Vertex Operator Algebra with Two τ-involutions Generating S_3

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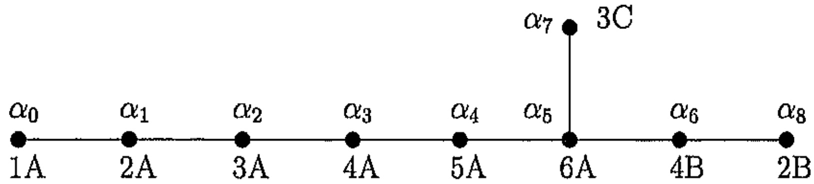
A dissertation submitted to the Doctoral Program
in Mathematics, the University of Tsukuba
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics

January 2003

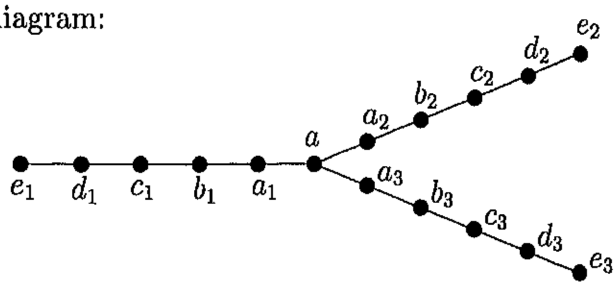
1 Introduction

The notion of vertex operator algebras (VOAs) is introduced in [B, FLM]. The most interesting example of vertex operator algebras is the Moonshine VOA $V^\natural = \oplus_{n=0}^{\infty} V_n^\natural$. It was constructed by Frenkel, Lepowsky and Meurman in [FLM] in order to solve the so-called McKay-Thompson conjecture about the representations of the Monster simple group $\mathbb{M}$, the largest member of the 26 sporadic simple finite groups, and some modular functions. The full automorphism group $\mathrm{Aut}(V^\natural)$ of $V^\natural$ is the Monster simple group $\mathbb{M}$ and the character $q^{-1}\sum_{n=0}^{\infty}(\dim V_n)q^n$ is the classical elliptic modular function $j(\tau) - 744$, where $q = e^{2\pi i\tau}$. The weight two subspace $V_2^\natural$ coincides with a commutative (non-associative) algebra (called the monstrous Griess algebra) of dimension 196884 with positive definite invariant bilinear form constructed by Griess [Gr] in order to construct the Monster simple group.

From the group theoretic point of view, one of the important results is that each $2A$-involution θ defines a unique idempotent e_θ (called an axis) of the monstrous Griess algebra such that the inner product $\langle e_\theta, e_\phi \rangle$ is uniquely determined by the conjugacy classes of a product $\theta\phi$ of $2A$-involutions θ and ϕ [C]. In fact, the conjugacy classes of a product of two $2A$-involutions are $1A$, $2A$, $3A$, $4A$, $5A$, $6A$, $3C$, $4B$ and $2B$, and the inner products of the corresponding axis vectors are $1/4$, $1/32$, $13/2^{10}$, $1/2^7$, $3/2^9$, $5/2^{10}$, $1/2^8$, $1/2^8$ and 0, respectively. There is also a mysterious relation with E_8 Dynkin diagram, which was observed by McKay [Mc]. Namely, the conjugacy classes of a product of two $2A$-involutions are corresponding to multiplicities of simple roots in the maximal root $\alpha_0 = -(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8)$ of E_8-root system:



Another interesting topic is the $Y_{5,5,5}$-diagram and the Bimonster $\mathbb{M} \wr \mathbb{Z}_2$. The Bimonster is presented by the single relation $(ab_1c_1ab_2c_2ab_3c_3)^{10} = 1$ in addition to the Coxeter relations of the $Y_{5,5,5}$-diagram:



where a vertex is an involution and an edge $x - y$ means $|xy| = 3$ and no edge between x and y implies $|xy| = 2$.

From the point of view of vertex operator algebras, irreducible highest weight modules $L(c, h)$ of the Virasoro algebra with central charge c and highest weight h are very useful. It was proved in [FZ] that $L(c, 0)$ is a VOA, which is the simplest example of VOAs. When $c = c_m = 1 - \frac{6}{(m+2)(m+3)}$, which is called the discrete series, it is proved by Wang [W] that $L(c_m, 0), m = 1, 2, \dots$, is rational and $L(c_m, h^m_{r,s}), 1 \le r \le m+1, 1 \le s \le m+2$, are exactly all the inequivalent irreducible modules of $L(c_m, 0)$, where $h^m_{r,s} = \frac{\{r(m+3)-s(m+2)\}^2-1}{4(m+2)(m+3)}$. Moreover, $L(c_m, 0)$ admits a non-trivial unitary form [GKO]. The study of the moonshine VOA as modules for a subalgebra isomorphic to a tensor product of rational Virasoro vertex operator algebras in the discrete series was initiated by Dong, Mason and Zhu [DMZ]. Along this line, it was shown by Miyamoto [M1] that a rational conformal vector with central charge $\frac{1}{2}$ define an involution on a VOA, which is called τ-involution or Miyamoto involution. If e is a rational conformal vector with central charge $\frac{1}{2}$, i.e., e generates a rational VOA $L(\frac{1}{2}, 0)$ called the Ising model, then one can define an involutive automorphism τ_e of V by

$$\tau_e : \quad \begin{cases} 1 & \text{on } W_0 \oplus W_{\frac{1}{2}} \\ -1 & \text{on } W_{\frac{1}{16}}, \end{cases}$$

where W_h denotes the sum of all irreducible VA(e)-submodules isomorphic to $L(\frac{1}{2}, h)$ and VA(e) is a subVOA generated by e. In the monstrous Griess algebra, a conformal vector e with central charge $\frac{1}{2}$ is corresponding to an axis and τ_e is a $2A$-involution of the Monster simple group.

A VOA V over $\mathbb{R}$ is referred to be *of moonshine type* if it admits a weight space decomposition $V = \oplus_{n=0}^{\infty} V_n$ with $V_0 = \mathbb{R}\mathbb{1}$ and $V_1 = 0$ and it possesses a definite invariant bilinear form $\langle \cdot, \cdot \rangle$ such that $\langle \mathbb{1}, \mathbb{1} \rangle = 1$. If V is of moonshine type, by defining $a \times b = a_{(1)}b$ for $a, b \in V_2$, $(V_2, \times)$ becomes a commutative (non-associative) algebra called a Griess algebra with a positive definite bilinear form $\langle \cdot, \cdot \rangle$ satisfying $\langle a \times b, c \rangle = \langle b, a \times c \rangle$ for $a, b, c \in V_2$, where $\langle a, b \rangle$ is given by $\langle a, b \rangle \mathbb{1} = a_{(3)}b \in \mathbb{R}\mathbb{1}$.

In [M3], Miyamoto studied a vertex subalgebra VA(e, f) of a VOA V of moonshine type generated by two conformal vectors e and f with central charge $\frac{1}{2}$ whose τ-involutions generate S_3. He determined that the possible inner products of such a pair of conformal vectors are $\frac{13}{2^{10}}$ or $\frac{1}{2^8}$. In the monstrous Griess algebra $V_2^\natural$, the inner products of such conformal vectors are $\frac{13}{2^{10}}$ and $\frac{1}{2^8}$ corresponding to $3A$-triality and $3C$-triality, respectively. In each case, the structure of a subalgebra of a (general) Griess algebra V_2 generated by such conformal vectors is uniquely determined. In the case where the inner product is $\frac{13}{2^{10}}$,

$\mathrm{VA}(e,f)$ is a VOA with central charge $\frac{58}{35}$ and $\dim(\mathrm{VA}(e,f))_2 = 4$. Moreover, $\mathrm{VA}(e,f)$ contains $L(\frac{4}{5},0)$ and $L(\frac{6}{7},0)$ as a subVOA, which are in the discrete series.

In this paper, we determine the VOA structure of $\mathrm{VA}(e,f)$ generated by two conformal vectors e and f with central charge $\frac{1}{2}$ such that $|\tau_e\tau_f| = 3$ and $\langle e, f\rangle = \frac{13}{2^{10}}$ and construct such a VOA.

We set $U = \mathrm{VA}(e,f)$. As shown in [M3], U_2 has mutually orthogonal conformal vectors ω^3 and ω^4 with central charge $\frac{4}{5}$ and $\frac{6}{7}$, respectively, and so contains $L(\frac{4}{5},0)\otimes L(\frac{6}{7},0)$ as a subVOA. Then, as a $L(\frac{4}{5},0)\otimes L(\frac{6}{7},0)$-module, we will show

$$\begin{aligned} U \simeq\ & L\left(\frac{4}{5},0\right)\otimes L\left(\frac{6}{7},0\right)\oplus L\left(\frac{4}{5},3\right)\otimes L\left(\frac{6}{7},0\right) \\ & \oplus L\left(\frac{4}{5},0\right)\otimes L\left(\frac{6}{7},5\right)\oplus L\left(\frac{4}{5},3\right)\otimes L\left(\frac{6}{7},5\right) \\ & \oplus L\left(\frac{4}{5},\frac{2}{3}\right)\otimes L\left(\frac{6}{7},\frac{4}{3}\right)\oplus L\left(\frac{4}{5},\frac{2}{3}\right)\otimes L\left(\frac{6}{7},\frac{4}{3}\right). \end{aligned} \tag{1.1}$$

In particular, U contains the 3-state Potts model $W(0) = L\left(\frac{4}{5},0\right)\oplus L\left(\frac{4}{5},0\right)$ and the tricritical 3-state Potts model $N(0) = L\left(\frac{6}{7},0\right)\oplus L\left(\frac{6}{7},5\right)$ as a subVOA, which are studied in [KMY] and [LY]. Then, viewing U as a $W(0)\otimes N(0)$-module, U has a $\mathbb{Z}_3$-grading

$$W(0)\otimes N(0)\oplus W\left(\frac{2}{3}\right)^{+}\otimes N\left(\frac{4}{3}\right)^{\pm}\oplus W\left(\frac{2}{3}\right)^{-}\otimes N\left(\frac{4}{3}\right)^{\mp}.$$

Hence, $\tau_e\tau_f$ is an automorphism defined by the $\mathbb{Z}_3$-symmetry of the fusion algebra for $W(0)$ [M2]. We will also show that all $\mathbb{Z}_3$-graded VOAs of this form are isomorphic to each other and so the VOA structure of U is uniquely determined. Therefore, the Moonshine VOA $V^\natural$ has a subVOA isomorphic to $W(0)$ and the automorphism determined by $W(0)$ is a $3A$ element of the Monster simple group.

The main idea of our construction is the GKO construction (or coset construction) of the unitary Virasoro VOA $L(c_m,0)$. We note that $c_3=\frac{4}{5}$ and $c_4=\frac{6}{7}$. First we obtain a decomposition of the lattice VOA $V_{A_1^{m+1}}$ as a module of a direct sum of certain copies of Virasoro algebras and the affine Lie algebra $\hat{sl}_2(\mathbb{C})$ by using the GKO construction, where A_1 is the root lattice of type A_1. We then extend A_1^5 to a larger lattice L and show that (1.1) is actually a commutant (or coset) subalgebra of V_L. We also show that all irreducible V-modules can be constructed using the similar method.

The organization of this paper is as follows: in Section 2.1 we recall some basic definitions about vertex operator algebras and modules. We also review Virasoro VOAs $L(c_m,0)$ in the discrete series in Section 2.2 and the GKO (coset) construction of these Virasoro VOAs and irreducible modules $L(c_m,h^m_{r,s})$ in Section 2.3. In Section 3.1, we determine the structure of a vertex algebra $U = \mathrm{VA}(e,f)$ generated by conformal vectors e and f with central charge $1/2$ in the case where τ_e and τ_f generate S_3 and the inner product $\langle e, f\rangle$ is $13/2^{10}$. In Section 3.2, we construct such a VOA U in the lattice VOA associated with a certain lattice L by using the GKO construction. In Section 3.3, we classify all irreducible U-modules and show that U is rational. In Section 3.4, we determine fusion rules among the irreducible U-modules, which is determined by fusion rules among irreducible U^0-submodules. In Section 3.5, we calculate all conformal vectors in U and determine the automorphism group of U.

I would like to express my sincere gratitude to Professor Masahiko Miyamoto for his helpful advice and encouragement. I am also very grateful to Doctor Hiroshi Yamauchi for useful informations and discussion in a joint work with him.

2 Preliminaries

2.1 Vertex Operator Algebra and Module

In this subsection, we recall some basic definitions for a vertex operator algebra and a module (cf. [B, DL, FHL, FLM]).

A *vertex operator algebra* (VOA) V is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$ equipped with a linear map $Y : V \to (\mathrm{End}\, V)[[z, z^{-1}]]$, $a \mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ for $a \in V$ such that $\dim V_n$ is finite for all integer n and that $V_n = 0$ for sufficiently small integer n (see [FLM]). For $a, b \in V$ and $m, n \in \mathbb{Z}$, we have the commutativity

$$[a_{(m)}, b_{(n)}] = \sum_{i=0}^{\infty} \binom{m}{i} (a_{(i)}b)_{(m+n-i)} \tag{2.2}$$

and the associativity

$$(a_{(m)}b)_{(n)} = \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} \left(a_{(m-i)}b_{(n+i)} - (-1)^m b_{(m+n-i)}a_{(i)}\right). \tag{2.3}$$

There are two distinguished vectors called the *vacuum vector* $\mathbb{1} \in V_0$ and the *Virasoro element* $\omega \in V_2$. By definition $Y(\mathbb{1}, z) = \mathrm{id}_V$, and the component operators $\{L_{(n)}\}$ of $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_{(n)} z^{-n-2}$ gives a representation of the Virasoro algebra on V with central charge $c \in \mathbb{C}$, that is, we have

$$[L_{(m)}, L_{(n)}] = (m-n)L_{(m+n)} + \delta_{m+n,0}\frac{m^3-m}{12}c \tag{2.4}$$

for $m, n \in \mathbb{Z}$. Each homogeneous space V_n is an eigenspace for $L_{(0)}$ with eigenvalue n. An element e of V is called a *conformal vector with central charge* c if operators $L^e_{(n)} = e_{(n+1)}$, $n \in \mathbb{Z}$, satisfy the Virasoro relation (2.4).

A $\mathbb{Z}_+$*-graded weak* V*-module* M is a $\mathbb{Z}_{\geq 0}$-graded vector space $M = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} M(n)$ equipped with a linear map $Y_M : V \to (\mathrm{End}\, M)\{z\}$, $a \mapsto Y_M(a,z) = \sum_{n\in\mathbb{Q}} a_{(n)} z^{-n-1}$, satisfying the commutativity (2.2) and the associativity (2.3), such that $a_{(m)}M(n) \subseteq M(k+n-m-1)$ for $a \in V_k$. We also have $Y_M(\mathbb{1}, z) = \mathrm{id}_M$, and the component operators $\{L^M_{(n)}\}$ of $Y_M(\omega, z) = \sum_{n\in\mathbb{Z}} L^M_{(n)} z^{-n-2}$ gives a representation of the Virasoro algebra on M with central charge c. If a $\mathbb{Z}_+$-graded weak V-module M is irreducible, for some $\lambda \in \mathbb{C}$, each $M(n)$ is an finite-dimensional eigenspace for $L_{(0)}$ with eigenvalue $n+\lambda$. A vertex operator algebra V is said to be *rational* if any admissible V-module is completely reducible.

Let W^1, W^2 and W^3 be V-modules. An V-intertwining operator of type $W^1 \times W^2 \to W^3$ is a linear map $I : W^2 \to (\mathrm{Hom}(W^2, W^3))\{z\}$, $u \mapsto I(u,z) = \sum_{n\in\mathbb{C}} u_{(n)} z^{-n-1}$, such that for $a \in V$, $u \in W^1$, $v \in W^2$ and $\alpha \in \mathbb{C}$, $u_{(\alpha+n)}v = 0$ for $n >> 0$, $I(L_{(-1)}u, z) = \frac{\mathrm{d}}{\mathrm{d}z} I(u,z)$ and the Jacobi identity holds:

$$\begin{aligned} & z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(a,z_1) I(u,z_2) - z_0^{-1}\delta\left(\frac{-z_1 + z_2}{z_0}\right) I(u,z_2) Y(a,z_1) \\ = \; & z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) I(Y(a,z_0)u, z_2), \end{aligned}$$

where $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$. We denote by $I_V \binom{W^3}{W^1\ W^2}$ the space of V-intertwining operators of type $W^1 \times W^2 \to W^3$ and $N^{W^3}_{W^1\ W^2} = \dim I_V \binom{W^3}{W^1\ W^2}$. We use an expression

$$W^1 \times W^2 = \sum_W N^W_{W^1\ W^2} W,$$

called the *fusion rule* or *fusion product*, where W runs over all irreducible V-modules. An irreducible V-module X is called a *simple current* V*-module* if it satisfies that for every irreducible V-module W, the fusion rule $X \times W$ is also irreducible.

An *automorphism* g of V is an endomorphism of V such that $Y(g(a),z)g = gY(a,z)$ for any $a \in V$. We denote by $\mathrm{Aut}(V)$ the group of all automorphisms of V. If g be an automorphism of V and (M, Y_M) a V-module, we can define a new vertex operator on M by $Y^g_M(a,z) = Y(g(a),z)$ for $a \in V$. Then, $g(M) = (M, Y^g_M)$ is a V-module.

A (symmetric) bilinear form $\langle\cdot,\cdot\rangle$ on V is said to be *invariant* if $\langle Y(a,z)u, v\rangle = \langle u, Y(e^{L_{(1)}}(-z^{-2})^{L_{(0)}}a, -z)v\rangle$ for $a, u, v \in V$. If $\dim(V_0/L_{(1)}V_1) = 1$, then V has a unique invariant bilinear form $\langle\cdot,\cdot\rangle$ satisfying $\langle\mathbb{1},\mathbb{1}\rangle = 1$ [Li].

2.2 The Discrete Series of the Virasoro VOAs

For any complex numbers c and h, denote by $L(c,h)$ the irreducible highest weight representation of the Virasoro algebra with central charge c and highest weight h. It is shown in [FZ] that $L(c,0)$ has a natural simple VOA structure. Let

$$c_m := 1 - \frac{6}{(m+2)(m+3)} \qquad (m = 1, 2, \dots), \tag{2.5}$$

$$h_{r,s}^{(m)} := \frac{\{r(m+3) - s(m+2)\}^2 - 1}{4(m+2)(m+3)} \tag{2.6}$$

for $r, s \in \mathbb{N}$, $1 \le r \le m+1$ and $1 \le s \le m+2$. It is shown in [W] that $L(c_m, 0)$ is rational and $L(c_m, h_{r,s}^{(m)})$, $1 \le s \le r \le m+1$, provide all irreducible $L(c_m, 0)$-modules (see also [DMZ]). The fusion rules for these modules are given by

$$L(c_m, h_{r_1,s_1}) \times L(c_m, h_{r_2,s_2}) = \sum_{i \in I, j \in J} L(c_m, h_{|r_1-r_2|+2i-1, |s_1-s_2|+2j-1}), \tag{2.7}$$

where

$$I = \{1, 2, \dots, \min\{r_1, r_2, m+2-r_1, m+2-r_2\}\},$$
$$J = \{1, 2, \dots, \min\{s_1, s_2, m+3-s_1, m+3-s_2\}\}.$$

In the case $m = 1$, $L(\frac{1}{2}, 0)$ is a rational VOA called an Ising model and has exactly three irreducible module $L\left(\frac{1}{2}, 0\right)$, $L\left(\frac{1}{2}, \frac{1}{2}\right)$ and $L\left(\frac{1}{2}, \frac{1}{16}\right)$. The fusion rules among these modules are given by

$$\begin{aligned} L\left(\tfrac{1}{2}, \tfrac{a}{2}\right) \times L\left(\tfrac{1}{2}, \tfrac{b}{2}\right) &= L\left(\tfrac{1}{2}, \tfrac{a+b}{2}\right), \quad a, b \in \mathbb{Z}_2, \\ L\left(\tfrac{1}{2}, \tfrac{a}{2}\right) \times L\left(\tfrac{1}{2}, \tfrac{1}{16}\right) &= L\left(\tfrac{1}{2}, \tfrac{1}{16}\right), \\ L\left(\tfrac{1}{2}, \tfrac{1}{16}\right) \times L\left(\tfrac{1}{2}, \tfrac{1}{16}\right) &= L\left(\tfrac{1}{2}, 0\right) + L\left(\tfrac{1}{2}, \tfrac{1}{2}\right). \end{aligned}$$

If a VOA V contains $L(\frac{1}{2}, 0)$ as a subVOA with the Virasoro element e, one can define an involutive automorphism τ_e of V by

$$\tau_e : \begin{cases} 1 & \text{on } W_0 \oplus W_{\frac{1}{2}} \\ -1 & \text{on } W_{\frac{1}{16}}, \end{cases}$$

where W_h denotes the sum of all irreducible $L(\frac{1}{2}, 0)$-submodules isomorphic to $L(\frac{1}{2}, h)$.

By (2.7), $L(c_m, h_{m+1,1}) \times L(c_m, h_{r,s}) = L(c_m, h_{m+1-r,s})$ and so $L(c_m, h_{m+1,1})$ is a simple current module. It is known that $V^{(m)} = L(c_m, 0) \oplus L(c_m, h_{m+1,1})$ is a simple VOA if $h_{m+1,1} \in \mathbb{Z}$ and a simple superVOA if $h_{m+1,1} \in \frac{1}{2} + \mathbb{Z}$ (cf. [LLY]).

In the case $m = 3$, a VOA $W(0) = L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3)$ is studied in [KMY, M2]. For $h = \frac{2}{3}, \frac{1}{15}$, since $L(\frac{4}{5}, 3) \times L(\frac{4}{5}, h) = L(\frac{4}{5}, h)$, $M = L(\frac{4}{5}, h)$ has two VOA structure (M, Y) and $(M, \tilde{Y})$ of $W(0)$ such that $Y(v, z) = \tilde{Y}(v, z)$ for $v \in L(\frac{4}{5}, 0)$ and $Y(v, z) = -\tilde{Y}(v, z)$ for $v \in L(\frac{4}{5}, 3)$, which we denote by $L(\frac{4}{5}, h)^{\pm}$.

Theorem 2.1. *[KMY] $W(0)$ is a rational VOA and has exactly six inequivalent irreducible modules:*

$$W(0) := L(\tfrac{4}{5}, 0) \oplus L(\tfrac{4}{5}, 3), \quad W(\tfrac{2}{3})^{\pm} := L(\tfrac{4}{5}, \tfrac{2}{3})^{\pm},$$
$$W(\tfrac{2}{5}) := L(\tfrac{4}{5}, \tfrac{2}{5}) \oplus L(\tfrac{4}{5}, \tfrac{7}{5}), \quad W(\tfrac{1}{15})^{\pm} := L(\tfrac{4}{5}, \tfrac{1}{15})^{\pm}.$$

Let θ_3 be an automorphism of $W(0)$ defined by 1 on $L(\frac{4}{5},0)$ and -1 on $L(\frac{4}{5},3)$. Then, $\theta_3(W(k)) = W(k)$ for $k = 0, \frac{2}{5}$ and $\theta_3(W(h)^{\pm}) = W(h)^{\mp}$ for $h = \frac{2}{3}, \frac{1}{15}$. The fusion algebra for $W(0)$ is determined in [M2]. For convenience, we use the following $\mathbb{Z}_3$-graded names. Define

$$\begin{array}{lll} A_0^0 := W(0), & A_0^1 := W(\frac{2}{3})^+, & A_0^2 := W(\frac{2}{3})^-, \\ A_1^0 := W(\frac{2}{5}), & A_1^1 := W(\frac{1}{15})^+, & A_1^2 := W(\frac{1}{15})^-. \end{array}$$

Theorem 2.2. *[M2] The fusion rules for irreducible $W(0)$-modules are given as*

$$A_0^i \times A_0^j = A_0^{i+j}, \quad A_0^i \times A_1^j = A_1^{i+j}, \quad A_1^i \times A_1^j = A_0^{i+j} + A_1^{i+j},$$

where $i,j \in \mathbb{Z}_3$. Therefore, the fusion algebra for $W(0)$ has a natural $\mathbb{Z}_3$-symmetry.

If a VOA V contains a subVOA W isomorphic to $W(0)$, by the $\mathbb{Z}_3$-symmetry of the fusion algebra for $W(0)$, we can define an automorphism σ_W of V with $\sigma_W^3 = 1$ as follows:

$$\sigma_W : \begin{cases} 1 & on \quad W(k), \quad k = 0, \frac{2}{5}, \\ e^{\pm 2\pi i/3} & on \quad W(h)^{\pm}, \quad h = \frac{2}{3}, \frac{1}{15}. \end{cases}$$

In the case $m = 4$, a VOA $N(0) = L(\frac{6}{7},0) \oplus L(\frac{6}{7},5)$ is studied in [LY, LLY]. For $h = \frac{4}{3}, \frac{1}{21}, \frac{10}{21}$, since $L(\frac{6}{7},5) \times L(\frac{6}{7},h) = L(\frac{6}{7},h)$, $M = L(\frac{6}{7},h)$ has two VOA structure (M,Y) and $(M,\tilde{Y})$ of $N(0)$ such that $Y(v,z) = \tilde{Y}(v,z)$ for $v \in L(\frac{6}{7},0)$ and $Y(v,z) = -\tilde{Y}(v,z)$ for $v \in L(\frac{6}{7},5)$, which we denote by $L(\frac{6}{7},h)^{\pm}$.

Theorem 2.3. *[LY, LLY] $N(0) = L(\frac{6}{7},0) \oplus L(\frac{6}{7},5)$ is a rational VOA and has exactly nine inequivalent irreducible modules:*

$$\begin{array}{ll} N(0) := L(\frac{6}{7},0) \oplus L(\frac{6}{7},5), & N(\frac{4}{3})^{\pm} := L(\frac{6}{7},\frac{4}{3})^{\pm}, \\ N(\frac{1}{7}) := L(\frac{6}{7},\frac{1}{7}) \oplus L(\frac{6}{7},\frac{22}{7}), & N(\frac{1}{21})^{\pm} := L(\frac{6}{7},\frac{1}{21})^{\pm}, \\ N(\frac{5}{7}) := L(\frac{6}{7},\frac{5}{7}) \oplus L(\frac{6}{7},\frac{12}{7}), & N(\frac{10}{21})^{\pm} := L(\frac{6}{7},\frac{10}{21})^{\pm}. \end{array}$$

Let θ_4 be an automorphism of $N(0)$ defined by 1 on $L(\frac{6}{7},0)$ and -1 on $L(\frac{6}{7},5)$. Then, $\theta_4(N(k)) = N(k)$ for $k = 0, \frac{1}{7}, \frac{5}{7}$ and $\theta_4(N(h)^{\pm}) = N(h)^{\mp}$ for $h = \frac{4}{3}, \frac{1}{21}, \frac{10}{21}$. The fusion algebra for $N(0)$ is also determined in [LY] and [LLY]. To state the fusion rules, we assign $\mathbb{Z}_3$-graded names to irreducible modules (cf. [LY]). Define

$$\begin{array}{lll} B_0^0 := N(0), & B_0^1 := N(\frac{4}{3})^+, & B_0^2 := N(\frac{4}{3})^-, \\ B_1^0 := N(\frac{1}{7}), & B_1^1 := N(\frac{10}{21})^+, & B_1^2 := N(\frac{10}{21})^-, \\ B_2^0 := N(\frac{5}{7}), & B_2^1 := N(\frac{1}{21})^+, & B_2^2 := N(\frac{1}{21})^-. \end{array}$$

Theorem 2.4. *[LY, LLY] The fusion rules for irreducible $N(0)$-modules are given as*

$$\begin{aligned} B_0^i \times B_s^j &= B_s^{i+j}, \quad s=0,1,2, \\ B_1^i \times B_1^j &= B_0^{i+j} + B_2^{i+j}, \\ B_1^i \times B_2^j &= B_1^{i+j} + B_2^{i+j}, \\ B_2^i \times B_2^j &= B_0^{i+j} + B_1^{i+j} + B_2^{i+j}, \end{aligned}$$

where $i, j \in \mathbb{Z}_3$. Therefore, the fusion algebra for $N(0)$ has a natural $\mathbb{Z}_3$-symmetry.

2.3 GKO Construction of the Virasoro VOA $L(c_m, 0)$

Let $\mathfrak{g}$ be the Lie algebra $sl_2(\mathbb{C})$ with generators e, f and α such that $[e,f]=\alpha, [\alpha, e] = 2e$ and $[\alpha, f] = -2f$. We use the standard invariant bilinear form on $sl_2(\mathbb{C})$ defined by $\langle \alpha, \alpha\rangle = 2, \langle e, f\rangle = 1$ and $\langle e,e\rangle = \langle f,f\rangle = \langle \alpha, e\rangle = \langle \alpha, f\rangle = 0$. Let $\hat{\mathfrak{g}} = sl_2(\mathbb{C})\otimes\mathbb{C}[t,t^{-1}]\oplus \mathbb{C}c \oplus \mathbb{C}d$ be the corresponding affine Lie algebra of type $A_1^{(1)}$ and $\Lambda_0 = d, \Lambda_1 = d + \frac{1}{2}\alpha$ the fundamental weights for $\hat{\mathfrak{g}}$. Then the dominant integral weights of $\hat{\mathfrak{g}}$ are given by

$$P_{++} = \left\{ (m-j)\Lambda_0 + j\Lambda_1 = md + \frac{1}{2}j\alpha \,\middle|\, m \in \mathbb{Z}^+, j \in \mathbb{Z}^+ \cup \{0\}, j \le m \right\}.$$

Let $\mathcal{L}(m,j) = \mathcal{L}((m-j)\Lambda_0 + j\Lambda_1)$ be the irreducible highest weight module of $\hat{\mathfrak{g}}$ of the weight $(m-j)\Lambda_0 + j\Lambda_1 \in P_{++}$. It was proved by Frenkel and Zhu [FZ] that $\mathcal{L}(m,0)$ is a rational simple VOAs for all $m > 0$ and $\{\mathcal{L}(m,j) \mid j = 0,1,\cdots,m\}$ is the set of all inequivalent irreducible $\mathcal{L}(m,0)$-modules. The Virasoro vector Ω^m of $\mathcal{L}(m,0)$ is given by

$$\Omega^m := \frac{1}{2(m+2)}\left(\frac{1}{2}\alpha_{(-1)}\alpha + e_{(-1)}f + f_{(-1)}e\right) \tag{2.8}$$

with central charge $3m/(m+2)$. Moreover, the fusion rules (cf. [FZ]) are given by

$$\mathcal{L}(m,j) \times \mathcal{L}(m,k) = \sum_{i=\max 0, j+k-m}^{\min j,k} \mathcal{L}(m, j+k-2i). \tag{2.9}$$

The weight 1 subspace of $\mathcal{L}(m,0)$ forms a Lie algebra isomorphic to $\mathfrak{g}$ under the 0-th product in $\mathcal{L}(m,0)$. Let α^1, e^1, f^1 be the generator of $\mathfrak{g}$ in $\mathcal{L}(1,0)_1$ and α^m, e^m, f^m those in $\mathcal{L}(m,0)_1$. Then $H = \alpha^m \otimes \mathbb{1} + \mathbb{1} \otimes \alpha^1$, $E = e^m \otimes \mathbb{1} + \mathbb{1}\otimes e^1$ and $F = f^m \otimes \mathbb{1} + \mathbb{1} \otimes f^1$ generate a sub VOA isomorphic to $\mathcal{L}(m+1,0)$ in $\mathcal{L}(m,0)\otimes\mathcal{L}(1,0)$ with the Virasoro vector Ω^{m+1} made from H, E and F by (2.8). It is shown in [DL] and [KR] that $\omega^m := \Omega^m \otimes \mathbb{1} + \mathbb{1} \otimes \Omega^1 - \Omega^{m+1}$ also gives a conformal vector with central charge $c_m = 1 - 6/(m+2)(m+3)$. Furthermore, Ω^{m+1} and ω^m are mutually commutative and ω^m generates a simple Virasoro VOA $L(c_m, 0)$. Hence, $\mathcal{L}(m,0)\otimes\mathcal{L}(1,0)$ contains a sub VOA isomorphic to $L(c_m, 0) \otimes \mathcal{L}(m+1, 0)$. Since both $L(c_m, 0)$ and $\mathcal{L}(m+1, 0)$ are rational, every $\mathcal{L}(m, 0) \otimes \mathcal{L}(1, 0)$-module can be decomposed into irreducible $L(c_m, 0) \otimes \mathcal{L}(m+1, 0)$-submodules. The following decomposition is obtained in [GKO]:

$$\mathcal{L}(m, i) \otimes \mathcal{L}(1, j) = \bigoplus_{\substack{0 \le s \le m+1 \\ s \equiv i+j \mod 2}} L(c_m, h^{(m)}_{j+1, s+1}) \otimes \mathcal{L}(m+1, s), \tag{2.10}$$

where $i = 0, 1$ and $0 \le j \le m$. Note that $h^{(m)}_{r,s} = h^{(m)}_{m+2-r, m+3-s}$. This is the famous GKO construction of the unitary Virasoro VOAs.

As a consequence, one knows that all the irreducible modules $L(c_m, h^{(m)}_{r,s})$, $1 \le r \le s \le m+1$ can be realized as certain submodules of $\mathcal{L}(m, j) \otimes \mathcal{L}(1, i)$ for $0 \le j \le m$ and $i = 0, 1$.

2.4 Griess Algebra

A VOA V over $\mathbb{R}$ is said to be *of moonshine type* if it admits a weight space decomposition $V = \oplus_{n=0}^{\infty} V_n$ with $V_0 = \mathbb{R}\mathbb{1}$ and $V_1 = 0$ and a (unique) invariant bilinear form $\langle \cdot, \cdot \rangle$ on V satisfying $\langle \mathbb{1}, \mathbb{1} \rangle = 1$ is a definite on V_n for each n.

If V is of moonshine type, by defining $a \times b = a_{(1)} b$ for $a, b \in V_2$, $(V_2, \times)$ becomes a commutative (non-associative) algebra called a Griess algebra with a (positive) definite invariant bilinear form $\langle \cdot, \cdot \rangle$, which is given by $\langle a, b \rangle \mathbb{1} = a_{(3)} b$ for $a, b \in V_2$. To simplify the notation, ab denotes $a \times b$ for $a, b \in V_2$. It is shown in [M1] that we have a one-to-one correspondence $e \mapsto e/2$ between the set of conformal vectors with central charge c in V and the set of idempotents with squared length $c/8$ in V_2.

3 VOA with two τ-involution generating S_3

Assume that a VOA V of moonshine type contains two distinct rational conformal vectors e and f with central charge $1/2$. In [M3], Miyamoto studied a vertex algebra $\mathrm{VA}(e, f)$ generated by e and f in the case where τ_e and τ_f generate S_3. In this section, we study $\mathrm{VA}(e, f)$ in the case where the inner product $\langle e, f \rangle$ is $13/2^{10}$. For convenience, we will treat the complexification $\mathbb{C} \otimes_{\mathbb{R}} V$ of V, which we also denote by V. We set $U = \mathrm{VA}(e, f)$ throughout this section.

3.1 Structures

In this subsection, we shall determine the VOA structure of U. Let $U^{\pm} = \{u \in U \mid \tau_e(u) = \pm u\}$ and $U_n^{\pm} = U_n \cap U^{\pm}$. As shown in [M3], the Griess algebra U_2 is of dimension 4 and we can choose a basis $\{\omega^3, \omega^4, \mathbf{u}, \mathbf{v}\}$ such that $U_2^+ = \mathbb{C}\omega^3 \perp \mathbb{C}\omega^4 \perp \mathbb{C}\mathbf{u}$ and $U_2^- = \mathbb{C}\mathbf{v}$, where $\omega^3 + \omega^4$ is the Virasoro vector of U and the multiplications and inner products in U are given as

$$\begin{aligned} &\omega^i_{(1)}\omega^i = 2\omega^i \ (i=3,4), \quad \omega^3_{(1)}\omega^4 = 0, \quad \mathbf{u}_{(1)}\mathbf{u} = \frac{5}{6}\omega^3 + \frac{14}{9}\omega^4 - \frac{10}{9}\mathbf{u}, \\ &\omega^3_{(1)}x = \frac{2}{3}x, \quad \omega^4_{(1)}x = \frac{4}{3}x \ \ (x \in \mathbb{C}\mathbf{u} + \mathbb{C}\mathbf{v}), \quad \mathbf{u}_{(1)}\mathbf{v} = \frac{10}{9}\mathbf{v}, \\ &\langle \omega^3, \omega^3 \rangle = \frac{2}{5}, \quad \langle \omega^4, \omega^4 \rangle = \frac{3}{7}, \quad \langle \mathbf{u}, \mathbf{u} \rangle = \frac{1}{2}, \quad \langle \mathbf{v}, \mathbf{v} \rangle = 1. \end{aligned} \tag{3.11}$$

In particular, ω^3 and ω^4 are conformal vectors with central charge $\frac{4}{5}$ and $\frac{6}{7}$, respectively. We set $L^i_{(n)} = \omega^i_{(n+1)}$ for $i = 3, 4$.

Let $T = \mathrm{VA}(\omega^3, \omega^4)$, which is isomorphic to $L\left(\frac{4}{5}, 0\right) \otimes L\left(\frac{6}{7}, 0\right)$, and $M(v)$ the T-submodule of V generated by $v \in V$. An element $v \neq 0$ of V is said to be a *highest weight vector* for T with highest weight (h, k) if $L^3_{(0)}v = hv$, $L^4_{(0)}v = kv$ and $L^i_{(n)}v = 0$ for $n \geq 1$ and $i = 3, 4$. Then we note that $h + k$ is an integer if and only if (h, k) is $(0, 0)$, $(3, 0)$, $(0, 5)$, $(3, 5)$ or $(\frac{4}{5}, \frac{6}{7})$,

Set

$$\begin{aligned} \mathbf{w}^3 &= \mathbf{u}_{(0)}\mathbf{v} - \frac{5}{9}(L^3_{(-1)} + L^4_{(-1)})\mathbf{v} \\ \mathbf{w}^5 &= \frac{5^2}{3^4}\left(\frac{11}{3}L^3_{(-3)} - 2L^3_{(-2)}L^3_{(-1)}\right)\mathbf{v} + \frac{5^2}{2^3 \cdot 3^2}\left(2L^3_{(-2)} - L^3_{(-1)}L^3_{(-1)}\right)L^4_{(-1)}\mathbf{v} \\ &\quad + \frac{7}{3^4}\left(\frac{20}{3}L^4_{(-3)} - L^4_{(-2)}L^4_{(-1)}\right)\mathbf{v} + \frac{7}{2^2 \cdot 3^2 \cdot 5}\left(8L^4_{(-2)} - L^4_{(-1)}L^4_{(-1)}\right)L^3_{(-1)}\mathbf{v} \\ &\quad - \frac{5}{2 \cdot 13}\left(\frac{1}{3}L^3_{(-2)} - \frac{3}{5}L^3_{(-1)}L^3_{(-1)}\right)\mathbf{w}^3 + \frac{28}{9}L^4_{(-2)}\mathbf{w}^3 - \mathbf{u}_{(-2)}\mathbf{v}. \end{aligned}$$

It is easy to see that $\mathbf{w}^3 \in U_3^-$ and $\mathbf{w}^5 \in U_5^-$.

Lemma 3.1. *(1)* $\mathbf{w}^3$ *is a highest weight vector for* T *with highest weight* $(3, 0)$ *and so* $M(\mathbf{w}^3) \simeq L(\frac{4}{5}, 3) \otimes L(\frac{6}{7}, 0)$.
(2) $\mathbf{w}^5$ *is a highest weight vector for* T *with highest weight* $(0, 5)$ *and so* $M(\mathbf{w}^3) \simeq L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 5)$.

Proof. By the commutativity of VOA, we can see that for $x, y \in V_2$ and $m, n \in \mathbb{Z}$,

$$\begin{aligned} &\left[x_{(m)}, y_{(n)}\right] \\ &= \left[x_{(1)}, y_{(m+n-1)}\right] + (m-1)(x_{(1)}y)_{(m+n-1)} + \delta_{m+n-2,0}\binom{m}{3}\langle x, y \rangle 1. \end{aligned} \tag{3.12}$$

Using this identity, for $n \geq 1$, $k \geq 0$ and $i = 3, 4$,

$$\begin{aligned} L^i_{(n)}\mathbf{u}_{(-k)}\mathbf{v} &= \left[L^i_{(n)}, \mathbf{u}_{(-k)}\right]\mathbf{v} \\ &= (\left[L^i_{(0)}, \mathbf{u}_{(n-k)}\right] + n(L^i_{(0)}\mathbf{u}))\mathbf{v} \\ &= (L^i_{(0)} + (n-1)k^i)\mathbf{u}_{(n-k)}\mathbf{v}, \quad (3.13) \\ \mathbf{u}_{(k+2)}\mathbf{u}_{(-k)}\mathbf{v} &= ((k+1)(\mathbf{u}_{(1)}\mathbf{u})_{(1)} + \binom{k+3}{3}\langle \mathbf{u}, \mathbf{u}\rangle)\mathbf{v} \\ &= (\frac{113}{81}(k+1) + \frac{1}{2}\binom{k+2}{3})\mathbf{v}, \quad (3.14) \end{aligned}$$

where $k^3 = \frac{2}{3}$ and $k^4 = \frac{4}{3}$. Hence, by (3.13) and (3.14), it is easy to check that $L^i_{(n)}\mathbf{w}^3 = 0$ for $n \geq 1, i = 3, 4$, and $\langle \mathbf{w}^3, \mathbf{w}^3\rangle = \frac{13}{81}$ since $\langle L^i_{(-n)}x, y\rangle = \langle x, L^i_{(n)}y\rangle$ and $\langle \mathbf{u}_{(n)}x, y\rangle = \langle x, \mathbf{u}_{(2-n)}y\rangle$ for $x, y \in V$. Therefore, $\mathbf{w}^3$ is a highest weight vector for T and $\dim U_3^- = 3$.

By using $L^i_{(n)}L^i_{(0)} = \left[L^i_{(n)}, L^i_{(0)}\right] + L^i_{(0)}L^i_{(n)} = (m + L^i_{(0)})L^i_{(n)}$ and (3.13),

$$\begin{aligned} &L^i_{(n_1)}L^i_{(n_2)} \cdots L^i_{(n_s)}\mathbf{u}_{(-k)}\mathbf{v} \\ &= (L^i_{(0)} + N_{s-1} + (n_s - 1)k^i) \cdots (L^i_{(0)} + N_1 + (n_2 - 1)k^i)(L^i_{(0)} + (n_s - 1)k^i)\mathbf{u}_{(N_s-k)}\mathbf{v} \end{aligned}$$

for $n_1, n_2, \cdots, n_s \geq 1$, where $N_j = \sum_{p=1}^{j} n_p$. Hence, by this and (3.14), we can check that

$$\begin{aligned} \langle L^{i_s}_{(-n_s)} \cdots L^{i_1}_{(-n_2)}L^i_{(-n_1)}x, \mathbf{w}^5\rangle &= 0 \\ \langle \mathbf{w}^5, \mathbf{w}^5\rangle &= \frac{58991}{2 \cdot 3^8 \cdot 13} \end{aligned}$$

for $n_t \geq 1$, $i_t = 3, 4$, $1 \leq t \leq s$ and $x = \mathbf{v}, \mathbf{w}^3$. Thus $L_{(n)}\mathbf{w}^5 \in (M(\mathbf{v}) \oplus M(\mathbf{w}^3))^{\perp} \cap U^-_{5-n} = 0$ for $n \geq 1$ since there is no highest weight vector for T in U_4. Therefore, $\mathbf{w}^5$ is a highest weight vectors for T. □

By the above lemma and fusion rules for $L\left(\frac{4}{5}, 0\right) \otimes L\left(\frac{6}{7}, 0\right)$,

Proposition 3.2. *We have a decomposition* $U = U^+ \oplus U^-$ *and as a* $L\left(\frac{4}{5}, 0\right) \otimes L\left(\frac{6}{7}, 0\right)$-*module*

$$\begin{aligned} U^+ &\simeq L\left(\frac{4}{5}, 0\right) \otimes L\left(\frac{6}{7}, 0\right) \bigoplus L\left(\frac{4}{5}, 3\right) \otimes L\left(\frac{6}{7}, 5\right) \bigoplus L\left(\frac{4}{5}, \frac{2}{3}\right) \otimes L\left(\frac{6}{7}, \frac{4}{3}\right), \\ U^- &\simeq L\left(\frac{4}{5}, 0\right) \otimes L\left(\frac{6}{7}, 5\right) \bigoplus L\left(\frac{4}{5}, 3\right) \otimes L\left(\frac{6}{7}, 0\right) \bigoplus L\left(\frac{4}{5}, \frac{2}{3}\right) \otimes L\left(\frac{6}{7}, \frac{4}{3}\right). \end{aligned}$$

Let $G = \langle \tau_e, \tau_f\rangle \simeq S_3$. The symmetric group G on three letters has 3 irreducible modules $\mathbb{C}(+)$, $\mathbb{C}(-)$ and B, where $\mathbb{C}(+)$ is a trivial module, τ_e and τ_f act on $\mathbb{C}(-)$ as

-1 and B is an irreducible module of dimension two. Therefore, U has a subVOA U^G such that U decomposes into the direct sum

$$U = U^G \oplus X \otimes \mathbb{C}(-) \oplus W \otimes B,$$

where X and W are irreducible U^G-modules, see [DM1]. Then, $\omega^3, \omega^4 \in U^G$ and it follows from Proposition 3.2 that $U^G \simeq L\left(\frac{4}{5},0\right) \otimes L\left(\frac{6}{7},0\right) \oplus L\left(\frac{4}{5},3\right) \otimes L\left(\frac{6}{7},5\right)$, $X \simeq L\left(\frac{4}{5},0\right) \otimes L\left(\frac{6}{7},5\right) \oplus L\left(\frac{4}{5},3\right) \otimes L\left(\frac{6}{7},0\right)$ and $W \simeq L\left(\frac{4}{5},\frac{2}{3}\right) \otimes L\left(\frac{6}{7},\frac{4}{3}\right)$ as a $L\left(\frac{4}{5},0\right) \otimes L\left(\frac{6}{7},0\right)$-module.

By Proposition 3.2, it is easy to see that U contains $W(0) = L\left(\frac{4}{5},0\right) \oplus L\left(\frac{4}{5},3\right)$ and $N(0) = L\left(\frac{6}{7},0\right) \oplus L\left(\frac{6}{7},5\right)$ as a subVOA.

Theorem 3.3. *A VOA U contains a sub VOA $W(0) \otimes N(0)$. As a $W(0) \otimes N(0)$-module, U is isomorphic to*

$$W(0) \otimes N(0) \oplus W\left(\frac{2}{3}\right)^+ \otimes N\left(\frac{4}{3}\right)^+ \oplus W\left(\frac{2}{3}\right)^- \otimes N\left(\frac{4}{3}\right)^- \qquad (3.15)$$

after fixing suitable choice of $\pm$-type of $N(\frac{4}{3})^\pm$.

Proof. Let $\rho = \tau_e \tau_f$ and $U^t = \left\{ v \in U \mid \rho v = e^{\frac{2\pi i t}{3}} v \right\}$. Then, U^0 is a subVOA of U isomorphic to $W(0) \otimes N(0)$ and $U = U^0 \oplus U^1 \oplus U^2$ is $\mathbb{Z}_3$-grading. Since $U^1 \simeq U^2 \simeq L\left(\frac{4}{5},\frac{2}{3}\right) \otimes L\left(\frac{6}{7},\frac{4}{3}\right)$ as $L\left(\frac{4}{5},0\right) \otimes L\left(\frac{6}{7},0\right)$-module, $U^i \simeq W(\frac{2}{3})^{\epsilon_i} \otimes N(\frac{4}{3})^{\xi_i}$ as a $W(0) \otimes N(0)$-module, where $\epsilon_i, \xi_i = \pm$. By fusion rule for $W(0) \otimes N(0)$, $\epsilon_1 \neq \epsilon_2$ and $\xi_1 \neq \xi_2$. □

Let D be a finite abelian group and V^0 a rational simple VOA. Assume that a set of irreducible V^0-modules $\{V^\alpha \mid \alpha \in D\}$ indexed by D is given. A *D-graded extension* V_D of V^0 is a simple VOA with the shape $V = \oplus_{\alpha \in D} V^\alpha$ whose vacuum element and Virasoro element are given by those of V^0 and vertex operations in V_D satisfies $Y(u^\alpha, z)v^\beta \in V^{\alpha+\beta}((z))$ for any $u^\alpha \in V^\alpha$ and $v^\beta \in V^\beta$. A D-graded extension $V_D = \oplus_{\alpha \in D} V^\alpha$ of V^0 is called a *simple current extension* if all V^α, $\alpha \in D$, are simple current V^0-modules. By Theorem 3.3, U is a $\mathbb{Z}_3$-graded simple current extension of $W(0) \otimes N(0)$.

Theorem 3.4. *All $\mathbb{Z}_3$-graded simple current extensions of $W(0) \otimes N(0)$ are isomorphic. In particular, U has a unique VOA structure.*

Proof. Let $V = \oplus_{s \in \mathbb{Z}_3} V^s$ and $\tilde{V} = \oplus_{s \in \mathbb{Z}_3} \tilde{V}^s$ be $\mathbb{Z}_3$-graded simple current extensions of $W(0) \otimes N(0)$. Then, $V^0 \simeq \tilde{V}^0 \simeq W(0) \otimes N(0)$ and V^s and $\tilde{V}^s$ are simple current $W(0) \otimes N(0)$-modules. Since the simple current $W(0) \otimes N(0)$-modules are $W(0) \otimes N(0)$ and $W(\frac{2}{3})^\pm \otimes N(\frac{4}{3})^\pm$, $V^1 \simeq W(\frac{2}{3})^\epsilon \otimes N(\frac{4}{3})^\xi$ and $\tilde{V}^1 \simeq W(\frac{2}{3})^{\epsilon'} \otimes N(\frac{4}{3})^{\xi'}$, where $\epsilon, \epsilon', \xi, \xi' = \pm$. Since $\theta_3(W(\frac{2}{3})^\pm) \simeq W(\frac{2}{3})^\mp$ and $\theta_4(N(\frac{4}{3})^\pm) \simeq N(\frac{4}{3})^\mp$, there is an automorphism

$\sigma \in \langle \theta_3, \theta_4 \rangle$ of $W(0) \otimes N(0)$ such that $\sigma(\tilde{V}^1) \simeq V^1$ and $\sigma(\tilde{V}^2) \simeq V^2$, where θ_3 and θ_4 are natural automorphisms of order 2 defined by the $\mathbb{Z}_2$-grading of $W(0)$ and $N(0)$, respectively. Then, $\sigma(\tilde{V}) = \oplus_{s \in \mathbb{Z}_3} \sigma(\tilde{V}^s)$ has a VOA structure isomorphic to $\tilde{V}$. It follows from Proposition 5.2 of [DM2] that $\sigma(\tilde{V})$ is isomorphic to V and so $V \simeq \tilde{V}$. □

3.2 Construction

In this subsection, we construct a VOA generated by two conformal vectors such that $\tau_e \tau_f$ is of order 3 and $\langle e, f \rangle = 13/2^{10}$.

First we recall a construction of the lattice VOA associated with an even lattice from [FLM]. Let L be an even lattice with a positive definite symmetric $\mathbb{Z}$-bilinear form $\langle \cdot, \cdot \rangle$ and $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$. Let $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t^{-1}, t^{-1}] \oplus \mathbb{C}C$ be the affinization of commutative Lie algebra $\mathfrak{h}$. For convenience, we denote $h \otimes t^n$ by $h(n)$ for $h \in \mathfrak{h}$ and $n \in \mathbb{Z}$. Let $\hat{L} = \{\pm e_\alpha | \alpha \in L\}$ be the canonical central extension of L by the cyclic group $\langle \pm 1 \rangle$ with commutator map $c(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle}$ for $\alpha, \beta \in L$. Let $L^* = \{x \in \mathbb{Q} \otimes_{\mathbb{Z}} L \mid \langle x, \alpha \rangle \in \mathbb{Z}\}$ be the dual lattice of L. Then there is an $\hat{L}$-module structure on $\mathbb{C}[L^*] = \oplus_{\alpha \in L^*} \mathbb{C}e^\alpha$, where -1 acts on $\mathbb{C}$ as multiplication by -1. Set $\mathbb{C}[M] = \oplus_{\alpha \in M} \mathbb{C}e^\alpha$ and $V_M = S(\hat{\mathfrak{h}}^-) \otimes \mathbb{C}[M]$ for a subset M of L^*, where $S(\hat{\mathfrak{h}}^-)$ is the symmetric algebra of $\hat{\mathfrak{h}}^- = \{h(-n) \mid h \in \mathfrak{h}, n > 0\}$. Then V_L is a simple vertex operator algebra and $V_{\alpha+L}$ is a irreducible V_L -module for $\alpha \in L^*$ (cf. [B, FLM]).

Let $A_1 = \mathbb{Z}\alpha$ with $\langle \alpha, \alpha \rangle = 2$ be the root lattice of type A_1 and V_{A_1} the lattice VOA associated with A_1. Then $A_1^* = A_1 \cup (\frac{1}{2}\alpha + A_1)$. It is well known that V_{A_1} and $V_{\frac{1}{2}\alpha + A_1}$ are both level 1 representations of $\hat{sl}_2(\mathbb{C})$ (cf. [DL, FLM]). In fact, $V_{A_1} \simeq \mathcal{L}(1, 0)$ and $V_{\frac{1}{2}\alpha + A_1} \simeq \mathcal{L}(1, 1)$. Let $A_1^{m+1} = \mathbb{Z}\alpha^0 \oplus \mathbb{Z}\alpha^1 \oplus \cdots \oplus \mathbb{Z}\alpha^m$ be the orthogonal sum of $m+1$ copies of A_1. Then we have

$$V_{A_1^{m+1}} \simeq V_{A_1} \otimes \cdots \otimes V_{A_1} \simeq \mathcal{L}(1,0) \otimes \cdots \otimes \mathcal{L}(1,0)$$

as a vertex operator algebra and

$$V_{\gamma_a + A_1^{m+1}} \simeq \mathcal{L}(1, a_0) \otimes \cdots \otimes \mathcal{L}(1, a_m)$$

as a module of $\mathcal{L}(1,0) \otimes \cdots \otimes \mathcal{L}(1,0)$, where $a = (a_0, a_1, \cdots, a_m) \in \{0,1\}^{m+1}$ and $\gamma_a = \frac{1}{2} \sum_{i=0}^m a_i \alpha^i$.

Let $H^j = \alpha^0(-1)\mathbf{1} + \cdots + \alpha^j(-1)\mathbf{1}$, $E^j = e^{\alpha^0} + \cdots + e^{\alpha^j}$ and $F^j = e^{-\alpha^0} + \cdots + e^{-\alpha^j}$. Then $\{H^j, E^j, F^j\}$ forms a simple Lie algebra $sl_2(\mathbb{C})$ inside the weight one space of $V_{A_1^{m+1}}$. It is shown in [DL, FZ] that H^j, E^j, F^j generates a simple VOA $\mathcal{L}(j+1, 0)$ of level $j+1$

and the Virasoro element of $\mathcal{L}(j+1,0)$ is given by

$$\begin{aligned}\Omega^j &= \frac{1}{2(j+3)}\left(\frac{1}{2}H^j_{(-1)}H^j + E^j_{(-1)}F^j + F^j_{(-1)}E^j\right)\\ &= \frac{1}{2(j+3)}\left\{\frac{3}{2}\sum_{p=0}^{j}\alpha^p(-1)^2\mathbb{1} + \frac{1}{2}\sum_{0\le p\ne q\le j}\alpha^p(-1)\alpha^q(-1)\mathbb{1} + 2\sum_{0\le p\ne q\le j} e^{\alpha^p-\alpha^q}\right\}\end{aligned}$$

and the central charge of Ω^j is $\frac{3(j+1)}{j+3}$. On the other hand, the Virasoro element of the lattice subVOA $V_{\mathbb{Z}\alpha^j}(\simeq V_{A_1})$ is given by $\frac{1}{4}\alpha^j(-1)\mathbb{1}$. By using the GKO construction, we know that $\omega^j = \frac{1}{4}\alpha^j(-1)\mathbb{1} + \Omega^{j-1} - \Omega^j$ generates a Virasoro subVOA $L(c_j,0)$ with central charge $c_j = 1-6/(j+2)(j+3)$. Thus, we know that we have an orthogonal decomposition of the Virasoro vector ω of $V_{A_1^{m+1}}$ into a sum of mutually commutative Virasoro vectors as $\omega = \omega^1 + \cdots + \omega^m + \Omega^m$ and the VOA $V_{A_1^{m+1}}$ contains a subVOA isomorphic to

$$T^{(m)} = L(c_1,0)\otimes L(c_2,0)\otimes\cdots\otimes L(c_m,0)\otimes\mathcal{L}(m+1,0).$$

Moreover, we have the following decomposition.

Lemma 3.5. *[LLY, Lemma 3.1] For $a=(a_0,a_1,\cdots,a_m)\in\{0,1\}^{m+1}$ define $b_j=\sum_{i=0}^{j}a_i$, then*

$$V_{\gamma_a+A_1^{m+1}} \simeq \bigoplus_{\substack{0\le k_j\le j+1\\ j=0,\cdots,m\\ k_j\equiv b_j \mod 2}} L(c_1,h^1_{k_0+1,k_1+1})\otimes\cdots\otimes L(c_m,h^1_{k_{m-1}+1,k_m+1})\otimes\mathcal{L}(m+1,k_m).$$

We consider the case $m=4$. Set $\gamma=\frac{1}{2}\alpha^0+\frac{1}{2}\alpha^1+\frac{1}{2}\alpha^2+\frac{1}{2}\alpha^3$ and $L=\langle A_1^5,\gamma\rangle = A_1^5\cup(\gamma+A_1^5)$. Then L is an even lattice so that we can construct a VOA V_L associated to L and we have $V_L = V_{A_1^5}\oplus V_{\gamma+A_1^5}$. By Lemma 3.5, we have the following:

Lemma 3.6. *As a module for $T^{(4)} = L\left(\frac{1}{2},0\right)\otimes L\left(\frac{7}{10},0\right)\otimes L\left(\frac{4}{5},0\right)\otimes L\left(\frac{6}{7},0\right)\otimes\mathcal{L}(5,0)$,*

$$\begin{aligned}&V_{k\gamma+A_1^5}\\ &\simeq \bigoplus_{\substack{i=0,1,2\\ s=0,1}}\ \bigoplus_{\substack{(h_1,h_2)\in X_s\\ (h_3,h_4)\in Y^k_{s,i}}} L\left(\frac{1}{2},h_1\right)\otimes L\left(\frac{7}{10},h_2\right)\otimes L\left(\frac{4}{5},h_3\right)\otimes L\left(\frac{6}{7},h_4\right)\otimes\mathcal{L}(5,2i)\end{aligned}$$

for $k=0,1$, where

$$\begin{aligned}
&X_0=\left\{(0,0),(\tfrac{1}{2},\tfrac{3}{2})\right\}, && X_1=\left\{(0,\tfrac{3}{5}),(\tfrac{1}{2},\tfrac{1}{10})\right\},\\
&Y^0_{0,0}=\left\{(0,0),(3,5),(\tfrac{2}{3},\tfrac{4}{3})\right\}, && Y^0_{1,0}=\left\{(\tfrac{7}{5},0),(\tfrac{2}{5},5),(\tfrac{1}{15},\tfrac{4}{3})\right\},\\
&Y^0_{0,1}=\left\{(0,\tfrac{5}{7}),(3,\tfrac{12}{7}),(\tfrac{2}{3},\tfrac{1}{21})\right\}, && Y^0_{1,1}=\left\{(\tfrac{7}{5},\tfrac{5}{7}),(\tfrac{2}{5},\tfrac{12}{7}),(\tfrac{1}{15},\tfrac{1}{21})\right\},\\
&Y^0_{0,2}=\left\{(0,\tfrac{22}{7}),(3,\tfrac{1}{7}),(\tfrac{2}{3},\tfrac{10}{21})\right\}, && Y^0_{1,2}=\left\{(\tfrac{7}{5},\tfrac{22}{7}),(\tfrac{2}{5},\tfrac{1}{7}),(\tfrac{1}{15},\tfrac{10}{21})\right\},\\
&Y^1_{0,0}=\left\{(0,5),(3,0),(\tfrac{2}{3},\tfrac{4}{3})\right\}, && Y^1_{1,0}=\left\{(\tfrac{2}{5},0),(\tfrac{7}{5},5),(\tfrac{1}{15},\tfrac{4}{3})\right\},\\
&Y^1_{0,1}=\left\{(0,\tfrac{12}{7}),(3,\tfrac{5}{7}),(\tfrac{2}{3},\tfrac{1}{21})\right\}, && Y^1_{1,1}=\left\{(\tfrac{2}{5},\tfrac{5}{7}),(\tfrac{7}{5},\tfrac{12}{7}),(\tfrac{1}{15},\tfrac{1}{21})\right\},\\
&Y^1_{0,2}=\left\{(0,\tfrac{1}{7}),(3,\tfrac{22}{7}),(\tfrac{2}{3},\tfrac{10}{21})\right\}, && Y^1_{1,2}=\left\{(\tfrac{2}{5},\tfrac{22}{7}),(\tfrac{7}{5},\tfrac{1}{7}),(\tfrac{1}{15},\tfrac{10}{21})\right\}.
\end{aligned}$$

We define

$$W = \{v \in V_L \mid \omega^1_{(1)}v = \omega^2_{(1)}v = \Omega^4_{(1)}v = 0\}.$$

Then, W is a subVOA of V_L with the Virasoro element $\omega^3 + \omega^4$ and, by Lemma 3.6, as a $L\left(\frac{4}{5}, 0\right) \otimes L\left(\frac{6}{7}, 0\right)$-module,

$$\begin{aligned} W &\simeq \bigoplus_{(h_3,h_4)\in Y^0_{0,0}\cup Y^1_{0,0}} L\left(\frac{4}{5}, h_3\right) \otimes L\left(\frac{6}{7}, h_4\right) \\ &\simeq L\left(\frac{4}{5}, 0\right) \otimes L\left(\frac{6}{7}, 0\right) \oplus L\left(\frac{4}{5}, 3\right) \otimes L\left(\frac{6}{7}, 0\right) \\ &\quad \oplus L\left(\frac{4}{5}, 0\right) \otimes L\left(\frac{6}{7}, 5\right) \oplus L\left(\frac{4}{5}, 3\right) \otimes L\left(\frac{6}{7}, 5\right) \\ &\quad \oplus L\left(\frac{4}{5}, \frac{2}{3}\right) \otimes L\left(\frac{6}{7}, \frac{4}{3}\right) \oplus L\left(\frac{4}{5}, \frac{2}{3}\right) \otimes L\left(\frac{6}{7}, \frac{4}{3}\right). \end{aligned} \tag{3.16}$$

We set

$$\begin{aligned} e &:= \frac{1}{16}\left((\alpha^3 - \alpha^4)_{(-1)}\right)^2 \mathbb{1} - \frac{1}{4}x^+_{\alpha^3-\alpha^4} \\ \mathbf{u} &:= \frac{5}{18}\omega^3 + \frac{7}{9}\omega^4 - \frac{16}{9}e, \\ \mathbf{v}_0 &:= \sum_{i=0,1,2} (\alpha_i + \alpha_0 + \alpha_1 + \alpha_2 - 4\alpha_4)x^-_{\frac{1}{2}(-2\alpha_i+\alpha_0+\alpha_1+\alpha_2-\alpha_3)} \\ &\quad -4\sum_{i=0,1,2} x^+_{\frac{1}{2}(-2\alpha_i+\alpha_0+\alpha_1+\alpha_2+\alpha_3-2\alpha_4)} \\ &\quad +12x^+_{\frac{1}{2}(\alpha_0+\alpha_1+\alpha_2+\alpha_3-2\alpha_4)}. \end{aligned}$$

where $x^\pm_\alpha = e^\alpha \pm e^{-\alpha}$ for $\alpha \in L$. Then we can show that e, $\mathbf{u}$ and $\mathbf{v}$ are contained in W_2 and $\omega^3, \omega^4, \mathbf{u}$ and $\mathbf{v}_0$ span W_2 since $\dim W_2 = 4$. Moreover, $\omega^3, \omega^4, \mathbf{u}$ and $\mathbf{v} = \frac{1}{\sqrt{\langle \mathbf{v}_0, \mathbf{v}_0\rangle}}\mathbf{v}_0$ satisfy the equation (3.11) and e is a conformal vector with central charge $1/2$. In particular, $\mathbf{u}$ and $\mathbf{v}$ are highest weight vectors for $T \simeq L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0)$ with highest weight $(\frac{2}{3}, \frac{4}{3})$. We also set

$$\begin{aligned} a &:= \frac{105}{2^8}(\omega^3 + \omega^4 - e), \quad b := \frac{3^2}{2^8}(-5\omega^3 + 7\omega^4 - 4e), \\ c &:= \frac{9\sqrt{6}}{2^6}\mathbf{v}, \qquad f := e + a + b + c. \end{aligned}$$

Then, a, b and c are eigenvectors of $e_{(1)}$ with eigenvalue 0, $\frac{1}{2}$ and $\frac{1}{16}$, respectively. Moreover, by direct calculations one can show that the multiplications and inner products in the

Griess algebra of U are given as follows:

$$
\begin{array}{lll}
e_{(1)}a = 0, & e_{(1)}b = \frac{1}{2}b, & e_{(1)}c = \frac{1}{16}c, \\
a_{(1)}a = \frac{105}{2^7}a, & a_{(1)}b = \frac{3^2\cdot 5\cdot 7}{2^9}b, & a_{(1)}c = \frac{31\cdot 105}{2^{12}}c, \\
b_{(1)}b = \frac{3^7}{2^{15}}e + \frac{3^3}{2^7}a, & b_{(1)}c = \frac{3^2\cdot 23}{2^{10}}c, & c_{(1)}c = \frac{3^5}{2^{13}}e + \frac{31}{2^5}a + \frac{23}{2^5}b, \\
\langle a, a\rangle = \frac{3^6\cdot 5\cdot 7}{2^{18}}, & \langle b, b\rangle = \frac{3^7}{2^{16}}, & \langle c, c\rangle = \frac{3^5}{2^{11}}.
\end{array}
$$

and f is a conformal vector with central charge $1/2$ and $\langle e, f\rangle = \frac{13}{2^{10}}$. Thus, W_2 is isomorphic to a Griess algebra given in [M3]

By the decomposition (3.16) and Theorem 3.4, we have

Theorem 3.7. *$W = \mathrm{VA}(e, f)$, that is, W is a simple VOA generated by two conformal vectors e and f with central charge $1/2$ such that $\langle e, f\rangle = 13/2^{10}$.*

3.3 Modules

In this subsection we will classify all irreducible U-modules. We may assume that $U = U^0 \oplus U^1 \oplus U^2$ with $U^0 = W(0)\otimes N(0)$, $U^1 = W(\frac{2}{3})^+ \otimes N(\frac{4}{3})^+$ and $U^2 = W(\frac{2}{3})^- \otimes N(\frac{4}{3})^-$.

Theorem 3.8. *U has exactly six inequivalent irreducible modules as follows:*

$$
\begin{aligned}
&W(0)\otimes N(0) \oplus W(\tfrac{2}{3})^+ \otimes N(\tfrac{4}{3})^+ \oplus W(\tfrac{2}{3})^- \otimes N(\tfrac{4}{3})^-,\\
&W(0)\otimes N(\tfrac{1}{7}) \oplus W(\tfrac{2}{3})^+ \otimes N(\tfrac{10}{21})^+ \oplus W(\tfrac{2}{3})^- \otimes N(\tfrac{10}{21})^-,\\
&W(0)\otimes N(\tfrac{5}{7}) \oplus W(\tfrac{2}{3})^+ \otimes N(\tfrac{1}{21})^+ \oplus W(\tfrac{2}{3})^- \otimes N(\tfrac{1}{21})^-,\\
&W(\tfrac{2}{5})\otimes N(0) \oplus W(\tfrac{1}{15})^+ \otimes N(\tfrac{4}{3})^+ \oplus W(\tfrac{1}{15})^- \otimes N(\tfrac{4}{3})^-,\\
&W(\tfrac{2}{5})\otimes N(\tfrac{1}{7}) \oplus W(\tfrac{1}{15})^+ \otimes N(\tfrac{10}{21})^+ \oplus W(\tfrac{1}{15})^- \otimes N(\tfrac{10}{21})^-,\\
&W(\tfrac{2}{5})\otimes N(\tfrac{5}{7}) \oplus W(\tfrac{1}{15})^+ \otimes N(\tfrac{1}{21})^+ \oplus W(\tfrac{1}{15})^- \otimes N(\tfrac{1}{21})^-.
\end{aligned}
\tag{3.17}
$$

Proof. Let $W_{(s,i)} = \left\{ v \in V_L \mid \omega^j_{(1)}v = h_j v, j = 1, 2,\ \Omega^4_{(1)}v = \frac{2i(i+1)}{7}v \right\}$ in Lemma 3.6, where $(h_1, h_2) \in X_s$. Then, $W_{(s,i)}$ is a W-submodule of V_L and as a $L\left(\frac{4}{5}, 0\right) \otimes L\left(\frac{6}{7}, 0\right)$-module $W_{(s,i)} \simeq \oplus_{(h_3,h_4)\in Y^0_{s,i}\cup Y^1_{s,i}} L\left(\frac{4}{5}, h_3\right) \otimes L\left(\frac{6}{7}, h_4\right)$. Therefore, all spaces in (3.17) are U-modules.

Let (M, Y) be an irreducible U-module and P^0 an irreducible U^0-submodule of M. It follows from the fusion rules for $U^0 = W(0)\otimes N(0)$-modules that $U^i \cdot P^0 \not\simeq U^j \cdot P^0$ as U^0-modules if $i \not\equiv j \mod 3$. Therefore, $M = P^0 \oplus P^1 \oplus P^2$ with $P^1 = U^1 \times P^0$ and $P^2 = U^2 \times P^0$ and M has a $\mathbb{Z}_3$-grading under the action of U. The vertex operators

$Y(\cdot,z)$ on M give U^0-intertwining operators of type $U^i \times P^j \to P^{i+j}$ for $i,j \in \mathbb{Z}_3$. The powers of z in an intertwining operator of type $U^i \times P^j \to P^{i+j}$ are contained in $-h_{U^i} - h_{P^i} + h_{P^{i+j}} + \mathbb{Z}$, where h_X denotes the top weight of a U^0-module X. Since the powers of z in $Y_M(\cdot,z)$ belong to $\mathbb{Z}$, by considering top weights M is isomorphic to one of (3.17) as a U^0-module. At last, we show that there exists a unique U-module structure on $\oplus_{s\in\mathbb{Z}_3} P^s$. Let $(\tilde{M} = \tilde{P}^0 \oplus \tilde{P}^1 \oplus \tilde{P}^2, \tilde{Y})$ be an U-module such that $P^s \simeq \tilde{P}^s$ as U^0-modules for all $s \in \mathbb{Z}_3$. By assumption, there exists U^0-isomorphism $\psi_s : P^s \to \tilde{P}^s$ such that $\tilde{Y}(a,z)\psi_s = \psi_s Y(a,z)$ for all $a \in U^0$. Then both $Y(\cdot,z)|_{U^s\times P^t}$ and $\psi_{s+t}^{-1}\tilde{Y}(\cdot,z)\psi_t|_{U^s\times P^t}$ are U^0-intertwining operators of type $U^s \times P^t \to P^{s+t}$ and hence there exist non-zero scalars $c(s,t) \in \mathbb{C}$ such that $\tilde{Y}(a,z)\psi_t = c(s,t)\psi_{s+t}Y(a,z)$ for all $a \in U^s$. Then, by the associativity we obtain

$$c(s+t,r) = c(s,t+r)c(t,r) \tag{3.18}$$

for $s,t,r \in \mathbb{Z}_3$. Define $\tilde{\psi} : M \to \tilde{M}$ by $\tilde{\psi}|_{P^s} = c(s,0)\psi_s$. Then, for $a \in U^s$, we have

$$\begin{aligned}
\tilde{Y}(a,z)\tilde{\psi}|_{P^t} &= c(t,0)\tilde{Y}(a,z)\psi_t \\
&= c(t,0)c(s,t)\psi_{s,t}Y(a,z) \\
&= c(s+t,0)\psi_{s,t}Y(a,z) \qquad \text{by (3.18)} \\
&= \tilde{\psi}|_{P^{s+t}}Y(a,z).
\end{aligned}$$

Therefore, $\tilde{\psi}$ defines a U-isomorphism between M and $\tilde{M}$. This completes the proof. □

Theorem 3.9. *U is rational.*

Proof. Let M be an admissible U-module. Take an irreducible U^0-submodule P. Since $U = U^0 \oplus U^1 \oplus U^2$, both $U^1 \cdot P$ and $U^2 \cdot P$ are non-trivial irreducible U^0-submodule of M. Since $U^i \cdot P \not\simeq U^j \cdot P$ if $i \not\equiv j \mod 3$, $P + (U^1 \cdot P) + (U^2 \cdot P) = P \oplus (U^1 \cdot P) \oplus (U^2 \cdot P)$ is an irreducible U-submodule of M. Hence, every irreducible U^0-submodule of M is contained in an irreducible U-submodule. Thus M is a completely reducible U-module. □

3.4 Fusion rules

Here we determine all fusion rules for irreducible U-modules. We will denote the fusion product of irreducible V-modules M^1 and M^2 by $M^1 \times_V M^2$. Recall the list of all irreducible U-modules shown in Theorem 3.8. each irreducible U-module contains one and only one of the following irreducible U^0-modules:

$$W(h) \otimes N(k),\ h = 0, \frac{2}{5},\ k = 0, \frac{1}{7}, \frac{5}{7}.$$

Therefore, seen as U^0-modules, all irreducible U-modules have the shapes

$$W(h)\otimes N(k)\oplus\{U^1\times_{U^0}(W(h)\otimes N(k))\}\oplus\{U^2\times_{U^0}(W(h)\otimes N(k))\}$$

with $h=0,\frac{2}{5}$ and $k=0,\frac{1}{7},\frac{5}{7}$, which we denote by $\mathrm{Ind}_{U^0}^U W(h)\otimes N(k)$ to emphasize that it is a U-module. Using this notation, the fusion products for irreducible U-modules can be computed as follows:

Theorem 3.10. *All fusion rules for irreducible U-modules are given by the following formula:*

$$\begin{aligned}&\dim_{\mathbb{C}} I_U\begin{pmatrix} & \mathrm{Ind}_{U^0}^U W(h_3)\otimes N(k_3) & \\ \mathrm{Ind}_{U^0}^U W(h_1)\otimes N(k_1) & & \mathrm{Ind}_{U^0}^U W(h_2)\otimes N(k_2)\end{pmatrix}\\ &\qquad = \quad \dim_{\mathbb{C}} I_{U^0}\begin{pmatrix} & U\times_{U^0}(W(h_3)\otimes N(k_3)) & \\ W(h_1)\otimes N(k_1) & & W(h_2)\otimes N(k_2)\end{pmatrix},\end{aligned}\tag{3.19}$$

where $h_1,h_2,h_3\in\{0,\frac{2}{5}\}$ and $k_1,k_2,k_3\in\{0,\frac{1}{7},\frac{5}{7}\}$.

Proof. Let X, W and T be irreducible U-modules and let X^0, W^0 and T^0 be irreducible U^0-submodules of X, W and T, respectively. Denote by $I_U\binom{T}{X\ W}$ the space of U-intertwining operators of type $X\times W\to T$. Then from [DL] by a restriction we obtain the following injection:

$$\pi: I_U\begin{pmatrix} T \\ X\ W\end{pmatrix}\ni I(\cdot,z)\mapsto I(\cdot,z)|_{X^0\times W^0}\in I_{U^0}\begin{pmatrix} T \\ X^0\ W^0\end{pmatrix}.$$

Since all irreducible U-modules are $\mathbb{Z}_3$-graded, by the same argument as in the proof of Lemma 5.3 in [LLY], for any U^0-intertwining operator $I(\cdot,z)$ of type $X^0\times W^0\to T^0$, there exists a U-intertwining operator $\tilde{I}(\cdot,z)$ of type $X\times W\to T$ such that $\pi(\tilde{I}(\cdot,z))=\tilde{I}(\cdot,z)|_{X^0\times W^0}=I(\cdot,z)$. Therefore, the assertion follows. □

3.5 Conformal vectors and Automorphisms

In this subsection, we classify all conformal vectors in U and determine the automorphism group of U.

The Griess algebra U_2 has three conformal vectors e, f and $f':=e^{\tau_f}=f^{\tau_e}$. As shown in [M3], $U_2=\mathbb{C}\omega+\mathbb{C}e+\mathbb{C}f+\mathbb{C}f'$ and

$$\begin{aligned}&e_{(1)}f=-\frac{105}{2^9}\omega+\frac{9}{2^5}e+\frac{9}{2^5}f+\frac{7}{2^5}f',\quad e_{(1)}f'=-\frac{105}{2^9}\omega+\frac{9}{2^5}e+\frac{7}{2^5}f+\frac{9}{2^5}f',\\ &f_{(1)}f'=-\frac{105}{2^9}\omega+\frac{7}{2^5}e+\frac{9}{2^5}f+\frac{9}{2^5}f',\quad \langle e,f\rangle=\langle e,f'\rangle=\langle f,f'\rangle=\frac{13}{2^{10}}.\end{aligned}$$

Set $g = -\frac{7}{18}\omega + \frac{14}{27}e + \frac{32}{27}f + \frac{32}{27}f'$. It is easy to check g is a conformal vector with central charge $\frac{4}{5}$. Set

$$C = \{e, f, f', g, g^{\tau_f}, g^{\tau_f\tau_e}, \omega^3, \omega^4, \omega\}.$$

We see that for any $x \in C$, x and $\omega - x$ are conformal vectors in U_2.

Theorem 3.11. *The set $C \cup (\omega - C)$ gives all conformal vectors in U_2, where $\omega - C = \{\omega - x \mid x \in C\}$. In particular, there are exactly three conformal vectors with central charge $1/2$ in U_2, namely e, f and f'.*

Proof. Let $x = \alpha\omega + \beta e + \gamma f + \delta f'$ be a conformal vector. Solving the equation $x_{(1)}x = 2x$ with respect to α, β, γ, δ by direct calculation, $(\alpha, \beta, \gamma, \delta)$, $(\alpha, \gamma, \delta, \beta)$ or $(\alpha, \delta, \beta, \gamma)$ is one of

$$\begin{aligned}&(0,0,0,0),\ (1,0,0,0),\ (0,1,0,0),\ (1,-1,0,0),\\ &(14/9, -32/27, -32/27, -32/27),\ \ (-7/18, 14/27, 32/27, 32/27),\\ &(-5/9, 32/27, 32/27, 32/27) \text{ and } (25/18, -14/27, -32/27, -32/27),\end{aligned}$$

which give elements in $C \cup (\omega - C)$. Thus, the assertion follows. □

Theorem 3.12. $\mathrm{Aut}(U) = \langle \tau_e, \tau_f \rangle$.

Proof. Let $g \in \mathrm{Aut}(U)$. Since U is generated by e and f, the action of g on U is completely determined by its actions on e and f. By Theorem 3.11, the set of conformal vectors with central charge $1/2$ in U is $\{e, f, f'\}$ so that we get an injection from $\mathrm{Aut}(U)$ to S_3. Since $\langle \tau_e, \tau_f \rangle$ acts on $\{e, f, f'\}$ as S_3, we obtain $\mathrm{Aut}(U) = \langle \tau_e, \tau_f \rangle$. □

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