type IIB strings. We have shown that the action is quadratic in $\theta$ for the plane-wave background obtained as the Penrose limit of an arbitrary solutions of supergravity.

Although, we have dealt with the background obtained as the Penrose limit, we have not understood whether the proof can be generalized further. In [10], it is proven that the light-cone gauge Green-Schwarz action of the Type IIB string is always quadratic with respect to $\theta$. in pp-wave with constant dilaton. Since we have used only the conditions (I) and (II') of the plane-wave backgrounds in section 8.2, we can generalize our proof a little bit to the pp-wave or any background satisfying the conditions (I) and (II'). Since the pp-wave background with constant dilaton satisfies the conditions (I) and (II'), it is included in the above general case. But we do not understand whether the conditions are always satisfied by the general pp-wave with non-constant dilaton or not.

Since the plane-wave background obtained by the Penrose limit can have Ramond-Ramond background fields, the fact that the action in quadratic in $\theta$ may be useful for quantizing the theory. In particular, the AdS/CFT correspondence relates the superstring theories with Ramond-Ramond background fields to super Yang-Mills theory in the large $N$ limit. Although the main interests are in the string theory in the AdS backgrounds (for example, $AdS_5 \times S^5$ background), the string theory in more general background may have some applications and is related to some more general super Yang-Mills theory.

For the Heterotic string, we studied the action in a general background, and presented the $\theta$-expansion explicitly. By using the result, we have checked whether the truncation of the Green-Schwarz action occurs or not in two different backgrounds; the plane-wave obtained by the Penrose limit, and the backgrounds which only depend on transverse coordinates. In the latter case it is proved that the light-cone gauge Green-Schwarz action is not generally quadratic in $\theta$ in the approximation $c_2 = 0$. This result gives a disproof of the result in [15]. Although this result itself is for the action in the very restricted background, essentially the same logic as the one used in [15] was used in a more recent paper [17], thus our disproof is meaningful as a caution for such mistakes.

If we were just investigating whether the light-cone gauge Green-Schwarz action is quadratic or not, we would not have to restrict ourselves to the case of vanishing gauge fields. The same dimensional analysis as in the Type IIB case might easily be applied to the Heterotic string with gauge fields and also to the Type IIA string, to prove that the light-cone gauge Green-Schwarz action of each string theory is quadratic in the plane-wave obtained by the Penrose limit.

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# Bibliography

[1] M. B. Green and J. H. Schwarz, "Covariant description of superstrings," Phys. Lett. B **136**, 367 (1984).

[2] M. B. Green, J. H. Schwarz and E. Witten, "Superstring theory. vol. 1: introduction," Cambridge University Press, Cambridge, 1987.

[3] A. Giveon and D. Kutasov, "Brane dynamics and gauge theory," Rev. Mod. Phys. **71**, 983 (1999) [arXiv:hep-th/9802067].

[4] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[5] R. R. Metsaev and A. A. Tseytlin, "Type IIB superstring action in AdS(5) x S(5) background," Nucl. Phys. B **533**, 109 (1998) [arXiv:hep-th/9805028].

[6] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, "A new maximally supersymmetric background of IIB superstring theory," JHEP **0201**, 047 (2002) [arXiv:hep-th/0110242].

[7] R. R. Metsaev, "Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background," Nucl. Phys. B **625**, 70 (2002) [arXiv:hep-th/0112044].

[8] R. R. Metsaev and A. A. Tseytlin, "Exactly solvable model of superstring in plane wave Ramond-Ramond Phys. Rev. D **65**, 126004 (2002) [arXiv:hep-th/0202109].

[9] R. Penrose, "Differential Geometry and Relativity," eds. M. Cahen and M. Flato (Reidel, Dordrecht, Netherlands, 1976).

[10] J. G. Russo and A. A. Tseytlin, "A class of exact pp-wave string models with interacting light-cone gauge JHEP **0209**, 035 (2002) [arXiv:hep-th/0208114].

[11] J. G. Russo and A. A. Tseytlin, "On solvable models of type IIB superstring in NS-NS and R-R plane wave JHEP **0204**, 021 (2002) [arXiv:hep-th/0202179].

[12] K. Sugiyama and K. Yoshida, "Type IIA string and matrix string on pp-wave," Nucl. Phys. B **644**, 128 (2002) [arXiv:hep-th/0208029].

[13] S. Hyun and H. Shin, "N = (4,4) type IIA string theory on pp-wave background," JHEP **0210**, 070 (2002) [arXiv:hep-th/0208074].

[14] S. Mizoguchi, T. Mogami and Y. Satoh, "Penrose limits and Green-Schwarz strings," Class. Quant. Grav. **20**, 1489 (2003) [arXiv:hep-th/0209043].

[15] J. J. Atick and A. Dhar, "Normal Coordinates, Theta Expansion And Strings On Curved Superspace," Nucl. Phys. B **284**, 131 (1987).

[16] S. J. J. Gates, P. Majumdar, B. Radak and S. Vashakidze, "Violation Of Lorentz Invariance In Type Ii Green-Schwarz Superstrings In Phys. Lett. B **226**, 237 (1989).

[17] V. Sahakian, "Strings in Ramond-Ramond backgrounds," arXiv:hep-th/0112063.

[18] R. Gueven, "Plane wave limits and T-duality," Phys. Lett. B **482**, 255 (2000) [arXiv:hep-th/0005061].

[19] M. B. Green and J. H. Schwarz, "Supersymmetrical string theories," Phys. Lett. B **109**, 444 (1982).

[20] D. Berenstein, J. M. Maldacena and H. Nastase, "Strings in flat space and pp waves from N = 4 super Yang Mills," JHEP **0204**, 013 (2002) [arXiv:hep-th/0202021].

[21] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, "Penrose limits and maximal supersymmetry," Class. Quant. Grav. **19**, L87 (2002) [arXiv:hep-th/0201081].

[22] J. J. Atick, A. Dhar and B. Ratra, "Superspace Formulation Of Ten-Dimensional N=1 Supergravity Coupled To N=1 Superyang-Mills Theory," Phys. Rev. D **33**, 2824 (1986).

[23] P. S. Howe and P. C. West, "The Complete N=2, D = 10 Supergravity," Nucl. Phys. B **238**, 181 (1984).

[24] J. J. Atick, A. Dhar and B. Ratra, "Superstring Propagation In Curved Superspace In The Presence Of Background Superyang-Mills Fields," Phys. Lett. B **169**, 54 (1986).

[25] M. T. Grisaru, P. S. Howe, L. Mezincescu, B. Nilsson and P. K. Townsend, "N=2 Superstrings In A Supergravity Background," Phys. Lett. B **162**, 116 (1985).