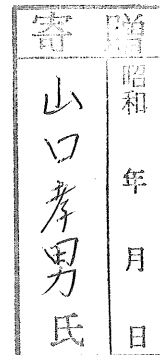


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FINITENESS AND UNIQUENESS OF DIFFEOMORPHISM CLASSES OF
RIEMANNIAN MANIFOLDS

BY

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THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Science in Mathematics in

The University of Tsukuba

December 1983

86318773

ACKNOWLEDGEMENT

It is with great pleasure that I acknowledge my indeptedness
to my advisor, Professor Katsuhiro Shiohama, for his advice,
encouragement and inspiration;

to Professor Hisao Nakagawa and Professor Tsunero Takahashi
for their helpful suggestion and encouragement;

to Nobuhiro Innami for his constructive criticism and advice.

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PREFACE

It is a main problem in Riemannian geometry to investigate to what extent differential geometric restrictions on a Riemannian manifold influence the topological structure of the manifold. Finiteness and uniqueness are the most typical theme relative to this problem. The study of uniqueness is to characterize up to topological type standard model spaces in terms of curvature or other geometric quantities. Rauch [18] and Klingenberg [15] gave a characterization of sphere, which is the so-called sphere theorem, and Berger [1] proved the rigidity theorem for compact rank one symmetric spaces. The differentiable version of the sphere theorem was also proved by Gromoll [9] Shikata [21] and Sugimoto and Shiohama [23].

On the other hand, the study of finiteness has been done by Cheeger [6] and Weinstein [24]. It is to investigate what bounds on the sizes of geometric quantities imply finiteness of topological types. Recently Gromov has developed this theory to a great extent.

In this paper, we shall first give explicit estimates for Cheeger's finiteness theorem and develop the idea used there to obtain uniqueness theorems up to diffeomorphism for sphere and the other rank one symmetric spaces. Finiteness is of course a rough conception. Our argument however shows that the idea of proving finiteness is very useful for the proof of uniqueness.

NOTATION

The following notation will be used throughout this paper:
For a Riemannian manifold M , we set

$d(., .)$ = the distance function induced from the Riemannian metric,
 K_M = the sectional curvature of M ,
 ∇R_M = the covariant derivative of the curvature tensor R_M of M ,
 $\text{Vol}(M)$ = the volume of M ,
 $\text{diam}(M)$ = the diameter of M ,
 $i(M)$ = the injectivity radius of exponential map of M .

For a point p of M and for a positive number r ,

M_p = the tangent space to M at p ,
 $B(p, r)$ = the open r -ball around p .

For a positive integer n , a positive number r and for a real number ,

$v(\lambda, r)$ = the volume of an r -ball in an n -dimensional complete
 simply connected space form of constant curvature λ .

$L()$ always denotes the length of curves.

CHAPTER I

FINITENESS OF DIFFEOMORPHISM CLASSES

1. Introduction

The study of finiteness for Riemannian manifolds, which has been done originally by Cheeger [6] and Weinstein [24], is to investigate what bounds on the sizes of geometrical quantities imply finiteness of topological types, - e.g., homotopy types, homeomorphism or diffeomorphism classes - of manifolds admitting metrics which satisfy the bounds. We recall the following

Cheeger's finiteness theorem I [6]. For given $n, \Lambda, V > 0$ there exist only finitely many pairwise non-diffeomorphic (non-homeomorphic) closed n ($\neq 4$)-manifolds M (4-manifolds) which admit metrics such that

$$|K_M| \leq \Lambda^2, \text{diam}(M) \leq 1, \text{Vol}(M) \geq V.$$

He proved directly finiteness up to homeomorphism for all dimension, and then for $n \neq 4$ used the results of Kirby and Siebenmann which show that finiteness up to homeomorphism implies finiteness up to diffeomorphism. For $n = 4$, he put a stronger bound on $\|\nabla R\|$. For given $n, \Lambda, \Lambda_1, V > 0$, we denote by $\mathcal{M}^n(\Lambda, \Lambda_1, V)$ a class of closed n -dimensional Riemannian manifolds M which satisfy the following bounds;

$$|K_M| \leq \Lambda^2, \|\nabla R_M\| \leq \Lambda_1, \text{diam}(M) \leq 1, \text{Vol}(M) \geq V,$$

and set

$$\mathcal{M}(\Lambda, \Lambda_1, V) = \bigcup_n \mathcal{M}^n(\Lambda, \Lambda_1, V).$$

Cheeger's finiteness theorem [6]. For given $n, \Lambda, \Lambda_1, V > 0$, the number $\#_{\text{diff}} \mathcal{M}^n(\Lambda, \Lambda_1, V)$ of diffeomorphism classes in $\mathcal{M}^n(\Lambda, \Lambda_1, V)$ is finite.

In the proof of the Cheeger finiteness theorem and our results as well, an estimate of the injectivity radius $i(M)$ of exponential map on M plays an important role. But since in his proof Ascoli's theorem is used essentially, it seems to us that it is impossible to bound the number $\#_{\text{diff}} \mathcal{M}^n(\Lambda, \Lambda_1, V)$ explicitly from above by using the proof as in [9]. In this chapter we show the existence of a finite upper bound for $\#_{\text{diff}} \mathcal{M}(\Lambda, \Lambda_1, V)$ and express upper bounds for $\#_{\text{diff}} \mathcal{M}^n(\Lambda, \Lambda_1, V)$ and $\#_{\text{diff}} \mathcal{M}(\Lambda, \Lambda_1, V)$ explicitly in terms of a priori given constants.

We first obtain the following theorem.

Theorem 1. For given $n, \Lambda, \Lambda_1, R > 0$, there exist $\varepsilon_1 = \varepsilon_1(n) > 0$ and $r_1 = r_1(n, \Lambda, \Lambda_1, R) > 0$ such that if complete n -manifolds M and \bar{M} satisfy the following conditions, then they are diffeomorphic;

$$1) |K_M|, |K_{\bar{M}}| \leq \Lambda^2, \|\nabla_{R_M}\|, \|\nabla_{R_{\bar{M}}}\| \leq \Lambda_1, i(M), i(\bar{M}) \geq R$$

2) for some r , $r \leq r_1$, and ε , $\varepsilon \leq \varepsilon_1$, there exist $2^{-(n+8)}r$ -dense and $2^{-(n+9)}r$ -discrete subsets $\{p_i\} \subset M$, $\{q_i\} \subset \bar{M}$ such that the correspondence $p_i \longrightarrow q_i$ is bijective and satisfies

$$(1 + \varepsilon)^{-1} \leq d(q_i, q_j)/d(p_i, p_j) \leq 1 + \varepsilon$$

for all p_i, p_j with $d(p_i, p_j) \leq 20r$.

We should note that ε_1 and r_1 can be written explicitly; e.g.,

$$\varepsilon_1 = 10^{-20} (n+1)^{-8} (n!)^{-2} 2^{-(2n^2+41n)}$$

$$r_1 = \min \{ R/140, \varepsilon_1/20\Lambda, \sqrt[3]{10^{-3} n^{-5} 2^{-\frac{n^2+7n}{2} \Lambda_1^{-1}}}, (10(2n^2 \Lambda^2 + 1))^{-1} \}.$$

For a metric space X a subset A is δ -dense iff for any $x \in X$, $d(x, A) < \delta$. A subset A is δ -discrete iff any two points of A have the distance at least δ .

Let ω_n denote the volume of the standard unit n -sphere. If we set

$$R = \min \{ \pi/\Lambda, (n-1)V/(2\omega_{n-2}e^{(n-1)\Lambda}) \},$$

then R gives a lower bound of $i(M)$ for all M in $\mathcal{M}^n(\Lambda, \Lambda_1, V)$, and every M in $\mathcal{M}(\Lambda, \Lambda_1, V)$ has the dimension at most n_0 , where

$$n_0 = 2 \max \{ [\log(k^{k+2}/k!V)], k \} + 3,$$

$$k = [\pi e^{2\Lambda+1}] + 1, \quad (\text{Lemma 20}).$$

Let $\varepsilon_1 = \varepsilon_1(n)$ and $r_1 = r_1(n, \Lambda, \Lambda_1, V)$ be as in Theorem 1. Using Theorem 1, we shall give the following estimates.

Theorem 2.

$$\#_{\text{diff}} \mathcal{M}^n(\Lambda, \Lambda_1, V) \leq (2^{2n+17}/\varepsilon_1 r_1^2)^{(N_0)+1} N_0,$$

$$\#_{\text{diff}} \mathcal{M}(\Lambda, \Lambda_1, V) \leq \sum_{n=2}^{n_0} (2^{2n+17}/\varepsilon_1 r_1^2)^{(N_0)+1} N_0,$$

where, $N_0 = [e^{\Lambda(n-1)}/(\Lambda 2^{-(n+9)} r_1)^n]$.

Recently M. Gromov [10], [11] states without giving detail of the proof that a similar result to Theorem 1 holds without the assumption for $\|\nabla R\|$. But our Theorem 1 is still valid for noncompact manifolds. However the assumption for $\|\nabla R\|$ is essential in the proof of Theorem 1. The proof is of course different from Gromov's one. The main tool of our proof is a technique of center of mass which is developed in [3] and [12].

The remainder of the chapter is organized as follows: Theorem 1 is proved in Section 2 — Section 4. The proof of Theorem 2 is given in Section 5, and another application of Theorem 1 is discussed in Section 6.

2. Construction of local diffeomorphisms

For given $n, \Lambda, R > 0$, set $R_0 = \frac{1}{2} \min \{R, \pi/\Lambda\}$ and let r and ε be adjustable parameters with $0 < r \leq R_0/70$, $0 < \varepsilon \leq 2^{-(n+14)}$. From now on we denote by M and \bar{M} complete n -dimensional Riemannian manifolds which satisfy the conditions in Theorem 1 for r and ε . In the final part of the proof, we shall set $r \leq r_1$ and $\varepsilon \leq \varepsilon_1$. We use the bound for $\|\nabla R\|$ actually only in Section 4. Let $\{p_i\} \subset M$ and $\{q_i\} \subset \bar{M}$ be $2^{-(n+8)}r$ -dense and $2^{-(n+9)}r$ -discrete subsets as in Theorem 1. Note that all δ -balls with $\delta \leq R_0$ in M and \bar{M} are convex and that by the Rauch comparison theorem (henceforth RCT), for any $v, w \in M_p$ with $\|v\|, \|w\| \leq t$, $t \leq R_0$

$$(\sin \Lambda t)/\Lambda t \leq d(\exp_p v, \exp_p w)/\|v - w\| \leq (\sinh \Lambda t)/\Lambda t.$$

The purpose of this section is to prove the following lemma.

Lemma 3. For each i there exists a linear isometry I_i from M_{p_i} to \bar{M}_{q_i} such that the composed map

$$F_i := \exp_{q_i} \circ I_i \circ \exp_{p_i}^{-1} : B(p_i, R_0) \longrightarrow B(q_i, R_0)$$

satisfies that $d(F_i(p_j), q_j) \leq \delta_1 r$ for every p_j with $d(p_i, p_j) \leq 10r$, where δ_1 is taken as

$$\delta_1 = 2(n+1)(6^{n+2} n! 2^{\frac{n}{2}+7})^{1/2} (40\Lambda r + 2\varepsilon)^{1/4}.$$

Proof. Set $\tilde{p}_j := \exp_{p_i}^{-1}(p_j)$ and $\tilde{q}_j := \exp_{q_i}^{-1}(q_j)$.

Then $\{\tilde{p}_j\}$ and $\{\tilde{q}_j\}$ are $2^{-(n+7)}r$ -dense and $2^{-(n+10)}r$ -discrete

subsets of the $10r$ -ball around 0 and satisfy

$$(1+\varepsilon)^{-1}e^{-20\Delta r} \leq \|q_j - q_k\|/\|p_j - p_k\| \leq (1+\varepsilon)e^{20\Delta r}$$

for all $j, k, j \neq k$. Hence Lemma 3 is a direct consequence of the following

Lemma 3'. Let $\{x_i\}$ be a $2^{-(n+7)}r$ -dense and $2^{-(n+10)}r$ -discrete subset of $B(0, r) \subset \mathbb{R}^n$ with $x_1=0$. If a system y_i of points in $B(0, r)$ with $y_1=0$ satisfies that

$$(1+\varepsilon)^{-1} \leq \|y_i - y_j\|/\|x_i - x_j\| \leq 1+\varepsilon$$

for every $i \neq j$, then there exists a linear isometry I of \mathbb{R}^n such that

$$\|I(x_i) - y_i\| \leq (n+1)(6^{n+2}n!2^{\frac{n}{2}+7}\varepsilon^{\frac{1}{2}})^{1/2} r$$

for every i .

For the proof of Lemma 3', it is convenient to introduce the following notion, a normal system, and to investigate some properties of a normal system. This is done in Lemma 5 — Lemma 7.

Definition 4. For $0 \leq \eta < 1$ and $r > 0$, we say that a system of n points $\{p_i\}_{1 \leq i \leq n}$ of \mathbb{R}^n is (r, η) -normal if

$$(1-\eta)r \leq \|p_i\| \leq r, \quad |\langle p_i, p_j \rangle| \leq \eta r^2$$

for every $i \neq j$.

Lemma 5. For every $L \geq n+1$, let $\{p_i\}_{1 \leq i \leq n}$ be an $(r, 2^{-L})$ -normal system for R^n . If we set

$$p'_i := p_i - \langle p_i, u_1 \rangle u_1 - \dots - \langle p_i, u_{i-1} \rangle u_{i-1}, \quad u_i := p'_i / \|p'_i\|$$

inductively, then

$$(1) \|p'_i\| \geq (1 - 2^{-(L-i)})^{1/2} r \geq (1 - 2^{-(L-i)})_r,$$

$$(2) |\langle p_k, u_i \rangle| \leq 2^{-(L-i)} r$$

for every i, k with $k > i$.

Proof. For $i=1$, (1) and (2) are trivial. Assume (1) and (2) for j , $1 \leq j \leq i$. Then we get

$$\begin{aligned} \|p'_{i+1}\|^2 &= \|p_{i+1}\|^2 - \langle p_{i+1}, u_1 \rangle^2 - \dots - \langle p_{i+1}, u_i \rangle^2 \\ &\geq ((1-2^{-L})^2 - 2^{-2(L-1)} - \dots - 2^{-2(L-i)}) r^2 \\ &\geq (1 - 2^{-(L-i-1)}) r^2 \geq (1 - 2^{-(L-i-1)})^2 r^2, \end{aligned}$$

and for $k > i+1$,

$$\begin{aligned} |\langle p_k, u_{i+1} \rangle| &\leq \|p'_{i+1}\|^{-1} (|\langle p_k, p_{i+1} \rangle| + |\langle p_{i+1}, u_1 \rangle| |\langle p_k, u_1 \rangle| + \dots \\ &\quad + |\langle p_{i+1}, u_i \rangle| |\langle p_k, u_i \rangle|) \\ &\leq 2(2^{-L} + 2^{-2(L-1)} + \dots + 2^{-2(L-i)}) r \\ &\leq 2^{-L+i+1} r. \end{aligned} \quad \text{Q.E.D.}$$

Thus for $L \geq n+1$, the Gram-Schmidt orthonormalization procedure yields the orthonormal basis $\{u_i\}$ for R^n via an $(r, 2^{-L})$ -normal system $\{p_i\}$.

Lemma 6. If $\{p_i\}_{1 \leq i \leq n}$ is an $(r, 2^{-L})$ -normal system for R^n , and if for some $\delta > 0$, x and y in R^n satisfy

$$\|x\|, \|y\| \leq r, \quad |\|x\| - \|y\|| \leq \delta, \quad |\|x - p_i\| - \|y - p_i\|| \leq \delta$$

for all i , $1 \leq i \leq n$, then $\|x - y\| \leq 3(n + 2^{-L+n+4})\delta$.

Proof. Notice that

$$|\langle p_i, x-y \rangle| = 2^{-1} |\|x\|^2 - \|y\|^2 + \|p_i - y\|^2 - \|p_i - x\|^2| \leq 3\delta r.$$

By induction, we show that

$$(*) \quad |\langle u_i, x-y \rangle| \leq 3(1 + 2^{-L+i+1})^2 \delta.$$

This is trivial for $i=1$. Assume $(*)$ for j , $1 \leq j \leq i$. Then we have with Lemma 5

$$\begin{aligned} |\langle u_{i+1}, x-y \rangle| &\leq \|p'_{i+1}\|^{-1} (|\langle p_{i+1}, x-y \rangle| + |\langle p_{i+1}, u_1 \rangle| |\langle u_1, x-y \rangle| + \dots \\ &\quad + |\langle p_{i+1}, u_i \rangle| |\langle u_i, x-y \rangle|) \\ &\leq 3(1+2^{-L+i+1})^{-1} (1+2^{-(L-1)}(1+2^{-L+2})^2 + \dots + 2^{-(L-i)}(1+2^{-L+i+1})^2) \delta \\ &\leq 3(1+2^{-L+i+2})(1+2^{-L+2} + \dots + 2^{-L+i+1}) \delta \\ &\leq 3(1+2^{-L+i+2})^2 \delta. \end{aligned}$$

Hence we conclude that

$$\|x-y\| \leq \sum_1^n |\langle u_i, x-y \rangle| \leq \sum_1^n 3(1+2^{-L+i+1})^2 \delta \leq 3(n+2^{-L+i+4})\delta. \quad \text{Q.E.D.}$$

Lemma 7. For k , $1 \leq k \leq n$, and $L \geq k+2$, let $\{e_i\}_{1 \leq i \leq k} \subset R^n$ be a $(1, 2^{-L})$ -normal system for the linear subspace spanned by $\{e_i\}$ with $\|e_i\| = 1$ for all i . If two unit vectors x and y which belong to $\text{Span}\{e_i\}_{1 \leq i \leq k}$ satisfy the following inequalities;

$|\angle(e_i, x) - \angle(e_i, y)| \leq \alpha \quad (1 \leq i \leq k-1), \quad \langle x, e_k \rangle \geq 3/4, \quad \langle y, e_k \rangle \geq 3/4,$
 then $\angle(x, y) \leq 6((k-1) + 2^{-L+k+3})\alpha$, where $\angle(x, y)$ denotes the angle between x and y .

Proof. Notice that $|\langle e_i, x \rangle - \langle e_i, y \rangle| \leq \alpha \quad (1 \leq i \leq k-1)$, and $2^{-1}\angle(x, y) \leq \sin \angle(x, y) \leq \|x-y\|$. Hence it suffices to estimate $\|x-y\|$ from above. Let $\{u_i\}$ be an orthonormal basis for $\text{Span}\{e_i\}$ obtained by the Gram-Schmidt process from $\{e_i\}$.

From Lemma 6 (*), we get

$$|\langle u_i, x-y \rangle| \leq (1 + 2^{-L+i+1})^2 \alpha \quad (1 \leq i \leq k-1).$$

By Lemma 5,

$$\begin{aligned} \langle u_k, x \rangle &\geq \|e_k\|^{-1} (\langle e_k, x \rangle - |\langle e_k, u_1 \rangle| |\langle u_1, x \rangle| - \dots = |\langle e_k, u_{k-1} \rangle| |\langle u_{k-1}, x \rangle|) \\ &\geq \langle e_k, x \rangle - 2^{-L+1} - \dots - 2^{-L+k-1} \\ &\geq 3/4 - 2^{-L+k} \geq 1/2. \end{aligned}$$

Hence the inequality;

$$|\langle u_k, x \rangle^2 - \langle u_k, y \rangle^2| = \left| \sum_1^{k-1} (\langle u_i, x \rangle^2 - \langle u_i, y \rangle^2) \right| \leq 2 \sum_1^{k-1} |\langle u_i, x-y \rangle|$$

implies

$$|\langle u_k, x-y \rangle| \leq 2 \sum_1^{k-1} |\langle u_i, x-y \rangle|,$$

and this implies that

$$\begin{aligned} \|x-y\| &\leq \sum_1^k |\langle u_i, x-y \rangle| \leq 3 \sum_1^{k-1} (1 + 2^{-L+i+1})^2 \alpha \\ &\leq 3((k-1) + 2^{-L+k+3})\alpha. \end{aligned} \quad \text{Q.E.D.}$$

From now on we return to the situation in Lemma 3'.

Let $\{x_i\}$ be a $2^{-(n+7)}$ -dense and $2^{-(n+10)}$ -discrete subset of $B(0, r)$ and let $\{y_i\}$ be a system of points in $B(0, r)$ with $y_1 = 0$ such that $(1+\varepsilon)^{-1} \leq \|y_i - y_j\| / \|x_i - x_j\| \leq 1+\varepsilon$ for every $i \neq j$. With these notations we have the following Lemma 8 and 9.

Lemma 8. $|\angle(x_i, x_j) - \angle(y_i, y_j)| \leq 2^{\frac{n}{2}+8} \varepsilon^{1/2}$ for every $i \neq j$.

Proof. Set $\alpha_{i,j} := \angle(x_i, x_j)$ and $\beta_{i,j} := \angle(y_i, y_j)$. First we show that $|\cos \alpha_{i,j} - \cos \beta_{i,j}| \leq 2^{(n+13)} \varepsilon$. Set $\kappa = 1+\varepsilon$, then we get

$$\begin{aligned} \cos \alpha_{i,j} &= (\|x_i\|^2 + \|x_j\|^2 - \|x_i - x_j\|^2) / (2\|x_i\|\|x_j\|) \\ &\leq (\kappa^2(\|y_i\|^2 + \|y_j\|^2) - \kappa^{-2}\|y_i - y_j\|^2) / (2\|x_i\|\|x_j\|) \\ &= (\kappa^2(2\|y_i\|\|y_j\|\cos \beta_{i,j} + \|y_i - y_j\|^2) - \kappa^{-2}\|y_i - y_j\|^2) / (2\|x_i\|\|x_j\|) \\ &= \kappa^2 \cos \beta_{i,j} \|y_i\|\|y_j\| / \|x_i\|\|x_j\| + (\kappa^2 - \kappa^{-2})\|y_i - y_j\|^2 / (2\|x_i\|\|x_j\|) \\ &\leq \kappa^4 \cos \beta_{i,j} + (\kappa^2 - \kappa^{-2})(2^{n+10} \kappa + \kappa^2), \\ \cos \alpha_{i,j} - \cos \beta_{i,j} &\leq (\kappa^4 - 1) \cos \beta_{i,j} + (\kappa^2 - \kappa^{-2})(2^{n+10} \kappa + \kappa^2) \\ &\leq 2^{n+13} \varepsilon. \end{aligned}$$

Hence we can get that $|\cos \alpha_{i,j} - \cos \beta_{i,j}| \leq 2^{n+13} \varepsilon$, and this yields

$$\begin{aligned} 2(\sin(|\alpha_{i,j} - \beta_{i,j}|/2))^2 &\leq 2^{n+13} \varepsilon, \\ |\alpha_{i,j} - \beta_{i,j}| &\leq 2 \sin^{-1}((2^{n+12} \varepsilon)^{1/2}) \\ &\leq 2^{\frac{n}{2}+8} \varepsilon^{1/2} \quad (\varepsilon \leq 2^{-(n+14)}). \quad \text{Q.E.D.} \end{aligned}$$

Lemma 9. There exist $\{x_{m_j}\}_{1 \leq j \leq n} \subset \{x_i\}$ and $\{y_{m_j}\}_{1 \leq j \leq n} \subset \{y_i\}$ such that they are $(r, 2^{-(n+4)})$ -normal systems for \mathbb{R}^n .

Proof. Take an orthogonal basis $\{w_j\}$ for R^n such that $\|w_j\| = (1 - 2^{-(n+6)})r$, and by denseness, take $\{x_{m_j}\}_{1 \leq j \leq n} \subset \{x_i\}$ such that $\|x_{m_j} - w_j\| \leq 2^{-(n+7)}r$. An easy calculation shows that $\{x_{m_j}\}_{1 \leq j \leq n}$ and the corresponding $\{y_{m_j}\}_{1 \leq j \leq n}$ have the required properties. Q.E.D.

Proof of Lemma 3'. Let $\{u_i\}$ and $\{v_i\}$ be the orthonormal bases for R^n obtained by applying the Gram-Schmidt process to $\{x_{m_i}\}$ and $\{y_{m_i}\}$ respectively. A required linear isometry I of R^n is defined by $I(u_i) = v_i$. If we set

$$X_k = I(x_{m_k})/\|I(x_{m_k})\|, \quad Y_k = I(y_{m_k})/\|I(y_{m_k})\|$$

then we have with Lemma 5 (1)

$$\langle v_k, X_k \rangle, \langle v_k, Y_k \rangle \geq 1 - 2^{-(n+4-k)}.$$

This yields

$$\begin{aligned} \langle X_k, Y_k \rangle &\geq \cos(\angle(X_k, v_k) + \angle(v_k, Y_k)) \\ &\geq 2 \cos^2 \theta - 1 \quad (\cos \theta := 1 - 2^{-(n+4-k)}) \\ &\geq 1 - 2^{-(n+2-k)} \geq 3/4. \end{aligned}$$

Assertion 1. $\angle(I(x_{m_k}), y_{m_k}) \leq (6k-5)6^{k-2}(k-1)!\epsilon'$. $\epsilon' := 2^{\frac{n}{2}+8}\epsilon^{\frac{1}{2}}$.

Proof. From the triangle inequality and Lemma 8, we have

$$\begin{aligned} \angle(y_{m_i}, I(x_{m_k})) &\leq \angle(I(x_{m_k}), I(x_{m_i})) + \angle(I(x_{m_i}), y_{m_i}) \\ &\leq \angle(y_{m_k}, y_{m_i}) + \angle(I(x_{m_i}), y_{m_i}) + \epsilon', \end{aligned}$$

and similarly,

$$\angle(y_{m_i}, I(x_{m_k})) \geq \angle(y_{m_k}, y_{m_i}) - \angle(I(x_{m_i}), y_{m_i}) - \epsilon',$$

hence,

$$|\Phi(y_{m_i}, I(x_{m_k})) - \Phi(y_{m_i}, y_{m_k})| \leq \Phi(I(x_{m_i}), y_{m_i}) + \varepsilon'.$$

Clearly, $\Phi(I(x_1), y_1) = 0$. Assume the assertion for i , $1 \leq i \leq k-1$.

Then we get for every i ($1 \leq i \leq k-1$)

$$\begin{aligned} |\Phi(y_{m_i}, I(x_{m_k})) - \Phi(y_{m_i}, y_{m_k})| &\leq (6i-5)6^{i-2}(i-1)!\varepsilon' + \varepsilon' \\ &\leq ((6k-11)6^{k-3}(k-2)! + 1)\varepsilon'. \end{aligned}$$

Notice that $\{y_{m_i}/\|y_{m_i}\|\}_{1 \leq i \leq k}$ is a $(1, 2^{-(n+3)})$ -normal system for its spanning subspace. Hence applying Lemma 7 to

$\{y_{m_i}/\|y_{m_i}\|\}_{1 \leq i \leq k}$, X_k and Y_k in place of $\{e_i\}_{1 \leq i \leq k}$, x and y , we conclude

$$\Phi(I(x_{m_k}), y_{m_k}) \leq (6k-5)6^{k-2}(k-1)!\varepsilon'. \quad \text{Q.E.D.}$$

Assertion 2. $|\|I(x_i) - y_{m_k}\| - \|y_i - y_{m_k}\|| \leq (2k!6^k\varepsilon')^{1/2}_r$
for every i and k , $1 \leq k \leq n$.

Together with Lemma 6, this completes the proof of Lemma 3'.

Proof of Assertion 2. Assertion 1 and the triangle inequality imply that

$$\begin{aligned} \Phi(I(x_i), y_{m_k}) &\leq \Phi(I(x_i), I(x_{m_k})) + \Phi(I(x_{m_k}), y_{m_k}) \\ &\leq \Phi(y_i, y_{m_k}) + ((6k-5)6^{k-2}(k-1)! + 1)\varepsilon', \end{aligned}$$

and similarly,

$$\Phi(I(x_i), y_{m_k}) \geq \Phi(y_i, y_{m_k}) - ((6k-5)6^{k-2}(k-1)! + 1)\varepsilon',$$

hence,

$$|\Phi(I(x_i), y_{m_k}) - \Phi(y_i, y_{m_k})| \leq ((6k-5)6^{k-2}(k-1)! + 1)\varepsilon'.$$

Therefore we have

$$\begin{aligned}
& | \|I(x_i) - y_{m_k}\|^2 - \|y_i - y_{m_k}\|^2 | \\
& \leq | \|I(x_i)\|^2 - \|y_i\|^2 | + 2\|y_{m_k}\| |\|y_i\| \cos \angle(y_i, y_{m_k}) - \|I(x_i)\| \cos \angle(I(x_i), y_{m_k})|,
\end{aligned}$$

where $| \|I(x_i)\|^2 - \|y_i\|^2 | \leq 2\varepsilon r^2$ and

$$\begin{aligned}
& | \|y_i\| \cos \angle(y_i, y_{m_k}) - \|I(x_i)\| \cos \angle(I(x_i), y_{m_k}) | \\
& \leq r(|\angle(y_i, y_{m_k}) - \angle(I(x_i), y_{m_k})| + \varepsilon) \\
& \leq ((6k-5)6^{k-2}(k-1)! + 2)\varepsilon' r.
\end{aligned}$$

Hence the inequality

$$| \|I(x_i) - y_{m_k}\| - \|y_i - y_{m_k}\| | \leq | \|I(x_i) - y_{m_k}\|^2 - \|y_i - y_{m_k}\|^2 |^{1/2}$$

implies the required estimate.

Q.E.D.

3. Reduction and C^0 -estimates.

In this section, we average the local diffeomorphisms F_i , constructed in the previous section, with a center of mass technique to obtain a smooth map $F:M \longrightarrow \bar{M}$ and control the C^0 error between F and F_i . Let ψ be a smooth function such that

$$\psi|_{[0, 4]} = 1, \quad \psi|_{[5, \infty)} = 0, \quad 0 \geq \psi' \geq -2.$$

For every $x \in M$, define the weight $\phi_i(x)$ of $F_i(x)$ by

$$\phi_i(x) := \psi(d(x, p_i)/r) / \sum_j \psi(d(x, p_j)/r).$$

Notice that all p_j with $d(x, p_j) \leq 5r$ are finite and the corresponding $F_j(x)$ are contained in some convex ball B . It is easy from a convexity argument to see that for a fixed $x \in M$, the function $C_x: \bar{M} \longrightarrow \mathbb{R}$ defined by

$$C_x(y) = \frac{1}{2} \sum_i \phi_i(x) d^2(y, F_i(x))$$

is C^∞ strongly convex on B , and has a unique minimum point on \bar{M} . Setting

$$F(x) := \text{the unique minimum point of } C_x$$

we define the map $F:M \longrightarrow \bar{M}$. We show that F is smooth.

Define a map V from a sufficiently small neighborhood of the graph of F in $M \times \bar{M}$ to the tangent bundle $T\bar{M}$ by

$$V(x, y) = - \sum_i \phi_i(x) \exp_y^{-1}(F_i(x)).$$

Since $V(x, y) = (\text{grad } C_x)(y)$, we have $V(x, F(x)) = 0$.

Let $K: T\bar{M} \longrightarrow T\bar{M}$ be the connection map, and define the map

$$D_2V(x,y): \bar{M}_y \longrightarrow \bar{M}_y \text{ by}$$

$$D_2V(x,y)(\dot{y}(0)) = \nabla_{\dot{y}(0)} V(x,y(t)),$$

where we consider $V(x,y(t))$ as a vector field along a smooth curve $y(t)$ with $y(0) = y$. Notice that

$K(d/dt V(x,y(t)))|_{t=0} = D_2V(x,y)(\dot{y}(0))$, and $D_2V(x,y)$ is a linear map. From a standard Jacobi fields estimate (See $(*)_9$) in the proof of Lemma 15), it follows that

$$\| D_2V(x,y)(\dot{y}(0)) - \dot{y}(0) \| \leq (30Ar)^2 \|\dot{y}(0)\| < \|\dot{y}(0)\|.$$

This yields that $D_2V(x,y)$ is a linear isomorphism, and hence for $y = F(x)$, the space spanned by $\{d/dt V(x,y(t))|_{t=0}\}$ and the (horizontal) tangent space of the zero-section of $T\bar{M}$ at $(F(x),0)$ span $(T\bar{M})_{(F(x),0)}$. Therefore the implicit function theorem implies the smoothness of F .

From now on we fix $x_0 \in M$ and set $y_0 := F(x_0)$.

Lemma 10. dF_{x_0} has maximal rank iff

$$(*)_2 \quad \sum_i d/dt \psi(d(x(t), p_i)/r)|_{t=0} \exp_{y_0}^{-1}(F_i(x_0)) \\ + \sum_i \psi(d(x_0, p_i)/r) d(\exp_{y_0}^{-1})(dF_i(\dot{x}(0))) \neq 0$$

for every smooth curve $x(t)$ with $x(0)=x_0$ and $\dot{x}(0) \neq 0$.

Proof. Differentiating the curve $V(x(t), F(x(t)))$ in the zero-section of $T\bar{M}$, we have

$$(\ast_3) \quad d/dt V(x(t), y_0)|_{t=0} + D_2 V(x_0, y_0)(dF(\dot{x}(0))) = 0.$$

Hence dF_{x_0} has maximal rank iff $d/dt V(x(t), y_0)|_{t=0} \neq 0$.

Since $V(x_0, y_0) = 0$,

$$\begin{aligned} (\ast_4) \quad d/dt V(x(t), y_0)|_{t=0} \\ = - \sum_i d/dt \psi(d(x(t), p_i)/r)|_{t=0} \exp_{y_0}^{-1}(F_i(x_0)) / \sum_j \psi(d(x_0, p_j)/r) \\ - \sum_i \phi_i(x_0) d(\exp_{y_0}^{-1})(dF_i(\dot{x}(0))). \end{aligned}$$

This completes the proof.

Q.E.D.

We shall show in Section 4 that in the above (\ast_4) , the norm of the first term is smaller than that of the second if r and ε are taken sufficiently small. To do this we must first estimate the numbers of the sum in each term.

Lemma 11. Set

$$N_1 = \#\{i; \psi(d(x_0, p_i)/r) = 1\},$$

$$N_2 = \#\{i; \psi(d(x_0, p_i)/r) \neq 0\}.$$

Then $N_2/N_1 \leq 6^n$.

Proof. Since $\{p_i\}$ is $2^{-(n+8)}r$ -dense, the union of the balls $B(p_i, 2^{-(n+8)}r)$ with $d(x_0, p_i) \leq 4r$ covers the $3.9r$ -ball around x_0 , and since $\{p_i\}$ is $2^{-(n+9)}r$ -discrete, the family of $B(p_i, 2^{-(n+10)}r)$ with $d(x_0, p_i) \leq 5r$ are disjoint and contained in the $5.1r$ -ball around x_0 . It follows from RCT that

$$N_1 \geq v(\Lambda^2, 3.9r)/v(-\Lambda^2, 2^{-(n+8)}r),$$

$$N_2 \leq v(-\Lambda^2, 5.1r)/v(\Lambda^2, 2^{-(n+10)}r)$$

Since

$$\begin{aligned} & v(\Lambda^2, r)/v(-\Lambda^2, s) \\ &= \int_0^r \sin^{n-1} \Lambda t \, dt \quad / \quad \int_0^s \sinh^{n-1} \Lambda t \, dt \end{aligned}$$

and $\Lambda r \leq \pi/140$, we can get the required explicit bound for N_2/N_1 . Q.E.D.

Now we fix i and k such that $d(x_0, p_i), d(x_0, p_k) \leq 5r$, and estimate the distance $d(F_i(x_0), F_k(x_0))$.

Lemma 12. $|d(q_j, F_k(x_0)) - d(q_j, F_i(x_0))| \leq \delta_2 r$

for every j with $d(p_i, p_j), d(p_k, p_j) \leq 10r$, where $\delta_2 = 2(\delta_1 + 600\Lambda r)$.

Proof. Notice that

$$e^{-20\Lambda r} \leq d(F_k(x_0), F_k(p_j))/d(x_0, p_j) \leq e^{20\Lambda r}.$$

By Lemma 3,

$$|d(q_j, F_k(x_0)) - d(F_k(p_j), F_k(x_0))| \leq \delta_1 r.$$

Hence the triangle inequality implies

$$(*_5) \quad |d(p_j, x_0) - d(q_j, F_k(x_0))| \leq (\delta_1 r + 40\Lambda r \, d(p_j, x_0)) \leq \delta_2 r/2.$$

From the same estimate for i , we have the required bound. Q.E.D.

Here we assume the following bounds on ϵ and r in order to bound $\delta_2 \leq 1/2$;

$$(\ast_6) \quad \varepsilon, 204r \leq 2^{-18}(n+1)^{-4}(6^{n+2}n!2^{\frac{n}{2}+7})^{-2}.$$

These bounds assure that $d(F_i(x_0), F_k(x_0)) \leq 2r/3$.

Lemma 13. $d(F_k(x_0), F_i(x_0)) \leq \delta_3 r$, where $\delta_3 = 8(n+1)\delta_2$.

Proof. Take a point $q_{m_0} \in \{q_i\}$ such that $d(q_{m_0}, F_k(x_0)) \leq 2^{-(n+8)}r$, and let x_k and x_i denote the images of $F_k(x_0)$ and $F_i(x_0)$ by $\exp_{q_{m_0}}^{-1}$. Then from the bound (\ast_6) we have

$\|x_k\|, \|x_i\| \leq r$. By Lemma 9, we can choose $\{q_{m_j}\}_{1 \leq j \leq n}$ out of $\{q_i\}$ such that if \tilde{q}_{m_j} denotes the image of q_{m_j} by $\exp_{q_{m_0}}^{-1}$, then $\{\tilde{q}_{m_j}\}_{1 \leq j \leq n}$ is an $(r, 2^{-(n+4)})$ -normal system for $\bar{M}_{q_{m_0}}$. Notice that $\{p_{m_j}\}_{1 \leq j \leq n}$ corresponding to $\{q_{m_j}\}_{1 \leq j \leq n}$ are contained in $B(p_k, 10r) \cap B(p_i, 10r)$. From Lemma 12, we get

$$|\|q_{m_j} - x_k\| - \|q_{m_j} - x_i\|| \leq 2\delta_2 r, \quad 0 \leq j \leq n,$$

and together with Lemma 6 this implies the required estimate.

Q.E.D.

From the definition of F it is clear that $d(F_k(x_0), F_i(x_0)) \leq \delta_3 r$ for every i with $d(x_0, p_i) \leq 5r$. Hence we have with Lemma 11

$$(\ast_7) \quad \left\| \sum_i \frac{d}{dt} \psi(d(x(t), p_i)/r) \Big|_{t=0} \exp_{y_0}^{-1}(F_i(x_0)) \right\| \\ \leq N_2(2/r)\delta_3 r \|\dot{x}(0)\| \leq 2 \cdot 6^n \delta_3 N_1 \|\dot{x}(0)\|.$$

4. C^1 -estimates, proof of Theorem 1.

To estimate the second term in Lemma 10 (\ast_2) from below, we must control the error between $d(\exp_{y_0}^{-1})(dF_i(\dot{x}(0)))$ and $d(\exp_{y_0}^{-1})(dF_k(\dot{x}(0)))$. To do this it is essential to estimate $\|dF_k(\dot{x}(0)) - PdF_i(\dot{x}(0))\|$ from above, where P denotes the parallel translation along the minimizing geodesic from $F_i(x_0)$ to $F_k(x_0)$. This is done in Lemma 17.

Lemma 14. For each $x \in \bar{M}$, let $\{q_{a_j}\} := \{q_i\} \cap B(x, r)$ and $N' := \#\{q_{a_j}\}$. The map $\Phi: B(x, r/2) \longrightarrow \mathbb{R}^{N'}$ defined by $\Phi^j(y) = d^2(q_{a_j}, y)$ satisfies the following;

- (1) Φ is an embedding, and $\|d\Phi(v)\| \geq r\|v\|$ for every tangent vector v on $B(x, r/2)$,
- (2) $N' \leq 2^{n(n+11)}$.

Proof. The convexity of each component Φ^i of Φ implies the injectivity of Φ . For a given non-zero tangent vector v on $B(x, r/2)$, let γ be a geodesic with $\dot{\gamma}(0) = v/\|v\|$. Take a point q_{a_j} such that $d(q_{a_j}, \gamma(r/2)) \leq 2^{-(n+8)}r$. Comparing the triangle with vertices $(\gamma(0), \gamma(r/2), q_{a_j})$ to a triangle with the same edge length in the sphere with constant curvature Λ^2 , we have that $\cos \angle(\dot{\gamma}(0), \dot{\sigma}(0)) \geq 1/2$, where σ denotes the unique minimizing geodesic from $\gamma(0)$ to q_{a_j} . This yields that

$$\|d\Phi(v)\| \geq |d\Phi^j(v)| \geq r\|v\|.$$

The same proof as in Lemma 11 implies (2).

Q.E.D.

We fix i and k with $d(p_i, x_0), d(p_k, x_0) \leq 5r$, and take an embedding $\Phi: B(F_k(x_0), r/2) \longrightarrow \mathbb{R}^{N'}$ defined in the previous lemma for $F_k(x_0)$, where we set $\{q_{aj}\} := \{q_{ij}\} \cap B(F_k(x_0), r)$. For a unit tangent vector v at x_0 , let γ, σ_k and σ_i be geodesics such that $\dot{\gamma}(0) = v, \dot{\sigma}_k(0) = dF_k(v)$ and $\dot{\sigma}_i(0) = dF_i(v)$. For every q_{aj} , we set

$$f_j(t) = d^2(p_{aj}, \gamma(t)), \quad g_{m,j}(t) = \Phi^j(F_m \circ \gamma(t)),$$

$$h_{m,j}(t) = \Phi^j(\sigma_m(t)), \quad m = k, i.$$

Lemma 15. On $[0, r/2]$,

$$(1) \quad 2(1 - \Lambda^2 f_j) \leq f_j'' \leq 2(1 + \Lambda^2 f_j),$$

$$2(1 - \Lambda^2 h_{m,j})e^{-20\Lambda r} \leq h_{m,j}'' \leq 2(1 + \Lambda^2 h_{m,j})e^{20\Lambda r},$$

$$(2) \quad |g_{m,j}'' - h_{m,j}''| \leq \Omega_1 r,$$

where $\Omega_1 = 82 + 10n^3 \Omega r$ and

$$\Omega = 60n(n-1)(10\Lambda_1 r^2 + 4\Lambda^2 r + 400n^{3/2}\Lambda(\Lambda r)^3 e^{10(2n^2\Lambda^2+1)r}.$$

(2) is the only place where we need the assumption for $\|\nabla R\|$.

Proof. We consider the geodesic variations

$$\alpha(t, s) = \exp_{q_{aj}} s(\exp_{q_{aj}}^{-1}(F_m(\gamma(t)))),$$

$$\beta(t, s) = \exp_{q_{aj}} s(\exp_{q_{aj}}^{-1}(\sigma_m(t))).$$

Then for a fixed t , we have the Jacobi fields

$$J_0(s) = \frac{\partial \alpha}{\partial t}(t, s), \quad J(s) = \frac{\partial \beta}{\partial t}(t, s),$$

and the second variation formula yields

$$\begin{aligned} g_{m,j}''(t) &= 2(\langle \nabla_{J_0} J_0, T_0 \rangle + \langle J_0, \nabla_{T_0} J_0 \rangle)(1), \\ h_{m,j}''(t) &= 2\langle J, \nabla_T J \rangle(1), \end{aligned}$$

where T_0 and T denote the vector fields $\frac{\partial \alpha}{\partial s}$ and $\frac{\partial \beta}{\partial s}$.

We assert that

$$(*_8) \quad (1 - \Lambda^2 \|T\|^2) \|J(1)\|^2 \leq \langle J, \nabla_T J \rangle(1) \leq (1 + \Lambda^2 \|T\|^2) \|J(1)\|^2,$$

which implies (1). Let τ be a geodesic with $\|\dot{\tau}\| = \|T\|$ in the n -sphere S^n with constant curvature Λ^2 and I a linear isometry from $\bar{M}_{q_{aj}}$ to $S^n_{\tau(0)}$ and W the vector field along τ defined by using the parallel translations along $\beta(t,)$ and τ and I .

Then a standard comparison argument implies

$$\langle J, J' \rangle(1) = I_0(J, J) \geq I_0(W, W) \geq I_0(V, V) = \langle V, V' \rangle(1),$$

where I_0 denotes the index form and V the Jacobi field along τ with $V(0) = 0$ and $V(1) = W(1)$. It is easy to check that

$$\|V(s)\|^2 = s^2 \|J^T(1)\|^2 + \frac{\sin^2 \Lambda \|T\| s}{\sin^2 \Lambda \|T\|} (\|J(1)\|^2 - \|J^T(1)\|^2),$$

$$\langle V, V' \rangle(1) = \|J^T(1)\|^2 + \Lambda \|T\| \cot \Lambda \|T\| (\|J(1)\|^2 - \|J^T(1)\|^2),$$

where J^T denote the tangential component of J . Hence we have that $\langle J, J' \rangle(1) \geq (1 - \Lambda^2 \|T\|^2) \|J(1)\|^2$. Let P be a parallel vector field along $\beta(t,)$, then we get

$$|\langle J(s) - s J'(s), P \rangle'| = |s \langle R(T, J)T, P \rangle| \leq 2\Lambda^2 \|T\|^2 \|J\| s.$$

The integration implies

$$(*_9) \quad \|J(1) - J'(1)\| \leq \Lambda^2 \|T\|^2 \|J(1)\|.$$

It follows from $(*_9)$ that

$$|\langle J, J' \rangle(1)| \leq \|J(1)\| \|J'(1)\| \leq (1 + \Lambda^2 \|T\|^2) \|J(1)\|^2.$$

For (2), we get with $(*_8)$

$$\begin{aligned} |g_{m,j}''(t) - h_{m,j}''(t)| &\leq 2|\langle J_0, J_0' \rangle(1) - \langle J, J' \rangle(1)| + 2|\langle \nabla_{J_0} J_0, T_0 \rangle(1)| \\ &\leq e^{20\Lambda r} (2+8\Lambda^2 r^2) - e^{-20\Lambda r} (2-8\Lambda^2 r^2) + 2 \|\nabla_{J_0}(1) J_0\| 2.3r \\ &\leq 82\Lambda r + 4.6r \|\nabla_{J_0}(1) J_0\|. \end{aligned}$$

Let $\{e_i\}$ be an orthonormal basis for M_{p_m} and $\{x_i\}, \{y_i\}$ the normal coordinate systems on $B(p_m, 10r), B(q_m, 10r)$ based on $\{e_i\}, \{I_m(e_i)\}$ respectively. Let $\Gamma_{i,j}^k$ and $\bar{\Gamma}_{i,j}^k$ be the Cristoffel symbols with respect to $\{x_i\}$ and $\{y_i\}$ and let $c := F_m \circ \gamma$. Note that

$$\begin{aligned} \dot{c} &:= \sum_i \dot{c}_i \frac{\partial}{\partial y_i}, \quad \ddot{c}_k + \sum_{i,j} \Gamma_{i,j}^k (\gamma(t)) \dot{c}_i \dot{c}_j = 0, \\ \nabla_{\dot{c}} \dot{c} &= \sum_k (\ddot{c}_k + \sum_{i,j} \bar{\Gamma}_{i,j}^k (\dot{c}(t)) \dot{c}_i \dot{c}_j) \frac{\partial}{\partial y_k} \\ &= \sum_{k,i,j} (-\Gamma_{i,j}^k (\gamma(t)) + \bar{\Gamma}_{i,j}^k (c(t))) \dot{c}_i \dot{c}_j \frac{\partial}{\partial y_k}. \end{aligned}$$

By RCT, we get

$$\begin{aligned} |\dot{c}_i| &\leq e^{10\Lambda r} \|\dot{c}\| \leq e^{30\Lambda r}, \quad \left\| \frac{\partial}{\partial y_k} \right\| \leq e^{10\Lambda r}, \\ |\Gamma_{i,j}^k| &\leq e^{10\Lambda r} \left\| \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right\|, \quad |\bar{\Gamma}_{i,j}^k| \leq e^{10\Lambda r} \left\| \nabla_{\frac{\partial}{\partial y_i}} \frac{\partial}{\partial y_j} \right\|, \end{aligned}$$

and from a Cheeger's result (See [6], Lemma 4.3), we can estimate with $(*_6)$ in Section 3

$$\|\nabla \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}\|, \quad \|\nabla \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j}\| \leq \Omega.$$

Therefore we conclude that $\|\nabla \dot{c}\| \leq 2n^3 e^{80\Lambda r} \Omega$,
and this yields (2). Q.E.D.

The following lemma is used in the proof of Lemma 17.

Lemma 16. Let $\varphi: [0, t] \longrightarrow \mathbb{R}$ be a C^2 -function such that $\varphi(0) = 0$ and $|\varphi(s)| \leq \alpha$, $|\varphi''(s)| \leq \kappa$ on $[0, t]$. Then
 $|\varphi'(0)| \leq \alpha/t + \kappa t/2$.

Lemma 17. $\|PdF_i(v) - dF_k(v)\| \leq 2^{n(n+11)/2} (11\delta_3 + \Omega_1 r/2)$,
where P denotes the parallel translation along the minimizing geodesic from $F_i(x_0)$ to $F_k(x_0)$.

Proof. Let τ be a geodesic with $\dot{\tau}(0) = PdF_i(v)$ and let $u_j(t) := \Phi^j(\tau(t))$. We apply the previous lemma to $h_{k,j} - u_j$. On $[0, r/2]$, we have with $(*)_5$ and Lemma 15 (2)

$$\begin{aligned} |h_{k,j} - h_{i,j}| &\leq |h_{k,j} - g_{k,j}| + |g_{k,j} - f_j| + |f_j - g_{i,j}| + |g_{i,j} - h_{i,j}| \\ &\leq 4\delta_2 r^2 + \Omega_1 r^3/4, \end{aligned}$$

and RCT implies

$$|h_{i,j} - u_j| \leq d(\sigma_i(0), \tau(0)) \cosh \Lambda r \cdot 4r \leq 5\delta_3 r^2,$$

hence

$$|h_{k,j} - u_j| \leq (4\delta_2 + 5\delta_3 + \Omega_1 r/4)r^2.$$

Together with Lemma 15 (1), Lemma 16 applied to $\varphi = h_{k,j} - u_j$ yields

$$\begin{aligned}
|d\Phi^j(\dot{\sigma}_k(0) - \dot{\tau}(0))| &\leq 2(4\delta_2 + 5\delta_3 + \Omega_1 r/4)r + 82\Lambda r^2/4 \\
&\leq (11\delta_3 + \Omega_1 r/2)r.
\end{aligned}$$

By Lemma 14, we obtain the required estimate.

Q.E.D.

Let P_k, P_i denote the parallel translations along the minimizing geodesics from y_0 to $F_k(x_0), F_i(x_0)$ respectively. For simplicity, set

$$v_m := dF_m(v), \quad \tilde{v}_m := d(\exp_{y_0}^{-1})(dF_m(v)), \quad m = i, k,$$

Lemma 18. $\|\tilde{v}_k - \tilde{v}_i\| \leq \delta_4$, where $\delta_4 = 2^{n(n+11)/2} (12\delta_3 + \Omega_1 r/2)$.

Proof. From the standard estimates of Jacobi equation and an easy comparison argument, we get

$$\|P_k \tilde{v}_k - v_k\|, \|P_i^{-1} v_i - \tilde{v}_i\|, \|P v_i - P_k P_i^{-1} v_i\| \leq \Lambda^2 r^2.$$

Together with Lemma 17, this yields

$$\begin{aligned}
\|\tilde{v}_k - \tilde{v}_i\| &= \|P_k \tilde{v}_k - P_k \tilde{v}_i\| \\
&\leq \|P_k \tilde{v}_k - v_k\| + \|v_k - P v_i\| + \|P v_i - P_k P_i^{-1} v_i\| + \|P_k P_i^{-1} v_i - P_k v_i\| \\
&\leq 2^{n(n+11)/2} (12\delta_3 + \Omega_1 r/2).
\end{aligned}$$

Q.E.D.

Proof of Theorem 1. By Lemma 18, we have

$$\left\| \sum_i \psi(d(x_0, p_i)/r) \tilde{v}_i - \sum_i \psi(d(x_0, p_i)/r) \tilde{v}_k \right\| \leq \delta_4 N_2,$$

hence with Lemma 11

$$\left\| \sum_i \psi(d(x_0, p_i)/r) \tilde{v}_i \right\| \geq (0.9 - 6^n \delta_4) N_1.$$

If we set $\varepsilon \leq \varepsilon_1$, $r \leq r_1$, then we get with $(*)_7$

$$\begin{aligned} & \| \sum_i \psi(d(x_0, p_i)/r) \tilde{v}_i \| \\ & > \| \sum_i d/dt \psi(d(\gamma(t), p_i)/r) |_{t=0} \exp_{y_0}^{-1}(F_i(x_0)) \| + 0.1 N_1. \end{aligned}$$

By Lemma 10, F is an immersion. Furthermore the above inequality and $(*)_4$ imply

$$\| d/dt V(\gamma(t), y_0) |_{t=0} \| > 0.1 N_1/N_2.$$

On the other hand, the Jacobi fields estimate $(*)_9$ yields

$$\| \nabla_{dF(v)} V(x_0, F(\gamma(t))) \| \leq 4N_2 \| dF(v) \|.$$

Hence we have with $(*)_3$ and Lemma 11

$$\begin{aligned} \| dF(v) \| & \geq N_1/40N_2^2 \\ & \geq v(\Lambda^2, 2^{-(n+10)}r)/40 \cdot 6^n \cdot v(-\Lambda^2, 5.1r) > 0. \end{aligned}$$

This conclude that F must be surjective, and hence injective since it is a homotopy equivalence by its construction. Q.E.D.

5. Estimate of the number of diffeomorphism classes

For a $\delta > 0$, a system of points $\{x_i\}$ in a metric space X is called a δ -maximal system of X if $\{x_i\}$ is maximal with respect to the property that the distance between any two of them is greater than or equal to δ . $\{x_i\}$ is a δ -maximal system if and only if it is δ -dense and δ -discrete.

Lemma 19. There exists a δ -maximal system of every Riemannian manifold M .

Proof. Take a sequence X_i of compact subsets of M such that $\bigcup_i X_i = M$, $\overset{\circ}{X}_{i+1} \supset X_i$, where $\overset{\circ}{A}$ denotes the interior of a set A . We denote by $i(X_k)$ the infimum of the injectivity radius of the exponential map at points of X_k , and set $r_k := \frac{1}{2} \min\{\delta, i(X_k)\}$. Take a δ -maximal system $\{p_i^1\}_{1 \leq i \leq N_1}$ of X_1 . Notice that since the balls $B(p_i^1, r_1)$ have compact closure, they are contained in some X_{k_1} . Together with the fact that $B(p_i^1, r_1)$ are disjoint, this implies

$$N_1 \leq \text{Vol}(X_{k_1}) / \min_i \text{Vol}(B(p_i^1, r_1)).$$

By induction, it is possible to take a δ -maximal system $\{p_i^k\}_{1 \leq i \leq N_k}$ of X_k such that $p_i^k = p_i^j$ for every $j < k$ and every i , $1 \leq i \leq N_j$. Then the system $\bigcup_{k=1}^{\infty} \{p_i^k\}_{N_{k-1}+1 \leq i \leq N_k}$ is a δ -maximal system of M , where $N_0 := 0$. Q.E.D.

To prove Theorem 2 we recall an injectivity radius estimate.

The following lemma is a dimension independent version of [6], [14] and [16].

Lemma 20. For given $\Lambda, V > 0$, there exist $n_0 = n_0(\Lambda, V)$ and $R_0 = R_0(\Lambda, V) > 0$ such that if M is an n -dimensional compact Riemannian manifold such that

$$|K_M| \leq \Lambda^2, \quad \text{diam}(M) \leq 1, \quad \text{Vol}(M) \geq V,$$

then

$$(1) \quad n = \dim M \leq n_0,$$

$$(2) \quad i(M) \geq \min \{ \pi/\Lambda, (n-1)V/(2\omega_{n-2}e^{(n-1)\Lambda}) \} \geq R_0,$$

where n_0 and R_0 can be written explicitly as

$$n_0 = 2 \max \{ \lceil \log(k^{k+2}/k!V) \rceil, k \} + 3, \quad k = \lceil \pi e^{2\Lambda+1} \rceil + 1,$$

$$R_0 = \min_{2 \leq n \leq n_0} \{ \pi/\Lambda, (n-1)V/(2\omega_{n-2}e^{(n-1)\Lambda}) \}.$$

Proof. For (1), RCT yields

$$V \leq \text{Vol}(M) \leq v(-\Lambda^2, 1) \leq \omega_{n-1}e^{(n-1)\Lambda},$$

where

$$\omega_{n-1} = \begin{cases} 2\pi^m/(m-1)! & (n=2m) \\ 2(2\pi)^m/(2m-1)(2m-3)\cdots 3\cdot 1 & (n=2m+1). \end{cases}$$

Notice that

$$\lim_{n \rightarrow \infty} \omega_{n-1}e^{(n-1)\Lambda} = 0.$$

It is an easy calculation to estimate such an n_0 that $\omega_{n-1}e^{(n-1)\Lambda} < V$ for all $n > n_0$. For (2), it suffices to bound the lengths of closed geodesics from below. Suppose that there is a closed geodesic with length l . RCT implies that $\text{Vol}(M)$ is not greater than the volume of the tubular neighborhood of

radius 1 of a geodesic segment with length 1 in the simply connected n -dimensional hyperbolic space with constant curvature $-\Lambda^2$.

Therefore we get

$$\begin{aligned} \text{Vol}(M) &\leq \omega_{n-2} \int_0^1 (\sinh \Lambda t / \Lambda)^{n-2} \cosh \Lambda t \, dt \\ &\leq \omega_{n-2} (\sinh \Lambda)^{n-1} / (n-1) \Lambda^{n-1} \\ &\leq \omega_{n-2} e^{(n-1)\Lambda} / (n-1). \end{aligned}$$

Hence $1 \geq (n-1)V/(\omega_{n-2} e^{(n-1)\Lambda})$, and this yields (2). Q.E.D.

Proof of Theorem 2. For each $M_\alpha \in \mathcal{M}^n(\Lambda, \Lambda_1, V)$, take a $2^{-(n+8)}r_1$ -maximal system $\{p_i^\alpha\}_i$ of M_α . Notice that since $\text{diam}(M_\alpha) \leq 1$,

$$\# \{p_i^\alpha\}_i \leq v(-\Lambda^2, 1)/v(\Lambda^2, 2^{-(n+9)}r_1) \leq [e^{(n-1)\Lambda}/(\Lambda 2^{-(n+9)}r_1)^n] = N_0.$$

Set

$$m := \#_{\text{diff}} \mathcal{M}^n(\Lambda, \Lambda_1, V), \quad L := 1/(2^{-(n+8)}r_1), \quad \varepsilon'_1 := \varepsilon_1/(2(1+\varepsilon_1)L).$$

Suppose that

$$m > (2^{2n+17}/\varepsilon_1 r_1^2)^{(\binom{N_0}{2}+1)} N_0 > ([L/2\varepsilon'_1]+1)^{(\binom{N_0}{2}+1)} N_0.$$

Then $\mathcal{M}^n(\Lambda, \Lambda_1, V)$ contains at least $\lfloor m/N_0 \rfloor$ pairwise non-diffeomorphic manifolds $\{M_\alpha\}_{\alpha \in A}$ with the $2^{-(n+8)}r_1$ -maximal systems whose cardinalities are all the same, say N_1 , $N_1 \leq N_0$. We consider the set

$$\Sigma = \{(i_k, j_k); 1 \leq k \leq \binom{N_1}{2}\} := N_1'$$

of all the distinct pairs of the indices i of the system $\{p_i^\alpha\}_i$ for $\{M_\alpha\}_{\alpha \in A}$. For each M_α and M_β ($\alpha, \beta \in A$), and for each

$(i_k, j_k) \in \Sigma$, we set

$$l(\alpha, \beta; k) = d(p_{i_k}^\beta, p_{j_k}^\beta) / d(p_{i_k}^\alpha, p_{j_k}^\alpha).$$

Notice that $L^{-1} \leq l(\alpha, \beta; k) \leq L$. We fix some $\alpha \in A$. For $(i_1, j_1) \in \Sigma$ there is a $t_1 \in [L^{-1}, L]$ such that if

$$A_1 := \{\beta \in A; l(\alpha, \beta; 1) \in [t_1 - \varepsilon'_1, t_1 + \varepsilon'_1]\}$$

then $\# A_1 \geq [m/N_0]([L/2\varepsilon'_1] + 1)^{-1}$. By induction, for $(i_k, j_k) \in \Sigma$ there is a $t_k \in [L^{-1}, L]$ such that if

$$A_k := \{\beta \in A_{k-1}; l(\alpha, \beta; k) \in [t_k - \varepsilon'_1, t_k + \varepsilon'_1]\}$$

then $\# A_k \geq [m/N_0]([L/2\varepsilon'_1] + 1)^{-k}$. By the assumption on m , it is possible to take distinct pair β and β' in $A_{N'_1}$. Then $|l(\alpha, \beta; k) - l(\alpha, \beta'; k)| \leq 2\varepsilon'_1$ for all k , $1 \leq k \leq N'_1$, and this implies

$$(1 + \varepsilon_1)^{-1} \leq l(\beta, \beta'; k) \leq 1 + \varepsilon_1.$$

This is a contradiction since by Theorem 1 M_β is diffeomorphic to $M_{\beta'}$. The estimate for $\#_{\text{diff}} \mathcal{M}(\Lambda, \Lambda_1, V)$ is an immediate consequence of the previous lemma (1) and the estimate for $\#_{\text{diff}} \mathcal{M}^n(\Lambda, \Lambda_1, V)$. Q.E.D.

6. An extension of Shikata's theorem.

Here we describe another application of Theorem 1. For a bi-Lipschitz map $f: X \longrightarrow Y$ between two metric spaces X and Y , set

$$l(f) = \inf \{L; L^{-1} \leq d(f(x), f(y))/d(x, y) \leq L \text{ for all } x, y \in X\}.$$

Definition 21. Define $\rho(X, Y)$ by

$$\rho(X, Y) = \begin{cases} \inf \{ \log l(f); f: X \longrightarrow Y \text{ is bi-Lipschitz} \} \\ \infty & \text{if any bi-Lipschitz map does not exist.} \end{cases}$$

It is clear that ρ is symmetric and satisfies the triangle inequality. In the case where X and Y are compact, Ascoli's theorem implies

$$\rho(X, Y) = 0 \text{ iff } X \text{ is isometric to } Y.$$

For a positive integer n , we denote by \mathcal{O}^n a class of complete n -dimensional Riemannian manifolds M with

$$|K_M| < \infty, \|\nabla_{R_M}\| < \infty, \quad i(M) > 0.$$

Of course, \mathcal{O}^n contains all compact Riemannian manifolds of dimension n . Conversely, according to [8] every noncompact n -manifold admits a metric which belongs to the class \mathcal{O}^n .

A theorem of Shikata [20] states that there exists an $\varepsilon(n) > 0$ depending only on n such that if compact n -dimensional Riemannian manifolds M and N satisfy $\rho(M, N) < \varepsilon(n)$, then they are diffeomorphic. We do not know whether ρ is a distance on \mathcal{O}^n , but can extend the Shikata theorem to the class \mathcal{O}^n .

Let $\varepsilon_1 = \varepsilon_1(n)$ be the constant as in Theorem 1.

Theorem 22. If M and N which belong to \mathcal{A}^n satisfy $\rho(M, N) < \log(1 + \varepsilon_1)$, then they are diffeomorphic.

Proof. By the assumption, there exists a bi-Lipschitz map $f: M \longrightarrow N$ such that $l(f) < 1 + \varepsilon_1(n)$. We may assume

$$|K_M|, |K_N| \leq \Lambda^2, \quad \|\nabla_{R_M}\|, \|\nabla_{R_N}\| \leq \Lambda_1, \quad i(M), i(N) \geq R$$

for some $\Lambda, \Lambda_1, R > 0$. Let $r_1 = r_1(n, \Lambda, \Lambda_1, R)$ be the constant as in Theorem 1. Take a $(1 + \varepsilon_1)2^{-(n+9)}r_1$ -maximal system $\{p_i\}$ of M . Since f is bi-Lipschitz, it is surjective. It follows that $\{f(p_i)\}$ is $2^{-(n+8)}r_1$ -dense and $2^{-(n+9)}r_1$ -discrete. Then Theorem 1 yields that M and N are diffeomorphic. Q.E.D.

Added in proof. Recently we have received a preprint, S. Peters "Cheeger's finiteness theorem for diffeomorphism classes of Riemannian manifolds", where the finiteness of diffeomorphism classes of Cheeger type is proved for all dimension without the assumption for $\|\nabla R\|$ by using a similar method to our Theorem 1.

CHAPTER II

UNIQUENESS FOR SYMMETRIC SPACES OF RANK ONE

1. Introduction

A main problem in Riemannian geometry is to investigate the influences of geometrical quantities of complete Riemannian manifolds on the topology. The pioneering work for this is the well known sphere theorem due to Rauch [18] which was improved by Klingenberg [15]. Take $M = S^n$ with the constant sectional curvature equal to 1. Then the theorem states that if \bar{M} is a complete simply connected n -manifold with the sectional curvature $\frac{1}{4} < K_{\bar{M}} \leq 1$, then \bar{M} is homeomorphic to S^n . A stronger assumption for curvature implies that \bar{M} must be diffeomorphic to S^n ([9], [20], [23]). The rigidity theorem by Berger[1] states that if $\frac{1}{4} \leq K_{\bar{M}} \leq 1$, then \bar{M} is either homeomorphic to S^n or isometric to a rank one symmetric space of the compact type.

Recently the other sphere theorems have been obtained by pinching diameter or volume in place of sectional curvature ([13], [22]). For example, a result of Shiohama[22] states that for given n , $-\Lambda^2$, there exists an $\varepsilon = \varepsilon(n, \Lambda) > 0$ such that if the Ricci and sectional curvature and the volume of a complete n -manifold \bar{M} satisfy

$$\text{Ric}_{\bar{M}} \geq 1, K_{\bar{M}} \geq -\Lambda^2, \text{Vol}(\bar{M}) \geq \text{Vol}(S^n) - \varepsilon,$$

then \bar{M} is homeomorphic to S^n .

But in the above situation, it was not known for M to be

diffeomorphic to the standard sphere. The first purpose of this chapter is to give a partial affirmative answer to the problem of this type.

Theorem 1. For given $n, \Lambda \geq 1, \Lambda_1 > 0$, there exists a positive constant $\delta = \delta(n, \Lambda, \Lambda_1)$ such that if a complete manifold \bar{M} of dimension n satisfies

$$1 \leq K_{\bar{M}} \leq \Lambda^2, \quad \|\nabla R_{\bar{M}}\| \leq \Lambda_1, \quad \text{Vol}(\bar{M}) \geq \text{Vol}(S^n) - \delta,$$

then \bar{M} is diffeomorphic to S^n .

On the other hand, the diameter or volume-pinching theorems for sphere can not be extended to more general model spaces without further assumptions since the curvature of such a space will not be constant. According to the classical Cartan-Ambrose-Hicks theorem (Cf. [7], Ch. 1, §12), the behavior of the curvature tensor under parallel translation determines the Riemannian metric.

Let M, \bar{M} be compact Riemannian manifolds of the same dimension n and $m \in M, \bar{m} \in \bar{M}$. Let $I: M_m \longrightarrow \bar{M}_{\bar{m}}$ be a linear isometry between the tangent spaces. For a geodesic γ emanating from m , let $\bar{\gamma}$ denote the geodesic emanating from \bar{m} such that $\bar{\gamma}'(0) = I(\gamma'(0))$, and P_γ the parallel translation along γ . Set $I_\gamma := P_{\bar{\gamma}} \circ I \circ P_\gamma^{-1}$. I_γ induces an isomorphism on tensor spaces. Cheeger[5] defined the following notion of pinching:

$$\tilde{\rho}(M, \bar{M}) = \inf_{m, \bar{m}, I} \left[\sup \{ \|R_M - I_\gamma^{-1}(R_{\bar{M}})\| ; L(\gamma) \leq 2 \text{ diam}(M) \} \right].$$

Let M be a simply connected compact rank one symmetric space

(henceforth SCROSS). Then one of his results states that there exists an $\varepsilon > 0$ such that if a compact simply connected manifold \bar{M} is ε -close to M with respect to $\tilde{\rho}$, then \bar{M} is piecewise linearly homeomorphic to M .

The second purpose of this chapter is both to consider diameter and volume pinching for SCROSSes using a somewhat weaker one than $\tilde{\rho}$, and to strengthen the topological conclusion to diffeomorphism.

For $m \in M$, we denote by \mathcal{G}_m the compact domain in M_m bounded by the tangent cut locus of m :

$$\mathcal{G}_m = \{v \in M_m; d(\exp_m v, m) = \|v\|\},$$

and by $U\mathcal{G}_m$ the set of all unit tangent vectors on \mathcal{G}_m . We define our pinching numbers by

Definition.

$$\rho_0(M, \bar{M}) = \inf_{m, \bar{m}, I} [\sup\{\|d \exp_{\bar{m}}^{-1}(v)\| - \|d \exp_m(v)\|\}; v \in U\mathcal{G}_m\},$$

$$\rho_1(M, \bar{M}) = \rho_0(M, \bar{M}) + |\text{diam}(M) - \text{diam}(\bar{M})| + |\text{Vol}(M) - \text{Vol}(\bar{M})|.$$

Then we get the following

Theorem 2. Let M be a SCROSS. Then given $\Lambda, \Lambda_1 > 0$, there exists an $\varepsilon > 0$ depending on M and Λ, Λ_1 such that if a compact manifold \bar{M} satisfies that $|K_{\bar{M}}| \leq \Lambda^2$, $\|\nabla R_{\bar{M}}\| \leq \Lambda_1$, $\rho_1(M, \bar{M}) < \varepsilon$, then \bar{M} is diffeomorphic to M .

It should be noted that ρ_0 is weaker than $\tilde{\rho}$ in the sense

that $\rho_0 \leq c\tilde{\rho}$ for some constant c , but the converse inequality does not hold. Furthermore in the case where M is a SCROSS, Klingenberg's injectivity radius estimate[15] implies that if $\tilde{\rho}(M, \bar{M}) \rightarrow 0$, then $\text{Vol}(\bar{M}) \rightarrow \text{Vol}(M)$, $\text{diam}(\bar{M}) \rightarrow \text{diam}(M)$ and hence $\rho_1(M, \bar{M}) \rightarrow 0$.

Our pinching constants can be estimated explicitly. But we will not do this in order to avoid non-essential complexity.

The remainder of this chapter is organized as follows: Theorem 1 is proved in Section 2 — Section 4, and the proof of Theorem 2 is achieved in Section 5 — Section 7.

2. Preliminary results

Let \bar{M} be an n -dimensional complete manifold with $K_{\bar{M}} \geq 1$. Then Bonnet's theorem implies that $\text{diam}(\bar{M}) \leq \pi$, hence \bar{M} is compact. For fixed points $p \in S^n$ and $x \in \bar{M}$, let I be a linear isometry from S^n_p to \bar{M}_x . We set $\mathcal{G}_x = \{v \in \bar{M}_x; d(\exp_x v, x) = \|v\|\}$ and $U = \exp_p(I^{-1}(\mathcal{G}_x))$. Let $C(x)$ denotes the cut locus of x . Then the map $F: U \longrightarrow \bar{M} - C(x)$ defined by $F = \exp_x \circ I \circ \exp_p^{-1}$ is a diffeomorphism and RCT implies that $\|dF\| \leq 1$.

Lemma 3. If a complete manifold \bar{M} of dimension n satisfies that $K_{\bar{M}} \geq 1$, $\text{Vol}(\bar{M}) \geq \text{Vol}(S^n) - \delta$, then $\text{Vol}(F(A)) \geq \text{Vol}(A) - \delta$ for every measurable subset A of U .

Proof. Since $C(x)$ has measure zero, RCT implies that $\text{Vol}(U-A) \geq \text{Vol}(\bar{M}-F(A))$. By the assumption we get

$$\begin{aligned} \text{Vol}(F(A)) &= \text{Vol}(\bar{M}) - \text{Vol}(\bar{M}-F(A)) \\ &\geq \text{Vol}(S^n) - \delta - \text{Vol}(U-A) \\ &\geq \text{Vol}(A) - \delta. \end{aligned} \qquad \text{Q.E.D.}$$

Remark. Owing to Bishop's result (See [2]), we can use the Ricci curvature instead of the sectional curvature in the previous lemma.

The following lemma will play an important role in the proof of our Theorem 1. For the proof, see Lemma 4.3 in [6].

Lemma 4. Let \bar{M} be a Riemannian manifold of dimension n and let $\{x_i\}$ be a normal coordinate system. Set $r = (\sum x_i^2)^{1/2}$. Then there exist continuous functions expressed explicitly

$$\Omega_1: R_+ \times R_+ \longrightarrow R_+, \quad \Omega_2: \mathbb{N} \times R \times R \longrightarrow R$$

such that if $|K_{\bar{M}}| \leq \Lambda^2$ and $\|\nabla_{R_{\bar{M}}}\| \leq \Lambda_1$ on the normal coordinate neighborhood, then

$$\|\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial x_i}\| \leq \Omega_1(\Lambda, r), \quad \|\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}\| \leq \Omega_2(n, \Lambda, \Lambda_1, r).$$

3. Estimate of injectivity radius

For a fixed point x in a complete manifold \bar{M} , let y realize the minimum distance from x to its cut locus. Then either there is a minimizing geodesic from x to y along which y is conjugate to x , or there are precisely two minimizing geodesics from x to y which form a geodesic loop at x . In the case $K_{\bar{M}} \geq 1$, we observe the influence of the distance $d(x, C(x))$ on the total volume of \bar{M} . From now on, all geodesics are assumed to be parametrized by arc length. We denote by $J(t)$ the Jacobian of the exponential map on S^n_p at a point of distance equal to t from the origin:

$$J(t) = \left(\frac{\sin t}{t}\right)^{n-1}.$$

Lemma 5. Let \bar{M} be an n -dimensional complete manifold with $K_{\bar{M}} \geq 1$ and let $x \in \bar{M}$ and $y \in C(x)$ satisfy $d(x, y) = d(x, C(x)) =: 1$. Suppose that there is a geodesic loop $\gamma: [0, 2] \longrightarrow \bar{M}$ at x

with $\gamma(1) = y$. Then

(1) if $l > \pi/2$, then $C(x)$ consists of the single point y .

In particular, M is homeomorphic to S^n and $\text{Vol}(\bar{M}) \leq v(1, l)$.

(2) If $l \leq \pi/2$, then $G_x \subset \bar{B}(0, \pi-l)$ and $\text{Vol}(\bar{M}) \leq v(1, \pi/2)$.

Proof. (1). For any $z \in C(x)$ let γ_0 be a minimizing geodesic from y to z . Set $l_1 = d(x, z)$, $l_2 = d(y, z)$. Since γ is a geodesic loop, we may assume that $\langle \dot{\gamma}_0(0), -\dot{\gamma}(1) \rangle \geq 0$. Then the Toponogov comparison theorem and the assumption $l_1 \geq l$ imply

$$(*_1) \quad \cos l \geq \cos l_1 \geq \cos l_2 \cos l.$$

It follows that if $l > \pi/2$, then $l_2 = 0$ since $\cos l (1 - \cos l_2) \geq 0$.

(2). Suppose that $l \leq \pi/2$ and $d(x, z) = l_1 > \pi - l$. Then it turns out that

$$-\cos l = \cos(\pi - l) > \cos l_1 \geq \cos l \cos l_2,$$

and hence $\cos l (1 + \cos l_2) < 0$. This is a contradiction.

Therefore $G_x \subset \bar{B}(0, \pi - l)$. We now prove the second part of (2).

For a unit tangent vector v at x , let t_v denote the distance from x to the cut point along the geodesic with direction v .

We show that $t_v + t_{-v} \leq \pi$. Set $\alpha := \angle(v, \dot{\gamma}(0))$, $l_2 := d(\exp t_v v, y)$.

Then the Toponogov comparison theorem implies

$$\cos l_2 \geq \cos l \cos t_v + \sin l \sin t_v \cos \alpha.$$

Together with $(*_1)$ this yields

$$\cos t_v \geq \cos l (\cos l \cos t_v + \sin l \sin t_v \cos \alpha),$$

and hence $\cot t_v \geq \cot l \cos \alpha$. Similarly $\cot t_{-v} \geq -\cot l \cos \alpha$.

Therefore we have

$$\sin(t_v + t_{-v}) = \sin t_v \sin t_{-v} (\cot t_v + \cot t_{-v}) \geq 0,$$

and $t_v + t_{-v} \leq \pi$. For a fixed v , we may assume that $t_v < \pi/2$, and $t_{-v} > \pi/2$. Then we get that for any $t \in [t_v, \pi/2]$,

$J(t) \geq J(\pi-t)$, and hence

$$\int_{t_v}^{\pi/2} J(t) dt \geq \int_{t_v}^{\pi/2} J(\pi-t) dt = \int_{\pi/2}^{\pi-t_v} J(t) dt \geq \int_{\pi/2}^{t_{-v}} J(t) dt.$$

It follows that

$$\text{Vol}(\bar{M}) \leq \int_{G_x} J(\|v\|) dv \leq \int_{\bar{B}(0, \pi/2)} J(\|v\|) dv = v(1, \pi/2).$$

Q.E.D.

Since by the previous lemma, $\text{Vol}(\bar{M}) > \frac{1}{2} \text{Vol}(S^n)$ implies that y is conjugate to x , we only need to treat the conjugate point case. To do this, we need upper bounds on $|K_{\bar{M}}|$ and $\|\nabla R_{\bar{M}}\|$.

Lemma 6. Let \bar{M} be a complete manifold of dimension n with $|K_{\bar{M}}| \leq \Lambda^2$, $\|\nabla R_{\bar{M}}\| \leq \Lambda_1$, and let $x \in \bar{M}$ and $y \in C(x)$ satisfy $d(x, y) = d(x, C(x)) =: l$. Suppose that there are $v \in M_x$ and a unit vector $w \in (\bar{M}_x)_{lv}$ such that $d(\exp_x)(w) = 0$. Let W be the parallel vector field on \bar{M}_x with $W(0) = w$. Then for an arbitrary $\varepsilon \in (0, \pi)$, there are $t_1 \in (1/2, 1)$ and $\eta = \eta(n, \Lambda, \Lambda_1, l, \varepsilon)$ such that

$$B(t_1 v, \eta) \subset \overset{\circ}{G}_x \text{ and } \|d(\exp_x)(W)\| \leq \varepsilon \text{ on } B(t_1 v, \eta).$$

η is monotone decreasing in n, Λ, Λ_1, l and monotone increasing in ε .

Proof. Let $\{e_i\}$ be an orthonormal frame at x with $e_1 = w$ and $\{x_i\}$ the normal coordinate system on $M-C(x)$ based on $\{e_i\}$.

By Lemma 4,

$$\|\nabla \frac{\partial}{\partial x_i}\| \leq \Omega_1(\Lambda, 1), \quad \|\nabla \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}\| \leq \Omega_2(n, \Lambda, \Lambda_1, 1) \quad \text{on } B(x, 1).$$

Notice that $\|\frac{\partial}{\partial x_1}(\gamma_v(t))\| = \|d(\exp_x)(W(tv))\| \rightarrow 0$ as $t \nearrow 1$.

Since $|\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_1}| \leq \|\nabla \frac{\partial}{\partial x_i}\|$, setting $t_1 = 1 - \varepsilon/(2\Omega_1(\Lambda, 1))$

we get that $\|\frac{\partial}{\partial x_1}(\gamma_v(t_1))\| \leq \varepsilon/2$. If $\eta := \varepsilon/(2n\Omega_2(n, \Lambda, \Lambda_1, 1))$,

then $B(t_1 v, \eta) \subset \mathring{G}_x$ and since $|\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_1}| \leq \|\nabla \frac{\partial}{\partial x_i}\|$, we get that for every $u \in B(t_1 v, \eta)$

$$\begin{aligned} |\|\frac{\partial}{\partial x_1}(\exp_x u)\| - \|\frac{\partial}{\partial x_1}(\exp_x t_1 v)\|| &\leq n\Omega_2(n, \Lambda, \Lambda_1, 1) \|u - t_1 v\| \\ &\leq n\Omega_2(n, \Lambda, \Lambda_1, 1) \eta \leq \varepsilon/2, \end{aligned}$$

hence

$$\|d(\exp_x)(W(u))\| = \|\frac{\partial}{\partial x_1}(\exp_x u)\| \leq \|\frac{\partial}{\partial x_1}(\exp_x t_1 v)\| + \varepsilon/2. \quad \text{Q.E.D.}$$

Proposition 7. For given $n, \Lambda \geq 1, \Lambda_1 > 0$ and for $\varepsilon \in (0, \pi/2)$ there exists a positive constant $\delta_1 = \delta_1(n, \Lambda, \Lambda_1, \varepsilon)$ such that if \bar{M} is a complete manifold of dimension n such that

$$1 \leq K_{\bar{M}} \leq \Lambda^2, \quad \|\nabla R_{\bar{M}}\| \leq \Lambda_1, \quad \text{Vol}(\bar{M}) \geq \text{Vol}(S^n) - \delta_1,$$

then $i(\bar{M}) > \pi - \varepsilon$.

Proof. Let \bar{M} satisfy $\text{Vol}(\bar{M}) \geq \text{Vol}(S^n) - \delta_1$ for some $\delta_1 > 0$. Suppose that $1 := i(\bar{M}) \leq \pi - \varepsilon$. Then two points $x, y \in \bar{M}$ with $y \in C(x)$, $d(x, y) = 1$ satisfy the following (1) or (2):

(1). There is a closed geodesic $\gamma: [0, 2l] \rightarrow \bar{M}$ such that

$$\gamma(0) = \gamma(2l) = x, \quad \gamma(l) = y.$$

(2). y is conjugate to x along a minimizing geodesic

$$\sigma: [0, 1] \longrightarrow \bar{M}.$$

In the case (1), Lemma 5 implies that $\text{Vol}(\bar{M}) \leq v(1, \pi - \varepsilon)$.

In the case (2), there is a unit vector $w \in (\bar{M}_x)_{1\dot{\sigma}(0)}$ such that $d(\exp_x)(w) = 0$. Let W be the parallel vector field on \bar{M}_x with $W(0) = w$. Then for $\varepsilon_1 := \frac{1}{2} J(\pi - \varepsilon)$, the constants t_1 , $1/2 < t_1 < 1$ and $\eta = \eta(n, \Lambda, \Lambda_1, \pi - \varepsilon, \varepsilon_1)$ as in Lemma 6 satisfy that

$$B(t_1 \dot{\sigma}(0), \eta) \subset \bar{G}_x \text{ and } \|d(\exp_x)(W)\| \leq \varepsilon_1 \text{ on } B(t_1 \dot{\sigma}(0), \eta).$$

Setting $q_1 = F^{-1}(\sigma(t_1))$, $\kappa = \eta \frac{\sin(\pi - \varepsilon)}{\pi - \varepsilon}$, we have

$$\exp^{-1}(F(B(q_1, \kappa))) \subset B(t_1 \dot{\sigma}(0), \eta),$$

where $F: U \longrightarrow \bar{M} - C(x)$ is the diffeomorphism constructed in Section 2. This implies

$$\begin{aligned} \text{Vol } F(B(q_1, \kappa)) &= \int_{B(q_1, \kappa)} \det(dF) \, dS^n \\ &< \varepsilon_1 v(1, \kappa) / J(t_1 + \kappa) < \varepsilon_1 v(1, \kappa) / J(\pi - \varepsilon) = \frac{1}{2} v(1, \kappa). \end{aligned}$$

On the other hand, Lemma 3 implies that $\text{Vol } F(B(q_1, \kappa)) \geq v(1, \kappa) - \delta_1$. It turns out that $\delta_1 > \frac{1}{2} v(1, \kappa)$. Thus the required δ_1 is obtained as $\delta_1 = \frac{1}{2} v(1, \kappa)$ since $v(1, \varepsilon) > \frac{1}{2} v(1, \kappa)$. Q.E.D.

Remark. By the remark in Section 2, an observation similar to Lemma 5 yields that the previous proposition holds for the following class of manifolds \bar{M} :

$$\text{Ric}_{\bar{M}} \geq 1, \quad |K_{\bar{M}}| \leq \Lambda^2, \quad \|\nabla R_{\bar{M}}\| \leq \Lambda_1.$$

The proof of Proposition 7 suggests that it is possible to bound $\|dF\|$ from below. This is done in Lemma 9.

Lemma 8. If a complete manifold \bar{M} satisfies that $K_{\bar{M}} \geq 1$ and $\bar{M} - B(x, \pi - \varepsilon) \neq \emptyset$ for an ε , then $\text{diam}(\bar{M} - B(x, \pi - \varepsilon)) \leq 2\varepsilon$.

Proof. For arbitrary two points y_1 and y_2 of $\bar{M} - B(x, \pi - \varepsilon)$ let γ_i be a minimizing geodesic from x to y_i , and let l_i denote the length $L(\gamma_i)$ of γ_i , $i = 1, 2$. Let $\bar{\gamma}_i$ denote a geodesic in S^n such that $\bar{\gamma}_1(0) = \bar{\gamma}_2(0)$, $\dot{\gamma}_1(0), \dot{\gamma}_2(0) = \dot{\bar{\gamma}}_1(0), \dot{\bar{\gamma}}_2(0)$. Then the Toponogov comparison theorem implies

$$d(y_1, y_2) = d(\gamma_1(l_1), \gamma_2(l_2)) \leq d(\bar{\gamma}_1(l_1), \bar{\gamma}_2(l_2)) \leq 2\varepsilon. \quad \text{Q.E.D.}$$

Lemma 9. For given $n, \Lambda \geq 1, \Lambda_1 > 0$ and for $\varepsilon \in (0, \pi/2)$, $\varepsilon' \in (0, 1)$, there exists a positive constant $\delta_2 = \delta_2(n, \Lambda, \Lambda_1, \varepsilon, \varepsilon')$ ($\leq \delta_1(n, \Lambda, \Lambda_1, \varepsilon/2)$) such that if \bar{M} is a complete manifold of dimension n such that

$$1 \leq K_{\bar{M}} \leq \Lambda^2, \quad \|\nabla R_{\bar{M}}\| \leq \Lambda_1, \quad \text{Vol}(\bar{M}) \geq \text{Vol}(S^n) - \delta_2$$

then $1 \geq \|dF\| \geq 1 - \varepsilon'$ on $B(p, \pi - \varepsilon) \subset S^n$, where $F: U \rightarrow \bar{M} - C(x)$ is the diffeomorphism constructed in Section 2.

Proof. Take a δ_2 with $\delta_2 \leq \delta_1(n, \Lambda, \Lambda_1, \varepsilon/2)$ and let \bar{M} satisfy the conditions. Suppose that there are $q_0 \in B(p, \pi - \varepsilon)$ and a unit vector $w \in S^n_{q_0}$ such that $\|dF(w)\| < 1 - \varepsilon'$, where $q_0 =: \exp_p t_0 u$, $\|u\| = 1$, $u \perp w$. Set $\tilde{w} = d(\exp_p^{-1})(w)$ and let $\{e_i\}$ be an orthonormal frame at x such that $e_1 = I(\tilde{w})/\|I(\tilde{w})\|$, $e_n = I(u)$. Let $\{x_i\}$ denote the normal coordinate system based on $\{e_i\}$. Then the comparison argument yields

$$\left\| \frac{\partial}{\partial x_1} \right\| \leq (1 - \varepsilon') \frac{\sin t_0}{t_0}, \quad \left\| \frac{\partial}{\partial x_i} \right\| \leq \frac{\sin t_0}{t_0}, \quad (2 \leq i \leq n-1) \text{ at } F(q_0),$$

hence, $\|\frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_n}\| \leq (1 - \varepsilon')J(t_0)$.

Let $\tau = \tau(n, \varepsilon, \varepsilon')$ be the solution of

$$(1 - \varepsilon')\left(\frac{\sin(\pi - \varepsilon)}{\pi - \varepsilon} + \tau\right)^{n-1} = (1 - \varepsilon'/2)J(\pi - \varepsilon),$$

and set

$$\eta := (1 - \varepsilon')\tau / (n \Omega_2(n, \Lambda, \Lambda_1, \pi - \varepsilon/2)).$$

Then for every $w \in B(t_0 I(u), \eta)$,

$$\begin{aligned} \left| \left\| \frac{\partial}{\partial x_i} (\exp_x w) \right\| - \left\| \frac{\partial}{\partial x_i} (F(q_0)) \right\| \right| &\leq n \Omega_2(n, \Lambda, \Lambda_1, \pi - \varepsilon/2) \|w - t_0 I(u)\| \\ &\leq (1 - \varepsilon')\tau, \end{aligned}$$

hence,

$$\begin{aligned} \left\| \frac{\partial}{\partial x_1} (\exp_x w) \right\| &\leq (1 - \varepsilon')\left(\frac{\sin t_0}{t_0} + \tau\right), \\ \left\| \frac{\partial}{\partial x_i} (\exp_x w) \right\| &\leq \frac{\sin t_0}{t_0} + \tau, \quad 2 \leq i \leq n-1. \end{aligned}$$

Since $\|\frac{\partial}{\partial x_n}\| \leq 1$, this implies that on $\exp_x(B(t_0 I(u), \eta))$

$$\left\| \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_n} \right\| < (1 - \varepsilon')\left(\frac{\sin t_0}{t_0} + \tau\right)^{n-1} \leq (1 - \varepsilon'/2)J(t_0).$$

Setting $\kappa_1 := \eta \frac{\sin(\pi - \varepsilon/2)}{\pi - \varepsilon/2}$, we have $\exp_x^{-1} F(B(q_0, \kappa_1)) \subset B(t_0 I(u), \eta)$.

Choose $\kappa_2 = \kappa_2(n, \varepsilon, \varepsilon')$ which satisfies the following inequality;

$$(1 - \varepsilon'/2)J(t)/J(t+s) \leq 1 - \varepsilon'/3$$

for every $t \in [0, \pi - \varepsilon]$ and every $s \in [0, \kappa_2]$. In fact, since the function

$$g(t) = \left(\frac{\sin t}{t}\right) / \left(\frac{\sin(t + \kappa_2)}{t + \kappa_2}\right) \quad (0 \leq t \leq \pi - \kappa_2)$$

is monotone increasing, κ_2 is obtained as the solution of

$g(\pi - \varepsilon) = \left(\frac{1 - \varepsilon'/3}{1 - \varepsilon'/2}\right)^{1/(n-1)}$. Thus if $\kappa = \min\{\kappa_1, \kappa_2\}$, then

$$\begin{aligned} \text{Vol } F(B(q_0, \kappa)) &= \int_{B(q_0, \kappa)} \det(dF) \, dS^n \\ &< (1 - \varepsilon'/2) J(t_0) v(1, \kappa) / J(t_0 + \kappa) \\ &\leq (1 - \varepsilon'/3) v(1, \kappa). \end{aligned}$$

On the other hand, Lemma 3 implies that $\text{Vol } F(B(q_0, \kappa)) \geq v(1, \kappa) - \delta_2$. Hence it appears that $\delta_2 > \varepsilon' v(1, \kappa) / 3$. Therefore the required δ_2 is given as $\delta_2 = \min\{\delta_1(n, \Lambda, \Lambda_1, \varepsilon/2), \varepsilon' v(1, \kappa) / 3\}$. Q.E.D.

4. Proof of Theorem 1

The proof of Theorem 1 is, roughly speaking, achieved as follows. In the situation of Lemma 9, if ε and ε' are taken sufficiently small, then $\text{diam}(M - B(x, \pi - \varepsilon))$ is also small, and then F is almost isometric. As a result, for a suitable choice of $\{p_i\} \subset S^n$, $\{p_i\}$ and $\{F(p_i)\}$ will satisfy the conditions in Theorem 1 in Chapter I.

Proof of Theorem 1. For n , $\Lambda \geq 1$, Λ_1 , $R = \pi/2$, let $\varepsilon_1 = \varepsilon_1(n)$, $r_1 = r_1(n, \Lambda, \Lambda_1, R)$ be the constants given in Theorem 1 in Chapter I. For $\eta := \frac{3}{4} 2^{-(n+8)} r_1$, set

$$\varepsilon := \eta \varepsilon_1 / (4(1 + \varepsilon_1)), \quad \varepsilon' := \varepsilon_1 / (4(1 + \varepsilon_1)).$$

Then the required constant δ is obtained as $\delta = \delta_2(n, \Lambda, \Lambda_1, \varepsilon, \varepsilon')$ where δ_2 is the constant given in Lemma 9. Let \bar{M} satisfy the conditions in Theorem 1. Take an η -maximal system $\{p_i\}$ of S^n and choose a point \bar{p} of S^n such that $d(\bar{p}, \{p_i\}) \geq \eta/2$.

Let p be the antipodal point of \bar{p} , and for a fixed point $x \in \bar{M}$, let $F:U \longrightarrow \bar{M}-C(x)$ be the diffeomorphism constructed in Section 2. Notice that $i(\bar{M}) > \pi - \varepsilon/2$ and $1 \geq \|dF\| \geq 1 - \varepsilon'$ on $B(p, \pi - \varepsilon)$.

Assertion. If $y_i := F(p_i)$, then $\{y_i\}$ is $2^{-(n+8)}r_1$ -dense and $2^{-(n+9)}r_1$ -discrete and satisfy that $(1 + \varepsilon_1)^{-1} \leq d(y_i, y_j)/d(p_i, p_j) \leq 1 + \varepsilon_1$ for every $i \neq j$.

Therefore by Theorem 1 in Chapter I, \bar{M} is diffeomorphic to S^n .

Proof of Assertion. (1) Denseness. Take an arbitrary point y in \bar{M} . If $y \in \bar{B}(x, \pi - \varepsilon)$, then we can take a point p_i with $d(p_i, F^{-1}(y)) \leq \eta$. If the minimizing geodesic from $F^{-1}(y)$ to p_i intersects the ball $\bar{B}(\bar{p}, \varepsilon)$, replacing the intersection by a minimizing curve in the boundary $\partial \bar{B}(\bar{p}, \varepsilon)$, we construct a curve from $F^{-1}(y)$ to p_i in $S^n - B(\bar{p}, \varepsilon)$ with length $\leq \eta + \pi\varepsilon$. Since F is length nonincreasing, $d(y, y_i) \leq \eta + \pi\varepsilon < 4\eta/3$. If $y \in \bar{M} - \bar{B}(x, \pi - \varepsilon)$, take $y' \in \partial B(x, \pi - \varepsilon)$ and p_i with $d(y, y') \leq \varepsilon$, $d(p_i, F^{-1}(y')) \leq \eta$. Then the above argument implies

$$d(y, y_i) \leq d(y, y') + d(y', y_i) \leq \varepsilon + (\eta + \pi\varepsilon) \leq 4\eta/3.$$

Hence $\{y_i\}$ is $2^{-(n+8)}r_1$ -dense.

(2) Discreteness. For any y_i, y_j , let σ be a minimizing geodesic from y_i to y_j , and let σ_1, σ_2 be the maximal geodesic segments of σ such that $\sigma_1(0) = y_i$, $\sigma_2(L(\sigma_2)) = y_j$, $\text{Int } \sigma_k \subset B(x, \pi - \varepsilon)$, $k = 1, 2$ (possibly $\sigma_1 = \sigma$). Notice that by Lemma 9

$$L(\sigma_k) \geq (1 - \varepsilon') L(F^{-1} \cdot \sigma_k) \geq (1 - \varepsilon')(\eta/2 - \varepsilon).$$

This yields

$$\begin{aligned} d(y_i, y_j) &\geq L(\sigma_1) + L(\sigma_2) \geq (1-\varepsilon')(L(F^{-1}\sigma_1) + L(F^{-1}\sigma_2)) \\ &\geq 2(1-\varepsilon')(\eta/2 - \varepsilon) > 2\eta/3. \end{aligned}$$

Hence $\{y_i\}$ is $2^{-(n+9)}r_1$ -discrete.

By estimating $d(y_i, y_j)/d(p_i, p_j)$, we shall complete the proof. Let γ be a minimizing geodesic from p_i to p_j , and let γ_1, γ_2 be the maximal geodesic segments of γ such that $\gamma_1(0) = p_i$, $\gamma_2(L(\gamma_2)) = p_j$, $\text{Int } \gamma_k \subset B(p, \pi - \varepsilon)$, $k = 1, 2$ (possibly $\gamma_1 = \gamma$). Then Lemma 8 implies

$$\begin{aligned} d(y_i, y_j)/d(p_i, p_j) &\leq (L(F\gamma_1) + L(F\gamma_2) + 2\varepsilon)/(L(\gamma_1) + L(\gamma_2)) \\ &\leq (L(\gamma_1) + L(\gamma_2) + 2\varepsilon)/(L(\gamma_1) + L(\gamma_2)) \\ &= 1 + 2\varepsilon/(L(\gamma_1) + L(\gamma_2)), \end{aligned}$$

where $L(\gamma_k) \geq d(p_i, \bar{p}) - \varepsilon \geq \eta/2 - \varepsilon$, hence

$$d(y_i, y_j)/d(p_i, p_j) \leq 1 + 2\varepsilon/(\eta - 2\varepsilon) < 1 + \varepsilon_1.$$

On the other hand, under the notation in (2), we have

$$\begin{aligned} d(y_i, y_j)/d(p_i, p_j) &\geq (L(\sigma_1) + L(\sigma_2))/(L(F^{-1}\sigma_1) + L(F^{-1}\sigma_2) + 2\varepsilon) \\ &\geq (1-\varepsilon')(L(\sigma_1) + L(\sigma_2))/(L(\sigma_1) + L(\sigma_2) + 2\varepsilon(1-\varepsilon')) \\ &= (1-\varepsilon') - 2\varepsilon(1-\varepsilon')^2/(L(\sigma_1) + L(\sigma_2) + 2\varepsilon(1-\varepsilon')) \\ &> 1 - (\varepsilon' + \varepsilon(1-\varepsilon')^2)/((1-\varepsilon)(\eta/2 - \varepsilon)) \\ &> (1 + \varepsilon_1)^{-1}. \end{aligned} \quad \text{Q.E.D.}$$

5. ε -mappings

Before proceeding to our pinching situation of Theorem 2, we begin with a general consideration. For given $n, \Lambda, \Lambda_1, R > 0$, we denote by $\mathcal{B}^n(\Lambda, \Lambda_1, R)$ the following class of n -dimensional Riemannian manifolds M (not necessary compact):

$$|K_M| \leq \Lambda^2, \quad \|\nabla R_M\| \leq \Lambda_1, \quad i(M) \geq R.$$

From now on, for given Λ, R , we shall set $R_0 := \frac{1}{2} \min \{\pi/\Lambda, R\}$ implicitly. Notice that if $r < R_0$, then the r -ball $B(p, r)$ around any $p \in M$ is convex and if r is taken sufficiently small, then $\exp_p|_{B(0, r)}$ is almost isometric.

Definition 10. We say that a map $f: X \longrightarrow Y$ between metric spaces X and Y is an ε -map if $|d(f(x), f(x')) - d(x, x')| < \varepsilon$ for all $x, x' \in X$.

Notice that f is not necessarily continuous and any inverse map $f^{-1}: f(X) \longrightarrow X$ is also an ε -map.

ε -mapping Theorem 11. There exists an $\varepsilon_0 = \varepsilon_0(n, \Lambda, \Lambda_1, R) > 0$ such that if $M, \bar{M} \in \mathcal{B}^n(\Lambda, \Lambda_1, R)$ and $f: M \longrightarrow \bar{M}$ is an ε_0 -map, then f can be approximated by a diffeomorphism.

In Lemma 3' of Chapter I, we have proved the following lemma essentially.

Lemma 12. Let $\alpha \leq 2^{-(n+7)}$, $\varepsilon \leq 2^{-(n+14)}$. Let $\{x_i\}_{i=1, \dots, N}$ be an αr -maximal system of $B(0, r) \subset \mathbb{R}^n$ with $x_1 = 0$. If a system

$\{y_i\}_{i=1, \dots, N}$ of points in $B(0, r)$ with $y_1 = 0$ satisfies

$$1 - \varepsilon \leq \|y_i - y_j\| / \|x_i - x_j\| \leq 1 + \varepsilon \quad \text{for every } i \neq j,$$

then there exist a linear isometry I of \mathbb{R}^n and some constant $c(n)$ such that $\|I(x_i) - y_i\| \leq c(n) \varepsilon^{1/2} r$ for every i , where $\varepsilon' = 4(3\varepsilon(1+2\varepsilon^{-2}))^{1/2}$.

Lemma 13. For given $n, \Lambda, \varepsilon > 0$, there exist $r, \delta > 0$ such that if complete n -manifolds M and \bar{M} with $|K_M|, |K_{\bar{M}}| \leq \Lambda^2$, $i(M), i(\bar{M}) \geq R$ admit a δ -map $f: M \rightarrow \bar{M}$, then for every $m \in M$ the following are satisfied:

- (1) $\exp_m|B(0, r)$ is a $4\Lambda r^2$ -map,
- (2) $f|B(m, r)$ admits an εr -approximation of the form $\exp_{f(m)} \circ I \circ \exp_m^{-1}$,
- (3) $f(M)$ is $2\varepsilon r$ -dense.

Proof. (1). By RCT, we may assume that $r < R_0$ is chosen so small that for every $x, y \in B(0, r)$

$$e^{-4\Lambda r} \leq \frac{\sin \Lambda r}{\Lambda r} \leq \frac{d(\exp_m x, \exp_m y)}{\|x - y\|} \leq \frac{\sinh \Lambda r}{\Lambda r} \leq e^{\Lambda r},$$

and hence $|d(\exp_m x, \exp_m y) - \|x - y\|| < 4\Lambda r^2$.

(2). (1) implies that the map $g := \exp_{f(m)}^{-1} \circ f \circ \exp_m|B(0, r)$ is a $(\delta + 8\Lambda r^2)$ -map.

Assertion. There exists a linear isometry $I: M_m \rightarrow \bar{M}_{f(m)}$ such that $\|g - I\| < \eta(n, \Lambda, \delta, r)r$ on $B(0, r)$, where $\eta \rightarrow 0$ as $r, \delta/r, r^3/\delta \rightarrow 0$.

Proof. Set $\alpha := \sqrt[4]{\delta/r}$ and take an αr -maximal system $\{x_i\}$ on $B(0, r)$ with $x_1 = 0$. Now

$$\|g(x_i) - g(x_j)\| - \|x_i - x_j\| < \delta + 8\Lambda r^2.$$

By discreteness, this implies

$$|\|g(x_i) - g(x_j)\| / \|x_i - x_j\| - 1| < \alpha^3 + 8\Lambda r / \alpha =: \varepsilon'.$$

By Lemma 12, if α and ε are taken sufficiently small, then there exists a linear isometry I such that

$$\|I(x_i) - g(x_i)\| \leq c(n) \varepsilon'^{1/2} r$$

for some constant $c(n)$, where $\varepsilon' = 4(3\varepsilon'(1+2\alpha^{-2}))^{1/2}$. Notice that $\varepsilon' \rightarrow 0$ as $r, \delta/r, r^3/\delta \rightarrow 0$. For any $x \in B(0, r)$, by denseness we may take an x_i such that $\|x_i - x\| < \alpha r$. Then we get

$$\begin{aligned} \|I(x) - g(x)\| &\leq \|I(x) - I(x_i)\| + \|I(x_i) - g(x_i)\| + \|g(x_i) - g(x)\| \\ &< (2\alpha + \alpha^3 + 8\Lambda r + c(n) \varepsilon'^{1/2}) r \\ &=: \eta(n, \Lambda, \delta, r) r. \end{aligned} \quad \text{Q.E.D.}$$

Now for any $p \in B(m, r)$, we have

$$\begin{aligned} d(f(p), \exp_{f(m)} \circ I \circ \exp_m^{-1}(p)) \\ < \|g(\exp_m^{-1}(p)) - I(\exp_m^{-1}(p))\| + 4\Lambda r^2 < (\eta + 4\Lambda r) r. \end{aligned}$$

Hence for the proof of (2), it suffices to choose r, δ so small that $\eta + 4\Lambda r < \varepsilon$.

(3). We first show that

$$(*)_2 \quad f(B(m, r)) \text{ is } 2\varepsilon r\text{-dense in } B(f(m), r) \text{ for every } m \in M.$$

Let $\{x_i\} \subset B(0, r)$ be the αr -maximal system as in the proof of (2).

For any $q \in B(f(m), r)$, there is an x_i such that $\exp_m x_i \in (\exp_{f(m)} \circ I \circ \exp_m^{-1})^{-1}(B(q, 2\alpha r))$. Then (2) yields

$$\begin{aligned} d(f(\exp_m x_i), q) &\leq d(f(\exp_m x_i), \exp_{f(m)} \circ I(x_i)) + d(\exp_{f(m)} \circ I(x_i), q) \\ &< \varepsilon r + 2\alpha r < 2\varepsilon r. \end{aligned}$$

By induction, we assert that $f(B(m, (2k-1)r))$ is $2\varepsilon r$ -dense in $B(f(m), kr)$ for $k=1, 2, \dots$. This will complete the proof of (3).

Suppose that the assertion is true for k . For any q in the ball $B(f(m), (k+1)r)$, take $q_1 \in B(f(m), kr)$ with $q \in B(q_1, r)$. The induction hypothesis assures the existence of such a point $p \in B(m, (2k-1)r)$ that $d(f(p), q_1) < 2\varepsilon r$. Since $d(f(p), q) < (1+2\varepsilon)r$, it is possible to take a point q_2 on the unique minimizing geodesic from $f(p)$ to q such that $d(f(p), q_2) < r$, $d(q_2, q) < 2\varepsilon r$. Then by (\star_2) there exists a $p' \in B(p, r)$ with $d(f(p'), q_2) < 2\varepsilon r$. Since

$$d(f(p'), q) \leq d(f(p'), q_2) + d(q_2, q) < 4\varepsilon r < r,$$

(\star_2) implies again the existence of such a point $p'' \in B(p', r)$ that $d(f(p''), q) < 2\varepsilon r$, where

$$d(m, p'') \leq d(m, p) + d(p, p') + d(p', p'') < (2k+1)r.$$

This completes the induction argument.

Q.E.D.

Proof of ε -mapping Theorem 11. Let $\varepsilon_1(n)$ and $r_1(n, \Lambda, \Lambda_1, R)$

be the constants as in Theorem 1 in Chapter I. We set

$r'_1 := 2^{-(n+8)} r_1$. For an $\varepsilon_0 < \frac{3}{4} r'_1 \varepsilon_1$, let $f: M \rightarrow \bar{M}$ be an ε_0 -map.

Take a $\frac{3}{4} r'_1$ -maximal system $\{p_i\}$ on M and set $q_i := f(p_i)$.

Now the inequality $|d(q_i, q_j) - d(p_i, p_j)| < \varepsilon_0$ implies

$$|d(q_i, q_j)/d(p_i, p_j) - 1| < \varepsilon_0 / (\frac{3}{4} r'_1) < \varepsilon_1.$$

In particular, the correspondence $p_i \mapsto q_i$ is bijective and $\{q_i\}$ is $r'_1/2$ -discrete. It remains to prove r'_1 -denseness of $\{q_i\}$.

In Lemma 13, take ε, r so small that $2\varepsilon r < r'_1/8$. Let δ be the constant given in the lemma. Then setting $\varepsilon_0 < \delta$ we see that for any $q \in \bar{M}$, there exists a $p \in M$ such that $d(q, f(p)) < 2\varepsilon r$. Taking p_i with $d(p, p_i) < \frac{3}{4} r'_1$, we have

$$d(q, q_i) \leq d(q, f(p)) + d(f(p), q_i) < 2\varepsilon r + \frac{3}{4} r'_1 + \varepsilon_0 < r'_1.$$

Hence $\{q_i\}$ is r'_1 -dense and therefore Theorem 1 in Chapter I implies the existence of a diffeomorphism $F: M \rightarrow \bar{M}$ such that $d(F(p_i), q_i) < \delta r'_1$, where $\delta \rightarrow 0$ as $\varepsilon_0, r'_1 \rightarrow 0$. Q.E.D.

It should be mentioned that Cheeger[5] showed the following: Let M be a compact Riemannian manifold. Then for given $\Lambda, R > 0$, there exists an $\varepsilon > 0$ such that if a compact manifold \bar{M} satisfies $|K_{\bar{M}}| \leq \Lambda^2$, $i(\bar{M}) \geq R$, then every ε -map from M to \bar{M} can be approximated by a piecewise linear homeomorphism. But it seems to us that the constant ε can not be estimated explicitly in terms of the given constants.

6. Property CM

From now on, let M, \bar{M} denote compact Riemannian manifolds of dimension n unless otherwise stated. We shall often assume that $m \in M, \bar{m} \in \bar{M}$ and $I: M_m \rightarrow \bar{M}_{\bar{m}}$ have been chosen so as to minimize the pinching number under consideration. Notice that the interior $\overset{\circ}{G}_m$ of G_m is mapped diffeomorphically onto $M - C(m)$ by the exponential map at m . Let $\exp_m^{-1}: M \rightarrow G_m$ be some extension of $(\exp_m|_{\overset{\circ}{G}_m})^{-1}$. We define the map $\Phi: M \rightarrow \bar{M}$ by $\Phi = \exp_{\bar{m}} \circ I \circ \exp_m^{-1}$. For an $\varepsilon \in \mathbb{R}$, we set $G_m^\varepsilon := \{(1 + \frac{\varepsilon}{\|v\|})v; v \in G_m\}$. Although \exp_m^{-1} is not continuous, we shall try to show that in case M is a SCROSS, Φ is an ε -map if \bar{M} is sufficiently close to M with respect to ρ . This is done in the next section.

Lemma 14. For given $M, \varepsilon_1, \varepsilon_2 > 0$, there exists a $\delta > 0$ such that $\rho_0(M, \bar{M}) < \delta$ implies that $\|d\Phi - 1\| < \varepsilon_2$ on $\exp_m(G_m^{-\varepsilon_1})$.

Proof. We may find an $\varepsilon > 0$ such that $\|d\exp_m\| > \varepsilon$ on $G_m^{-\varepsilon_1}$. Set $\delta := \varepsilon \varepsilon_2$. Then for any unit vector v on $G_m^{-\varepsilon_1}$, we have

$$\left| \frac{\|d\exp_{\bar{m}} I(v)\|}{\|d\exp_m(v)\|} - 1 \right| < \frac{\rho(M, \bar{M})}{\varepsilon} < \varepsilon_2.$$

For any unit vector u on $\exp_m(G_m^{-\varepsilon_1})$, setting $w := d\exp_m^{-1}(u)$, we may rewrite $\|d\Phi\|$ as:

$$\|d\Phi(u)\| = \|d\exp_{\bar{m}} I(\frac{w}{\|w\|})\| / \|d\exp_m(\frac{w}{\|w\|})\|.$$

This implies the required estimate.

Q.E.D.

In order to obtain some more information about Φ , we require the model space M to have the following property.

Definition 15. The pair (M, m) is said to have property CM if for any $p, q \in M - C(m)$ and for any $\epsilon > 0$, there exists a curve h_ϵ from p to q which does not intersect $C(m)$ such that $L(h_\epsilon) < d(p, q) + \epsilon$. If (M, m) has property CM for all $m \in M$, then M is said to have property CM.

We show that if (M, m) has property CM, then every geodesic emanating from m minimizes up to its first conjugate point, in particular M must be simply connected. Otherwise, there is a geodesic emanating from m along which the cut point, say p , of m is not conjugate to m . Then some ball $B(p, r)$ is the diffeomorphic image by \exp_m , and is divided into two connected components by $C(m)$. For $q, q' \in B(p, r/2)$ in the distinct components, let h be a curve from q to q' which does not meet $C(m)$. Then it turns out that $L(h) > 2r > d(q, q') + r$. Hence (M, m) does not have property CM.

All geodesics will be assumed to be parametrized by arc length, and the diameter of the model space M will be denoted by D for simplicity.

Lemma 16. Let M have property CM. Then for a given ϵ there exists a $\delta > 0$ such that $\rho_0(M, \bar{M}) < \delta$ implies that $d(\Phi(p), \Phi(q)) < d(p, q) + \epsilon$ for all $p, q \in M$.

Proof. There are the unique minimizing geodesics γ, σ from m to p, q which are compatible with the choice of \exp_m^{-1} .

Set $p' := \gamma(d(m,p) - \varepsilon)$, $q' := \sigma(d(m,q) - \varepsilon)$. By the compactness of $\exp_m(\mathfrak{G}_m^{-\varepsilon})$, we may find an $\varepsilon' > 0$ such that for all $p'', q'' \in \exp_m(\mathfrak{G}_m^{-\varepsilon})$, there exists a curve h_ε from p'' to q'' such that

$$L(h_\varepsilon) < d(p'', q'') + \varepsilon, \quad d(h_\varepsilon, C(m)) > \varepsilon'.$$

By Lemma 14, we may choose a $\delta > 0$ such that $\rho_0(M, \bar{M}) < \delta$ implies

$$\begin{aligned} d(\Phi(p'), \Phi(q')) &\leq L(\Phi \cdot h_\varepsilon) < L(h_\varepsilon)(1 + \varepsilon/(D+1)) \\ &< d(p', q') + 3\varepsilon, \end{aligned}$$

and therefore,

$$\begin{aligned} d(\Phi(p), \Phi(q)) &\leq d(\Phi(p'), \Phi(q')) + 2\varepsilon \\ &< d(p', q') + 5\varepsilon \\ &< d(p, q) + 7\varepsilon. \end{aligned} \quad \text{Q.E.D.}$$

For completeness, we give the proof of the following lemma which was proved in [5] where the assumption $d(p, q) < R_0$ was not assumed. But it seems to us that the proof of the part is incomplete.

Lemma 17. For given $n, \Lambda, R, \varepsilon_1$, there exists a $\delta > 0$ such that for a given integer N there exists some $\eta > 0$ such that the following is true: Let M be a complete n -manifold such that $|K_M| \leq \Lambda^2$, $i(M) \geq R$ and let a subset $C \subset M$ admit a cover by balls $\{B(p_i, r_i)\}_{i=1, \dots, N}$ with $\sum_1^N r_i^{n-1} < \delta$. Then if $p, q \in M$, $d(p, C), d(q, C) > \varepsilon_1$, $d(p, q) < R_0$, then there exists a curve h from p to q such that

- (1) $L(h) < d(p, q) + \varepsilon_1$,
- (2) $d(h, C) > \eta$.

Proof. We denote by d_E the Euclidean distance on $B(p, R_0)$ with respect to a normal coordinate system at p . RCT implies

$$e^{-\Lambda s} \leq d(x, y)/d_E(x, y) \leq e^{-\Lambda s}$$

for all $x, y \in B(p, s)$.

Case 1). $d(p, q) < \varepsilon_1$. By the triangle inequality, the unique minimizing geodesic γ from p to q satisfies $d(\gamma, C) > \varepsilon_1/2$. Hence it suffices to set $h := \gamma$, $\eta = \varepsilon_1/2$.

Case 2). $R_0 > d(p, q) \geq \varepsilon_1$. Set $s := d(p, q)$. We denote by d_S the distance on the sphere $S = S(p, s)$ of radius s around p induced from d_E . Clearly

$$1 \leq d_S(x, y)/d_E(x, y) \leq \pi/2.$$

Let $\phi: B(p, s) - B(p, \varepsilon_1/2) \longrightarrow S$ be the radial projection from p . Then RCT implies the existence of $\Omega(\Lambda, s, \varepsilon_1) > 0$ such that

$$(*_3) \quad d_S(\phi(x), \phi(y)) < \Omega d(x, y).$$

Let A_t denote the Euclidean volume of a t -ball in S and \mathcal{Q} denote the Euclidean volume measure on S . Set $\varepsilon := \varepsilon_1/2$, $\delta := \frac{1}{2} A_\varepsilon / \Omega^{n-1}$. From the given balls $\{B(p_i, r_i)\}_{i=1, \dots, N}$, we choose such balls that intersect $C \cap B(p, s)$, say $B(p_1, r_1), \dots, B(p_k, r_k)$. Set

$$B := (B(p_1, r_1) \cup \dots \cup B(p_k, r_k)) \cap B(p, s).$$

Then $(*_3)$ yields

$$\begin{aligned} \mathcal{Q}(\phi(B)) &\leq \sum_{i=1}^k \mathcal{Q}(\phi(B(p_i, r_i) \cap B(p, s))) \\ &< \sum A(\Omega r_i) < \sum (\Omega r_i)^{n-1} < \frac{1}{2} A_\varepsilon. \end{aligned}$$

Then we assert that there exists an $\eta_1(n, N, \varepsilon, s) > 0$ independent of p_i, r_i such that there exists a $q' \in S \cap B(q, \varepsilon)$ with

$$d(q', \phi(C \cap B(p, s))) > d(q', \phi(B)) > \eta_1.$$

In order to see this, define a compact subset L of \mathbb{R}^N by

$$L := \{(t_1, \dots, t_N); 0 \leq t_i, \sum_1^N t_i^{n-1} \leq \frac{1}{2} A_\varepsilon\}.$$

Consider the following function $f: \underbrace{S \times \dots \times S}_N \times L \rightarrow [0, \pi s]$,

$$f(q_1, \dots, q_N, t_1, \dots, t_N) := \sup \left\{ d(x, \bigcup_1^N B(q_i, t_i)); x \in B^S(q, \varepsilon) \right\},$$

where B^S denotes a ball in S . Notice that $f > 0$. Since

$\phi(B(p_i, r_i) \cap B(p, s)) \subset B^S(\phi(p_i), \Omega r_i)$, it suffices to observe that f is continuous and hence takes the positive minimum $\eta_1(n, N, \varepsilon, s)$.

Now let γ be the minimizing geodesic from p to q' . Then we have

$$d(\gamma, C) > \min \{ \varepsilon, d(\gamma, C \cap B(p, s)) \} \geq \min \{ \varepsilon, \eta_1 / \Omega \} = \eta_1 / \Omega =: \eta.$$

Since $\varepsilon_1 \leq s < R_0$, we may choose η independent of s . Let σ be the minimizing geodesic from q' to q . Then the required curve h is given as the broken geodesic $\gamma \cup \sigma$. Q.E.D.

The previous lemma will be very useful in the case where the $n-1$ dimensional Hausdorff measure $H^{n-1}(C(m))$ of $C(m)$ is equal to zero.

Corollary 18. If $H^{n-1}(C(m)) = 0$, then (M, m) has property CM.

Proof. For given $p, q \in M - C(m)$, and $\varepsilon > 0$, we take a minimizing geodesic from p to q and choose $p_0 = p, p_1, \dots, p_k = q$

on the geodesic so that $d(p_i, p_{i+1}) < R_0$, $k < [D/R_0] + 1$.

Take $p'_i \in B(p_i, \varepsilon R_0/6D) - C(m)$, $i = 1, \dots, k-1$. For each pair (p'_i, p'_{i+1}) , by Lemma 17 there exists a curve h_i from p'_i to p'_{i+1} such that

$$L(h_i) < d(p'_i, p'_{i+1}) + \varepsilon R_0/4D, \quad h_i \cap C(m) = \emptyset.$$

Set $h := h_0 \cup h_1 \cup \dots \cup h_{k-1}$. Then we have

$$\begin{aligned} L(h) &< \sum d(p'_i, p'_{i+1}) + \varepsilon/3 \\ &< \sum (d(p_i, p_{i+1}) + \varepsilon R_0/3D) + \varepsilon/3 \\ &< d(p, q) + \varepsilon. \end{aligned} \quad \text{Q.E.D.}$$

It is well known that the cut locus $C(m)$ in a SCROSS is a submanifold of codimension ≥ 2 , in particular $H^{n-1}(C(m)) = 0$. More generally, it is also known in [5] that the following classes of manifolds satisfy $H^{n-1}(C(m)) = 0$:

- 1) simply connected symmetric spaces of the compact type,
- 2) simply connected manifolds with the property that all geodesics emanating from m have the first conjugate points of order ≥ 2 .

7. Uniqueness Theorems

In our pinching situation, we would like to show that Φ is an ε -map if \bar{M} is sufficiently close to M . For this, it will be needed that the tangent cut locus of m is close to that of \bar{m} . For this reason, we adopt a SCROSS as the model space M . A crucial property which M possesses is that $D = \text{diam}(M) = i(M)$.

We prepare an estimate for Jacobi fields.

Lemma 19. Let \bar{M} be a manifold with $|K_{\bar{M}}| \leq \Lambda^2$ and $J(t)$ a Jacobi field along a geodesic γ in \bar{M} such that $J(0) = 0$, $\|J'(0)\| = 1$. Then for $0 < a < b$, there exists an $\Omega(\Lambda, a, b) > 0$ such that

$$|(\|J(t)\|/t)'| \leq \Omega \quad \text{on } [a, b].$$

Proof. The estimate from the theory of ordinary differential equations as in [5], §2 implies the existence of $\Omega_1(\Lambda, b) > 0$ such that $\|J(t)\| \leq \Omega_1 t$ on $[0, b]$. Then the Jacobi equation: $J'' = R_{\bar{M}}(\gamma', J)\gamma'$ implies that on $[0, b]$

$$\begin{aligned} \|J'(t)\| &\leq \|J'(0)\| + \int_0^t \|J''(t)\| \, dt \\ &\leq 1 + \int_0^t \|R_{\bar{M}}\| \|J(t)\| \, dt \leq 1 + \Omega_1 \Lambda^2 t^2. \end{aligned}$$

Therefore we have

$$\begin{aligned} |(\|J(t)\|/t)'| &= |\langle J(t)/\|J(t)\|, J'(t) \rangle / t - t^{-2} \|J(t)\| | \\ &< \Omega_1 \Lambda^2 b + (1 + \Omega_1)/a =: \Omega. \quad \text{Q.E.D.} \end{aligned}$$

Lemma 20. Let M be a SCROSS. Then for given $\Lambda, \varepsilon > 0$, there exists a $\delta > 0$ such that $|K_{\bar{M}}| \leq \Lambda^2$, $\rho(M, \bar{M}) < \delta$ imply that $d(\bar{m}, C(\bar{m})) \geq D - \varepsilon$.

Proof. We may find a $\delta_1 > 0$ such that $\rho_0(M, \bar{M}) < \delta_1$ implies that $d \exp_{\bar{m}}$ is non-singular on $I(\bar{\zeta}_m^{-\varepsilon})$. Now suppose that $d(\bar{m}, C(\bar{m})) =: l < D - \varepsilon$. Then there exists a geodesic loop $\sigma: [0, 2l] \rightarrow \bar{M}$ at \bar{m} such that $\sigma(l) =: x \in C(\bar{m})$. We observe the influence of the existence of σ on the total volume of \bar{M} . For any $y \in C(\bar{m})$, let $\sigma_1: [0, l_1] \rightarrow \bar{M}$ be a minimizing geodesic from \bar{m} to y . Set $\theta := \angle(\dot{\sigma}(0), \dot{\sigma}_1(0))$, $l_2 := d(x, y)$. Then the Toponogov comparison theorem implies

$$\cosh \Lambda l_2 \leq \cosh \Lambda l \cosh \Lambda l_1 - \sinh \Lambda l \sinh \Lambda l_1 \cos \theta.$$

If τ denotes a minimizing geodesic from x to y , then we have immediately

$$\angle(\dot{\tau}(0), \dot{\sigma}(l)) \leq \pi/2, \text{ or } \angle(\dot{\tau}(0), -\dot{\sigma}(l)) \leq \pi/2.$$

Hence the Toponogov comparison theorem implies again

$$\cosh \Lambda l_1 \leq \cosh \Lambda l \cosh \Lambda l_2.$$

The above inequalities yield $\cos \theta \leq \coth \Lambda l_1 \tanh \Lambda l$.

Now if $l_1 \geq D - \varepsilon/2$, then

$$\cos \theta < \coth \Lambda(D - \varepsilon/2) \tanh \Lambda(D - \varepsilon) < 1.$$

Define $\theta_0 = \theta_0(\Lambda, \varepsilon)$ by $\cos \theta_0 = \coth \Lambda(D - \varepsilon/2) \tanh \Lambda(D - \varepsilon)$ and set

$$C := \{v \in M_{\bar{m}}; \angle(v, I^{-1}(\dot{\sigma}(0))) \leq \theta_0, D - \varepsilon/2 \leq \|v\| \leq D\}.$$

We have just verified that $I(C)$ does not meet $\bar{\zeta}_m$. Set

$v(\varepsilon) = \text{Vol}(\exp_m(C))$. Then we may find an $\varepsilon_1 > 0$ such that

$$\text{Vol}(\exp_m(\mathcal{G}_m^{\varepsilon_1} - \mathcal{G}_m^{-\varepsilon_1})) < \frac{1}{4} v(\varepsilon).$$

Necessarily $\varepsilon_1 < \varepsilon$. By Lemma 19 we may choose a $\delta_2 > 0$ such that $\rho_0(M, \bar{M}) < \delta_2$ implies

$$\text{Vol}(\exp_{\bar{m}} \circ I(\mathcal{G}_m^{\varepsilon_1} - \mathcal{G}_m^{-\varepsilon_1})) < \frac{1}{3} v(\varepsilon).$$

Taking δ_2 smaller if necessary, we may assume that

$$\text{Vol}(\exp_{\bar{m}} \circ I(\mathcal{G}_m^{-\varepsilon_1} - C)) < \text{Vol}(\exp_m(\mathcal{G}_m^{-\varepsilon_1} - C)) + \frac{1}{3} v(\varepsilon).$$

Now if $\text{diam}(\bar{M}) < D + \varepsilon_1$, then $\mathcal{G}_{\bar{m}} \subset I(\mathcal{G}_m^{\varepsilon_1})$. Hence we get

$$\begin{aligned} \text{Vol}(\bar{M}) &< \text{Vol}(\exp_{\bar{m}} \circ I(\mathcal{G}_m^{-\varepsilon_1} - C) \cup \exp_{\bar{m}} \circ I(\mathcal{G}_m^{\varepsilon_1} - \mathcal{G}_m^{-\varepsilon_1})) \\ &< \text{Vol}(\exp_{\bar{m}} \circ I(\mathcal{G}_m^{-\varepsilon_1} - C)) + \text{Vol}(\exp_{\bar{m}} \circ I(\mathcal{G}_m^{\varepsilon_1} - \mathcal{G}_m^{-\varepsilon_1})) \\ &< \text{Vol}(\exp_m(\mathcal{G}_m^{-\varepsilon_1} - C)) + \frac{2}{3} v(\varepsilon) \\ &< \text{Vol}(M) - \frac{1}{3} v(\varepsilon). \end{aligned}$$

Therefore the required δ is obtained as:

$$\delta = \min \{ \delta_1, \delta_2, \varepsilon_1, \frac{1}{3} v(\varepsilon) \}. \quad \text{Q.E.D.}$$

Lemma 21. Let M be a SCROSS. Then for given $\Lambda, \varepsilon > 0$, there exists a $\delta > 0$ such that $|K_{\bar{M}}| \leq \Lambda^2$ and $\rho(M, \bar{M}) < \delta$ implies that Φ is an ε -map.

Proof. Since we may assume that $\text{diam}(\bar{M}) < 2D$, $\text{Vol}(\bar{M}) > \frac{1}{2} \text{Vol}(M)$, Cheeger's injectivity radius estimate [6] shows that there exists an R independent of \bar{M} such that $i(\bar{M}) \geq R$. For n, Λ, R and $\varepsilon_1 := \varepsilon / (30(\lfloor D/R_0 \rfloor + 1))$, let δ_1 be as in Lemma 17. Since

$H^{n-1}(C(m)) = 0$, there is a covering $\{B(p_i, r_i)\}_{i=1, \dots, N}$ of $C(m)$ such that $\sum_1^N (2r_i)^{n-1} < \delta_1$. Let η be the constant given in Lemma 17. Set $r := \min\{r_i; 1 \leq i \leq N\}$. By Lemma 17, we may find $\delta > 0$ such that $\rho(M, \bar{M}) < \delta$ implies that $d(\Phi(p), \Phi(q)) < d(p, q) + r/2$ for all $p, q \in M$. On the other hand, by Lemma 20 we may assume that

$$\partial \mathcal{G}_{\bar{m}} \subset B(0, D+r/4) - B(0, D-r/4) \subset \bar{M}_{\bar{m}}.$$

Hence we can conclude that the balls $\{B(\Phi(p_i), 2r_i)\}_{i=1, \dots, N}$ cover $C(m)$. Now for any $p, q \in M$, take the points $q_0 = \Phi(p)$, $q_1, \dots, q_k = \Phi(q)$ on a minimizing geodesic from $\Phi(p)$ to $\Phi(q)$ such that $d(q_i, q_{i+1}) < R_0$, $k \leq [D/R_0] + 1$. Let γ_0, γ_k be the minimizing geodesics from m to p, q which are compatible with the choice of \exp_m^{-1} . Set $\bar{\gamma}_0 := \Phi \circ \gamma_0$, $\bar{\gamma}_k := \Phi \circ \gamma_k$. Let $\bar{\gamma}_i: [0, l_i] \rightarrow \bar{M}$ be a minimizing geodesic from \bar{m} to q_i , $1 \leq i \leq k-1$. Set $q'_i := \bar{\gamma}_i(l_i - 2\varepsilon_1)$. Choosing δ smaller if necessary, we may assume that $d(q'_i, C(\bar{m})) > \varepsilon_1$. Hence by Lemma 17 there exists a curve h_i from q'_i to q'_{i+1} such that

$$L(h_i) < d(q'_i, q'_{i+1}) + \varepsilon_1, \quad d(h_i, C(\bar{m})) > \eta.$$

We may assume that $d(\bar{m}, C(\bar{m})) \geq D - \eta$, and that by Lemma 14, for $\varepsilon_2 := \varepsilon/(2(D+R_0))$, $L(\Phi^{-1} h_i) < (1 + \varepsilon_2)L(h_i)$. It follows that

$$\begin{aligned} d(p, q) &\leq 4\varepsilon_1 + \sum_0^{k-1} L(\Phi^{-1} h_i) \\ &< 4\varepsilon_1 + \sum (1 + \varepsilon_2)L(h_i) \\ &< 4\varepsilon_1 + \sum (1 + \varepsilon_2)(d(q'_i, q'_{i+1}) + \varepsilon_1) \\ &< 4\varepsilon_1 + \sum (1 + \varepsilon_2)(d(q_i, q_{i+1}) + 5\varepsilon_1) \\ &< d(\Phi(p), \Phi(q)) + \varepsilon. \end{aligned}$$

Together with Lemma 16, this completes the proof.

Q.E.D.

Theorem 2 is an immediate consequence of ε -mapping Theorem 11 and Lemma 21.

We denote by \mathcal{M}_C^n the class of compact n -manifolds M such that $H^{n-1}(C(m)) = 0$ for all $m \in M$. It contains simply connected symmetric spaces of the compact type, and simply connected manifolds with the property that all geodesics have the first conjugate points of order ≥ 2 , as stated before.

Theorem 22. Let $M \in \mathcal{M}_C^n$. If \bar{M} satisfies that $\rho_0(M, \bar{M}) = 0$, $\text{Vol}(M) = \text{Vol}(\bar{M})$, then \bar{M} is isometric to M .

Proof. Notice that $\Phi|_{M-C(m)}$ is a local isometry and that $\mathcal{G}_{\bar{m}} \subset I(\mathcal{G}_m)$. But the assumption $\text{Vol}(M) = \text{Vol}(\bar{M})$ implies that $\mathcal{G}_{\bar{m}} = I(\mathcal{G}_m)$. Hence by Lemma 21, $\Phi: M \rightarrow \bar{M}$ is a "0-map", that is an isometry. Q.E.D.

Here we consider more general model spaces than SCROSSes. In this cases, of course, a more strict pinching will be needed.

Lemma 23. Let $M \in \mathcal{M}_C^n$. Then for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $\tilde{\rho}(M, \bar{M}) + |\text{Vol}(M) - \text{Vol}(\bar{M})| < \delta$ implies that Φ is an ε -map.

Proof. Let $x \in \mathcal{G}_m$, and v_1, \dots, v_{n-1} an orthonormal basis for the orthogonal complement to the radial line at x . Notice that

$$\text{Jacobian of } (d \exp_m)|_x = \|d \exp_m(v_1) \wedge \dots \wedge d \exp_m(v_{n-1})\|,$$

Jacobian of $(d \exp_{\bar{m}})|_{I(x)} = \|d \exp_{\bar{m}} I(v_1) \wedge \dots \wedge d \exp_{\bar{m}} I(v_{n-1})\|$.

Now for a given $\varepsilon_1 > 0$, by compactness, there exist some $A > B > 0$, $\pi > \alpha > \beta > 0$ such that

$$A \geq \|d \exp_m(v_i)\| \geq B, \quad \alpha \geq \angle(d \exp_m(v_i), d \exp_m(v_j)) \geq \beta \quad \text{on } \mathcal{G}_m^{-\varepsilon_1}.$$

On the other hand, by Lemma 3.2 in [5], for a given ε there exists $\delta > 0$ such that $\tilde{\rho}(M, \bar{M}) < \delta$ implies that $\mathcal{G}_{\bar{m}} \subset I(\mathcal{G}_m^\varepsilon)$. We may assume that $|K_{\bar{M}}| \leq 2 \max |K_M|$. Therefore we may find $\delta > 0$ in the same way as in Lemma 20 such that

$$\tilde{\rho}(M, \bar{M}) + |Vol(M) - Vol(\bar{M})| < \delta \quad \text{implies} \quad \partial \mathcal{G}_{\bar{m}} \subset I(\mathcal{G}_m^\varepsilon - \mathcal{G}_m^{-\varepsilon}).$$

Thus the proof will complete in the same way as in Lemma 21. Q.E.D.

Together with ε -mapping Theorem 11, we have just proved the following

Theorem 24. Let $M \in \mathcal{M}_C^n$. Then for a given $\Lambda_1 > 0$, there exists an $\varepsilon > 0$ such that $\|\nabla R_{\bar{M}}\| \leq \Lambda_1$ and $\tilde{\rho}(M, \bar{M}) + |Vol(M) - Vol(\bar{M})| < \varepsilon$ implies that \bar{M} is diffeomorphic to M .

Final remark. If the assumption for $\|\nabla R\|$ in Theorem 1 in Chapter I could be removed, then the parameter Λ_1 will be negligible.

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