Crystal Bases, Path Models, and a Twining Character Formula for Demazure Modules

Daisuke SAGAKI

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0 Introduction.

In [FRS] and [FSS], they introduced new character-like quantities corresponding to a graph automorphism of a Dynkin diagram, called twining characters, for certain Verma modules and integrable highest weight modules over a symmetrizable Kac-Moody algebra, and gave twining character formulas for them. Recently, the notion of twining characters has naturally been extended to various modules, and formulas for them has been given ([KN], [KK], [N1]–[N4]).

The purpose of this paper is to give a twining character formula for Demazure modules over a symmetrizable Kac-Moody algebra. Our formula is an extension of one of the main results in [KN], which describes the twining characters of Demazure modules over a finite-dimensional semi-simple Lie algebra. While their proof is an algebro-geometric one, we give a combinatorial proof by using the theories of path models and crystal bases.

Let us explain our formula more precisely. Let $\mathfrak{g} = \mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be a symmetrizable Kac-Moody algebra over $\mathbb{Q}$ associated to a generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$ of finite size, where $\mathfrak{h}$ is the Cartan subalgebra, $\mathfrak{n}_+$ the sum of positive root spaces, and $\mathfrak{n}_-$ the sum of negative root spaces, and let $\omega : I \to I$ be a (Dynkin) diagram automorphism, that is, a bijection $\omega : I \to I$ satisfying $a_{\omega(i),\omega(j)} = a_{ij}$ for all $i, j \in I$. It is known that a diagram automorphism induces a Lie algebra automorphism $\omega \in \mathrm{Aut}(\mathfrak{g})$ that preserves the triangular decomposition of $\mathfrak{g}$. Then we define a linear automorphism $\omega^* \in GL(\mathfrak{h}^*)$ by $(\omega^*(\lambda))(h) := \lambda(\omega(h))$ for $\lambda \in \mathfrak{h}^*$, $h \in \mathfrak{h}$. We set $(\mathfrak{h}^*)^0 := \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\}$, and call its elements symmetric weights. We also set $\widetilde{W} := \{w \in W \mid w\,\omega^* = \omega^* w\}$.

Further we define a "folded" matrix $\widehat{A}$ associated to ω, which is again a symmetrizable GCM if ω satisfies a certain condition, called the linking condition (we assume it throughout this paper). The Kac-Moody algebra $\widehat{\mathfrak{g}} = \mathfrak{g}(\widehat{A})$ associated to $\widehat{A}$ is called the orbit Lie algebra. We denote by $\widehat{\mathfrak{h}}$ the Cartan subalgebra of $\widehat{\mathfrak{g}}$ and by $\widehat{W}$ the Weyl group of $\widehat{\mathfrak{g}}$. Then there exist a linear isomorphism $P_\omega^* : \widehat{\mathfrak{h}}^* \to (\mathfrak{h}^*)^0$ and a group isomorphism $\Theta : \widehat{W} \to \widetilde{W}$ such that $\Theta(\widehat{w}) = P_\omega^* \circ \widehat{w} \circ (P_\omega^*)^{-1}$ for all $\widehat{w} \in \widehat{W}$.

Let λ be a dominant integral weight. Denote by $L(\lambda) = \bigoplus_{\chi \in \mathfrak{h}^*} L(\lambda)_\chi$ the irreducible highest weight $\mathfrak{g}$-module of highest weight λ. Then, for $w \in W$, we define the Demazure module $L_w(\lambda)$ of lowest weight $w(\lambda)$ in $L(\lambda)$ by $L_w(\lambda) := U(\mathfrak{b})L(\lambda)_{w(\lambda)}$, where $U(\mathfrak{b})$ is the universal enveloping algebra of the Borel subalgebra $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}_+$ of $\mathfrak{g}$. If λ is symmetric, then we have a (unique) linear automorphism $\tau_\omega : L(\lambda) \to L(\lambda)$ such that

$$\tau_\omega(xv) = \omega^{-1}(x)\tau_\omega(v) \quad \text{for all } x \in \mathfrak{g},\ v \in L(\lambda)$$

and $\tau_\omega(u_\lambda) = u_\lambda$ with u_λ a (nonzero) highest weight vector of $L(\lambda)$. Then it is easily seen that the Demazure module $L_w(\lambda)$ with $w \in \widetilde{W}$ is τ_ω-stable. Here we define the twining character $\mathrm{ch}^{\omega}(L_w(\lambda))$ of $L_w(\lambda)$ by:

$$\mathrm{ch}^{\omega}(L_w(\lambda)) := \sum_{\chi \in (\mathfrak{h}^*)^0} \mathrm{tr}\big(\tau_\omega|_{L_w(\lambda)_\chi}\big) e(\chi).$$

Our main theorem is the following:

Theorem. *Let λ be a symmetric dominant integral weight and $w \in \widetilde{W}$. Set $\widehat{\lambda} := (P_\omega^*)^{-1}(\lambda)$ and $\widehat{w} := \Theta^{-1}(w)$. Then we have*

$$\mathrm{ch}^{\omega}(L_w(\lambda)) = P_\omega^*(\mathrm{ch}\, \widehat{L}_{\widehat{w}}(\widehat{\lambda})),$$

where $\widehat{L}_{\widehat{w}}(\widehat{\lambda})$ is the Demazure module of lowest weight $\widehat{w}(\widehat{\lambda})$ in the irreducible highest weight module $\widehat{L}(\widehat{\lambda})$ of highest weight $\widehat{\lambda}$ over the orbit Lie algebra $\widehat{\mathfrak{g}}$.

The starting point of this work is the main result in [NS1]. Denote by $\mathbb{B}(\lambda)$ the set of Lakshmibai-Seshadri paths (L-S paths for short) of class λ, where L-S paths of class λ are, by definition, piecewise linear, continuous maps $\pi : [0,1] \to \mathfrak{h}^*$ parametrized by sequences of elements in $W\lambda$ and rational numbers with a certain condition, called the chain condition. In [Li1], Littelmann showed that there exists a subset $\mathbb{B}_w(\lambda)$ of $\mathbb{B}(\lambda)$ such that

$$\sum_{\pi \in \mathbb{B}_w(\lambda)} e(\pi(1)) = \mathrm{ch}\, L_w(\lambda).$$

For $\pi \in \mathbb{B}(\lambda)$, we deifne a path $\omega^*(\pi) : [0,1] \to \mathfrak{h}^*$ by $(\omega^*(\pi))(t) := \omega^*(\pi(t))$. If λ is symmetric and $w \in \widetilde{W}$, then $\mathbb{B}_w(\lambda)$ is ω^*-stable. We denote by $\mathbb{B}_w^0(\lambda)$ the set of all elements of $\mathbb{B}_w(\lambda)$ fixed by ω^*. Then we see from the main result of [NS1] that

$$\sum_{\pi \in \mathbb{B}_w^0(\lambda)} e(\pi(1)) = P_\omega^*(\mathrm{ch}\, \widehat{L}_{\widehat{w}}(\widehat{\lambda})).$$

In this paper, we prove that the left-hand side is, in fact, equal to $\mathrm{ch}^{\omega}(L_w(\lambda))$.

In order to prove the equality $\mathrm{ch}^{\omega}(L_w(\lambda)) = \sum_{\pi \in \mathbb{B}_w^0(\lambda)} e(\pi(1))$, we introduce a "quantum version" of twining characters, called q-twining characters. Let $U_q(\mathfrak{g})$ be the quantum group associated to the Kac-Moody algebra $\mathfrak{g}$ over the field $\mathbb{Q}(q)$ of rational functions in q, and $V(\lambda) = \bigoplus_{\chi \in \mathfrak{h}^*} V(\lambda)_\chi$ the irreducible highest weight $U_q(\mathfrak{g})$-module of highest weight λ. For $w \in W$, the quantum Demazure module $V_w(\lambda)$ is defined by $V_w(\lambda) := U_q^+(\mathfrak{g}) V(\lambda)_{w(\lambda)}$, where $U_q^+(\mathfrak{g})$ is the "positive part" of $U_q(\mathfrak{g})$. A diagram automorphism ω induces a $\mathbb{Q}(q)$-algebra automorphism ω_q of $U_q(\mathfrak{g})$. Assume that λ is symmetric. Then we get a $\mathbb{Q}(q)$-linear automorphism τ_{ω_q} of $V(\lambda)$ that has the same properties as τ_ω in the Lie algebra case. Since $V_w(\lambda)$ is stable under τ_{ω_q} if $w \in \widetilde{W}$, we can define the q-twining character $\mathrm{ch}_q^{\omega}(V_w(\lambda))$ of $V_w(\lambda)$ by

$$\mathrm{ch}_q^{\omega}(V_w(\lambda)) := \sum_{\chi \in (\mathfrak{h}^*)^0} \mathrm{tr}\big(\tau_{\omega_q}|_{V_w(\lambda)_\chi}\big) e(\chi),$$

where the traces are naively elements of $\mathbb{Q}(q)$ (in fact, they are elements of $\mathbb{Q}[q, q^{-1}]$). We show that the specialization of the q-twining character $\mathrm{ch}_q^{\omega}(V_w(\lambda))$ by $q = 1$ is equal to the (ordinary) twining character $\mathrm{ch}^{\omega}(L_w(\lambda))$, that is,

$$\mathrm{ch}_q^{\omega}(V_w(\lambda))\Big|_{q=1} = \mathrm{ch}^{\omega}(L_w(\lambda)).$$

The advantage of considering a quantum version is the existence of a basis of $V_w(\lambda)$ compatible with τ_{ω_q}. Let $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ be the (lower) crystal base of $V(\lambda)$. In [Kas3], Kashiwara showed that, for each $w \in W$, there exists a subset $\mathcal{B}_w(\lambda)$ of $\mathcal{B}(\lambda)$ such that

$$V_w(\lambda) := \bigoplus_{b \in \mathcal{B}_w(\lambda)} \mathbb{Q}(q) G_\lambda(b),$$

where $G_\lambda(b)$ denotes the (lower) global base introduced in [Kas2]. We prove that τ_{ω_q} stabilizes the basis $\{G_\lambda(b) \mid b \in \mathcal{B}_w(\lambda)\}$ of $V_w(\lambda)$.

By combining these facts and the equivalence theorem between path models $\mathbb{B}(\lambda)$ and crystal bases $\mathcal{B}(\lambda)$, which was proved by Kashiwara [Kas5] *et al.*, we can obtain the desired equality above, and hence the our main theorem.

This paper is organized as follows. In §1 we review some facts about Kac-Moody algebras, diagram automorphisms, orbit Lie algebras, quantum groups, crystal bases, and path models. There we also define an algebra automorphism of the quantum group $U_q(\mathfrak{g})$ induced from a diagram automorphism. In §2, we recall the definition of the twining characters of $L(\lambda)$ and $L_w(\lambda)$, and then introduce the q-twining characters of $V(\lambda)$ and $V_w(\lambda)$. Furthermore, we show that the q-twining characters of $V(\lambda)$ and $V_w(\lambda)$ are q-analogues of the twining characters of $L(\lambda)$ and $L_w(\lambda)$, respectively. In §3 we give a proof of our main theorem by calculating the q-twining character of $V_w(\lambda)$.

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1 Preliminaries.

1.1 Kac-Moody Algebras and Diagram Automorphisms. In this subsection, we review some basic facts about Kac-Moody algebras from [Kac] and [MP], and about diagram automorphisms from [FRS] and [FSS].

Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix (GCM for short) indexed by a finite set I. Then there exists a diagonal matrix $D = \mathrm{diag}(\varepsilon_i)_{i \in I}$ with $\varepsilon_i \in \mathbb{Q}_{>0}$ such that $D^{-1}A$ is a symmetric matrix. Let $\omega : I \to I$ be a diagram automorphism of order N, that is, a bijection $\omega : I \to I$ of order N such that $a_{\omega(i),\omega(j)} = a_{ij}$ for all $i, j \in I$.

Remark 1. Set

$$D' = \mathrm{diag}(\varepsilon_i')_{i\in I} := \mathrm{diag}\left(\frac{1}{\sum_{k=0}^{N-1}\varepsilon_{\omega^k(i)}^{-1}}\right)_{i\in I}.$$

Then we see that $\varepsilon'_{\omega(i)} = \varepsilon'_i$ and $(D')^{-1}A$ is a symmetric matrix. Hence, by replacing D with D' above if necessary, we may (and will henceforth) assume that $\varepsilon_{\omega(i)} = \varepsilon_i$ (see also [N1, §3.1]).

We take a realization $(\mathfrak{h}, \Pi, \Pi^\vee)$ of the GCM $A = (a_{ij})_{i,j\in I}$ over $\mathbb{Q}$ and linear automorphisms $\omega : \mathfrak{h} \to \mathfrak{h}$ and $\omega^* : \mathfrak{h}^* \to \mathfrak{h}^*$ as follows (cf. [Kac, Exercises 1.15 and 1.16]). Let $\mathfrak{h}'$ be an n-dimensional vector space over $\mathbb{Q}$ with $\Pi^\vee := \{\alpha_i^\vee\}_{i\in I}$ a basis. We define a $\mathbb{Q}$-linear automorphism $\omega' : \mathfrak{h}' \to \mathfrak{h}'$ by $\omega'(\alpha_i^\vee) = \alpha_{\omega(i)}^\vee$, and $\omega'' : (\mathfrak{h}')^* \to (\mathfrak{h}')^*$ by $(\omega''(\lambda))(h) := \lambda((\omega')^{-1}(h))$ for $\lambda \in (\mathfrak{h}')^*$ and $h \in \mathfrak{h}'$. We also define $\varphi : \mathfrak{h}' \to (\mathfrak{h}')^*$ by $(\varphi(\alpha_i^\vee))(\alpha_j^\vee) = a_{ij}$. It can be readily seen that $\omega'' \circ \varphi = \varphi \circ \omega'$. This means that $\mathrm{Im}\,\varphi$ is ω''-stable, and hence we can take a complementary subspace $\mathfrak{h}''$ of $\mathrm{Im}\,\varphi$ in $(\mathfrak{h}')^*$ that is also ω''-stable. Now set $\mathfrak{h} := \mathfrak{h}' \oplus \mathfrak{h}''$, and $\Pi := \{\alpha_i\}_{i\in I}$, where $\alpha_i \in \mathfrak{h}^*$ is defined by

$$\alpha_i\left(\sum_{j\in I} c_j\alpha_j^\vee + h''\right) := \sum_{j\in I} c_j(\varphi(\alpha_j^\vee))(\alpha_i^\vee) + h''(\alpha_i^\vee) \quad \text{for} \quad h'' \in \mathfrak{h}''. \tag{1.1}$$

Then we see that Π is a linearly independent subset of $\mathfrak{h}^*$. Furthermore, since $\dim_{\mathbb{Q}} \mathfrak{h}'' = \#I - \dim_{\mathbb{Q}} \mathrm{Im}\,\varphi = \#I - \mathrm{rank}\,A$, we have $\dim_{\mathbb{Q}} \mathfrak{h} = 2\#I - \mathrm{rank}\,A$. Hence $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a (minimal) realization of the GCM A. We define a $\mathbb{Q}$-linear automorphism $\omega : \mathfrak{h} \to \mathfrak{h}$ by $\omega(h'+h'') := \omega'(h')+\omega''(h'')$ for $h' \in \mathfrak{h}'$ and $h'' \in \mathfrak{h}''$, and the transposed map $\omega^* : \mathfrak{h}^* \to \mathfrak{h}^*$ by $(\omega^*(\lambda))(h) = \lambda(\omega(h))$ for $\lambda \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$. Then we can check, by using (1.1), that $\omega^*(\alpha_i) = \alpha_{\omega^{-1}(i)}$ for each $i \in I$.

Here, as in [Kac, §2.1], we define the (standard) nondegenerate symmetric bilinear form $(\cdot\,,\cdot)$ on $\mathfrak{h}$ associated to the decomposition $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$ above. We set

$$\begin{cases} (\alpha_i^\vee, h) := \alpha_i(h)\varepsilon_i & \text{for} \quad i \in I,\ h \in \mathfrak{h}, \\ (h, h') := 0 & \text{for} \quad h,\ h' \in \mathfrak{h}''. \end{cases}$$

Then it follows from the construction above and Remark 1 that $(\omega(h), \omega(h')) = (h, h')$ for all $h,\ h' \in \mathfrak{h}$. We denote also by $(\cdot\,,\cdot)$ the nondegenerate symmetric bilinear form on $\mathfrak{h}^*$ induced from the bilinear form on $\mathfrak{h}$. Then $(\omega^*(\lambda), \omega^*(\lambda')) = (\lambda, \lambda')$ for all $\lambda,\ \lambda' \in \mathfrak{h}^*$.

We set

$$(\mathfrak{h}^*)^0 := \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\}, \qquad \mathfrak{h}^0 := \{h \in \mathfrak{h} \mid \omega(h) = h\}. \tag{1.2}$$

Elements of $(\mathfrak{h}^*)^0$ are called symmetric weights. Note that $(\mathfrak{h}^*)^0$ can be identified with $(\mathfrak{h}^0)^*$ in a natural way.

Remark 2. Let ρ be a Weyl vector, i.e., an element of $\mathfrak{h}^*$ such that $\rho(\alpha_i^\vee) = 1$ for all $i \in I$. Then, by replacing ρ with $(1/N)\sum_{k=0}^{N-1}(\omega^*)^k(\rho)$ if necessary, we may (and will henceforth) assume that a Weyl vector ρ is a symmetric weight.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the Kac-Moody algebra over $\mathbb{Q}$ associated to the GCM A with $\mathfrak{h}$ the Cartan subalgebra, $\Pi = \{\alpha_i\}_{i\in I}$ the set of simple roots, and $\Pi^\vee = \{\alpha_i^\vee\}_{i\in I}$ the set of simple coroots. Denote by $\{x_i, y_i \mid i \in I\}$ the Chevalley generators, where x_i (resp. y_i) spans the root space of $\mathfrak{g}$ corresponding to α_i (resp. $-\alpha_i$). The Weyl group W of $\mathfrak{g}$ is defined by $W := \langle r_i \mid i \in I\rangle$, where r_i is the simple reflection with respect to α_i. The following lemma is obvious from the definitions of Kac-Moody algebras and the linear map $\omega : \mathfrak{h} \to \mathfrak{h}$ above (see also [FSS, §3.2]).

Lemma 1.1. *The $\mathbb{Q}$-linear map $\omega : \mathfrak{h} \to \mathfrak{h}$ above can be extended to a Lie algebra automorphism $\omega \in \mathrm{Aut}(\mathfrak{g})$ of order N such that $\omega(x_i) = x_{\omega(i)}$ and $\omega(y_i) = y_{\omega(i)}$.*

Let λ be a dominant integral weight. Denote by $L(\lambda) = \bigoplus_{\chi\in\mathfrak{h}^*} L(\lambda)_\chi$ the irreducible highest weight $\mathfrak{g}$-module of highest weight λ, where $L(\lambda)_\chi$ is the χ-weight space of $L(\lambda)$. We set $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_+$ is the sum of positive root spaces of $\mathfrak{g}$. For $w \in W$, the Demazure module $L_w(\lambda) \subset L(\lambda)$ of lowest weight $w(\lambda)$ is defined by $L_w(\lambda) := U(\mathfrak{b})L(\lambda)_{w(\lambda)}$, where $U(\mathfrak{b})$ is the universal enveloping algebra of $\mathfrak{b}$. In addition, for each $i \in I$, we define the Demazure operator D_i by

$$D_i(e(\lambda)) := \frac{e(\lambda+\rho) - e(r_i(\lambda+\rho))}{1 - e(-\alpha_i)} e(-\rho) \quad \text{for } \lambda \in \mathfrak{h}^*. \tag{1.3}$$

By [Kas3], [Ku], and [M], we know the following character formula for Demazure modules.

Theorem 1.2. *Let λ be a dominant integral weight and $w \in W$. Assume that $w = r_{i_1} r_{i_2} \cdots r_{i_k}$ is a reduced expression of w. Then we have*

$$\operatorname{ch} L_w(\lambda) = D_{i_1} \circ D_{i_2} \circ \cdots \circ D_{i_k}(e(\lambda)). \tag{1.4}$$

Remark 3. The Demazure operators $\{D_i\}_{i\in I}$ satisfy the braid relations (see [D]). Hence the right-hand side of (1.4) above does not depend on the choice of a reduced expression of w.

1.2 Orbit Lie Algebras.

In this subsection, we review the notion of orbit Lie algebras. For details, see [FRS] and [FSS].

We set

$$c_{ij} := \sum_{k=0}^{N_j-1} a_{i,\omega^k(j)} \quad \text{for } i, j \in I \quad \text{and} \quad c_i := c_{ii} \quad \text{for } i \in I, \tag{1.5}$$

where N_i is the number of elements of the ω-orbit of $i \in I$ in I. From now on, we assume that a diagram automorphism ω satisfies

$$c_i = 1 \text{ or } 2 \quad \text{for each } i \in I. \tag{1.6}$$

This condition is called the linking condition. Here we choose a complete set $\widehat{I}$ of representatives of the ω-orbits in I, and define a matrix $\widehat{A} = (\widehat{a}_{ij})_{i,j\in\widehat{I}}$ by

$$\widehat{A} = (\widehat{a}_{ij})_{i,j\in\widehat{I}} := (2c_{ij}/c_j)_{i,j\in\widehat{I}}. \tag{1.7}$$

Proposition 1.3 ([FSS, §2.2]). *The matrix $\widehat{A}$ is a symmetrizable* GCM.

The Kac-Moody algebra $\widehat{\mathfrak{g}} := \mathfrak{g}(\widehat{A})$ over $\mathbb{Q}$ associated to the GCM $\widehat{A}$ is called the orbit Lie algebra (associated to the diagram automorphism ω). Denote by $\widehat{\mathfrak{h}}$ the Cartan subalgebra of $\widehat{\mathfrak{g}}$, and by $\widehat{\Pi} = \{\widehat{\alpha}_i\}_{i\in\widehat{I}}$ and $\widehat{\Pi}^\vee = \{\widehat{\alpha}_i^\vee\}_{i\in\widehat{I}}$ the set of simple roots and simple coroots of $\widehat{\mathfrak{g}}$, respectively.

As in [FRS, §2], we have a $\mathbb{Q}$-linear isomorphism $P_\omega : \mathfrak{h}^0 \longrightarrow \widehat{\mathfrak{h}}$ such that

$$\begin{cases} P_\omega\left(\dfrac{1}{N_i}\displaystyle\sum_{k=0}^{N_i-1} \alpha_{\omega^k(i)}^\vee\right) = \widehat{\alpha}_i^\vee & \text{for each } i \in \widehat{I}, \\ (P_\omega(h), P_\omega(h')) = (h, h') & \text{for all } h,\, h' \in \mathfrak{h}^0, \end{cases}$$

where we denote also by $(\cdot\,,\cdot)$ the (standard) nondegenerate symmetric bilinear form on $\widehat{\mathfrak{h}}$. Let $P_\omega^* : \widehat{\mathfrak{h}}^* \longrightarrow (\mathfrak{h}^0)^* \cong (\mathfrak{h}^*)^0$ be the transposed map of P_ω defined by

$$(P_\omega^*(\widehat{\lambda}))(h) := \widehat{\lambda}(P_\omega(h)) \quad \text{for } \widehat{\lambda} \in \widehat{\mathfrak{h}}^*,\ h \in \mathfrak{h}^0. \tag{1.8}$$

Proposition 1.4 ([FRS, Proposition 3.3]). *Set $\widetilde{W} := \{w \in W \mid w\,\omega^* = \omega^* w\}$. Then there exists a group isomorphism $\Theta : \widehat{W} \to \widetilde{W}$ such that $\Theta(\widehat{w}) = P_\omega^* \circ \widehat{w} \circ (P_\omega^*)^{-1}$ for each $\widehat{w} \in \widehat{W}$.*

1.3 Quantum Groups. From now on, we take the bilinear form $(\cdot\,,\cdot)$ in such a way that $(\alpha_i, \alpha_i) \in \mathbb{Z}_{>0}$ for all $i \in I$. Let $P \subset \mathfrak{h}^*$ be an ω^*-stable integral weight lattice such that $\alpha_i \in P$ for all $i \in I$, and set $P_+ := \{\lambda \in P \mid \lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I\}$. Notice that the dual lattice $P^\vee := \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$ is stable under ω. The quantum group (or quantized universal enveloping algebra) $U_q(\mathfrak{g})$ associated to $\mathfrak{g}$ is, by definition, the algebra generated by the symbols X_i, Y_i and q^h ($h \in P^\vee$) over the field $\mathbb{Q}(q)$ of rational functions in q with the following defining relations:

$$\begin{cases} q^0 = 1,\ q^{h_1}q^{h_2} = q^{h_1+h_2} & \text{for } h_1,\, h_2 \in P^\vee, \\ q^h X_i q^{-h} = q^{\alpha_i(h)} X_i,\quad q^h Y_i q^{-h} = q^{-\alpha_i(h)} Y_i & \text{for } i \in I,\ h \in P^\vee, \\ [X_i, Y_i] = \delta_{ij}\dfrac{t_i - t_i^{-1}}{q_i - q_i^{-1}} & \text{for } i \in I, \\ \displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k X_i^{(k)} X_j X_i^{(1-a_{ij}-k)} = 0 & \text{for } i,\, j \in I \text{ with } i \neq j, \\ \displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k Y_i^{(k)} Y_j Y_i^{(1-a_{ij}-k)} = 0 & \text{for } i,\, j \in I \text{ with } i \neq j. \end{cases} \tag{1.9}$$

Here we have used the following notation:

$$q_i = q^{(\alpha_i,\alpha_i)}, \quad t_i = q^{(\alpha_i,\alpha_i)\alpha_i^\vee},$$

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n}[k]_i, \quad \text{and} \quad X_i^{(n)} = \frac{X_i^n}{[n]_i!}, \quad Y_i^{(n)} = \frac{Y_i^n}{[n]_i!}.$$

Lemma 1.5. *There exists a unique $\mathbb{Q}(q)$-algebra automorphism ω_q of $U_q(\mathfrak{g})$ such that $\omega_q(X_i) = X_{\omega(i)}$, $\omega_q(Y_i) = Y_{\omega(i)}$, and $\omega_q(q^h) = q^{\omega(h)}$.*

Proof. We need only show that the images of the generators by ω_q also satisfy the defining relations (1.9). However it can be easily checked by using the equalities $q_{\omega(i)} = q_i$, $[n]_{\omega(i)} = [n]_i$, and $t_{\omega(i)} = t_i$. □

Let $\lambda \in P_+$. Denote by $V(\lambda) = \bigoplus_{\chi\in\mathfrak{h}^*} V(\lambda)_\chi$ the irreducible highest weight $U_q(\mathfrak{g})$-module of highest weight λ, where $V(\lambda)_\chi$ is the χ-weight space of $V(\lambda)$. It is known (cf. [Kas1, (1.2.7)]) that

$$V(\lambda) \cong U_q^-(\mathfrak{g}) \Big/ \left(\textstyle\sum_{i\in I} U_q^-(\mathfrak{g}) Y_i^{1+\lambda(\alpha_i^\vee)}\right), \tag{1.10}$$

where $U_q^-(\mathfrak{g})$ is the $\mathbb{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by $\{Y_i\}_{i\in I}$. For each $w \in W$, we define the quantum Demazure module $V_w(\lambda)$ by $V_w(\lambda) := U_q^+(\mathfrak{g})V(\lambda)_{w(\lambda)}$, where $U_q^+(\mathfrak{g})$ is the $\mathbb{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by $\{X_i\}_{i\in I}$.

1.4 Crystal Bases and Global Bases. In this subsection, we review the notions of (lower) crystal bases and (lower) global bases. For details, see [Ja] and [Kas1]–[Kas3].

First let us recall the definition of the Kashiwara operators E_i, F_i on $V(\lambda)$. It is known that each element $u \in V(\lambda)_\chi$ can be uniquely written as $u = \sum_{k\geq 0} Y_i^{(k)} u_k$, where $u_k \in (\ker X_i) \cap V(\lambda)_{\chi+k\alpha_i}$. We define the $\mathbb{Q}(q)$-linear operators E_i, F_i on $V(\lambda)$ by

$$E_i u := \sum_{k\geq 0} Y_i^{(k-1)} u_k, \qquad F_i u := \sum_{k\geq 0} Y_i^{(k+1)} u_k. \tag{1.11}$$

Denote by A_0 the subring of $\mathbb{Q}(q)$ consisting of the rational functions in q regular at $q = 0$, and by $\mathcal{L}_0(\lambda)$ the A_0-submodule of $V(\lambda)$ generated by all elements of the form $F_{i_1}F_{i_2}\cdots F_{i_k}u_\lambda$, where u_λ is a (nonzero) highest weight vector of $V(\lambda)$. Let $\mathcal{B}(\lambda) \subset \mathcal{L}_0(\lambda)/q\mathcal{L}_0(\lambda)$ be the set of nonzero images of $F_{i_1}F_{i_2}\cdots F_{i_k}u_\lambda$ by the canonical map $^-$: $\mathcal{L}_0(\lambda) \to \mathcal{L}_0(\lambda)/q\mathcal{L}_0(\lambda)$. Then it is known from [Kas1, Theorem 2] that $(\mathcal{L}_0(\lambda), \mathcal{B}(\lambda))$ is a (lower) crystal base of $V(\lambda)$, i.e.,

(1) $V(\lambda) = \mathbb{Q}(q) \otimes_{A_0} \mathcal{L}_0(\lambda)$,

(2) $\mathcal{L}_0(\lambda) = \bigoplus_{\chi\in\mathfrak{h}^*} \mathcal{L}_0(\lambda)_\chi$, where $\mathcal{L}_0(\lambda)_\chi = \mathcal{L}_0(\lambda) \cap V(\lambda)_\chi$,

(3) $E_i\mathcal{L}_0(\lambda) \subset \mathcal{L}_0(\lambda)$ and $F_i\mathcal{L}_0(\lambda) \subset \mathcal{L}_0(\lambda)$,

(4) $\mathcal{B}(\lambda)$ is a basis of the $\mathbb{Q}$-vector space $\mathcal{L}_0(\lambda)/q\mathcal{L}_0(\lambda)$,

(5) $E_i\mathcal{B}(\lambda) \subset \mathcal{B}(\lambda) \cup \{0\}$ and $F_i\mathcal{B}(\lambda) \subset \mathcal{B}(\lambda) \cup \{0\}$,

(6) $\mathcal{B}(\lambda) = \cup_{\chi \in \mathfrak{h}^*} \mathcal{B}(\lambda)_\chi$ (disjoint union), where $\mathcal{B}(\lambda)_\chi = \mathcal{B}(\lambda) \cap (\mathcal{L}_0(\lambda)_\chi / q\mathcal{L}_0(\lambda)_\chi)$,

(7) For b_1, $b_2 \in \mathcal{B}(\lambda)$, $b_1 = F_i b_2$ if and only if $b_2 = E_i b_1$.

Note that, by (3), we have the operators on $\mathcal{L}_0(\lambda)/q\mathcal{L}_0(\lambda)$ induced from E_i, F_i, which are also denoted by E_i, F_i (cf. (5), (7)).

Next we recall the notion of (lower) global bases. Set $V_{\mathbb{Q}}(\lambda) := U_q^{\mathbb{Q}}(\mathfrak{g})u_\lambda \subset V(\lambda)$, where $U_q^{\mathbb{Q}}(\mathfrak{g})$ is the $\mathbb{Q}[q, q^{-1}]$-subalgebra of $U_q(\mathfrak{g})$ generated by all $X_i^{(n)}$, $Y_i^{(n)}$, q^h, and

$$\begin{Bmatrix} q^h \\ n \end{Bmatrix} := \prod_{k=1}^{n} \frac{q^{1-k}q^h - q^{k-1}q^{-h}}{q^k - q^{-k}}$$

for $i \in I$, $n \in \mathbb{Z}_{\geq 0}$, $h \in P^*$. We define a $\mathbb{Q}$-algebra automorphism $\psi : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ by

$$\begin{cases} \psi(X_i) := X_i, \ \psi(Y_i) := Y_i & \text{for } \ i \in I, \\ \psi(q) := q^{-1}, \ \psi(q^h) := q^{-h} & \text{for } \ h \in P^*. \end{cases} \tag{1.12}$$

By virtue of (1.10), we have a $\mathbb{Q}$-linear automorphism ψ of $V(\lambda)$ defined by $\psi(xu_\lambda) := \psi(x)u_\lambda$ for $x \in U_q^-(\mathfrak{g})$. Let $\mathcal{L}_\infty(\lambda)$ be the image of $\mathcal{L}_0(\lambda)$ by ψ. Then it is known (see, for example, [Kas2]) that the restriction of the canonical map $^-$ to $E(\lambda) := V_{\mathbb{Q}}(\lambda) \cap \mathcal{L}_0(\lambda) \cap \mathcal{L}_\infty(\lambda)$ is an isomorphism from $E(\lambda)$ to $\mathcal{L}_0(\lambda)/q\mathcal{L}_0(\lambda)$ as $\mathbb{Q}$-vector spaces. We denote by G_λ the inverse of this isomorphism. Then we have

$$V(\lambda) = \bigoplus_{b \in \mathcal{B}(\lambda)} \mathbb{Q}(q)G_\lambda(b). \tag{1.13}$$

Moreover we have the following.

Theorem 1.6 ([Kas3, Proposition 3.2.3])**.** *Let $\lambda \in P_+$ and $w \in W$. Then there exists a subset $\mathcal{B}_w(\lambda)$ of $\mathcal{B}(\lambda)$ such that*

$$V_w(\lambda) = \bigoplus_{b \in \mathcal{B}_w(\lambda)} \mathbb{Q}(q)G_\lambda(b). \tag{1.14}$$

1.5 Path Models. Let $\lambda \in P_+$. For $\mu, \nu \in W\lambda$, we write $\mu \geq \nu$ if there exist a sequence $\mu = \lambda_0, \lambda_1, \ldots, \lambda_s = \nu$ of elements in $W\lambda$ and a sequence $\beta_1, \ldots, \beta_s$ of positive real roots such that $\lambda_k = r_{\beta_k}(\lambda_{k-1})$ and $\lambda_{k-1}(\beta_k^\vee) < 0$ for $k = 1, 2, \ldots, s$, where for a positive real root β, we denote by r_β the reflection with respect to β, and by $\beta^\vee$ the dual root of β. Then we define $\mathrm{dist}(\mu, \nu)$ to be the maximal length s among all possible such sequences.

Remark 4. Assume that $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$. It immediately follows that $\mu \geq \nu$ if and only if $\omega^*(\mu) \geq \omega^*(\nu)$. Moreover, we have $\mathrm{dist}(\omega^*(\mu), \omega^*(\nu)) = \mathrm{dist}(\mu, \nu)$ when $\mu \geq \nu$.

Let $\lambda \in P_+$, μ, $\nu \in W\lambda$ with $\mu \geq \nu$, and $0 < a < 1$ a rational number. An a-chain for (μ, ν) is, by definition, a sequence $\mu = \lambda_0 > \lambda_1 > \cdots > \lambda_r = \nu$ of elements in $W\lambda$ such that $\mathrm{dist}(\lambda_i, \lambda_{i-1}) = 1$ and $\lambda_i = r_{\beta_i}(\lambda_{i-1})$ for some positive real root β_i, and such that $a\lambda_{i-1}(\beta_i^\vee) \in \mathbb{Z}$ for all $i = 1, 2, \ldots, r$.

Here let us consider a pair $\pi = (\underline{\nu}\,;\, \underline{a})$ of a sequence $\underline{\nu} : \nu_1 > \nu_2 > \cdots > \nu_s$ of elements in $W\lambda$ and a sequence $\underline{a} : 0 = a_0 < a_1 < \cdots < a_s = 1$ of rational numbers such that for each $i = 1, 2, \ldots, s-1$, there exists an a_i-chain for (ν_i, ν_{i+1}). Then we associate to $\pi = (\underline{\nu}\,;\, \underline{a})$ the following path $\pi : [0,1] \to \mathfrak{h}^*$:

$$\pi(t) := \sum_{i=1}^{j-1}(a_i - a_{i-1})\nu_i + (t - a_{j-1})\nu_j \quad \text{for} \quad a_{j-1} \leq t \leq a_j. \tag{1.15}$$

Such a path is called a Lakshmibai-Seshadri path (L-S path for short) of class λ. Denote by $\mathbb{B}(\lambda)$ the set of L-S paths of class λ.

Let us recall the raising and lowering root operators (cf. [Li1]–[Li4]). For convenience, we introduce an extra element θ that is not a path. For $\pi \in \mathbb{B}(\lambda)$ and $i \in I$, we set

$$h_i^\pi(t) := \big(\pi(t)\big)(\alpha_i^\vee), \qquad m_i^\pi := \min\big\{h_i^\pi(t) \mid t \in [0,1]\big\}. \tag{1.16}$$

First we define the raising root operator e_i with respect to the simple root α_i. We define $e_i\theta := \theta$, and $e_i\pi := \theta$ for $\pi \in \mathbb{B}(\lambda)$ with $m_i^\pi > -1$. If $m_i^\pi \leq -1$, then we can take the following points:

$$\begin{aligned} t_1 &:= \min\big\{t \in [0,1] \mid h_i^\pi(t) = m_i^\pi\big\}, \\ t_0 &:= \max\big\{t' \in [0,t_1] \mid h_i^\pi(t) \geq m_i^\pi + 1 \text{ for all } t \in [0,t']\big\}. \end{aligned} \tag{1.17}$$

We set

$$(e_i\pi)(t) := \begin{cases} \pi(t) & \text{if } 0 \leq t \leq t_0, \\ \pi(t) - (h_i^\pi(t) - m_i^\pi - 1)\alpha_i & \text{if } t_0 \leq t \leq t_1, \\ \pi(t) + \alpha_i & \text{if } t_1 \leq t \leq 1. \end{cases} \tag{1.18}$$

The lowering root operator f_i is defined in a similar fashion. We define $f_i\theta := \theta$, and $f_i\pi := \theta$ for $\pi \in \mathbb{B}(\lambda)$ with $h_i^\pi(1) - m_i^\pi < 1$. If $h_i^\pi(1) - m_i^\pi \geq 1$, then we can take the following points:

$$\begin{aligned} t_0 &:= \max\big\{t \in [0,1] \mid h_i^\pi(t) = m_i^\pi\big\}, \\ t_1 &:= \min\big\{t' \in [t_0,1] \mid h_i^\pi(t) \geq m_i^\pi + 1 \text{ for all } t \in [t',1]\big\}. \end{aligned} \tag{1.19}$$

We set

$$(f_i\pi)(t) := \begin{cases} \pi(t) & \text{if } 0 \leq t \leq t_0, \\ \pi(t) - (h_i^\pi(t) - m_i^\pi)\alpha_i & \text{if } t_0 \leq t \leq t_1, \\ \pi(t) - \alpha_i & \text{if } t_1 \leq t \leq 1. \end{cases} \tag{1.20}$$

Then we know the following.

Theorem 1.7 ([Li1] and [Li2]). *Let $\pi \in \mathbb{B}(\lambda)$. If $e_i\pi \neq \theta$ (resp. $f_i\pi \neq \theta$), then $e_i\pi \in \mathbb{B}(\lambda)$ (resp. $f_i\pi \in \mathbb{B}(\lambda)$). Hence the set $\mathbb{B}(\lambda) \cup \{\theta\}$ is stable under the action of the root operators. Moreover, every element $\pi \in \mathbb{B}(\lambda)$ is of the form $\pi = f_{i_1} f_{i_2} \cdots f_{i_k} \pi_\lambda$ for some $i_1, i_2, \ldots, i_k \in I$, where $\pi_\lambda := (\lambda; 0, 1) = t\lambda$ is the only element of $\mathbb{B}(\lambda)$ such that $e_i \pi_\lambda = \theta$ for all $i \in I$. Furthermore, we have*

$$\sum_{\pi \in \mathbb{B}(\lambda)} e(\pi(1)) = \operatorname{ch} L(\lambda), \qquad \sum_{\pi \in \mathbb{B}_w(\lambda)} e(\pi(1)) = \operatorname{ch} L_w(\lambda), \tag{1.21}$$

where $\mathbb{B}_w(\lambda) := \{(\nu_1, \ldots, \nu_s\,;\, \underline{a}) \in \mathbb{B}(\lambda) \mid \nu_1 \leq w(\lambda)\}$ for each $w \in W$.

It is known from [Kas5] *et al.* that $\mathbb{B}(\lambda)$ has a natural crystal structure isomorphic to $\mathcal{B}(\lambda)$. Namely, we have the following theorem (see [La2] for the second assertion).

Theorem 1.8. *There exists a unique bijection $\Phi : \mathcal{B}(\lambda) \xrightarrow{\sim} \mathbb{B}(\lambda)$ such that*

$$\Phi(F_{i_1} F_{i_2} \cdots F_{i_k} \overline{u}_\lambda) = f_{i_1} f_{i_2} \cdots f_{i_k} \pi_\lambda. \tag{1.22}$$

Moreover, $\Phi(\mathcal{B}_w(\lambda)) = \mathbb{B}_w(\lambda)$ for each $w \in W$.

At the end of this subsection, we recall the main result of [NS1]. Let $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$. For $\pi \in \mathbb{B}(\lambda)$, we define a path $\omega^*(\pi) : [0,1] \to \mathfrak{h}^*$ by $(\omega^*(\pi))(t) := \omega^*(\pi(t))$. Then we deduce that $\mathbb{B}(\lambda)$ and $\mathbb{B}_w(\lambda)$ with $w \in \widetilde{W}$ are ω^*-stable (cf. Remark 4 and [NS1, Lemma 3.1.1]). Denote by $\mathbb{B}^0(\lambda)$ the set of L-S paths that are fixed by ω^*, and set $\mathbb{B}^0_w(\lambda) := \mathbb{B}_w(\lambda) \cap \mathbb{B}^0(\lambda)$ for each $w \in \widetilde{W}$.

Theorem 1.9 ([NS1, Theorem 3.2.4]). *Let $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$, and $w \in \widetilde{W}$. Set $\widehat{\lambda} := (P^*_\omega)^{-1}(\lambda)$ and $\widehat{w} := \Theta^{-1}(w)$. Then we have*

$$\mathbb{B}^0(\lambda) = P^*_\omega(\widehat{\mathbb{B}}(\widehat{\lambda})), \qquad \mathbb{B}^0_w(\lambda) = P^*_\omega(\widehat{\mathbb{B}}_{\widehat{w}}(\widehat{\lambda})), \tag{1.23}$$

where we denote by $\widehat{\mathbb{B}}(\widehat{\lambda})$ the set of all L-S paths of class $\widehat{\lambda}$ for the orbit Lie algebra $\widehat{\mathfrak{g}}$, and set $\widehat{\mathbb{B}}_{\widehat{w}}(\widehat{\lambda}) := \{(\widehat{\nu}_1, \ldots, \widehat{\nu}_s\,;\, \underline{a}) \in \widehat{\mathbb{B}}(\widehat{\lambda}) \mid \widehat{\nu}_1 \preceq \widehat{w}(\widehat{\lambda})\}$ with $\preceq$ the relative Bruhat order on $\widehat{W}\widehat{\lambda}$. Here, for $\widehat{\pi} \in \widehat{\mathbb{B}}(\widehat{\lambda})$, we define a path $P^*_\omega(\widehat{\pi}) : [0,1] \to (\mathfrak{h}^*)^0$ by $(P^*_\omega(\widehat{\pi}))(t) := P^*_\omega(\widehat{\pi}(t))$.

2 Twining Characters and q-twining Characters.

2.1 The Twining Characters. From now on, we always assume that $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$ and $w \in \widetilde{W}$. First we consider the linear automorphism $\omega^{-1} \otimes \mathrm{id}$ of the Verma module $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{Q}(\lambda)$ of highest weight λ over $\mathfrak{g}$, where $\mathbb{Q}(\lambda)$ is the one-dimensional $\mathfrak{b}$-module on which $h \in \mathfrak{h}$ acts by the scalar $\lambda(h)$ and $\mathfrak{n}_+$ acts trivially. Since this map stabilizes the (unique) maximal proper $\mathfrak{g}$-submodule $N(\lambda)$ of $M(\lambda)$, we obtain an induced

$\mathbb{Q}$-linear automorphism $\tau_\omega : L(\lambda) \to L(\lambda)$, where $L(\lambda) = M(\lambda)/N(\lambda)$. It is easily seen that τ_ω has the following properties:

$$\tau_\omega(xv) = \omega^{-1}(x)\tau_\omega(v) \quad \text{for } x \in \mathfrak{g},\ v \in L(\lambda)$$

and $\tau_\omega(u_\lambda) = u_\lambda$, where u_λ is a (nonzero) highest weight vector of $L(\lambda)$.

Remark 5. From [N1, Lemma 4.1] (or [NS2, Lemma 2.2.3]), we know that τ_ω is a unique endomorphism of $L(\lambda)$ with the properties above.

The twining character $\mathrm{ch}^\omega(L(\lambda))$ of $L(\lambda)$ is defined to be the formal sum

$$\mathrm{ch}^\omega(L(\lambda)) := \sum_{\chi \in (\mathfrak{h}^*)^0} \mathrm{tr}\big(\tau_\omega|_{L(\lambda)_\chi}\big) e(\chi). \tag{2.1}$$

Since $\tau_\omega(L(\lambda)_\chi) = L(\lambda)_{\omega^*(\chi)}$ for all $\chi \in \mathfrak{h}^*$ and $\dim L(\lambda)_{w(\lambda)} = 1$ for all $w \in W$, we see that the Demazure module $L_w(\lambda)$ is τ_ω-stable for all $w \in \widetilde{W}$. Hence we can define the twining character $\mathrm{ch}^\omega(L_w(\lambda))$ of $L_w(\lambda)$ by

$$\mathrm{ch}^\omega(L_w(\lambda)) := \sum_{\chi \in (\mathfrak{h}^*)^0} \mathrm{tr}\big(\tau_\omega|_{L_w(\lambda)_\chi}\big) e(\chi). \tag{2.2}$$

2.2 The q-twining Characters. In this subsection, we introduce the q-twining characters of $V(\lambda)$ and $V_w(\lambda)$, which are, in fact, q-analogues of $\mathrm{ch}^\omega(L(\lambda))$ and $\mathrm{ch}^\omega(L_w(\lambda))$, respectively (see Proposition 2.1 below).

By (1.10), we have a $\mathbb{Q}(q)$-linear automorphism $\tau_{\omega_q} : V(\lambda) \to V(\lambda)$ induced from $\omega_q^{-1} : U_q^-(\mathfrak{g}) \to U_q^-(\mathfrak{g})$. As in the usual Lie algebra case in §2.1, τ_{ω_q} has the following properties:

$$\tau_{\omega_q}(xv) = \omega_q^{-1}(x)\tau_{\omega_q}(v) \quad \text{for } x \in U_q(\mathfrak{g}),\ v \in V(\lambda)$$

and $\tau_{\omega_q}(u_\lambda) = u_\lambda$, where u_λ is a (nonzero) highest weight vector of $V(\lambda)$.

Remark 6. In a similar way to the proof of [N1, Lemma 4.1], we can show that τ_{ω_q} is a unique endomorphism of $V(\lambda)$ with the properties above.

The q-twining character $\mathrm{ch}_q^\omega(V(\lambda))$ of $V(\lambda)$ is defined to be the formal sum

$$\mathrm{ch}_q^\omega(V(\lambda)) := \sum_{\chi \in (\mathfrak{h}^*)^0} \mathrm{tr}\big(\tau_{\omega_q}|_{V(\lambda)_\chi}\big) e(\chi). \tag{2.3}$$

We easily see that the quantum Demazure module $V_w(\lambda)$ is τ_{ω_q}-stable for every $w \in \widetilde{W}$. Hence we can define the q-twining character $\mathrm{ch}_q^\omega(V_w(\lambda))$ of $V_w(\lambda)$ by

$$\mathrm{ch}_q^\omega(V_w(\lambda)) := \sum_{\chi \in (\mathfrak{h}^*)^0} \mathrm{tr}\big(\tau_{\omega_q}|_{V_w(\lambda)_\chi}\big) e(\chi). \tag{2.4}$$

Here let us recall some facts from [Ja, §§5.12–5.15]. Let $V(\lambda)_\mathbb{Q}$ (resp. $V(\lambda)_{\chi,\mathbb{Q}}$) be the $\mathbb{Q}[q, q^{-1}]$-submodule of $V(\lambda)$ generated by all elements of the form $Y_{i_1}Y_{i_2}\cdots Y_{i_k}u_\lambda$

(resp. with $\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_k} = \lambda - \chi$). It is clear that all $V(\lambda)_{\chi,\mathbb{Q}}$ are finitely generated, torsion free $\mathbb{Q}[q, q^{-1}]$-modules. Therefore they are free $\mathbb{Q}[q, q^{-1}]$-modules of finite rank because $\mathbb{Q}[q, q^{-1}]$ is a principal ideal domain. We also know that the natural map $\mathbb{Q}(q) \otimes_{\mathbb{Q}[q,q^{-1}]} V(\lambda)_{\mathbb{Q}} \to V(\lambda)$ (given by $a \otimes v \to av$) is a $\mathbb{Q}(q)$-linear isomorphism.

Now we consider $\mathbb{Q}$ as a $\mathbb{Q}[q, q^{-1}]$-module by the evaluation at $q = 1$. Set $V := \mathbb{Q} \otimes_{\mathbb{Q}[q,q^{-1}]} V(\lambda)_{\mathbb{Q}}$ and $V_\chi := \mathbb{Q} \otimes_{\mathbb{Q}[q,q^{-1}]} V(\lambda)_{\chi,\mathbb{Q}}$. It follows from [Ja, Lemma 5.12] that $V(\lambda)_{\mathbb{Q}}$ is stable under the actions of X_i, Y_i, and $(q^h - q^{-h})/(q - q^{-1})$ for $i \in I$, $h \in P^\vee$. Thus we obtain endomorphisms x_i, y_i, and h of V defined by

$$x_i := 1 \otimes X_i, \quad y_i := 1 \otimes Y_i, \quad \text{and} \quad h := 1 \otimes (q^h - q^{-h})/(q - q^{-1}),$$

respectively. From [Ja, Lemmas 5.13 and 5.14], we know that the endomorphisms x_i, y_i, and h of V satisfy the Serre relations, and hence that these endomorphisms make V into a $\mathfrak{g}$-module. Moreover, $V \cong L(\lambda)$ as $\mathfrak{g}$-modules, and the image of V_χ by this $\mathfrak{g}$-module isomorphism is $L(\lambda)_\chi$ for all $\chi \in \mathfrak{h}^*$. Taking these facts into account, we show the following proposition.

Proposition 2.1. *Let $\chi \in (\mathfrak{h}^*)^0$ and $w \in \widetilde{W}$. Then $\mathrm{tr}(\tau_{\omega_q}|_{V(\lambda)_\chi})$ and $\mathrm{tr}(\tau_{\omega_q}|_{V_w(\lambda)_\chi})$ are elements of $\mathbb{Q}[q, q^{-1}]$. Moreover, we have*

$$\mathrm{tr}(\tau_{\omega_q}|_{V(\lambda)_\chi})\Big|_{q=1} = \mathrm{tr}(\tau_\omega|_{L(\lambda)_\chi}), \qquad \mathrm{tr}(\tau_{\omega_q}|_{V_w(\lambda)_\chi})\Big|_{q=1} = \mathrm{tr}(\tau_\omega|_{L_w(\lambda)_\chi}), \tag{2.5}$$

and hence

$$\mathrm{ch}_q^\omega(V(\lambda))\Big|_{q=1} = \mathrm{ch}^\omega(L(\lambda)), \qquad \mathrm{ch}_q^\omega(V_w(\lambda))\Big|_{q=1} = \mathrm{ch}^\omega(L_w(\lambda)). \tag{2.6}$$

Proof. It can be easily checked that $V(\lambda)_{\mathbb{Q}}$ is τ_{ω_q}-stable, and the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Q}(q) \otimes_{\mathbb{Q}[q,q^{-1}]} V(\lambda)_{\mathbb{Q}} & \xrightarrow{\sim} & V(\lambda) \\ {\scriptstyle 1\otimes(\tau_{\omega_q}|_{V(\lambda)_{\mathbb{Q}}})}\downarrow & & \downarrow{\scriptstyle \tau_{\omega_q}} \\ \mathbb{Q}(q) \otimes_{\mathbb{Q}[q,q^{-1}]} V(\lambda)_{\mathbb{Q}} & \xrightarrow{\sim} & V(\lambda). \end{array}$$

Since $V(\lambda)_{\chi,\mathbb{Q}}$ is a free $\mathbb{Q}[q, q^{-1}]$-module, we can define the trace of $\tau_{\omega_q}|_{V(\lambda)_{\chi,\mathbb{Q}}}$ for each $\chi \in (\mathfrak{h}^*)^0$. Note that a basis of $V(\lambda)_{\chi,\mathbb{Q}}$ over $\mathbb{Q}[q, q^{-1}]$ is also a basis of $V(\lambda)_\chi$ over $\mathbb{Q}(q)$. We obtain from the commutative diagram above that

$$\mathrm{tr}(\tau_{\omega_q}|_{V(\lambda)_\chi}) = \mathrm{tr}(\tau_{\omega_q}|_{V(\lambda)_{\chi,\mathbb{Q}}}) \in \mathbb{Q}[q, q^{-1}] \quad \text{for all } \chi \in (\mathfrak{h}^*)^0. \tag{2.7}$$

Now let $w \in \widetilde{W}$, and take $u_{w(\lambda)} \in V(\lambda)_{w(\lambda),\mathbb{Q}} \setminus \{0\}$. Here we remark that the rank of the free $\mathbb{Q}[q, q^{-1}]$-module $V(\lambda)_{w(\lambda),\mathbb{Q}}$ is one. We define $V_w(\lambda)_{\mathbb{Q}}$ to be the $\mathbb{Q}[q, q^{-1}]$-submodule of $V(\lambda)$ generated by the elements of the form $X_{i_1} X_{i_2} \cdots X_{i_k} u_{w(\lambda)}$. It is clear that $V_w(\lambda)_{\mathbb{Q}}$ is

τ_{ω_q}-stable. Since $V(\lambda)_{\mathbb{Q}}$ is stable under the action of X_i, we see that $V_w(\lambda)_{\mathbb{Q}}$ is a $\mathbb{Q}[q,q^{-1}]$-submodule of $V(\lambda)_{\mathbb{Q}}$. We set $V_w(\lambda)_{\chi,\mathbb{Q}} := V_w(\lambda)_{\mathbb{Q}} \cap V(\lambda)_{\chi,\mathbb{Q}}$. Then we immediately obtain the following commutative diagram:

$$\begin{array}{ccc} \mathbb{Q}(q) \otimes_{\mathbb{Q}[q,q^{-1}]} V_w(\lambda)_{\mathbb{Q}} & \xrightarrow{\sim} & V_w(\lambda) \\ {\scriptstyle 1\otimes(\tau_{\omega_q}|_{V_w(\lambda)_{\mathbb{Q}}})}\downarrow & & \downarrow{\scriptstyle \tau_{\omega_q}} \\ \mathbb{Q}(q) \otimes_{\mathbb{Q}[q,q^{-1}]} V_w(\lambda)_{\mathbb{Q}} & \xrightarrow{\sim} & V_w(\lambda). \end{array}$$

Hence, in the same way as above, we have

$$\mathrm{tr}\big(\tau_{\omega_q}|_{V_w(\lambda)_\chi}\big) = \mathrm{tr}\big(\tau_{\omega_q}|_{V_w(\lambda)_{\chi,\mathbb{Q}}}\big) \in \mathbb{Q}[q,q^{-1}] \quad \text{for all } \chi \in (\mathfrak{h}^*)^0,$$

thereby completing the proof of the first assertion.

Next we show the equalities (2.5). Note that the $\mathbb{Q}$-linear automorphism $\tau'_\omega := 1 \otimes (\tau_{\omega_q}|_{V(\lambda)_{\mathbb{Q}}})$ of $V := \mathbb{Q} \otimes_{\mathbb{Q}[q,q^{-1}]} V(\lambda)_{\mathbb{Q}}$ satisfies $\tau'_\omega(xv) = \omega^{-1}(x)\tau'_\omega(v)$ for $x \in \mathfrak{g}$, $v \in V$, and $\tau'_\omega(1 \otimes u_\lambda) = 1 \otimes u_\lambda$. Hence it follows from Remark 5 that the following diagram commutes:

$$\begin{array}{ccc} V = \mathbb{Q} \otimes_{\mathbb{Q}[q,q^{-1}]} V(\lambda)_{\mathbb{Q}} & \xrightarrow{\sim} & L(\lambda) \\ {\scriptstyle \tau'_\omega = 1\otimes(\tau_{\omega_q}|_{V(\lambda)_{\mathbb{Q}}})}\downarrow & & \downarrow{\scriptstyle \tau_\omega} \\ V = \mathbb{Q} \otimes_{\mathbb{Q}[q,q^{-1}]} V(\lambda)_{\mathbb{Q}} & \xrightarrow{\sim} & L(\lambda). \end{array}$$

Remark that, for all $\chi \in (\mathfrak{h}^*)^0$,

$$\mathrm{tr}\big(\tau_\omega|_{L(\lambda)_\chi}\big) = \mathrm{tr}\big(\tau'_\omega|_{V_\chi}\big) = 1 \otimes_{\mathbb{Q}[q,q^{-1}]} \mathrm{tr}\big(\tau_{\omega_q}|_{V(\lambda)_{\chi,\mathbb{Q}}}\big) = \mathrm{tr}\big(\tau_{\omega_q}|_{V(\lambda)_{\chi,\mathbb{Q}}}\big)\Big|_{q=1}, \tag{2.8}$$

since we regard $\mathbb{Q}$ as a $\mathbb{Q}[q,q^{-1}]$-module by the evaluation at $q=1$. Combining (2.8) with (2.7), we obtain

$$\mathrm{tr}\big(\tau_\omega|_{L(\lambda)_\chi}\big) \overset{(2.8)}{=} \mathrm{tr}\big(\tau_{\omega_q}|_{V(\lambda)_{\chi,\mathbb{Q}}}\big)\Big|_{q=1} \overset{(2.7)}{=} \mathrm{tr}\big(\tau_{\omega_q}|_{V(\lambda)_\chi}\big)\Big|_{q=1} \quad \text{for all } \chi \in (\mathfrak{h}^*)^0,$$

which proves the first equality of (2.5). By considering $V_w := \mathbb{Q} \otimes_{\mathbb{Q}[q,q^{-1}]} V_w(\lambda)_{\mathbb{Q}}$ for $w \in \widetilde{W}$, we also obtain

$$\mathrm{tr}\big(\tau_{\omega_q}|_{V_w(\lambda)_\chi}\big)\Big|_{q=1} = \mathrm{tr}\big(\tau_\omega|_{L_w(\lambda)_\chi}\big) \quad \text{for all } \chi \in (\mathfrak{h}^*)^0$$

in the same way. This completes the proof of Proposition 2.1. □

3 Twining Character Formula for Demazure Modules.

The main result of this paper is the following.

Theorem 3.1. *Let $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$ and $w \in \widetilde{W}$. Set $\widehat{\lambda} := (P^*_\omega)^{-1}(\lambda)$ and $\widehat{w} := \Theta^{-1}(w)$. Then we have*

$$\mathrm{ch}^\omega(L_w(\lambda)) = P^*_\omega(\mathrm{ch}\,\widehat{L}_{\widehat{w}}(\widehat{\lambda})), \tag{3.1}$$

where $\widehat{L}_{\widehat{w}}(\widehat{\lambda})$ is the Demazure module of lowest weight $\widehat{w}(\widehat{\lambda})$ in the irreducible highest weight module $\widehat{L}(\widehat{\lambda})$ of highest weight $\widehat{\lambda}$ over the orbit Lie algebra $\widehat{\mathfrak{g}}$.

We need some lemmas in order to prove this theorem.

Lemma 3.2. *For each $i \in I$, we have $\tau_{\omega_q} \circ E_i = E_{\omega^{-1}(i)} \circ \tau_{\omega_q}$ and $\tau_{\omega_q} \circ F_i = F_{\omega^{-1}(i)} \circ \tau_{\omega_q}$.*

Proof. We show only $\tau_{\omega_q} \circ E_i = E_{\omega^{-1}(i)} \circ \tau_{\omega_q}$ since the proof of $\tau_{\omega_q} \circ F_i = F_{\omega^{-1}(i)} \circ \tau_{\omega_q}$ is similar. Let $u = \sum_{k \geq 0} Y_i^{(k)} u_k \in V(\lambda)$, where $u_k \in (\ker X_i) \cap V(\lambda)_{\chi + k\alpha_i}$. Since $\omega_q^{-1}(Y_i^{(k)}) = Y_{\omega^{-1}(i)}^{(k)}$, we have

$$\tau_{\omega_q} \circ E_i(u) = \sum_{k \geq 0} Y_{\omega^{-1}(i)}^{(k-1)} \tau_{\omega_q}(u_k).$$

On the other hand, $\tau_{\omega_q}(u) = \sum_{k \geq 0} Y_{\omega^{-1}(i)}^{(k)} \tau_{\omega_q}(u_k) \in V(\lambda)_{\omega^*(\chi)}$. Here we note that $\tau_{\omega_q}(u_k) \in (\ker X_{\omega^{-1}(i)}) \cap V(\lambda)_{\omega^*(\chi) + k\alpha_{\omega^{-1}(i)}}$. Hence, by the uniqueness of the expression of $\tau_{\omega_q}(u)$, we have

$$E_{\omega^{-1}(i)} \circ \tau_{\omega_q}(u) = \sum_{k \geq 0} Y_{\omega^{-1}(i)}^{(k-1)} \tau_{\omega_q}(u_k).$$

Therefore we obtain $\tau_{\omega_q} \circ E_i(u) = E_{\omega^{-1}(i)} \circ \tau_{\omega_q}(u)$ for all $u \in V(\lambda)$, thereby completing the proof. $\square$

This lemma implies that $\mathcal{L}_0(\lambda)$ is τ_{ω_q}-stable. Hence we have the $\mathbb{Q}$-linear automorphism $\overline{\tau}_{\omega_q}$ of $\mathcal{L}_0(\lambda)/q\mathcal{L}_0(\lambda)$ induced from τ_{ω_q}. Then, by the definition of $\overline{\tau}_{\omega_q}$ and Lemma 3.2, we can easily check that the set $\mathcal{B}(\lambda)$ is $\overline{\tau}_{\omega_q}$-stable. Moreover, by Theorem 1.8, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{B}(\lambda) & \xrightarrow{\Phi} & \mathbb{B}(\lambda) \\ {\scriptstyle \overline{\tau}_{\omega_q}} \downarrow & & \downarrow {\scriptstyle \omega^*} \\ \mathcal{B}(\lambda) & \xrightarrow[\Phi]{} & \mathbb{B}(\lambda). \end{array} \tag{3.2}$$

Here we have used the fact that $\omega^* \circ e_i = e_{\omega^{-1}(i)} \circ \omega^*$ and $\omega^* \circ f_i = f_{\omega^{-1}(i)} \circ \omega^*$ (see [NS1, Lemma 3.1.1]). The next lemma immediately follows from the commutative diagram (3.2) and Theorem 1.8, since $\mathbb{B}_w(\lambda)$ is ω^*-stable for all $w \in \widetilde{W}$.

Lemma 3.3. *Let $w \in \widetilde{W}$. Then $\mathcal{B}_w(\lambda)$ is stable under $\overline{\tau}_{\omega_q}$. Hence we obtain the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{B}_w(\lambda) & \xrightarrow{\Phi} & \mathbb{B}_w(\lambda) \\ {\scriptstyle \overline{\tau}_{\omega_q}} \downarrow & & \downarrow {\scriptstyle \omega^*} \\ \mathcal{B}_w(\lambda) & \xrightarrow[\Phi]{} & \mathbb{B}_w(\lambda). \end{array} \tag{3.3}$$

Because $\psi \circ \tau_{\omega_q} = \tau_{\omega_q} \circ \psi$, we see that $\mathcal{L}_\infty(\lambda)$ is also τ_{ω_q}-stable. Since $V_{\mathbb{Q}}(\lambda)$ is obviously τ_{ω_q}-stable, we deduce that $E(\lambda)$ is τ_{ω_q}-stable.

Lemma 3.4. $\tau_{\omega_q} \circ G_\lambda = G_\lambda \circ \overline{\tau}_{\omega_q}$.

Proof. Remark that $\{G_\lambda(b) \mid b \in \mathcal{B}(\lambda)\}$ is a basis of the $\mathbb{Q}$-vector space $E(\lambda)$. Hence, for $b \in \mathcal{B}(\lambda)$, we have $\tau_{\omega_q}(G_\lambda(b)) = \sum_{b' \in \mathcal{B}(\lambda)} c_{b'}\, G_\lambda(b')$ for some $c_{b'} \in \mathbb{Q}$ since $E(\lambda)$ is τ_{ω_q}-stable. Then we obtain $\overline{\tau}_{\omega_q}(b) = \sum_{b' \in \mathcal{B}(\lambda)} c_{b'}\, b'$ in $\mathcal{L}_0(\lambda)/q\mathcal{L}_0(\lambda)$. Put $b'' := \overline{\tau}_{\omega_q}(b) \in \mathcal{B}(\lambda)$. Because $\mathcal{B}(\lambda)$ is a basis of the $\mathbb{Q}$-vector space $\mathcal{L}_0(\lambda)/q\mathcal{L}_0(\lambda)$, we see that $c_{b''} = 1$ and $c_{b'} = 0$ for all $b' \in \mathcal{B}(\lambda)$, $b' \neq b''$. Hence we obtain $\tau_{\omega_q}(G_\lambda(b)) = G_\lambda(b'') = G_\lambda(\overline{\tau}_{\omega_q}(b))$, as desired. □

Proof of Theorem 3.1. By combining Lemmas 3.3 and 3.4, we see that the set $\{G_\lambda(b) \mid b \in \mathcal{B}_w(\lambda) \cap \mathcal{B}(\lambda)_\chi\}$ is τ_{ω_q}-stable. Because $\{G_\lambda(b) \mid b \in \mathcal{B}_w(\lambda)\}$ is a basis of $V_w(\lambda)_\chi$ over $\mathbb{Q}(q)$ (see (1.14)), we obtain

$$\mathrm{tr}\big(\tau_{\omega_q}|_{V_w(\lambda)_\chi}\big) = \#\big\{G_\lambda(b) \mid \tau_{\omega_q}(G_\lambda(b)) = G_\lambda(b),\ b \in \mathcal{B}_w(\lambda) \cap \mathcal{B}(\lambda)_\chi\big\}$$

for $\chi \in (\mathfrak{h}^*)^0$ (note that if an endomorphism f on a finite-dimensional vector space V stabilizes a basis of V, then the trace of f on V is equal to the number of basis elements fixed by f). By Lemma 3.4 again, we get

$$\mathrm{tr}\big(\tau_{\omega_q}|_{V_w(\lambda)_\chi}\big) = \#\big\{b \in \mathcal{B}_w(\lambda) \cap \mathcal{B}(\lambda)_\chi \mid \overline{\tau}_{\omega_q}(b) = b\big\},$$

and hence

$$\mathrm{ch}_q^\omega(V_w(\lambda)) = \sum_{b \in \mathcal{B}_w^0(\lambda)} e(\mathrm{wt}(b)), \tag{3.4}$$

where $\mathrm{wt}(b) := \chi$ if $b \in \mathcal{B}(\lambda)_\chi$, and $\mathcal{B}_w^0(\lambda)$ is the set of elements of $\mathcal{B}_w(\lambda)$ fixed by $\overline{\tau}_{\omega_q}$. The commutative diagram (3.3) implies that

$$\mathrm{ch}_q^\omega(V_w(\lambda)) \overset{(3.4)}{=} \sum_{b \in \mathcal{B}_w^0(\lambda)} e(\mathrm{wt}(b)) \overset{(3.3)}{=} \sum_{\pi \in \mathbb{B}_w^0(\lambda)} e(\pi(1)).$$

We see from Theorems 1.7 and 1.9 that the right-hand side of the above equality coincides with $P_\omega^*(\mathrm{ch}\,\widehat{L}_{\widehat{w}}(\widehat{\lambda}))$, where $\widehat{\lambda} := (P_\omega^*)^{-1}(\lambda)$ and $\widehat{w} := \Theta^{-1}(w)$. Therefore we obtain

$$\mathrm{ch}_q^\omega(V_w(\lambda)) = P_\omega^*(\mathrm{ch}\,\widehat{L}_{\widehat{w}}(\widehat{\lambda})).$$

Notice that the right-hand side is independent of q. Hence we find that $\mathrm{ch}_q^\omega(V_w(\lambda))\Big|_{q=1} = P_\omega^*(\mathrm{ch}\,\widehat{L}_{\widehat{w}}(\widehat{\lambda}))$. Combining this with (2.6), we finally arrive at the conclusion that

$$\mathrm{ch}^\omega(L_w(\lambda)) = P_\omega^*(\mathrm{ch}\,\widehat{L}_{\widehat{w}}(\widehat{\lambda})).$$

Thus we have proved Theorem 3.1. □

Remark 7. By replacing $V_w(\lambda)$ by $V(\lambda)$ and $L_w(\lambda)$ by $L(\lambda)$ in the arguments above, we can give another proof of the twining character formula for the integrable highest weight module $L(\lambda)$, which is the main result of [FSS] ([FRS]).

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