CONTRIBUTIONS TO PRIME NUMBER THEORY

—— GAPS BETWEEN PRIMES ——

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THESIS

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1. Introduction.

In this memoir we consider the problems surrounding three old conjectures in prime number theory.

(A) There exist infinitely many prime twins.

(B) There exists a prime in the intervals $[x, x + c\log^2 x]$ for all x.

(C) The least prime in an arithemetic progression to modulus q is less than $cq\log^2 q$ for all reduced residue classes.

Here c is some positive constant. These are, at present, far from our reach, however. We have never known how one of these may be deduced from plausible hypotheses.

For a positive integer k, let

$$\Psi(x,2k) = \sum_{2k<n\le x} \Lambda(n)\Lambda(n-2k)$$

where Λ is the von Mangoldt function. Essentially Ψ counts the prime pairs. In 1923 G.H.Hardy and J.E.Littlewood [10] conjectured that Ψ would be asymptotically equal to

$$H(x,2k) = 2 \prod_{p>2} \left(1-\frac{1}{(p-1)^2}\right) \prod_{2<p|k} \left(\frac{p-1}{p-2}\right)\cdot(x-2k).$$

After I.M.Vinogradov [37] it has been well known that this is true in a mean value sense. [1,21].

Recently D.Wolke [39] demonstrated that, for any constant $A>0$,

$$\sum_{k\leq x} (\Psi(y,2k)-H(y,2k))^2 \ll xy^2\log^{-A}y$$

providing

$$2x \leq y \leq x^{8/5-\varepsilon} \quad (\varepsilon>0).$$

This means that the formula

$$E(y,2k) = \Psi(y,2k)-H(y,2k) \ll y\log^{-B}y$$

is valid for "almost all" $k \leq x$. He is the first to extend the range of validity to $[2x, x^{\theta}]$ with $\theta>1$. Moreover he remarked that if the density hypothesis for L-series is true, then the exponent $8/5$ could be replaced by 2.

Our main attempt is to improve this exponent beyond 2.

THEOREM 1.

Let ε and $A>0$ be given. If $2x \leq y \leq x^{3-\varepsilon}$, then we have

$$\sum_{k\leq x} E(y,2k)^2 \ll xy^2\log^{-A}y$$

where the implied constant depends only on ε and A.

THEOREM 2.

Let ε and $A>0$ be given. If $x^{1/6+\varepsilon} \leq y \leq 2x$, then we have

$$\sum_{k\leq x} E(2k+y,2k)^2 \ll xy^2\log^{-A}x$$

where the implied constant depends only on ε and A.

This may be regarded as dual version of [39], and also of Theorem 1. It is of some interest to compare Theorem 2 with the result of

M.N.Huxley [17]: if $y>x^{1/6+\varepsilon}$, then

$$\int_1^x \Big(\sum_{t<n\le t+y} \Lambda(n)-y \Big)^2 dt \ll xy^2\log^{-A}x.$$

In 1943 A.Selberg [34] showed, under the Riemann hypothesis, that there exists a prime in the intervals

$$[\ n\ ,\ n + g(n)\log^2 n\ ]$$

for almost all n. Here "almost all" means that the number of exceptional n's not exceeding x is $o(x)$, and g is any positive function with $g(x)\to\infty$ as $x\to\infty$. It follows from the above mentioned formula of Huxley that, unconditionally, the intervals $[\ n,\ n+n^{1/6+\varepsilon}]$ contain a prime for almost all n.

Let P_r denote integers with at most r prime factors counted according to multiplicity. Several authors considered the analogous problem for P_2. D.R.Heath-Brown [12] proved that there exists a P_2 in the intervals

$$[\ n,\ n + n^{1/11+\varepsilon}] \quad (\varepsilon>0)$$

for almost all n. Y.Motohashi [30] replaced the exponent $1/11+\varepsilon$ by ε, by a simple analytic trick. D.Wolke [38] reduced the length of intervals to the powers of $\log n$. By refining Wolke's argument, G.Harman [11] showed that the intervals $[\ n,\ n + \log^{7+\varepsilon} n\ ]$ contain a P_2 for almost all n.

We take the alternative approach to give a modest improvement.

THEOREM 3.

There exists a P_2 in the intervals

$$[\ n,\ n + g(n)\log^5 n\ ]$$

for almost all n.

In 1936 P.Turán [35] showed, under the generalized Riemann hypothesis, that there exists a prime p such that

$$p \equiv a \pmod q, \quad p \le q\log^{2+\varepsilon} q \quad (\varepsilon>0)$$

for almost all reduced residue classes a modulo q. The terminology "almost all" means that the number of exceptional reduced classes modulo q is $o(\varphi(q))$ as $q\to\infty$.

Motohashi [28] considered the corresponding problem for P_2, and showed that there exists a P_2 such that

$$P_2 \equiv a \pmod q, \quad P_2 \le q^{11/10}$$

for almost all a. He remarked that, if the q-analogue of Lindelöf hypothesis is true, then the exponent $11/10$ could be replaced by $1+\varepsilon$, $\varepsilon>0$. Moreover he in [30] noted that his trick [30] does not work in this case.

In contrast, our method in the proof of Theorem 3 makes an improvement upon this problem as well.

THEOREM 4.

There exists a P_2 such that

$$P_2 \equiv a \pmod q, \quad P_2 \le g(q)q\log^5 q$$

for almost all reduced classes a modulo q.

In 1975 Motohashi [29] proved that there exists a P_3 such that

$$P_3 \equiv a \pmod q, \quad P_3 \ll q\log^{70} q$$

for any fixed non-zero integer a and "almost all" q with $(q,a)=1$.

This is the dual problem of [28], and also of Theorem 4.

The generalized Riemann hypothesis is not capable of showing the existence of a prime p with

$$p \equiv a \pmod q, \quad p \le q^2,$$

even in an averaged sense over q. So the above bound for P_3 is interesting.

Our final task is to extend this result to that for P_2.

THEOREM 5.

Let Q be a large parameter and a be any fixed integer with $0<|a|\le Q$. Then, except for $O(Q\log^{-1}Q)$ moduli q with $Q < q \le 2Q$ and $(q,a)=1$, there exists a P_2 such that

$$P_2 \equiv a \pmod q, \quad P_2 \le \tau(a) q \log^7 q$$

where the implied O-constant is absolute and τ denotes the divisor function.

Our notation and convention are standard. For example, $\bar{r}$, used in either $\bar{r}/s$ or congruence (mod.s) means $\bar{r}r \equiv 1$ (mod.s). $*$ in $\sum^*_{1\le x\le y}$ stands for the restriction $(x,y)=1$. ε is any small positeve number. A is any large constant. Both are not necessarily the same at each occasion. The implied constant in O- and $\ll$- notation may depend on ε and A, when these are appeared in the symbol. For real x, $e(x)= e^{2\pi i x}$, $\psi(x)= [x]-x+\frac{1}{2}$ and $\|x\| = \min_{n\in\mathbb{Z}} |x-n|$. $n\sim N$ means that $N\le N_1<n\le N_2\le 2N$ for some N_1 and N_2. C is some absolute positive constant.

"Theorems" come from the results of my study during the last five years and are found in [22-25]. I would like to thank Professor Saburô Uchiyama for continual encouragement and great generosity with my stubbornness. I would also like to thank the authors whose published articles had inspired my work. I am very glad to express my gratitude to teachers Takeshi Kano at Okayama and Akio Miyai at Morioka.

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2. Analytic method.

In this section we consider the distribution of primes in short arithmetic progressions. We appeal to the classical theory of L-functions. See [6], for example. Throughout this section we assume

$$q \le \log^C N, \quad N^{1/6+2\varepsilon} \le \Delta \le N^{1-2\varepsilon}.$$

LEMMA 1 [32, Kap.IV.Satz 8.2, Kap.VIII.Satz 6.2.].

There is no zero of $L(s,\chi)$, $s=\sigma+it$, $\chi(\mathrm{mod.}q)$, in the region

$$\sigma \ge 1-c_0\log^{-4/5}x, \quad |t| \le x$$

where c_0 is some positive constant.

Let $N(\alpha,T,\chi)$ denote the number of zeros of $L(s,\chi)$, $s=\sigma+it$, $\chi(\mathrm{mod.}q)$, in the rectangle $\alpha \le \sigma < 1$, $|t| \le T$.

LEMMA 2 [26, Chap.12] [18].

Suppose $b \ge 12/5+\delta$ for any $\delta>0$. Then,

$$\sum_{\chi(\mathrm{mod.}q)} N(\alpha,T,\chi) \ll (qT)^{b(1-\alpha)}\log^{14}qT.$$

LEMMA 3.

Let $T=N/q\Delta$. For any constant A,

$$\sup_{\alpha\ge 1/2} N^{2(\alpha-1)} \sum_{\chi(\mathrm{mod.}q)} N(\alpha,T,\chi) \ll \log^{-A}N.$$

Proof.

By Lemma 1, $N(\alpha,T,\chi)=0$ for $\alpha \ge \alpha_0 = 1-c_0\log^{-4/5}T$. So the supremum may be taken over $1/2 \le \alpha \le \alpha_0$. By Lemma 2, we have

$$N^{2(\alpha-1)} \sum_{\chi(\mathrm{mod.}q)} N(\alpha,T,\chi) \ll N^{2(\alpha-1)}(qT)^{b(1-\alpha)}\log^{14}qT$$
$$\ll N^{-\varepsilon'(1-\alpha)}\log^{14}N \quad (\varepsilon'>0)$$

since

$$\frac{1}{b}+2\varepsilon > 1-\frac{2}{12/5+\varepsilon}+\varepsilon \geq 1-\frac{2}{b}+\varepsilon,$$

also

$$(qT)^b=\left(\frac{qN}{q\Delta}\right)^b < \left(\frac{N}{N^{1-2/b+\varepsilon}}\right)^b = (N^{2/b-\varepsilon})^b = N^{2-\varepsilon'}.$$

Then, the supremum is attained at $\alpha=\alpha_0$. Hence the expression in question is

$$\ll \exp(-\varepsilon''\log^{1/5}N)\log^{14}N$$
$$\ll \log^{-A}N.$$

Here we use the convention; # in $\sum_n^{\#}\chi(n)\Lambda(n)$ means that when χ is principal $\chi(n)\Lambda(n)$ is replaced by $\Lambda(n)-1$.

LEMMA 4.

$$\mathcal{J} = \sum_{\chi(\mathrm{mod}.q)} \int_N^{2N} \Big| \sum_{t<n\leq t+q\Delta}^{\#} \chi(n)\Lambda(n)\Big|^2 dt \ll (qT)^2 N\log^{-A}N.$$

Proof.

By the explicit formula [32, Kap.Ⅶ.Satz 4.6.] we have

$$\sum_{t<n\leq t+q\Delta}^{\#} \chi(n)\Lambda(n) = \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma|\leq N}} \frac{(t+q\Delta)^{\rho}-t^{\rho}}{\rho} + O(\log^2 N)$$

$$= \sum_{|\gamma|\leq T} t^{\rho}\int_1^{1+q\Delta/N} y^{\rho-1}dy + \sum_{j=0,1}(-1)^{j+1}\sum_{T<|\gamma|\leq N}\frac{(t+jq\Delta)^{\rho}}{\rho} + O(\log^2 N)$$

$$= \Sigma_1 + \Sigma_2 + O(\log^2 N), \text{ say,}$$

where $\rho=\rho_\chi=\beta_\chi+i\gamma_\chi=\beta+i\gamma$ runs over the non-trivial zeros of $L(s,\chi)$, and $T=N/q\Delta$. First the contribution of O-term is

$$\ll qN\log^4 N,$$

which is negligible. Next we consider the contribution of Σ_1 to $\mathcal{J}$. Since

$$|\Sigma_1| \le \int_1^{1+q\Delta/N} \Big| \sum_{|\gamma|\le T} t^{\rho} y^{\rho-1} \Big| dy$$

or

$$|\Sigma_1|^2 \le T^{-1} \int_1^{1+T^{-1}} \Big| \sum_{|\gamma|\le T} t^{\rho} y^{\rho-1} \Big|^2 dy,$$

we infer that

$$\int_N^{2N} |\Sigma_1|^2 dt << T^{-2} \sup_{1\le y\le 1+T^{-1}} \int_N^{2N} \Big| \sum_{|\gamma|\le T} t^{\rho} y^{\rho-1} \Big|^2 dt$$

(1) $$= T^{-2} \sup_y I_y, \text{ say.}$$

We proceed to the estimation of I_y. Expanding the spuare we have

$$I_y \le \sum_{|\gamma'|,} \sum_{|\gamma|\le T} |y^{\rho'+\bar{\rho}-2}| \Big| \int_N^{2N} t^{\rho'+\bar{\rho}} dt \Big|$$

$$<< \sum_{|\gamma|\le T} N^{1+2\beta} \sum_{|\gamma'|\le T} \frac{1}{1+|\gamma'-\gamma|} .$$

Since $\sum_{x<\gamma\le x+1} 1 << \log x$, we see

$$\sum_{|\gamma'|\le T} \frac{1}{1+|\gamma'-\gamma|} << \log^2 T.$$

Thus,

$$I_y << \log^2 T \sum_{\substack{\beta\ge 1/2 \\ |\gamma|\le T}} N^{1+2\beta} + N^2 \log^2 N \cdot T \log T.$$

Combining this with (1), we obtain

$$\sum_{\chi(\mathrm{mod}\ q)} \int_N^{2N} |\Sigma_1|^2 dt$$

$$\ll T^{-2}\log^2 N \sum_{\chi(q)} \sum_{\substack{\beta_\chi \geq 1/2 \\ |\gamma_\chi| \leq T}} N^{1+2\beta_\chi} + qT^{-2}N^2 \log^3 N\cdot T$$

$$\ll T^{-2}\log^2 N \int_{1/2}^{1} N^{1+2\sigma}\, d_\sigma\Big(\sum_{\chi}\sum_{\beta_\chi \geq \sigma} 1\Big) + q^2\Delta\, N\log^3 N$$

$$\ll \left(\frac{q\Delta}{N}\right)^2 \log^3 N\cdot \sup_{\sigma\geq 1/2} N^{1+2\sigma} \sum_{\chi(q)} N(\sigma,T,\chi) + (q\Delta)^2 N\Delta^{-1}\log^3 N$$

$$\ll (q\Delta)^2 N\log^3 N \{ \sup_{\sigma\geq 1/2} N^{2(\sigma-1)} \sum_{\chi(q)} N(\sigma,T,\chi) + \Delta^{-1}\}.$$

Hence, by Lemma 3, we have that Σ_1 contributes to $\mathcal{J}$

$$\ll (q\Delta)^2 N \log^{-A} N.$$

Finally we consider Σ_2. By a similar argument to Σ_1, we get the following chain of inequalities.

$$\int_N^{2N} |\Sigma_2|^2 dt \ll \int_N^{3N} \Big| \sum_{T<|\gamma|\leq N} \frac{t^\rho}{\rho} \Big|^2 dt$$

$$\ll \sum_{T<|\gamma'|,\,|\gamma|\leq N} \frac{1}{|\rho'||\rho|} \Big| \int_N^{3N} t^{\rho'+\bar{\rho}} dt \Big|$$

$$\ll \log^2 N \sum_{\substack{\beta\geq 1/2 \\ T<|\gamma|\leq N}} \frac{N^{1+2\beta}}{|\rho|^2} + T^{-1}N^2\log^4 N$$

or

$$\sum_{\chi(\mathrm{mod}\ q)} \int_N^{2N} |\Sigma_2|^2 dt \ll \log^3 N \sup_{\sigma\geq 1/2} N^{1+2\sigma} \sum_{\chi(q)} \sum_{\substack{\beta_\chi \geq \sigma \\ T<|\gamma_\chi|\leq N}} |\gamma|^{-2} + q^2\Delta\, N\log^4 N$$

$$= \log^3 N \sup_{\sigma\geq 1/2} N^{1+2\sigma} S_\sigma + q^2\Delta\, N\log^4 N, \text{ say.}$$

Here,

$$S_\sigma \ll \log N \sup_{T<T_1<N} T_1^{-2} \sum_{\chi(q)} \sum_{\substack{|\gamma_\chi|\sim T_1 \\ \beta_\chi \geq \sigma}} 1$$

$$\ll \log N \sup_{T<T_1<N} T_1^{-2} \sum_{\chi(q)} N(\sigma,T_1,\chi).$$

Since $N(\sigma,T,\chi) \ll T \log T$, the supremum is attained at $T_1=T$. Thus, by Lemma 3, the contribution of Σ_2 to $\mathcal{J}$ is

$$\ll \log^4 N\ T^{-2} \sup N^{1+2\sigma} \sum_{\chi(q)} N(\sigma,T,\chi) + q^2\Delta\ N\log^4 N$$

$$\ll (q\Delta)^2 N\log^4 N \{ \sup N^{2(\sigma-1)} \sum_{\chi(q)} N(\sigma,T,\chi) + \Delta^{-1}\}$$

$$\ll (q\Delta)^2 N \log^{-A} N,$$

as required.

3. Circle method.

In this section we evaluate the contribution from major arc. We begin modifying [5, Lemma 1]. For complex (c_n), $I=(a,b]$ and $2< \Delta < b/2$, define

$$\mathfrak{L} = \Delta^2 \int_{-1/2\Delta}^{1/2\Delta} \Big|\sum_{n\in I} c_n e(\beta n)\Big|^2 d\beta .$$

LEMMA 5.

$$\mathfrak{L} \ll \Delta \sum_{n\in I} |c_n|^2 + 2\mathrm{Re} \sum_{0<r\le\Delta} (\Delta-r) \sum_{n,n+r\in I} \bar{c}_n c_{n+r} .$$

Proof.

Put $K^{\Delta}(x)=(\frac{\sin \pi\Delta x}{\pi x})^2$, then

$$\int_{-\infty}^{\infty} K^{\Delta}(x)e(xy)dx = \max(0,\ \Delta-|n_1-n_2|).$$

Since $K^{\Delta}(x) \gg \Delta^2$ if $|x|\le 1/2\Delta$, we have

$$\mathfrak{L} \ll \int_{-\infty}^{\infty} K^{\Delta}(\beta)\Big|\sum_{n\in I} c_n e(\beta n)\Big|^2 d\beta$$

(1) $$= \sum_{n',n\in I}\sum c_{n'}\bar{c}_n \max(0,\ \Delta-|n'-n|).$$

Writing $n'=n+r$, we get Lemma 5.

LEMMA 6.

$$\mathfrak{L} \ll \int_I \Big|\sum_{t<n\le t+\Delta} c_n\Big|^2 dt + \Delta^3 (\sup_{n\in I} |c_n|)^2 .$$

Proof. Put

$$\mathbb{1}(t,n)=\begin{cases} 1, & t<n\le t+\Delta \\ 0, & \text{otherwise}\end{cases}$$

then, we see

$$\max(0,\ \Delta-|n'-n|)= \int_{-\infty}^{\infty} \mathbb{1}(t,n')\mathbb{1}(t,n)dt.$$

Combining this with (1), we obtain

$$\mathfrak{L} \ll \int_{-\infty}^{\infty} \Big| \sum_{n\in I} c_n \mathbb{1}(t,n) \Big|^2 dt$$

$$= \left(\int_{a-\Delta}^{a} + \int_{a}^{b-\Delta} + \int_{b-\Delta}^{b} \right) \Big| \sum_{\substack{a<n\le b \\ t<n\le t+\Delta}} c_n \Big|^2 dt$$

$$\ll \int_I \Big| \sum_{t<n\le t+\Delta} c_n \Big|^2 dt + \Delta^3 \big(\sup_{n\in I} |c_n| \big)^2 .$$

Next, by using Lemma 6, we estimate the remainder term from major arc. This method is due to [27]. The following Lemma is new and competent for saving the order of Farey dissection.

LEMMA 7.

Suppose $q \le \log^C x$ and $x^{1/6+\varepsilon} \le \Delta \le x^{1-\varepsilon}$. For any constant A,

$$\mathcal{J} = \sum_{a=1}^{q}{}^{*} \int_{-1/q\Delta}^{1/q\Delta} \Big| \sum_{n\le x} \Lambda(n) e(\tfrac{a}{q}n) e(\beta n) - \frac{\mu(q)}{\varphi(q)} \sum_{n\le x} e(\beta n) \Big|^2 d\beta \ll x \log^{-A} x .$$

Proof.

We replace the summation condition $n\le x$ in $\mathcal{J}$ by $n\sim N$ and denote the resulting expression by $\mathcal{J}_N$. Then,

$$\mathcal{J} \ll \log^2 N \sup_{N\le x} \mathcal{J}_N .$$

Trivially,

$$\mathcal{J}_N \ll \int_{1/q\Delta}^{1+1/q\Delta} \Big| \sum_{n\sim N} \Lambda(n) e(\alpha n) \Big|^2 d\alpha + \frac{\mu^2(q)}{\varphi(q)} \int_{-1/2}^{1/2} \Big| \sum_{n\sim N} e(\beta n) \Big|^2 d\beta$$

$$\ll N \log N .$$

Thus, Lemma follows from the inequality

$$\mathcal{J}_N \ll N \log^{-A} N$$

for $N > x^{1-\varepsilon'}$. In this case, $N^{1/6+\varepsilon'} \le \Delta \le N^{1-\varepsilon'}$.

Now,

$$\sum_{n\sim N} \Lambda(n)e(\frac{a}{q}n)e(\beta n)$$

$$= \frac{1}{\varphi(q)} \sum_{\chi(\mathrm{mod}.q)} \chi(a)\tau(\overline{\chi}) \sum_{n\sim N} \chi(n)\Lambda(n)e(\beta n) + O(\log^2 N)$$

$$= \frac{\mu(q)}{\varphi(q)}\sum_{n\sim N} e(\beta n) + \frac{\mu(q)}{\varphi(q)}\sum_{n\sim N} (\Lambda(n)-1)e(\beta n) +$$

$$+ \frac{1}{\varphi(q)}\sum_{\chi\ne\chi_0(\mathrm{mod}.q)} \chi(a)\tau(\overline{\chi}) \sum_{n\sim N} \chi(n)\Lambda(n)e(\beta n) + O(\log^2 N).$$

The integrand of $\mathcal{J}_N$ is equal to

$$\frac{1}{\varphi(q)}\sum_{\chi(\mathrm{mod}.q)} \chi(a)\tau(\overline{\chi}) \sideset{}{^\#}\sum_{n\sim N}\chi(n)\Lambda(n)e(\beta n) + O(\log^2 N).$$

By the orthogonal relation of characters,

$$\mathcal{J}_N << \frac{1}{\varphi^2(q)} \sum_{\chi(q)} \varphi(q)|\tau(\overline{\chi})|^2 \int_{-1/q\Delta}^{1/q\Delta} |\sideset{}{^\#}\sum_{n\sim N}\chi(n)\Lambda(n)e(\beta n)|^2 d\beta + \Delta^{-1}\log^4 N.$$

Thus, by Lemmas 6 and 4, we get

$$\mathcal{J}_N << \frac{q}{\varphi(q)}\sum_{\chi(q)} \{(q\Delta)^{-2} \int_N^{2N} | \sideset{}{^\#}\sum_{t<n\le t+q\Delta/2} \chi(n)\Lambda(n)|^2 dt + q\Delta\log^2 N \} + \Delta^{-1}\log^4 N$$

$$<< \frac{q}{\varphi(q)}(q\Delta)^{-2} \sum_{\chi(q)} \int_N^{2N} | \sideset{}{^\#}\sum_{t<n\le t+q\Delta/2} \chi(n)\Lambda(n)|^2 dt + q^2\Delta \log^3 N$$

$$<< N \log^{-A} N,$$

as required.

Finally we evaluate the main contribution from major arc.

LEMMA 8.

$$\sum_{a=1}^{q}{}^{*}\int_{-1/q\Delta}^{1/q\Delta}\left|\frac{\mu(q)}{\varphi(q)}\sum_{n\le x}e(\beta n)\right|^2 e\left(-2k\left(\frac{a}{q}+\beta\right)\right)d\beta$$
$$= \frac{\mu^2(q)}{\varphi^2(q)}\, c_q(-2k)(x-2k)+ O\left(\frac{q\Delta}{\varphi(q)}\right).$$

Proof. The left hand side is equal to

$$\frac{\mu^2(q)}{\varphi^2(q)}c_q(-2k)\int_{-1/q\Delta}^{1/q\Delta}\left|\sum_{n\le x}e(\beta n)\right|^2 e(-2k\beta)\,d\beta$$

$$= \frac{\mu^2(q)}{\varphi^2(q)}c_q(-2k)\Big\{\sum_{\substack{n',n\le x\\ n'-n=2k}} 1 + O(q\Delta)\Big\}$$

$$= \frac{\mu^2(q)}{\varphi^2(q)}c_q(-2k)(x-2k)+ O\left(\frac{q\Delta}{\varphi(q)}\right).$$

4. Trigonometric sums over primes.

This section is devoted to estimate a mean value for the exponential sums. Throughout this section, apart from Lemma 17, we assume that

$$(1)\qquad |\alpha - \frac{a}{q}| \le q^{-2} \quad \text{with } (a,q)=1 \quad \text{and} \quad q < \Delta < N/2.$$

Let f and g be arbitrary real sequences such that $|f(n)| \le \log n$ and $|g(n)| \le \tau_5(n) \log n$. Moreover, let U and V be parameters and define

$$J\mathrm{I}_U = \Delta^2 \int_{-1/2\Delta}^{1/2\Delta} \Big| \sum_{\substack{mn\sim N \\ m>U}} g(n) e((\alpha+\beta)mn) \Big|^2 d\beta$$

$$J\mathrm{II}_{U,V} = \Delta^2 \int_{-1/2\Delta}^{1/2\Delta} \Big| \sum_{d\ell \sim N} \Big(\sum_{\substack{mn=d \\ m\le U \\ n\le V}} g(n) \Big) e((\alpha+\beta)d\ell) \Big|^2 d\beta$$

$$J\mathrm{III}_U = \Delta^2 \int_{-1/2\Delta}^{1/2\Delta} \Big| \sum_{\substack{mn\sim N \\ m\sim U}} f(m) g(n) e((\alpha+\beta)mn) \Big|^2 d\beta .$$

In order to estimate the above integrals we use the elementary lemma:

LEMMA 9. If $1 < X \le Y$, then

$$\sum_{n\le X} \min\Big(\frac{Y}{m}, \frac{1}{\|\alpha m\|} \Big) \ll \Big(\frac{Y}{q} + X + q \Big) \log qX .$$

LEMMA 10.

$$J\mathrm{I}_U \ll \log^{325} N \, \{ \Delta N \, (\Delta q^{-1/2} + (q\Delta)^{1/2}) + \Delta^2 (\tfrac{N}{U})^2 + \Delta^3 \}$$

Proof. By Lemma 5,

$$\mathcal{J}I \ll \Delta \sum_{k\sim N} \Big(\sum_{mn=k} g(n)\Big)^2 + 2\mathrm{Re} \sum_{0<r\le\Delta} (\Delta-r)e(\alpha r) \sum_{\substack{m'n'-mn=r\\ m',m\ge U\\ m'n',mn\sim N}} g(n')g(n)$$

$$= O\Big(\Delta \sum_{k\sim N} \tau_6(k)^2\log^2 k \Big) + 2\mathrm{Re} \sum_{0<r\le\Delta} (\Delta-r)e(\alpha r) \sum_{n',n} g(n')g(n) \sum_{\substack{m'n'-mn=r\\ m',m> U\\ m'n',mn\sim N}} 1$$

(1) $$= O(\Delta N\log^{37}N) + 2\mathrm{Re} \sum_{0<r\le\Delta} (\Delta-r)e(\alpha r)\Phi(r), \text{ say.}$$

The inner sum in Φ is equal to

$$\#\{m' : N(r) < m'n' \le 2N, \quad m'n' \equiv r \pmod n\},$$

where $N(r) = \max(Un', Un+r, N+r)$. The above conguence is soluble if and only if $(n',n)|r$. Write $n^* = n/(n',n)$ and $r^* = r/(n',n)$. Then, the sum over m' is

$$= \#\{m' : \frac{N(r)}{n'} < m' \le \frac{2N}{n'}, \quad m' \equiv \overline{n'^*}r^* \pmod{n^*}\}$$

$$= \frac{2N-N(r)}{[n',n]} + O(1)$$

$$= \frac{2N-N(0)}{[n',n]} + O\Big(\frac{r}{[n',n]} + 1\Big).$$

Thus,

$$\Phi(r) = \sum_{\substack{n',n<2N/U\\ (n',n)|r}} g(n')g(n)\frac{2N-N(0)}{[n',n]} + O\Bigg(r \sum_{n',n<\frac{2N}{U}} \frac{|g(n')g(n)|}{[n',n]} + \Big(\sum_{n<\frac{2N}{U}} |g(n)|\Big)^2 \Bigg)$$

$$= (2N-N(0)) \sum_{\substack{n',n<2N/U\\ (n',n)|r}} \frac{g(n')g(n)}{[n',n]} + O\Bigg(\Delta \sum_{d<\frac{2N}{U}} d\Big(\sum_{d|n} \frac{|g(n)|}{n}\Big)^2 + \Big(\sum_{n<\frac{2N}{U}} |g(n)|\Big)^2 \Bigg)$$

(2) $$= \Phi_1(r) + O\Big(\Delta\log^{37}N + \big(\tfrac{N}{U}\big)^2\log^{10}N \Big), \text{ say.}$$

The contribution of $\Phi_1(r)$ to $\mathcal{J}I$ is

$$2\mathrm{Re}(2N-N(0)) \sum_{n',n<\frac{2N}{U}} \frac{g(n')g(n)}{[n',n]} \sum_{\substack{0<r\le\Delta \\ (n',n)|r}} (\Delta-r)e(\alpha r)$$

$$= 2\mathrm{Re}(2N-N(0)) \int_1^{\Delta} \sum_{n',n<\frac{2N}{U}} \frac{g(n')g(n)}{[n',n]} \sum_{\substack{0<r\le t \\ (n',n)|r}} e(\alpha r)\ dt$$

$$<< N \int_1^{\Delta} \sum_{\substack{n',n<2N/U \\ (n',n)\le t}} \frac{|g(n')g(n)|}{[n',n]} \Big| \sum_{\substack{0<r\le t \\ (n',n)|r}} e(\alpha r) \Big| dt$$

$$<< N\Delta \sum_{d\le\Delta} d \left(\sum_{\substack{n<2N/U \\ d|n}} \frac{|g(n)|}{n} \right)^2 \min\left(\frac{\Delta}{d}, \frac{1}{\|\alpha d\|} \right)$$

$$<< \Delta\ N\ \log^{12} N \sum_{d\le N} \frac{\tau_5(d)^2}{d} \min\left(\frac{\Delta}{d}, \frac{1}{\|\alpha d\|} \right)$$

$$<< \Delta\ N\ \log^{12} N \left(\sum_{d\le\Delta} \frac{\tau_5(d)^4}{d} \right)^{\frac{1}{2}} \left(\sum_{d\le\Delta} \frac{1}{d} \min\left(\frac{\Delta}{d}, \frac{1}{\|\alpha d\|} \right) \right)^{\frac{1}{2}}$$

$$<< \Delta\ N\ \log^{12} N \ \{ \Delta \log^{625} N\ (\frac{\Delta}{q} + \log N + q) \}^{1/2}$$

$$(3) \qquad << \Delta\ N\ \log^{325} N\ (\Delta q^{-1/2} + (q\Delta)^{1/2})$$

by partial summation and Lemma 9. In conjunction with (1),(2) and (3), we have

$$JI << \Delta N \log^{38} N + \Delta N \log^{325} N (\Delta q^{-1/2} + (q\Delta)^{1/2}) + \Delta^2 (\Delta \log^{37} N + (\frac{N}{U})^2 \log^{10} N)$$

$$<< \log^{325} N\ \{ \Delta N\ (\Delta q^{-1/2} + (q\Delta)^{1/2}) + \Delta^2 (\frac{N}{U})^2 + \Delta^3 \}.$$

We turn to $J\mathbb{I}$. We use the following lemmas.

LEMMA 11.

For arbitrary real numbers x_m, and $H>0$, we have

$$|\sum_{m\sim M}\psi(x_m)| \ll \frac{M}{H} + \sum_{0<h\leq H}\frac{1}{h}|\sum_{m\sim M}e(hx_m)|.$$

LEMMA 12 [14,16].

For any $\varepsilon>0$, we have

$$\sum_{\substack{n\sim N\\(n,d)=1}} e\left(k\frac{\overline{n}}{d}\right) \ll (k,d)^{1/2}d^{1/2+\varepsilon}\left(1+\frac{N}{d}\right).$$

LEMMA 13.

$$J\mathbb{I}_{U,V} \ll \log^{663}N\ \{\Delta N(\Delta q^{-1/2}+(q\Delta)^{1/2})+\Delta^3\}+\Delta^2(N^{1-\varepsilon}+N^{8\varepsilon}U^{3/2}V^3).$$

Proof. Write $\displaystyle G_d = \sum_{\substack{mn=d\\ m\leq U,\, n\leq V}} g(n)$.

By the similar argument to that in Lemma 10, we have

$$J\mathbb{I} \ll \log^{663}N\ \{\Delta N(\Delta q^{-1/2}+(q\Delta)^{1/2})+\Delta^3\}+R$$

where

$$R = \Delta\sum_{0<r\leq\Delta}|\sum_{(d',d)|r} G_{d'}G_d\theta(d'^*,d^*,r^*)|$$

with

$$\theta(d'^*,d^*,r^*) = \psi\left(\frac{2N}{[d',d]}-r^*\frac{\overline{d'^*}}{d^*}\right)-\psi\left(\frac{N+r}{[d',d]}-r^*\frac{\overline{d'^*}}{d^*}\right).$$

We proceed to estimate R. By the definition of G_d, we have

$$R = \Delta \sum_{0<r\le\Delta} \Big| \sum_{\delta|r} \sum_{ab=\delta} \sum_{\substack{am\le U\\(mn,d)=1}} \sum_{bn\le V} \sum g(bn) G_{\delta d} \Theta(mn,d,k) \Big|$$

$$\ll \Delta \sum_{0<\delta k\le\Delta} \sum_{ab=\delta} \sum_{\delta d\le UV} \sum_{bn\le V} |g(bn)G_{\delta d}| \Big| \sum_{\substack{m\le U/a\\(mn,d)=1}} \Theta(mn,d,k) \Big|$$

$$\ll \Delta^2 N^{\varepsilon} \sup_{\substack{D\le UV, L\le V, M\le U\\ k\le\Delta, T\le 2N}} \sum_{d\sim D} \sum_{n\sim L} \Big| \sum_{\substack{m\sim M\\(mn,d)=1}} \psi\Big(\frac{T}{dmn} - k\frac{\overline{mn}}{d}\Big) \Big|$$

(4) $$= \Delta^2 N^{\varepsilon} \sup R_1(D,L,M,k,T), \quad \text{say.}$$

We now appeal to Lemmas 11 and 12. When $DLM \le N^{1-3\varepsilon}$, trivially,

(5) $$R_1 \ll N^{1-3\varepsilon}.$$

Henceforce we assume $DLM > N^{1-3\varepsilon}$. Choose $H = DLMN^{4\varepsilon-1}$ in Lemma 11. Then, $H>2$. Thus, by Lemma 11, we have

$$R_1 \ll \frac{DLM}{H} + \sum_{0<h\le H} \frac{1}{h} \sum_d \sum_n \Big| \sum_{\substack{m\sim M\\(mn,d)=1}} e\Big(\frac{hT}{dmn}\Big) e\Big(-hk\frac{\overline{mn}}{d}\Big) \Big|$$

(6) $$= N^{1-4\varepsilon} + \sum_h \frac{1}{h} S(h), \quad \text{say.}$$

Lemma 12 yields

$$S(h) \ll \sum_d \sum_h \Big(1 + \frac{hT}{DLM}\Big)(hk,d)^{1/2} d^{1/2+\varepsilon}\Big(1 + \frac{M}{d}\Big)$$

$$\ll L\Big(1 + \frac{HT}{DLM}\Big) N^{\varepsilon} \Big(\sum_d \frac{(hk,d)}{d} \Big)^{1/2} \Big\{ \Big(\sum_d d^2 \Big)^{1/2} + M\Big(\sum_d 1 \Big)^{1/2} \Big\}$$

$$\ll L(1+N^{4\varepsilon}) N^{2\varepsilon} (D^{3/2} + MD^{1/2})$$

$$\ll N^{6\varepsilon} U^{3/2} V^{5/2}.$$

Combining this with (4),(5) and (6), we have

$$R \ll \Delta^2 (N^{8\varepsilon} U^{3/2} V^{5/2} + N^{1-3\varepsilon}).$$

This gives the required bound for $\mathcal{J}_{\text{II}}$.

LEMMA 14. If $U<\Delta$, then

$$\mathcal{J}_{\text{III}_U} \ll \Delta N \log^{28} N \left(U + \frac{\Delta}{q} + \frac{\Delta}{U} + q \right).$$

Proof.

We may impose the restriction $n \in I = [\frac{N}{3U}, \frac{3N}{U}]$. Then,

$$\mathcal{J}_{\text{III}} \le \Delta^2 \int_{-1/2\Delta}^{1/2\Delta} \sum_{m' \sim U} f(m')^2 \cdot \sum_{m \sim U} \Big| \sum_{\substack{mn \sim N \\ n \in I}} g(n) e((\alpha+\beta)mn) \Big|^2 d\beta$$

$$\ll U \log^2 U \cdot \sum_{m \sim U} \Delta^2 \int_{-1/2\Delta}^{1/2\Delta} \Big| \sum_{\substack{mn \sim N \\ n \in I}} g(n) e((\alpha+\beta)mn) \Big|^2 d\beta$$

(7) $$= U \log^2 N \cdot \mathcal{J}, \quad \text{say.}$$

By Lemma 5, we have

$$\mathcal{J} \ll \sum_{m \sim U} \Big\{ \Delta \sum_{mn \sim N} g(n)^2 + 2\mathrm{Re} \sum_{0<mr \le \Delta} (\Delta - mr) e(\alpha mr) \sum_{\substack{n'-n=r \\ n', n \in I \\ mn', mn \sim N}} g(n') g(n) \Big\}$$

$$= O(\Delta N \log^{26} N) + 2\mathrm{Re} \sum_{0<r \le \Delta} \sum_{\substack{n'-n=r \\ n', n \in I}} g(n') g(n) \sum_{\substack{0<mr \le \Delta \\ m \sim U \\ mn', mn \sim N}} (\Delta - mr) e(\alpha mr)$$

(8) $$= O(\Delta N \log^{26} N) + \mathcal{J}_1, \quad \text{say.}$$

We proceed to $\mathcal{J}_1$. Since $U < m \le \Delta / r$ in the innermost sum, we see $r \le \Delta/U$. Thus, by Lemma 9,

$$\mathcal{J}_1 \ll \sum_{0<r\le\Delta/U} \sum_{\substack{n'-n=r\\ n',n\sim N/U}} |g(n')g(n)|\ r\int_1^{\Delta/r} \Big| \sum_{\substack{0<m\le t\\ m\sim U\\ mn',mn\sim N}} e(\alpha m r)\Big| dt$$

$$\ll \sum_{r\le\Delta/U} \sum_{\substack{n'-n=r\\ n',n\sim N/U}} |g(n')g(n)|\ \Delta \min\Big(\frac{\Delta}{r}, \frac{1}{\|\alpha r\|}\Big)$$

$$\ll \Delta \sum_{n\sim N/U} g(n)^2 \cdot \sum_{r\le\Delta/U} \min\Big(\frac{\Delta}{r}, \frac{1}{\|\alpha r\|}\Big)$$

$$\ll \Delta \frac{N}{U} \log^{26} N \Big(\frac{\Delta}{q} + \frac{\Delta}{U} + q \Big).$$

Combining this with (7) and (8), we get

$$J_{\mathrm{III}} \ll U \log^2 U \cdot \Big\{ \Delta N \log^{26} N + \Delta \frac{N}{U} \log^{26} N \Big(\frac{\Delta}{q} + \frac{\Delta}{U} + q \Big)\Big\}$$

$$\ll \Delta N \log^{28} N \Big(U + \frac{\Delta}{q} + \frac{\Delta}{U} + q \Big),$$

as required.

LEMMA 15. For real α, define

$$J = J(\alpha,\Delta) = \Delta^2 \int_{-1/2\Delta}^{1/2\Delta} \Big| \sum_{N<n\le 2N} \Lambda(n) e((\alpha+\beta)n)\Big|^2 d\beta.$$

Suppose $|\alpha - \frac{a}{q}| \le q^{-2}$ with $(a,q)=1$. Then, for any small $\varepsilon>0$, we have

$$J \ll \log^{665} N \{ \Delta N(N^{1/3} + \Delta q^{-1/2} + (q\Delta)^{1/2}) + \Delta^2 N^{1-\varepsilon} + \Delta^3 \}.$$

To prove Lemma 15 we use the combinatrial identity;

LEMMA 16 [13].

Let k be a positive integer. If $n \le X$, then

$$\Lambda(n)=\sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}\sum_{\substack{n_1\cdots n_j n_{j+1}\cdots n_{2j}=n\\ n_{j+1},\cdots,n_{2j}\le X^{1/k}}}(\log n_1)\mu(n_{j+1})\cdots\mu(n_{2j}).$$

Proof of Lemma 15.

Unless

(9) $$N^{1/3}<\Delta/2 \quad\text{and}\quad q<\Delta<N/2,$$

then Lemma 15 is trivial. So we may assume (9). Since $n\le 3N$, we appeal to Lemma 16 with $X=8N$ and $k=3$. $\Lambda(n)$ is decomposed into a linear combination of $O(1)$ sums

$$\Lambda^*(n)=\sum_{\substack{n_1n_2n_3n_4n_5n_6=n\\ n_4,n_5,n_6\le 2N^{1/3}}}(\log n_1)\mu(n_4)\mu(n_5)\mu(n_6).$$

It is sufficient to show Lemma 15 with Λ^* in place of Λ. Moreover, we may assume $\min(n_1,n_2,n_3)=n_3$, for the other cases are similarly treated. We then see that

$$n_i\le(3N)^{1/3} \quad \text{for } i=3,4,5,6.$$

Put $n'=n_3n_4n_5n_6$. Let $v>2$ be a parameter, and $z=v^4$. We divide the integrand of J according to the following three cases.

(1) $n'\le z$ and $n_1>N^{1/2}v$,

(2) $n'\le z$ and $n_1\le N^{1/2}v$,

(3) $n'>z$.

Let $\Sigma(i)$ denote the corresponding sum to case (i).

In case (1), we may write

$$\Lambda^*(n)=\sum_{\substack{n_1n'=n\\ n_1>N^{1/2}v}}(\log n_1)g(n')$$

with $|g(n)|\le\tau_5(n)$. By partial summation and Cauchy's inequality, we have

$$\Delta^2\int_{-1/2\Delta}^{1/2\Delta}|\Sigma(1)|^2dt \ll \log^2 N \sup_{u\ge N^{1/2}v} \mathcal{J}\mathrm{I}_u .$$

In case (2),

$$\begin{aligned}\Lambda^*(n)&=\sum_{n_1n_2n'=n}(\log n_1)g(n')\\ &=\sum_{n_2n''=n}\Big(\sum_{\substack{n_1n'=n''\\ n_1\le N^{1/2}v,\ n'\le z=v^4}}(\log n_1)g(n')\Big)\end{aligned}$$

with $|g(n)|\le\tau_4(n)$. Hence,

$$\Delta^2\int_{-1/2\Delta}^{1/2\Delta}|\Sigma(2)|^2dt \ll \log^2 N \sup_{u\le N^{1/2}v} \mathcal{J}\mathrm{II}_{u,v^4} .$$

In case (3), since

$$v^4=z<n'=n_3n_4n_5n_6\le(\max_{i=3,4,5,6.} n_i)^4,$$

there exists an index i such that

$$v<n_i\le(3N)^{1/3}.$$

So we may write

$$\Lambda^*(n)=\sum_{\substack{n_in''=n\\ v<n_i\le(3N)^{1/3}}} f(n_i)g(n'')$$

with $|f(n)|\le1$, $|g(n)|\le\tau_5(n)\log n$. Decomposing this interval into the sum of $[2^j,2^{j+1}]$ type intervals, we see

$$\Delta^2\int_{-1/2\Delta}^{1/2\Delta}|\Sigma(3)|^2dt \ll \log^2 N \sup_{v<u<2N^{1/3}} \mathcal{J}\mathrm{III}_u.$$

By the above argument, we have

$$\mathcal{J} \ll \{\sup_{u>N^{1/2}v}\mathcal{J}\mathrm{I}_u + \sup_{u\le N^{1/2}v}\mathcal{J}\mathrm{II}_{u,v^4} + \sup_{v<u<2N^{1/3}}\mathcal{J}\mathrm{III}_u\} \log^2 N.$$

Because of (9), all of the assumptions in (1) and Lemma 14 are satisfied. We choose $v=N^{2\varepsilon}$ with any $0<\varepsilon<1/200$. Thus, by Lemmas 10, 13 and 14, we get

$$\begin{aligned} J &\ll \log^{665}N \,\{\, \Delta N(\, \Delta q^{-1/2}+(\Delta q)^{1/2} \,) + \Delta^3 \,\} + \\ &\quad + \Delta^2 \log^{327}N\, (N/N^{1/2}v)^2 + \Delta^2 N^{1-\varepsilon} + \Delta^2 N^{8\varepsilon}\, (N^{1/2}v)^{3/2}\, (v^4)^3 + \\ &\quad + \Delta N \log^{30}N \,(\, N^{1/3} + \frac{\Delta}{q} + \frac{\Delta}{v} + q \,) \\ &\ll \log^{665}N \,\{\, \Delta N(\, \Delta q^{-1/2}+(\Delta q)^{1/2}+ N^{1/3} \,)+ \Delta N^{1-\varepsilon}+ \Delta^3 \,\}, \end{aligned}$$

as required.

Finally we quote the well known bound for trigonometric sums over primes.

LEMMA 17 [36].

Suppose $|\alpha- \frac{a}{q}|\le q^{-2}$ with $(a,q)=1$. Then,

$$\sum_{n\le x} \Lambda(n)e(\alpha n) \ll \{\, xq^{-1/2}+ x^{4/5}+ (qx)^{1/2}\} \log^4 x.$$

5. Proof of Theorem 1.

Let $\varepsilon>0$ and $A>0$ be given. Define

$$2x\le y\le x^{3-\varepsilon},\qquad k\le x,$$

$$S(\alpha)=\sum_{n\le y}\Lambda(n)e(\alpha n),$$

$$Q_1=\log^D x,\qquad Q=y^{1/4},$$

$$M=\bigcup_{q\le Q_1}\ \bigcup_{a=1}^{q}{}^{*}\ I_{q,a},\qquad I_{q,a}=[\tfrac{a}{q}-\tfrac{1}{qQ},\ \tfrac{a}{q}+\tfrac{1}{qQ}],$$

$$\mathfrak{m}=[Q^{-1},\ 1+Q^{-1}]\setminus M,$$

where D is some constant, which will be specified later. Then,

$$\Psi(y,2k)=\int_{Q^{-1}}^{1+Q^{-1}}|S(\alpha)|^2e(-2k\alpha)d\alpha.$$

We begin with the major arc M. For $\alpha=\frac{a}{q}+\beta\in I_{q,a}$, we write

$$T(\alpha)=\frac{\mu(q)}{\varphi(q)}\sum_{n\le y}e(\beta n),$$

$$U(\alpha)=S(\alpha)-T(\alpha).$$

Then, by Lemma 8, we have

$$\int_M|T(\alpha)|^2e(-2k\alpha)d\alpha=\sum_{q\le Q_1}\ \sum_{a=1}^{q}{}^{*}\int_{-1/qQ}^{1/qQ}\Big|\frac{\mu(q)}{\varphi(q)}\sum_{n\le y}e(\beta n)\Big|^2e(-2k(\tfrac{a}{q}+\beta))d\beta$$

$$=\sum_{q\le Q_1}\Big\{\frac{\mu^2(q)}{\varphi^2(q)}c_q(-2k)(y-2k)+O(\frac{qQ}{\varphi(q)})\Big\}$$

(1)
$$=\sum_{q\le Q_1}\frac{\mu^2(q)}{\varphi^2(q)}c_q(-2k)(y-2k)+O(Q_1Q\log Q_1).$$

Also,

$$\int_{M} |T(\alpha)|^2 d\alpha \le \sum_{q\le Q_1} \sum_{a=1}^{q}{}^{*} \int_{-1/2}^{1/2} \frac{\mu^2(q)}{\varphi^2(q)} \Big| \sum_{n\le y} e(\beta n)\Big|^2 d\beta$$

$$= y \sum_{q\le Q_1} \frac{\mu^2(q)}{\varphi(q)}$$

(2) $$\ll y \log Q_1 .$$

Lemma 7 yields

$$\int_{M} |U(\alpha)|^2 d\alpha = \sum_{q\le Q_1} \sum_{a=1}^{q}{}^{*} \int_{-1/qQ}^{1/qQ} \Big| \sum_{n\le y} \Lambda(n) e(\tfrac{a}{q}n) e(\beta n) - \frac{\mu(q)}{\varphi(q)} \sum_{n\le y} e(\beta n)\Big|^2 d\beta$$

(3) $$\ll Q_1 y \log^{-E} y \qquad (E=1+D+A).$$

Hence, by (1),(2) and (3),

$$\int_{M} |S(\alpha)|^2 e(-2k\alpha) d\alpha = \int_{M} |T(\alpha)+U(\alpha)|^2 e(-2k\alpha) d\alpha$$

$$= \int_{M} |T(\alpha)|^2 e(-2k\alpha) d\alpha \; +$$

$$+ \; O\Big(\Big(\int_{M} |T(\alpha)|^2 d\alpha + \int_{M} |U(\alpha)|^2 d\alpha\Big)^{1/2}\Big) + O\Big(\int_{M} |U(\alpha)|^2 d\alpha\Big)$$

$$= \sum_{q\le Q_1} \frac{\mu^2(q)}{\varphi^2(q)} c_q(-2k)(y-2k) + O(Q_1 Q \log y) \; +$$

$$+ \; O((y\log y \cdot Q_1 y \log^{-E} y)^{1/2}) + O(Q_1 y \log^{-E} y)$$

(4) $$= \sum_{q\le Q_1} \frac{\mu^2(q)}{\varphi^2(q)} c_q(-2k)(y-2k) + O(y \log^{-A/2} y).$$

For square-free q,

$$c_q(-2k) = \mu\Big(\frac{q}{(q,2k)}\Big)\varphi((q,2k)).$$

So the singular series

$$\sum_{q=1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} c_q(-2k)$$

is absolutely convergent and equal to $\mathfrak{S}(2k)$. Moreover, since

$$s(2k) = \sum_{q>Q_1} \frac{\mu^2(q)}{\varphi^2(q)} c_q(-2k) \ll \log 2k,$$

we see

$$\sum_{k\le x} s(2k)^2 \ll \log 2k \sum_{q>Q_1} \frac{\mu^2(q)}{\varphi^2(q)} \sum_{d|q} d \sum_{\substack{k\le x \\ d|2k}} 1$$

$$\ll x\log x \sum_{q>Q_1} \frac{\mu^2(q)}{\varphi^2(q)} \tau(q)$$

$$\ll xQ_1^{-1}\log^2 x. \tag{5}$$

We proceed to the minor arc $\mathfrak{m}$. Put $\Delta = x^{-1}Q_1$.

$$\sum_{k\le x} |\int_{\mathfrak{m}}|^2 = \sum_{k\le x} | \int_{\mathfrak{m}} |S(\alpha)|^2 e(-2k\alpha)d\alpha|^2$$

$$\ll \int_{\mathfrak{m}} \int_{\mathfrak{m}} |S(\alpha')|^2 |S(\alpha)|^2 \min(x, \frac{1}{\|\alpha'-\alpha\|}) d\alpha' d\alpha.$$

The contribution arising from the range $\|\alpha'-\alpha\| > 1/2\Delta$ is

$$\ll \Delta \left(\int_{\mathfrak{m}} |S(\alpha)|^2 d\alpha \right)^2$$

$$\ll xQ_1^{-1} y^2 \log^2 y.$$

As for the remaining range, we write $\alpha'-\alpha=\beta$. Since the exponential sum $S(\alpha)$ has the period 1, we get

$$\sum_{k\le x} |\int_{\mathfrak{m}}|^2 \ll x \int_{\alpha\in\mathfrak{m}} |S(\alpha)|^2 \left(\int_{\|\beta\|\le 1/2\Delta} |S(\alpha+\beta)|^2 d\beta \right) d\alpha + xQ_1^{-1} y^2 \log^2 y$$

(6) $$\ll xy\log y\cdot \sup_{\alpha\in \mathfrak{m}} \int_{|\beta|\le 1/2\Delta} |S(\alpha+\beta)|^2 d\beta + xQ_1^{-1}y^2\log^2 y.$$

For any $\alpha\in\mathfrak{m}$, there exist a and q with $(a,q)=1$ such that

$$|\alpha-\frac{a}{q}|\le q^{-2} \quad \text{and} \quad Q_1< q\le Q.$$

Since

$$Q= y^{1/4}\ll x\log^{-2D}x \ll \Delta Q_1^{-1},$$

$$y\ll x^{3-\varepsilon}\ll x^3\log^{-6D}x \ll (\Delta Q_1^{-1})^3,$$

Lemma 15 yields

$$\int_{|\beta|\le 1/2\Delta} |S(\alpha+\beta)|^2 d\beta$$

$$\ll \Delta^{-2}\sup_{Q_1<q\le Q}\{\Delta y(y^{1/3}+\Delta q^{-1/2}+(q\Delta)^{1/2})+\Delta^2 y^{1-\varepsilon}+\Delta^3\}\log^{666}y$$

$$\ll yQ_1^{-1/2}\log^{666}y,$$

uniformly for $\alpha\in\mathfrak{m}$. Combining this with (6) we get

$$\sum_{k\le x}|\int_{\mathfrak{m}}|^2\ll xy\log y\cdot yQ_1^{-1/2}\log^{666}y + xy^2Q_1^{-1}\log^2 y$$

(7) $$\ll xy^2\log^{-A}y.$$

Here we choosed $2D= 667+A$.

In conjunction with (4),(5) and (7), we conclude

$$\sum_{k\le x}E(y,2k)^2=\sum_{k\le x}(s(2k)+O(y\log^{-A/2}y)+\int_{\mathfrak{m}})^2$$

$$\ll \sum_{k\le x}s(2k)^2+xy^2\log^{-A}y+\sum_{k\le x}|\int_{\mathfrak{m}}|^2$$

$$\ll xy^2\log^{-A}y.$$

This completes the proof of Theorem 1.

6. Proof of Theorem 2.

Let $\varepsilon>0$ and $A>0$ be given. Let D be a positive constant, which will be specified later. In fact, D= 9+A. Define

$$x^{1/6+\varepsilon}\le y \le x, \quad k\le x,$$

$$S(\alpha)=\sum_{k\le y}\Lambda(n)e(\alpha n), \quad T(\alpha)=\sum_{n\le 3x}\Lambda(n)e(\alpha n),$$

$$Q_1=\log^D x, \quad Q=x^{1/6+\varepsilon/2}< yQ_1^{-5},$$

$$M=\bigcup_{q\le Q_1}\bigcup_{a=1}^{q\,*} I_{q,a}, \quad I_{q,a}=[\frac{a}{q}+\frac{1}{qQ},\ \frac{a}{q}+\frac{1}{qQ}],$$

$$\mathfrak{m}=[Q^{-1},\ 1+Q^{-1}]\setminus M.$$

Then,

$$\sum_{2k<n\le 2k+y}\Lambda(n)\Lambda(n-2k)=\int_{Q^{-1}}^{1+Q^{-1}} S(-\alpha)T(\alpha)e(-2k\alpha)d\alpha. \tag{1}$$

On putting

$$U(\alpha)=T(\alpha)-\frac{\mu(q)}{\varphi(q)}\sum_{n\le 3x}e((\alpha-\frac{a}{q})n) \quad \text{for } \alpha\in I_{q,a},$$

we see that

$$\int_{Q^{-1}}^{1+Q^{-1}} = P + R \tag{2}$$

where

$$P=\sum_{q\le Q_1}\sum_{a=1}^{q\,*}\int_{-1/qQ}^{1/qQ} S(-(\frac{a}{q}+\beta))\frac{\mu(q)}{\varphi(q)}\sum_{n\le 3x}e(\beta n)e(-2k(\frac{a}{q}+\beta))d\beta$$

$$R=R(k)=\int_M S(-\alpha)U(\alpha)e(-2k\alpha)d\alpha+\int_{\mathfrak{m}} S(-\alpha)T(\alpha)e(-2k\alpha)d\alpha.$$

First we estimate the mean square for R. By Lemmas 7 and 17, we have

$$\sum_{k\le x} R(k)^2 \ll \int_{\mathfrak{M}} |S(\alpha)|^2 |U(\alpha)|^2 d\alpha + \int_{\mathfrak{m}} |S(\alpha)|^2 |T(\alpha)|^2 d\alpha$$

$$\ll y^2 \int_{\mathfrak{M}} |U(\alpha)|^2 d\alpha + \sup_{Q_1<q\le Q} (y^2 q^{-1} + y^{8/5} + qy)\log^8 y \cdot \int_{\mathfrak{m}} |T(\alpha)|^2 d\alpha$$

$$\ll y^2 Q_1 x \log^{-2D} x + y^2 Q_1{}^{-1} x \log^9 x$$

$$\ll xy^2 Q_1{}^{-1} \log^9 x. \tag{3}$$

Next we evaluate P. We extend the range of integral to $[-1/2, 1/2]$. Then the resulting error is

$$\ll \sum_{q\le Q_1} \frac{\mu(q)}{\varphi(q)} \sum_{a=1}^{q}{}^{*} \left(\int_{1/qQ}^{1/2} |S(\tfrac{a}{q}+\beta)|^2 d\beta \right)^{1/2} \left(\int_{1/qQ}^{1/2} |\sum_{n\le 3x} e(\beta n)|^2 d\beta \right)^{1/2}$$

$$\ll \sum_{q\le Q_1} \mu^2(q) (y \log y \cdot qQ)^{1/2}$$

$$\ll yQ_1{}^{-1} \log y.$$

Since $m+2k\le y+2k \le 3x$, we see that for all $m\le y$

$$\sum_{n\le 3x} \int_{-1/2}^{1/2} e((-m+n-2k)\beta) d\beta = 1.$$

Thus, by the Siegel-Walfisz theorem, we have

$$P = \sum_{q\le Q_1} \frac{\mu(q)}{\varphi(q)} \sum_{m\le y} \Lambda(m) c_q(-m-2k) + O(yQ_1{}^{-1} \log y)$$

$$= \sum_{q\le Q_1} \frac{\mu(q)}{\varphi(q)} c_q(-b-2k) \sum_{\substack{m\le y \\ m\equiv b(q)}} \Lambda(m) + O(yQ_1{}^{-1} \log y)$$

$$= y \sum_{q\le Q_1} \frac{\mu^2(q)}{\varphi^2(q)} c_q(-2k) + O(yQ_1{}^{-1} \log y). \tag{4}$$

By (5.5),

$$\sum_{k\le x} s(2k)^2 = \sum_{k\le x} \Big(\sum_{q>Q_1} \frac{\mu^2(q)}{\varphi^2(q)} c_q(-2k) \Big)^2$$

$$\ll xQ_1^{-1}\log^2 y. \tag{5}$$

From (1),(2),(3),(4) and (5), we infer that

$$\sum_{k\le x} E(2k+y,2k)^2 = \sum_{k\le x} \Big(\sum_{2k<n\le 2k+y} \Lambda(n)\Lambda(n-2k) - \mathfrak{S}(2k)y \Big)^2$$

$$= \sum_{k\le x} \big(s(2k)y + O(yQ_1^{-1}\log y) + R(k) \big)^2$$

$$\ll y^2 \sum_{k\le x} s(2k)^2 + xy^2Q_1^{-2}\log^2 x + \sum_{k\le x} R(k)^2$$

$$\ll xy^2Q_1^{-1}\log^2 x + xy^2Q_1^{-1}\log^9 x$$

$$\ll xy^2\log^{9-D} x.$$

On taking D= 9+D, we obtain Theorem 2.

7. Kloostermania.

This section is devoted to estimate the bilinear form involving ψ-function. The method in this section has been created by [14,16,20], and influenced diverse problems in number theory. [2,3,4]. Let ρ, κ, ω, ξ, η, υ and ν be positive integers $\le T$, $(\omega,\xi)=(\upsilon,\nu)=1$. Define

$$K= K(T_1,T_2,M_1,N_1,M_2,N_2;\ \alpha,\beta,\gamma,\delta;\ \rho,\kappa,\omega,\xi,\eta,\upsilon,\nu)$$

$$= \sum_{\substack{m_1\sim M_1 \\ (m_1n_1,\rho\eta\omega\upsilon)=1 \\ (m_2n_2,\rho\eta\omega\nu)=1 \\ (m_1n_1,m_2n_2)=1}} \sum_{\substack{n_1\sim N_1 \\ \mu^2(n_1n_2)=1}} \sum_{m_2\sim M_2} \sum_{n_2\sim N_2} \alpha(m_1)\beta(n_1)\gamma(m_2)\delta(n_2)\sum_{j=1,2}(-1)^j\psi\left(\frac{T_j}{m_1n_2m_2n_2}+\theta\left(\frac{m_1n_1}{m_2n_2}\right)\right)$$

where

$$\theta\left(\frac{r}{s}\right)= \theta(r,s;\ \rho,\kappa,\omega,\xi,\upsilon,\nu)= -\rho\frac{\overline{rs\xi}}{\omega} + \rho\kappa\frac{\overline{r\nu}}{s\upsilon}.$$

Also,

$$K=K(T,M,N;\ \rho,\kappa,\omega,\xi,\eta,\upsilon,\nu)$$

$$= \sup_{\substack{|\alpha|,|\beta|,|\gamma|,|\delta|\le 1 \\ M_1,M_2\le M \quad N_1,N_2\le N \\ T_1<T_2\le T \\ M_1N_1M_2N_2>T^{1-2\varepsilon}}} K(T_1,T_2,M_1,N_1,M_2,N_2;\ \alpha,\beta,\gamma,\delta;\ \rho,\kappa,\omega,\xi,\eta,\upsilon,\nu).$$

Moreover,

$$\mathscr{E} = \mathscr{E}(M_1,N_1,M_2,N_2,H;\ c;\ \rho,\kappa,\omega,\xi,\eta,\upsilon,\nu)$$

$$= \sum_{\substack{m_1\sim M_1 \\ (m_1,m_2)=1 \\ (m_1,\omega\upsilon)=1 \\ (m_2,\rho\omega\nu)=1}} \sum_{m_2\sim M_2} \left| \sum_{0<h\le H} \sum_{\substack{n_1\sim N_1 \\ (m_1n_2,m_2n_2)=1 \\ (n_1,\rho\eta\omega\upsilon)=(n_2,\rho\eta\omega\nu)=1 \\ \mu^2(n_1n_2)=1}} \sum_{n_2\sim N_2} c(h,n_1,n_2)e\left(h\theta\left(\frac{m_1n_1}{m_2n_2}\right)\right)\right|^2.$$

LEMMA 18.

If $\upsilon\nu(MN)^{2/3} < T^{1-8\varepsilon}$, then

$$K \ll T \sup (M_1N_1M_2N_2)^{-1}(M_1M_2 \mathscr{E})^{1/2}$$

where $H = M_1N_1M_2N_2T^{3\varepsilon-1}$ and the supremum is taken over all sequences (c) with $|c|\le 1$, and $M_1, M_2 \le M$, $N_1, N_2 \le N$.

In order to prove Lemma 18 we appeal to the following tools.

LEMMA 19. Let $H>2$. Then,

$$\psi(t) = \sum_{0<|h|\le H} \frac{e(ht)}{2\pi ih} + O(\min(1, \frac{1}{H\|t\|})).$$

LEMMA 20.

$$\min(1, \frac{1}{H\|t\|}) = \sum_{h\in\mathbb{Z}} C_h e(ht),$$

where

$$|C_h| \ll \min(\frac{\log H}{H}, \frac{1}{h}, \frac{H}{h^2}).$$

LEMMA 21 [16].

For any $\varepsilon>0$,

$$\sum_{\substack{m\sim M \\ m\equiv x (\mathrm{mod}\ y) \\ (m,cd)=1}} e\left(l\frac{\bar{m}}{d}\right) \ll \tau(c)(l,d)^{\frac{1}{2}} d^{\frac{1}{2}+\varepsilon}\left(1+\frac{M}{d}\right).$$

LEMMA 22. For integer m,

$$\sum_{n\sim N} \frac{(m,n)}{n} \ll \tau(m).$$

Proof of Lemma 18.

By Lemma 19, ψ-function is expressed as a sum of the trigonometric polynomial, ψ_1 say, and the tail of Fourier expansion, ψ_2 say.

Let K_i, $i=1,2$, be the corresponding sum to ψ_i, respectively. First we deal with K_1. To simplify the notation, let ! denote the condition in K.

$$\psi_1 = \sum_{0<|h|\le H} \frac{1}{2\pi i h}\left\{ e\left(\frac{hT_2}{m_1n_1m_2n_2}\right) - e\left(\frac{hT_1}{m_1n_1m_2n_2}\right)\right\} e\left(h\theta\left(\frac{m_1n_1}{m_2n_2}\right)\right)$$

$$= \sum_{0<|h|\le H} \int_{T_1/m_1m_2}^{T_2/m_1m_2} \frac{1}{n_1n_2} e\left(\frac{ht}{n_1n_2}\right) dt \cdot e\left(h\theta\left(\frac{m_1n_1}{m_2n_2}\right)\right) .$$

Also,

$$K_1 = \sum_{m_1,n_1,m_2,n_2}{}^{!} \alpha(m_1)\beta(n_1)\gamma(m_2)\delta(n_2)\psi_1$$

$$= \frac{1}{N_1N_2} \sum_{m_1,m_2}{}^{!} \alpha(m_1)\gamma(m_2) \int_{T_1/m_1m_2}^{T_2/m_1m_2} \sum_{0<|h|\le H} \sum_{n_1,n_2}{}^{!} \beta(n_1)\delta(n_2)\frac{N_1N_2}{n_1n_2} e\left(\frac{ht}{n_1n_2}\right) e\left(h\theta\left(\frac{m_1n_1}{m_2n_2}\right)\right) dt$$

$$\ll \frac{1}{N_1N_2} \int_0^{T/M_1M_2} \sum_{\substack{(m_1,m_2)=1\\(m_1,\rho\eta\omega\upsilon)=1\\(m_2,\rho\eta\omega\upsilon)=1}} \left| \sum_{0<h\le H} \sum_{n_1,n_2}{}^{!} \beta(n_1)\delta(n_2)\frac{N_1N_2}{n_1n_2} e\left(\frac{ht}{n_1n_2}\right) e\left(h\theta\left(\frac{m_1n_1}{m_2n_2}\right)\right) \right| dt$$

$$\ll \frac{T}{M_1M_2N_1N_2} \sup_{|c|\le 1} \sum\sum_{\substack{(m_1,m_2)=1\\(m_1,\omega\upsilon)=1\\(m_2,\rho\omega\upsilon)=1}} \left| \sum_{0<h\le H} \sum\sum_{\substack{\mu^2(n_1n_2)=1\\(m_1n_1,m_2n_2)=1\\(n_1,\rho\eta\omega\upsilon)=(n_2,\rho\eta\omega\upsilon)=1}} c(h,n_1,n_2) e\left(h\theta\left(\frac{m_1n_1}{m_2n_2}\right)\right) \right|$$

$$\ll \frac{T}{M_1N_1M_2N_2} \sup_{|c|\le 1} (M_1M_2\mathscr{E})^{1/2}.$$
(1)

We turn to K_2. By Lemma 20,

$$K_2 \ll T^{\varepsilon} \sum_{j=1,2} \sum_{\substack{r\sim M_1N_1\\ s\sim M_2N_2}}^{!} \min\left(1, \frac{1}{H\left\| \frac{T_j}{rs} + \theta(\frac{r}{s})\right\|}\right)$$

$$\ll T^{\varepsilon} \sum_{j=1,2} \sum_{h\in\mathbb{Z}} |c_h| \left| \sum_{\substack{(r\nu,s\upsilon)=1\\ (r,\rho\eta\omega\upsilon)=1\\ (s,\rho\eta\omega\nu)=1}} \sum e\left(\frac{hT_j}{rs}\right) e(h\theta(\frac{r}{s}))\right|$$

(2) $$= T^{\varepsilon} \sum_{h\in\mathbb{Z}} |c_h| \ S(h), \quad \text{say.}$$

We proceed to the estimation of $S(h)$. Trivially,

(3) $$S(h) \ll M_1N_1M_2N_2.$$

By partial summation,

$$S(h) \ll \left(1+ \frac{hT}{M_1N_1M_2N_2}\right) \left| \sum_{r,s}^{!} e(h\theta(\frac{r}{s}))\right|$$

$$= \left(1+ \frac{hT}{M_1N_1M_2N_2}\right) S_1(h), \quad \text{say.}$$

Lemma 21 yields

$$S_1(h) \le \sum_{\substack{(s,\rho\eta\omega\nu)=1\\ (\nu,\upsilon)=1}} \sum_{l=1}^{\omega}{}^{*} \left| \sum_{\substack{r\equiv l(\omega)\\ (r,s\rho\eta\omega\upsilon)=1}} e(h\theta(\frac{r}{s}))\right|$$

$$\le \sum_{\substack{s\sim M_2N_2\\ (s,\rho)=1}} \sum_{l=1}^{\omega}{}^{*} \left| \sum_{\substack{r\sim M_1N_1\\ r\equiv l(\omega)\\ (r,\rho\eta s\upsilon)=1}} e(h\rho\kappa \frac{\overline{r\nu}}{s\upsilon})\right|$$

$$\ll \sum_{(s,\rho)=1} \omega\cdot\tau(\rho\eta)(h\rho\kappa,s\upsilon)^{1/2}(s\upsilon)^{1/2+\varepsilon}\left(1+\frac{M_1N_1}{s\upsilon}\right)$$

$$\ll T^{\varepsilon}\,\omega\upsilon \sum_{s\sim M_2N_2} (h\kappa,s)^{1/2}\left(s^{1/2}+\frac{M_1N_1}{s^{1/2}}\right)$$

$$\ll T^{\varepsilon}\omega\upsilon\left(\sum_{s} \frac{(h\kappa,s)}{s}\right)^{1/2} \left\{\left(\sum_{s} s^2\right)^{1/2} + M_1N_1\left(\sum_{s} 1\right)^{1/2}\right\}.$$

By Lemma 22, we have

$$S(h) \ll \left(1+ \frac{hT}{M_1N_1M_2N_2}\right)T^{2\varepsilon}\omega\upsilon\ \tau(h)\{(M_2N_2)^{3/2}+M_1N_1(M_2N_2)^{1/2}\}$$

$$\ll \left(1+ \frac{hT}{M_1N_1M_2N_2}\right)\tau(h)T^{2\varepsilon}\omega\upsilon(MN)^{3/2}. \tag{4}$$

Now we estimate K_2. By (2),

$$K_2 \ll T^{\varepsilon}\Big(\sum_{h=0} + \sum_{0<|h|\le H} + \sum_{H<|h|\le HM_2N_2} + \sum_{|h|>HM_2N_2} \Big)|C_h|\ S(h)$$

$$= T^{\varepsilon}(\ \Sigma_0+ \Sigma_1+ \Sigma_2+ \Sigma_3\ ),\quad \text{say.} \tag{5}$$

By (3),

$$\Sigma_0+\Sigma_3 \ll M_1N_1M_2N_2\Big(\frac{\log H}{H} + \sum_{h>HM_2N_2} \frac{H}{h^2} \Big)$$

$$\ll T^{1-3\varepsilon}\log H + MN\ . \tag{6}$$

Also, by (4),

$$\Sigma_1+\Sigma_2 \ll \omega\upsilon(MN)^{3/2}\ T^{2\varepsilon}.$$

$$\cdot\Big\{ \sum_{0<h\le H}\frac{1}{h}\left(1+ \frac{hT}{M_1N_1M_2N_2}\right)\tau(h)+ \sum_{H<h\le HM_2N_2} \frac{H}{h^2}\left(1+ \frac{hT}{M_1N_1M_2N_2}\right)\tau(h)\Big\}$$

$$\ll \omega\upsilon(MN)^{3/2}\ T^{2\varepsilon}\left(1+ \frac{HT}{M_1N_1M_2N_2}\right)\ \log^2T$$

$$\ll \omega\upsilon(MN)^{3/2}\ T^{5\varepsilon}\ \log^2T. \tag{7}$$

From (5),(6) and (7) we infer that

$$K_2 \ll T^{\varepsilon}\{\ T^{1-3\varepsilon}\log H + \omega\upsilon(MN)^{3/2}\ T^{5\varepsilon}\log^2T\ \}$$

$$\ll T^{1-2\varepsilon}\log H + \omega\upsilon(MN)^{3/2}\ T^{6\varepsilon}\log^2T$$

$$\ll T^{1-\varepsilon}.$$

Combining this with (1) we obtain Lemma 18.

LEMMA 23.

Let $0 < k, h_1, h_2 \le T$. Then,

$$\Sigma = \sum_{n_1,n_3\sim N_1} \sum_{n_2,n_4\sim N_2} \sum \sum \sum_{\substack{m\sim M \\ h_1n_3n_4-h_2n_1n_2=r(n_2,n_3)(n_1,n_4)\neq 0 \\ m_2n_2n_4=d(n_2,n_3)(n_1,n_4)}} \frac{(kr,d)}{d} \ll T^{\varepsilon}N_1{}^2.$$

Proof. Write

$$\delta_1=(n_1,n_4) \qquad \delta_2=(n_2,n_3)$$

$$n_1=\delta_1\nu_1 \quad n_2=\delta_2n_2{}^* \quad n_3=\delta_2\nu_3 \quad n_4=\delta_1n_4{}^*.$$

Furthermore,

$$\gamma=(n_2{}^*,n_4{}^*)$$

$$n_2{}^*=\gamma\nu_2 \qquad n_4{}^*=\gamma\nu_4.$$

Then, we have

$$\Sigma= \sum_{n_1} \sum_{n_3} \sum_{\delta_1\nu_1=n_1} \sum_{\delta_2\nu_2=n_3} \sum_{m_2} \sum_{\gamma} \sum_{\nu_2} \sum_{\substack{\nu_4 \\ (\nu_1,\gamma\nu_4)=(\nu_3,\gamma\nu_2)=(\nu_2,\nu_4)=1 \\ h_1\nu_3\nu_4\neq h_2\nu_1\nu_2}} \frac{(k(h_1\nu_3\nu_4-h_2\nu_1\nu_2),m\gamma\nu_2\nu_4)}{m\gamma\nu_2\nu_4}.$$

Put $B= k(h_1\nu_3\nu_4-h_2\nu_1\nu_2)\neq 0$. Since $(\nu_2,\nu_4)=1$,

$$(B,\nu_2)=(kh_1\nu_3\nu_4,\nu_2)=(kh_1\nu_3,\nu_2).$$

Also,

$$(B,\nu_4)=(kh_2\nu_1\nu_2,\nu_4)=(kh_2\nu_1,\nu_4).$$

Thus, by Lemma 22,

$$\Sigma \le \sum_{n_1} \sum_{n_3} \sum_{\delta_1 \nu_1 = n_1} \sum_{\delta_2 \nu_2 = n_3} \sum_{\nu_2} \sum_{\nu_4} \frac{(B,\nu_2)(B,\nu_4)}{\nu_2 \quad \nu_4} \sum_{m} \frac{(B,m)}{m} \sum_{\gamma} \frac{(B,\gamma)}{\gamma}$$

$$<< T^{\varepsilon} \sum_{n_1} \sum_{n_3} \sum_{\delta_1 \nu_1 = n_1} \sum_{\delta_2 \nu_2 = n_3} \sum_{\nu_2} \frac{(kh_1\nu_3,\nu_2)}{\nu_2} \sum \frac{(kh_2\nu_1,\nu_4)}{\nu_4}$$

$$<< T^{2\varepsilon} \sum_{n_1} \sum_{n_3} \tau(n_1)\tau(n_2)$$

$$<< T^{3\varepsilon} N_1^{\,2}.$$

LEMMA 24.

$$\mathcal{G} << H^2 T^{1-3\varepsilon} \log^8 T + \omega\upsilon T^{2\varepsilon} H^2 (M_1+M_2) M_2^{\,1/2} N_1^{\,2} N_2^{\,3}.$$

Proof.

Expanding the square and changing the order of summation, we have

$$\mathcal{G} = \sum_{0<h_1,h_2\le H} \sum \sum_{\substack{n_j \ (j=1,2,3,4) \\ (n_1n_3,\rho\eta\omega\upsilon)=(n_2n_4,\rho\eta\omega\upsilon)=1 \\ \mu^2(n_1n_2)=\mu^2(n_3n_4)=1}} c(h_1,n_1n_2)\bar{c}(h_2,n_3,n_4) \sum \sum_{\substack{(m_1n_1,m_2n_2)=1 \\ (m_1n_3,m_2n_4)=1 \\ (m_1,\omega\upsilon)=1 \\ (m_2,\rho\omega\upsilon)=1}} e\left(h_1\theta\left(\frac{m_1n_1}{m_2n_2}\right)-h_2\theta\left(\frac{m_1n_3}{m_2n_4}\right)\right)$$

$$\le \sum_{0<h_1,h_2\le H} \sum \sum_{\substack{n_j \ (j=1,2,3,4) \\ (n_1n_3,\rho\eta\omega\upsilon)=(n_2n_4,\rho\eta\omega\upsilon)=(m_2,n_1n_3\rho\omega\upsilon)=1 \\ \mu^2(n_1n_2)=\mu^2(n_3n_4)=1}} \sum_{m_2} \sum_{l=1}^{\omega}{}^{*} \left| \sum_{\substack{m_1\sim M_1 \\ m_1\equiv l(\omega) \\ (m_1,m_2n_2n_4\upsilon)=1}} e\left(\rho\kappa\left(h_1\frac{\overline{m_1n_1\upsilon}}{m_2n_2\upsilon}-h_2\frac{\overline{m_1n_3\upsilon}}{m_2n_4\upsilon}\right)\right)\right| \tag{8}$$

We proceed to treat the argument in the above exponential sums.

We have

$$h_1\frac{\overline{m_1n_1\upsilon}}{m_2n_2\upsilon}-h_2\frac{\overline{m_1n_3\upsilon}}{m_2n_4\upsilon}=\frac{g}{m_2\frac{n_2}{(n_2,n_3)}\frac{n_4}{(n_1,n_4)}\upsilon}$$

where

$$g = h_1\frac{1}{(n_2,n_3)}\frac{n_4}{(n_1,n_4)}(m_1n_1\nu)^{*} - h_2\frac{1}{(n_1,n_4)}\frac{n_2}{(n_2,n_3)}(m_1n_3\nu)^{**}$$

with $x^{*}x\equiv 1(\mathrm{mod.}\ m_2n_2\nu)$ and $x^{**}x\equiv 1(\mathrm{mod.}\ m_2n_4\nu)$. We then find, with certain integers k_1 and k_2, that

$$m_1n_1n_3\nu g = h_1\frac{n_3}{(n_2,n_3)}\frac{n_4}{(n_1,n_4)}(1+k_1m_2n_2\nu) - h_2\frac{n_1}{(n_1,n_4)}\frac{n_2}{(n_2,n_3)}(1+k_2m_2n_4\nu)$$

$$\equiv h_1\frac{n_3\quad n_4}{(n_2,n_3)(n_1,n_4)} - h_2\frac{n_1\quad n_2}{(n_1,n_4)(n_2,n_3)}\ \left(\mathrm{mod.}\ m_2\frac{n_2\quad n_4}{(n_2,n_3)(n_1,n_4)}\nu\right)$$

$$= r, \quad \text{say.}$$

Since $(m_1n_1,m_2n_2)=(m_1n_3,m_2n_4)=1$, $\mu^2(n_1n_2)=\mu^2(n_3n_4)=1$ and $(\nu,M_2n_2n_4\nu)=(m_1n_1n_3,\nu)=(\nu,\nu)=1$, we have

$$\left(m_1n_1n_3\nu,\ m_2\frac{n_2}{(n_2,n_3)}\frac{n_4}{(n_1,n_4)}\nu\right)=1,$$

whence

$$g\equiv \overline{m_1n_1n_3\nu}\ r \quad \left(\mathrm{mod.}\ m_2\frac{n_2}{(n_2,n_3)}\frac{n_4}{(n_1,n_4)}\nu\right).$$

Thus, writing $d= m_2\frac{n_2}{(n_2,n_3)}\frac{n_4}{(n_1,n_4)}$, we get

$$\mathscr{G}\le \sum_{h_1}\sum_{h_2}\sum_{\substack{n_j(j=1,2,3,4)\\ h_1n_3n_4-h_2n_1n_2=r(n_2,n_3)(n_1,n_4)}}\ \sum_{\substack{m_2\\ m_2n_2n_4=d(n_2,n_3)(n_1,n_4)\\ (d,\rho)=1}}\ \sideset{}{^*}\sum_{l=1}^{\omega}\left|\sum_{\substack{m_1\sim M_1\\ m_1\equiv l(\omega)\\ (m_1,d(n_2,n_3)(n_1,n_4)\nu)=1}} e\left(\rho\kappa r\frac{\overline{m_1n_1n_3\nu}}{d\nu}\right)\right| .$$

Lemma 21 yields

$$\mathscr{E} \ll \sum_{h_1 n_3 n_4 = h_2 n_1 n_2} M_1 M_2 + \sum_{\substack{r \neq 0 \\ (d,\rho)=1}} \omega \cdot \tau((n_2,n_3)(n_1,n_4))(\rho\kappa r, d\upsilon)^{\frac{1}{2}} (d\upsilon)^{\frac{1}{2}+\varepsilon} (1+\frac{M_1}{d\upsilon})$$

$$\ll M_1 M_2 \sum_{s \le 4HN_1N_2} \tau_3(s)^2 + \omega\upsilon T^{\varepsilon} \sum_{r \neq 0} (\kappa r, d)^{1/2} d^{1/2} (1+ \frac{M_1}{d})$$

$$\ll M_1 M_2 H N_1 N_2 \log^8 T + \omega\upsilon T^{\varepsilon} \sum_{h_1, h_2} (\sum_d \frac{(\kappa r, d)}{d})^{1/2} \{ (\sum_d d^2)^{1/2} + M_1 (\sum 1)^{1/2} \}.$$

By Lemma 23, we have

$$\mathscr{E} \ll H^2 T^{1-3\varepsilon} \log^8 T + \omega\upsilon T^{2\varepsilon} H^2 N_1 \{ (N_1{}^2 (M_2 N_2{}^2)^3)^{1/2} + M_1 (N_1{}^2 M_2 N_2{}^2)^{1/2} \}$$

$$\ll H^2 \; T^{1-3\varepsilon} \log^8 T + \omega\upsilon T^{2\varepsilon} H^2 (M_1 + M_2) M_2{}^{1/2} N_1{}^2 N_2{}^3.$$

LEMMA 25.

If $M \le T^{\frac{1}{2}-4\varepsilon}$ and $\omega\upsilon N^5 \le T^{\frac{1}{4}-4\varepsilon}$, then $K \ll T^{1-\varepsilon}$.

Proof.

By Lemma 18,

(9) $$K \ll T \sup_{\substack{M_1, M_2 \le M \\ N_1, N_2 \le N}} (M_1 N_1 M_2 N_2)^{-1} (M_1 M_2 \; \mathscr{E} \;)^{1/2} + T^{1-\varepsilon},$$

since the conditions in this lemma implies that in Lemma 18.

Lemma 24 yields

$$\frac{M_1 M_2 \; \mathscr{E}}{(M_1 N_1 M_2 N_2)^2} \ll \frac{T^{6\varepsilon - 2}}{H^2} M_1 M_2 \; \mathscr{E}$$

$$\ll T^{4\varepsilon - 1} M_1 M_2 + T^{8\varepsilon - 2} M_1 M_2 \omega\upsilon (M_1 + M_2) M_2{}^{1/2} N_1{}^2 N_2{}^3$$

$$\ll T^{4\varepsilon-1}M^2 + T^{8\varepsilon-2}M^{3/2}\omega\upsilon N^5$$

$$\ll T^{1-2\varepsilon}.$$

Combining this with (9), we obtain Lemma 25.

LEMMA 26. If $\omega\upsilon \le T^{\varepsilon}$, then

$$K(T, T^{\frac{1}{2}-4\varepsilon}, T^{\frac{1}{20}-\varepsilon}; \rho, \kappa, \omega, \xi, \eta, \upsilon, \nu) \ll T^{1-\varepsilon}.$$

Proof.

This immediately follows from Lemma 25.

8. Sieve method.

In this section we give a lower bound sieve inequality for almost-primes of type P_2. We use Richert's weight [33,8] combined with Rosser-Iwaniec linear sieve [19,20,31]. As for a sophisticated form of the weighted sieve, see [7,9]. Let $\mathcal{A}$ be a finite sequence of integers and $\mathcal{P}$ be a set of primes.
Put, for $d\geq 1$, $z>2$,

$$P(z)=\prod_{\substack{p<z\\ p\in\mathcal{P}}} p\ ,\qquad \mathcal{A}_d=\{n\in\mathcal{A}:\ n\equiv 0 \pmod d\},$$

and

$$S(\mathcal{A},\mathcal{P},z)=\#\{n\in\mathcal{A}:\ (n,P(z))=1\}.$$

Suppose that, for square-free d composed by primes $\in\mathcal{P}$,

$$\#\mathcal{A}_d=\frac{\omega(d)}{d}X+r_d(\mathcal{A}),$$

where X is some positive number independent of d and $\omega(d)$ is multiplicative. Moreover we assume that, for any $2<w<z$,

$$(1)\qquad \prod_{\substack{w\leq p<z\\ p\in\mathcal{P}}}\left(1-\frac{\omega(p)}{p}\right)\leq\frac{\log z}{\log w}\left(1+\frac{K}{\log w}\right),$$

and

$$(2)\qquad \sum_{\substack{w\leq p<z\\ p\in\mathcal{P}}}\ \sum_{\alpha\geq 2}\frac{\omega(p^{\alpha})}{p^{\alpha}}\leq\frac{L}{\log 3w}\ ,$$

with some constants K, $L>1$. Write

$$V(z)=\prod_{\substack{p<z\\ p\in\mathcal{P}}}\left(1-\frac{\omega(p)}{p}\right).$$

LEMMA 27 [8,19,31].

Let $z>2$, $D>2$. For any $\eta>0$ we have

$$S(\mathcal{A},\mathcal{P},z)\le V(z)X\{F(s)+E\}+R^{+}$$

$$S(\mathcal{A},\mathcal{P},z)\ge V(z)X\{f(s)-E\}-R^{-}$$

where $s=\log D/\log z$, $E=c\eta+O((\log D)^{-1/3})$ with some constant c. The functions $F(s)$ and $f(s)$ are the continuous solutions of some system of differential-difference equations. In particular,

$$sF(s)=2e^{\gamma} \qquad (0<s\le 3)$$

$$sf(s)=\begin{cases} 0 & (0<s\le 2) \\ 2e^{\gamma}\log(s-1) & (2<s\le 4)\end{cases}$$

where γ is the Euler constant. The remainder term $R^{\pm}$ has the form

$$R^{\pm}=\sum_{d\mid P(z)}\mu_d^{\pm}(D,\eta)r_d(\mathcal{A})$$

where the sequence $(\mu_d^{\pm})=(\mu_d^{\pm}(D,\eta))$ has the properties:

$$\mu_d^{\pm}=0 \qquad \text{if } d\ge 0,$$

$$|\mu_d^{\pm}|\le \mu^{2}(d),$$

and, for any $M,N>1$, $MN=D$,

(3) $$\mu_d^{\pm}=\sum_{l\le\log z}\ \sum_{\substack{m\le M\\ mn=d}}\ \sum_{n\le N} a_{m,l}(M,N,\eta)b_{n,l}(M,N,\eta)$$

with $|a_{m,l}|,|b_{n,l}|\le 1$.

We proced to state the weighted sieve. For a given $\mathcal{A}$, let

$$\mathcal{P}=\{p:\ p\mid n \quad \text{for all } n\in\mathcal{A}\ \}$$

and $\max_{n\in\mathcal{A}} n\le x$. Set

$$\alpha = \frac{11}{20} - 4\varepsilon, \qquad \frac{1}{u} = \frac{1}{2} - 4\varepsilon, \qquad \frac{1}{v} = \frac{\alpha}{4} = \frac{11}{80} - \varepsilon,$$

then,

$$\frac{1}{\alpha} < u < v, \qquad \alpha v = 4, \qquad u < 3, \qquad \frac{2}{u} + \frac{1}{v} > 1.$$

Write

$$D = x^{\alpha}, \qquad y = x^{1/u}, \qquad z = x^{1/v}.$$

We define the weight w;

$$w(n) = 1 - \frac{1}{3-u} \sum_{\substack{p \mid n \\ z \le p < y \\ p \in \mathscr{P}}} \left(1 - u\frac{\log p}{\log x}\right).$$

LEMMA 28.

$$\sum_{\substack{n \in \mathscr{A} \\ (n, P(z)) = 1}} w(n) > \nabla V(z) X - \sum_{d \mid P(z)} \lambda_d r_d(\mathscr{A}).$$

Here $\nabla > \frac{e^{\gamma}}{20}(\log 10 - 11 \log \frac{40}{11} - \varepsilon)$. The sieving weights λ_d have the following property;

$$\lambda_d = 0 \qquad \text{if} \quad d > D = x^{11/20 - 5\varepsilon},$$

$$|\lambda_d| \le \mu^2(d),$$

(3) $$\lambda_d = \sum_{l \le \log^2 x} \sum_{\substack{m \le x^{1/2 - 4\varepsilon} \\ mn = d}} \sum_{n \le x^{1/20 - \varepsilon}} a_{m,l} b_{n,l},$$

where $|a_{m,l}|, |b_{n,l}| \le 1$.

Proof.

$$\sum_{\substack{n \in \mathscr{A} \\ (n, P(z)) = 1}} w(n) = \sum_{\substack{n \in \mathscr{A} \\ (n, P(z)) = 1}} \left\{1 - \frac{1}{3-u} \sum_{\substack{q \mid n \\ z \le q < y \\ q \in \mathscr{P}}} \left(1 - u\frac{\log q}{\log x}\right)\right\}$$

(4) $$= S(\mathscr{A}, \mathscr{P}, z) - \frac{1}{3-u} \sum_{z \le q < y} \left(1 - u\frac{\log q}{\log x}\right) S(\mathscr{A}_q, \mathscr{P}, z).$$

By Lemma 27,

$$(5)\qquad S(\mathscr{A},\mathscr{P},z)>V(z)X\{f(\frac{\log D}{\log z})-\varepsilon\}-\sum_{d|P(z)}\mu_d^-(D)r_d(\mathscr{A}),$$

and

$$(6)\qquad S(\mathscr{A}_q,\mathscr{P},z)<\frac{\omega(q)}{q}V(z)X\{F(\frac{\log(D/P)}{\log z})+\varepsilon\}+\sum_{e|P(z)}\mu_e^+(\frac{D}{P})r_{eq}(\mathscr{A}),$$

where P is defind by the relation, for $z\le q<y$, $P\le q<\min(y,2P)$ and $P= z2^{j-1}$ $(1\le j\le J\ ,\ z2^{J-1}<y\le z2^J)$. In conjunction with (4),(5) and (6) we have

$$\sum_n w(n)> V(z)X\{f(\frac{\log D}{\log z})-\frac{1}{3-u}\sum_P\sum_q(1-u\frac{\log q}{\log x})\frac{\omega(q)}{q}F(\frac{\log(D/P)}{\log z})-\varepsilon\}$$

$$-\sum_{d|P(z)}\mu_d^-(D)r_d(\mathscr{A})+\frac{1}{3-u}\sum_P\sum_q(1-u\frac{\log q}{\log x})\sum_{e|P(z)}\mu_e^+(\frac{D}{P})r_{eq}(\mathscr{A}).$$

$$= V(z)X\ \nabla - R_1+ R_2,\quad \text{say}.$$

First we consider ∇. Since F is monotonically decreasing,

$$\nabla > f(\frac{\log D}{\log z})-\frac{1}{3-u}\sum_{\substack{q\in\mathscr{P}\\ z\le q<y}}(1-u\frac{\log q}{\log x})\frac{\omega(q)}{q}F(\frac{\log(D/q)}{\log z})\ -\varepsilon$$

$$= f(\alpha v)-\frac{1}{3-u}\int_u^v F(v(\alpha-\frac{1}{t}))(1-\frac{u}{t})\frac{dt}{t}-\varepsilon$$

$$=\frac{2e^\gamma}{\alpha v}\{\log(\alpha v-1)-\alpha u\log\frac{v}{u}+(\alpha u-1)\log(\frac{\alpha v-1}{\alpha u-1})\}-\varepsilon$$

$$=\frac{e^\gamma}{20}(\log 10 - 11\log\frac{40}{33})-\varepsilon$$

by partial summation. Next we turn to R_j. By Lemma 27, μ^- in R_1 is decomposed into the sum of type (3). Moreover,

$$R_2=\sum_{d|P(y)}\left(\sum_{\substack{e|P(z)\\ eq=d}}\ \sum_{\substack{q\in\mathcal{P}\\ z\le q<y}}(1-u\frac{\log q}{\log x})\sum_P \lambda_e^+(\frac{D}{P})\right) r_d(\mathcal{A})$$

$$=\sum_{d|P(y)}\lambda'_d r_d(\mathcal{A}), \quad \text{say.}$$

Then, simply,

$$\lambda'_d=0 \quad \text{if} \quad d>2PM'N=x^{11/20-5\varepsilon},$$

$$|\lambda'_d|\le \mu^2(d).$$

By Lemma 27, with $\frac{D}{P}=M'N$, $M'=\frac{x^{1/2-4\varepsilon}}{2P}$, $N=x^{1/20-\varepsilon}$, we see

$$\lambda'_d=\sum_P\sum_{l'}\sum_{\substack{p\sim P\\ pm'n=d}}\ \sum_{\substack{pm'\le x^{1/2-4\varepsilon}\\ m'n|P(z)}}\ \sum_{n\le x^{1/20-\varepsilon}} a_{m',l'}(P)b_{n,l'}(P)$$

$$=\sum_{l\le\log^2 x}\ \sum_{\substack{m\le x^{1/2-4\varepsilon}\\ mn=d}}\ \sum_{n\le x^{1/20-\varepsilon}} a_{m,l}\,b_{n,l}$$

where $|a_{m,l}|,|b_{n,l}|\le 1$. Hence we get Lemma 28.

LEMMA 29.

Let $\mathcal{S}=\{n\in\mathcal{A}:\ (n,P(z))=1,\ \ p^2|n \text{ for all } p \text{ with } z\le p<y\}$.
We have that if $n\in\mathcal{S}$ and $\Omega(n)>2$, then $w(n)\le 0$.

Proof. Put

$$v(n)=\#\{p:\ p|n,\ p\in\mathcal{P},\ z\le p<y\}.$$

Firstly, if $v(n)\ge 3$ then

$$w(n)=1-\frac{1}{3-u}(v(n)-u\frac{\sum_{p|n}\log p}{\log x})$$

$$\le 1-\frac{1}{3-u}(v(n)-u)$$

$\leq 0.$

Seconcly, if $\upsilon(n)=2$ then $n=pp'm$ with $z\leq p,p'<y$ and $(m,P(z))=1$. Since $m=\frac{n}{pp'}\leq xz^{-2}=x^{1-\frac{1}{v}}<x^{\frac{2}{u}}=y^2$, m is a prime. Hence $m\geq y$, or $pp'\leq\frac{x}{m}\leq\frac{x}{y}=x^{1-\frac{1}{u}}$. Thus,

$$\begin{aligned} w(n) &= 1-\frac{1}{3-u}\left(2-u\frac{\log pp'}{\log x}\right) \\ &\leq 1-\frac{1}{3-u}\left(2-u\left(1-\frac{1}{u}\right)\right) \\ &= 0. \end{aligned}$$

Thirdly, if $\upsilon(n)=1$ then $n=pm$ with $z\leq p<y$ and $(m,P(z))=1$. Since $m=\frac{n}{p}\leq\frac{x}{z}=x^{1-\frac{1}{v}}<x^{\frac{2}{u}}=y^2$, m is a prime. So, $\Omega(n)=2$. Finally, if $\upsilon(n)=0$ then $(n,P(y))=1$. Since $n\leq x<x^{3/u}=y^3$, $\Omega(n)\leq 2$. From the above we conclude Lemma 29.

LEMMA 30.

$$\#\{\, P_2\in\mathcal{A}\,\} > CV(z)X-\sum_{d\mid P(y)}\lambda_d r_d(\mathcal{A})-\sum_{n\in\mathcal{A}}\sum_{\substack{p^2\mid n \\ x^{1/8}\leq p\leq x^{1/2-4\varepsilon}}} 1 .$$

where λ_d's have the property given in Lemma 28.

Proof. Since $w(n)\leq 1$, we have

$$\begin{aligned} \sum_{\substack{n\in\mathcal{A} \\ (n,P(z))=1}} w(n) &= \sum_{\substack{n\in\mathcal{S} \\ \Omega(n)\leq 2}} w(n)+\sum_{\substack{n\in\mathcal{S} \\ \Omega(n)>2}} w(n)+\sum_{\substack{n\in\mathcal{A} \\ (n,P(z))=1 \\ n\in\mathcal{S}}} w(n) \\ &\leq \sum_{\substack{n\in\mathcal{S} \\ \Omega(n)\leq 2}} 1+\sum_{n\in\mathcal{A}}\sum_{\substack{p^2\mid n \\ z\leq p<y}} 1, \end{aligned}$$

by Lemma 29. Combining this with Lemma 28, we get Lemma 30.

9. Proof of Theorem 3.

Let $\mathscr{A} = \{n : y-\Delta < n \le y\}$

$\mathscr{P} = \{$ all of primes $\}$

$r_d(\mathscr{A}) = \#\mathscr{A}_d - \frac{\Delta}{d}.$

Then, by Lemma 30, we have

(1) $$\#\{ P_2 : y-\Delta < P_2 \le y \} > \frac{C\Delta}{\log y} - R(y)$$

where

$$R(y) = \sum_d \lambda_d r_d(\mathscr{A}) + \sum_{\substack{y-\Delta < p^2 m \le y \\ y^{4\varepsilon} < p \le y^{1/2-4\varepsilon}}} 1$$

$$= R_1(y) + R_2(y), \quad \text{say.}$$

Let $\mathscr{E} = \mathscr{E}(x) = \{n : x < n \le 2x,\ P_2 \in (n, n+g(n)\log^5 n\,]\}.$

Put $\Delta = \min_{x<n\le 2x} g(n)\log^5 n$, then

(2) $$\Delta^{-1}\log^5 x \to \infty \quad (x\to\infty).$$

Since n is an integer, if $n\in\mathscr{E}$ then there exists an interval I_n such that

$$n\in I_n = [u, u+\tfrac{1}{3}] \quad \text{and} \quad P_2 \in [y, y+\Delta] \quad \text{for all } y\in I_n.$$

Thus, by (1), we see that

$$R(y) > \frac{C\Delta}{\log y} \quad \text{for all } y\in I_n,\ n\in\mathscr{E}.$$

Hence, we get

(3) $$\#\mathscr{E}\left(\frac{\Delta}{\log x}\right)^2 \ll \sum_{n\in\mathscr{E}} \int_{I_n} R(y)^2 dy$$

$$< \int_{x-1}^{2x+1} R(y)^2 dy$$

since the intervals I_n are mutually disjoint. Suppose

$$(4) \qquad \int_x^{2x} R(y)^2 dy \ll \Delta x \log^3 x + \Delta^2 x \log^{-3} x.$$

From (4), (3) and (2), we infer that

$$\#\mathcal{E} \ll \left(\frac{\log x}{\Delta}\right)^2 (\Delta x \log^3 x + \Delta^2 x \log^{-3} x + \Delta^2 x^{\varepsilon})$$

$$\ll \Delta^{-1} x \log^5 x + x \log^{-1} x$$

or

$$(5) \qquad \#\mathcal{E}(x) = o(x).$$

Let $\mathcal{E}_0$ denote the exceptional set of intervals, namely

$$\mathcal{E}_0 = \bigcup_{j=1}^{\infty} \mathcal{E}(x2^{-j}).$$

By (5) and the trivial bound $\mathcal{E}(x) \le x$, we obtain

$$\#\mathcal{E}_0 = \sum_{j \le \frac{\log x}{2\log 2}} o(x2^{-j}) + \sum_{j > \frac{\log x}{2\log 2}} O(x2^{-j}) = o(x).$$

It remains to establish (4). We begin with R_1. By the definiiton of $r_d(\mathcal{A})$,

$$R_1(y) = \sum_{y-\Delta<n\le y} \Big(\sum_{d|n} \lambda_d\Big) - y\Big(\sum_d \frac{\lambda_d}{d}\Big).$$

On putting

$$a_n = \Big(\sum_{d|n} \lambda_d\Big) - \Big(\sum_d \frac{\lambda_d}{d}\Big) = f(n) - F, \quad \text{say},$$

we see

$$R_1(y) = \sum_{y-\Delta<n\le y} a_n + O(\log D).$$

By Lemma 6, we have

$$\int_x^{2x} R_1(y)^2 dy << \Delta \sum_{n\sim x} a_n^2 + \sum_{0<r\le\Delta} 2(\Delta-r)\sum_{n\sim x} a_n a_{n+r} + \Delta^3(\sup|a_n|)^2 + x\log^2 x$$

(6) $$= O(\Delta x\log^3 x) + Q + O(\Delta^3 x^{\varepsilon}).$$

Also,

$$Q = \sum_{0<r\le\Delta} 2(\Delta-r)\sum_{n\sim x} f(n)f(n+r) - (\Delta^2+O(\Delta))F^2x + O(\Delta^2 D\log D)$$

(7) $$= Q_1 - \Delta^2F^2x + O(\ \Delta x\log^2 x + \Delta^2 D\log x), \quad \text{say.}$$

We proceed to evaluate Q_1.

$$Q_1 = \sum_{0<r\le\Delta} 2(\Delta-r) \sum_{d'}\sum_{d} \lambda_{d'}\lambda_d \#\{n:\ \begin{matrix} x<n\le 2x \\ n\equiv 0(d') \\ n+r\equiv 0(d)\end{matrix}\ \}.$$

The above simultaneous congruences are soluble if and only if $(d',d)|r$, and, in this case, reduce to the single congruence $n\equiv b\ (\text{mod.}[d',d])$ where

(8) $$\left\{ \begin{matrix} b\equiv 0 & (\text{mod. } d') \\ b\equiv -r & (\text{mod. } d^*) \end{matrix} \right.$$

with $d^* = d/(d',d)$. The inner sum is equal to

$$\#\{n:\ x<n\le 2x,\quad n\equiv b\ (\text{mod.}[d',d])\}$$

$$= \#\{m:\ x< b+[d',d]m \le 2x\ \}$$

$$= \frac{x}{[d',d]} + \psi\left(\frac{2x-b}{[d',d]}\right) - \psi\left(\frac{x-b}{[d',d]}\right)$$

$$= \frac{x}{[d',d]} + \phi\left(2x,x;\frac{-b}{[d',d]}\right), \quad \text{say.}$$

Thus,

$$Q_1 = \sum_{0<r\le\Delta} 2(\Delta-r) \sum\sum_{(d',d)|r} \lambda_{d'}\lambda_d \left\{ \frac{x}{[d',d]} + \phi \right\}$$

(9) $$= Q_2 + Q_3, \quad \text{say.}$$

First we carry out the summation over r in Q_2.

$$Q_2 = x \sum \sum \frac{\lambda_{d'}\lambda_d}{[d',d]} \sum_{\substack{0<r\le\Delta \\ (d',d)|r}} 2(\Delta - r).$$

We may assume $(d',d)|r$, for otherwise the inner sum is empty. Then, the sum over r is equal to

$$\frac{\Delta^2}{(d',d)} + O(\Delta).$$

Hence,

$$Q_2 = \Delta^2 x \sum_{(d',d)\le\Delta} \sum \frac{\lambda_{d'}\lambda_d}{[d',d]} + O(\Delta x \sum_{(d',d)\le\Delta} \sum \frac{1}{[d',d]})$$

$$= \Delta^2 x\{ (\sum_d \frac{\lambda_d}{d})^2 + O(\sum_{(d',d)>\Delta} \sum \frac{1}{d'd})\} + O(\Delta x \log^3 x)$$

$$(10) \qquad = \Delta^2 F^2 x + O(\Delta x \log^3 x).$$

We proceed to estimate Q_3. By the definition (8) of b and the reciprocity relation

$$\frac{\overline{m}}{n} + \frac{\overline{n}}{m} \equiv \frac{1}{mn} \pmod{1} \quad \text{for } (m,n)=1,$$

we have

$$-\frac{b}{[d',d]} \equiv -\frac{b\overline{d^*}}{d'} - \frac{b\overline{d'}}{d^*} \equiv r\frac{\overline{d'}}{d^*} \equiv \frac{r}{(d',d)}\frac{\overline{d'^*}}{d^*} \pmod{1}$$

since $\mu^2(d') = \mu^2(d) = 1$ and $(d',d)|r$. Next we decompose (λ_d).

$$\lambda_d = \sum_{l\le\log^2 D} \sum_{\substack{em\le M \\ ef=(d',d)}} \sum_{\substack{fn\le N \\ mn=d^*}} a_{em,l}(M,N) b_{fn,l}(M,N)$$

where $M = (2X)^{1/2-4\varepsilon}$ and $N = (2x)^{1/20-\varepsilon}$. Thus,

$$Q_3 = \sum_{0<r\le\Delta} 2(\Delta-r) \sum_{(d',d)\mid r} \sum_{\substack{l\le\log^2 D\\ l'\le\log^2 D}} \sum_{\substack{em,e'm'\le M\\ ef=e'f'=(d',d)\\ mn=d^*}} \sum_{\substack{fn,f'n'\le N\\ m'n'=d'^*}} \cdot$$

$$\cdot a_{em,l} b_{fn,l} a_{e'm',l'} b_{f'n',l'} \cdot \phi(2x,x;\frac{r}{(d',d)} \frac{\overline{d'^*}}{d^*}).$$

$$<< \Delta \sum_{0<\delta k\le\Delta} \tau(\delta)^2 \log^4 D \sup_{\substack{|a|,|b|\le 1\\ |a'|,|b'|\le 1}} \Big| \sum_{\substack{m,m'\le M\\ n,n'\le N\\ (mn,m'n')=1}} a(m(b(n)a'(m')b'(n')\phi(2x,x;k\frac{\overline{m'n'}}{mn}) \Big|$$

$$<< \Delta \sum_{0<\delta k\le\Delta} \tau(\delta)^2 \log^8 D \sup_{\substack{|a|,|b|,|a'|,|b'|\le 1\\ M,M'\le M \quad N,N'\le N}} \cdot$$

$$\cdot \Big| \sum_{\substack{m\sim M\\ n\sim N}} \sum_{\substack{m'\sim M'\\ n'\sim N'\\ (mn,m'n')=1}} a(m)b(n)a(m')b'(n')\phi(2x,x;k\frac{\overline{m'n'}}{mn}) \Big|$$

$$<< \Delta^2 \log^{12} x \cdot \{ x^{1-2\varepsilon} + K(2x,M,N;\ 1,k,1,1,1,1,1) \}$$

with the notation in section 7. Lemma 26 yields

(11) $$Q_3 << \Delta^2 x^{1-\varepsilon}.$$

In conjunction with (6),(7),(9),(10) and (11) we obtain

(12) $$\int_x^{2x} R_1(y)^2 dy << \Delta x \log^3 x + \Delta^2 x^{1-\varepsilon}.$$

We turn to R_2. Since

$$R_2(y) < \sum_{x^{2\varepsilon}\le p\le x^{1/2-2\varepsilon}} \sum_{y-\Delta<p^2 m\le y} 1 << x^{\varepsilon}\Delta,$$

we have

$$\int_x^{2x} R_2(y)^2 dy << x^{\varepsilon}\Delta^2 \sum_{x^{2\varepsilon}\le p\le x^{1/2-2\varepsilon}} \sum_{x-\Delta<p^2 m\le 2x} 1$$

$$\ll x^{\varepsilon}\Delta^{2} \sum_{x^{2\varepsilon}\le p\le x^{1/2-2\varepsilon}} \left(\frac{x}{p^{2}} + 1\right)$$

$$\ll \Delta^{2}x^{1-\varepsilon}.$$

combining this with (12), we get the required bound in (4). This completes the proof of Theorem 3.

10. Proof of Theorem 4.

Let
$$\mathscr{A} = \{n:\ x<n\le 2x,\quad n\equiv a \ (\mathrm{mod.}q)\}$$
$$\mathscr{P} = \{p:\ p\mid q\ \}$$
$$r_d(\mathscr{A}) = \#\mathscr{A}_d - \frac{x}{qd}.$$

Then, by Lemma 30, we have

$$(1)\qquad \#\{\ P_2\in\mathscr{A}\ \} > \frac{Cx}{\varphi(q)\log 2x} - R(a)$$

where

$$R(a)=R(x;q,a)= \sum_d \lambda_d r_d(\mathscr{A}) + \sum_{x^{2\varepsilon}\le p\le x^{1/2-2\varepsilon}} \sum_{p^2 m\in\mathscr{A}} 1$$

$$= R_1(a) + R_2(a), \quad \text{say.}$$

Let $\mathscr{E}$ denote the exceptional set of reduced classes modulo q, namely,

$$\mathscr{E} = \{a: \begin{matrix}1\le a\le q\\ (a,q)=1\end{matrix}\ ,\ \#\{P_2: \begin{matrix}P_2\equiv a\ (\mathrm{mod.}q)\\ P_2\le g(q)q\log^5 q\end{matrix}\}=0\ \}.$$

Put $2X= g(q)q\log^5 q$ in (1). It follows from (1) that

$$(2)\qquad R(a) > \frac{Cx}{\varphi(q)\log 2X} \qquad \text{for all } a\in\mathscr{E}$$

since the left hand side of (1) is zero. Suppose

$$(3)\qquad \sum_{a=1}^{q}{}^{*} R(a)^2 << x\log^3 x + x^2 q^{-1}\log^{-3} x.$$

Then we infer from this assumption and (2) that

$$\#\mathscr{E}\left(\frac{x}{\varphi(q)\log x}\right)^2 << \sum_{a\in\mathscr{E}} R(a)^2 \le \sum_{a=1}^{q}{}^{*} R(a)^2$$
$$<< x\log^3 x + x^2 q^{-1}\log^{-3} x$$

or

$$\frac{\#\mathcal{E}}{\varphi(q)} \ll \varphi(q)x^{-1}\log^5 x + \log^{-1} x$$

$$\ll g(q)^{-1} + \log^{-1} q.$$

This gives, apart from the verification of (3), the proof of Theorem 4.

We proceed to prove (3). We begin with R_1. By the definition of $r_d(\mathcal{A})$ we have

$$R_1(a) = \sum_{n\in\mathcal{A}} \Big(\sum_{\substack{d|n \\ (d,q)=1}} \lambda_d \Big) - \frac{x}{q} \sum_{(d,q)=1} \frac{\lambda_d}{d} = \sum_{n\in\mathcal{A}} f(n) - \frac{x}{q}F, \quad \text{say.}$$

Then,

$$\sum_{a=1}^{q}{}^{*} R_1(a)^2 \le \sum_{a=1}^{q} \Big(\sum_{n\in\mathcal{A}} f(n) - \frac{x}{q}F \Big)^2$$

$$= \sum_{a=1}^{q} \sum_{\substack{n,n'\sim x \\ n,n'\equiv a(q)}} f(n)f(n') - 2\frac{x}{q}F \sum_{a=1}^{q} \sum_{\substack{n\sim x \\ n\equiv a(q)}} f(n) + \Big(\frac{x}{q}F\Big)^2 q$$

(4) $$= W - 2V + U, \quad \text{say.}$$

First we consider V.

$$V = \frac{x}{q}F \sum_{(d,q)=1} \lambda_d \sum_{\substack{n\sim x \\ d|n}} 1$$

$$= \frac{x}{q}Fx \sum_{(d,q)=1} \frac{\lambda_d}{d} + O\Big(\frac{x}{q}|F|D\Big)$$

(5) $$= U + O\Big(\frac{x}{q}D\log D\Big).$$

Next,

$$W = \sum_{\substack{n,n'\sim x \\ n\equiv n'(q)}} f(n)f(n')$$

$$= \sum_{n\sim x} f(n)^2 + 2 \sum_{\substack{n<n'\sim x \\ n\equiv n'(q)}} \sum f(n)f(n')$$

$$= O\Big(\sum_{n\sim x} \tau(n)^2\Big) + 2 \sum_{0<ql\le x} \sum_{x<n\le 2x-ql} f(n)f(n+ql)$$

(6) $$= O(x\log^3 x) + W_1, \quad \text{say.}$$

Also,

$$W_1 = 2 \sum_{0<l\le x/q} \sum_{(d'd,q)=1} \sum \lambda_{d'}\lambda_d \ \#\{n:\ \begin{matrix} x<n\le 2x \\ n\equiv 0(d') \\ n+ql\equiv 0(d) \end{matrix}\}.$$

The above simultaneous congruences are soluble if and only if $(d',d)\mid l$ and reduce to the single congruence $n\equiv b \pmod{[d',d]}$ where

$$\begin{cases} b\equiv 0 & (\mathrm{mod.}\, d') \\ b\equiv -ql & (\mathrm{mod.}\, d^*) \end{cases}$$

with $d^* = d/(d',d)$. Then,

$$\#\{n:\ \begin{matrix} x<n\le 2x-ql \\ n\equiv b([d',d]) \end{matrix}\} = \#\{m:\ \frac{x-b}{[d',d]} < m \le \frac{2x-ql-b}{[d',d]}\}$$

$$= \frac{x-ql}{[d',d]} + \psi\Big(\frac{2x-ql-b}{[d',d]}\Big) - \psi\Big(\frac{x-b}{[d',d]}\Big)$$

$$= \frac{x-ql}{[d',d]} + \Phi, \quad \text{say.}$$

Also,

(7) $$W_1 = 2 \sum_{l\le x/q} \sum_{(d',d)\mid l} \sum \lambda_{d'}\lambda_d\Big\{\frac{x-ql}{[d',d]} + \Phi\Big\} = W_2 + W_3, \quad \text{say.}$$

We carry out the summation over l in W_2:

$$W_2 = \sum_{d'} \sum_{\substack{d \\ (d'd,q)=1}} \frac{\lambda_{d'}\lambda_d}{[d',d]} \sum_{\substack{l\le x/q \\ (d',d)\mid l}} 2(x-ql).$$

If $(d',d) > x/q$ then the inner sum is empty, so we may assume $(d',d)\le x/q$. The sum over l is equal to

$$\frac{x^2}{q(d',d)} + O(x).$$

Hence,

$$W_2 = \sum_{\substack{(d'd,q)=1 \\ (d',d)\le x/q}} \sum \frac{\lambda_{d'}\lambda_d}{[d',d]} \{\frac{x^2}{q(d',d)} + O(x)\}$$

$$= \frac{x^2}{q} \sum_{\substack{(d',d)\le x/q \\ (d'd,q)=1}} \sum \frac{\lambda_{d'}\lambda_d}{d'd} + O(x \sum_{(d',d)\le x/q} \sum \frac{|\lambda_{d'}\lambda_d|}{[d',d]})$$

$$= \frac{x^2}{q}\{(\sum_{(d,q)=1} \frac{\lambda_d}{d})^2 + O(\sum_{(d',d)>x/q} \sum \frac{1}{d'd})\} + O(x\log^3 x)$$

(8) $$= U + O(x\log^3 x).$$

We turn to W_3:

$$W_3 = 2 \sum_{l} \sum_{(d',d)|l} \sum \lambda_{d'}\lambda_d \, \Phi(2x-ql, x; -\frac{b}{[d',d]}).$$

By the definition of b (mod.[d',d]) and the reciprocity relation we have

$$-\frac{b}{[d',d]} \equiv q\frac{l}{(d',d)} \frac{\overline{d'^*}}{d^*} \text{ (mod.1)}$$

since $\mu^2(d')=\mu^2(d)=1$ and $(d',d)|l$. By a similar argument to that of the previous section we have

$$W_3 \ll \frac{x}{q} \log^{12} x \{ x^{1-2\varepsilon} + K(2x, x^{1/2-4\varepsilon}, x^{1/20-\varepsilon}; q,k,1,1,1,1,1)\}$$

with the notation in section 7. Lemma 26 yields

(9) $$W_3 \ll \frac{x^{2-\varepsilon}}{q} .$$

In conjunction with)4),(5),(6),(7),(8) and (9) we get

$$\sum_{a=1}^{q}{}^{*} R_1(a)^2 \le O(x\log^3 x) + U + O\left(\frac{x^{2-\varepsilon}}{q}\right) - 2U + O\left(\frac{x}{q} D\log D\right) + U \tag{10}$$

$$<< x\log^3 x + x^2 q^{-1}\log^{-3} x.$$

It remains to deal with R_2. Since $R_2(a) << x^{\varepsilon}\frac{x}{q}$, we have

$$\sum_{a=1}^{q}{}^{*} R_2(a)^2 << x^{\varepsilon}\frac{x}{q} \sum_{x^{2\varepsilon} \le p \le x^{1/2-2\varepsilon}} \sum_{p^2 m \sim x} 1$$

$$<< x^{2-\varepsilon} q^{-1}.$$

Combining this with (10),

$$\sum_{a=1}^{q}{}^{*} R(a)^2 << x\log^3 x + x^2 q^{-1}\log^{-3} x.$$

This is the required bound in (7). Our proof is complete.

11. Dispersion method.

In this section we consider the dispersion

$$\mathfrak{D}=\sum_{\substack{Q<q\le 2Q\\(q,a)=1}}\Bigl|\sum_{\substack{a<n\le x\\ n\equiv a(q)\\(n,a)=1}}\Bigl(\sum_{\substack{d|n\\(d,q)=1}}\lambda_d\Bigr)-\Bigl(\sum_{(d,aq)=1}\frac{\lambda_d}{d}\Bigr)\sum_{\substack{a<n\le x\\ n\equiv a(q)\\(n,a)=1}}1\Bigr|^2$$

where (λ_d) are the sieving weights of level D. We follow the argument of [15] with a minor modification. We use the elementary lemma;

LEMMA 31. For $(c,d)=1$,

$$\#\{m:\ \substack{\alpha<m\le\beta\\ m\equiv e(\mathrm{mod}.d)\\(m,c)=1}\}=\frac{\varphi(c)}{c}\,\frac{\beta-\alpha}{d}+O(\tau(c)).$$

Expanding the square, we see

(1) $$\mathfrak{D}=W-2V+U,\quad \text{say.}$$

First we consider W. By the definition,

$$W=\sum_{\substack{Q<q\le 2Q\\(q,a)=1}}\ \sum_{\substack{a<n\le x\\ n\equiv a(q)\\(n,a)=1}}\Bigl(\sum_{\substack{d|n\\(d,q)=1}}\lambda_d\Bigr)\sum_{\substack{a<n'\le x\\ n'\equiv a(q)\\(n',a)=1}}\Bigl(\sum_{\substack{d'|n'\\(d',n')=1}}\lambda_{d'}\Bigr)$$

$$=\sum_{\substack{Q<q\le 2Q\\(q,a)=1}}\ \sum_{\substack{(dd',aq)=1\\ n\ne n'}}\sum\lambda_d\lambda_{d'}\sum_{\substack{a<n\le x\\ n\equiv a(q)\\(n,a)=1}}1\sum_{\substack{a<n'\le x\\ n'\equiv a(q)\\(n',a)=1}}1+O\Bigl(\sum_q\sum_{\substack{a<n\le x\\ n\equiv a(q)}}\tau(n)^2\Bigr).$$

We express the congruential condition $n\equiv a(q)$, $n'\equiv a(q)$ as $n=a+q\ell$, $n'=a+q\ell'$. Then, the condition for ℓ and ℓ' is

$$\ell\ne\ell'\le\frac{x-a}{q},\qquad\begin{cases}a+q\ell\equiv 0\ (\mathrm{mod}.d)\\ a+q\ell'\equiv 0\ (\mathrm{mod}.d')\end{cases},\qquad (\ell\ell',a)=1.$$

Since $(\ell\ell',a)=1$, $(\ell,d)=(\ell',d')=1$. Changing the order of summation, we have

$$W = \sum_{(dd',a)=1}\sum \lambda_d \lambda_{d'} \sum_{\substack{\ell \neq \ell' \leq L \\ (\ell\ell',a)=1 \\ (\ell,d)=(\ell',d')=1}}\sum \#\left\{q:\ \begin{matrix} Q<q\leq Q'(\ell,\ell') \\ q\equiv -a\ell\ (d) \\ q\equiv -a\ell'(d') \\ (q,add')=1 \end{matrix}\right\} + O\Big(\sum_{a<n\leq x} \tau(n)^2\tau(n-a)\Big)$$

where $L=\frac{x-a}{Q}$ and $Q'=Q'(\ell,\ell')=\min(2Q, \frac{x-a}{\ell}, \frac{x-a}{\ell'})$. We consider the innermost sum. The simultaneous congruences are soluble if and only if $\ell\equiv \ell'(\mathrm{mod.}(d,d'))$ and expressed as the single congruence

$$q\equiv b\ (\mathrm{mod.}[d,d'])$$

where

$$(2)\qquad \begin{cases} b\equiv -a\overline{\ell}\ \ (\mathrm{mod.}\,d) \\ b\equiv -a\overline{\ell'}\ \ (\mathrm{mod.}\,d'^{*}). \end{cases}$$

with $d'^{*}\equiv d'/(d,d')$. This congruence then absorbs the condition $(q,dd')=1$. Hence the sum over q is, under the restriction $\ell\equiv \ell'\ (\mathrm{mod.}(d,d'))$, equal to

$$\#\left\{q:\ \begin{matrix} Q<q\leq Q'(\ell,\ell') \\ q\equiv b\ ([d,d']) \\ (q,a)=1 \end{matrix}\right\}$$

$$= \frac{\varphi(a)}{a}\,\frac{Q'-Q}{[d,d']} + \Xi, \qquad \text{say.}$$

Thus,

$$W = \sum_{(dd',a)=1}\sum \lambda_d\lambda_{d'} \sum_{\substack{\ell\neq\ell'\leq L \\ (\ell,ad)=(\ell',ad')=1 \\ \ell\equiv\ell'((d,d'))}}\sum \left(\frac{\varphi(a)}{a}\,\frac{Q'-Q}{[d,d']} + \Xi\right) + O(x\log^4 x)$$

$$(3)\qquad = W_0 + \Gamma + O(x\log^4 x), \qquad \text{say.}$$

Now we carry out the summation over ℓ and ℓ' in W_0. Write

$$\sum_{\ell\neq\ell'} = \sum\sum_{\substack{\ell-\ell'\leq L \\ (\ell,ad)=(\ell',ad')=1 \\ \ell\equiv\ell'((d,d'))}} \left(\min\left(2Q,\ \frac{x-a}{\ell},\ \frac{x-a}{\ell'}\right) - Q\right) = \sum_{\ell<\ell'} + \sum_{\ell>\ell'}.$$

Then,

$$\sum_{\ell<\ell'} = \sum_{\substack{\ell'\leq L \\ (\ell',ad')=1}} \left(\min\left(2Q,\ \frac{x-a}{\ell'}\right) - Q\right) \#\left\{\ell:\ \begin{matrix}\ell<\ell' \\ \ell\equiv\ell'((d,d')) \\ (\ell,ad)=1\end{matrix}\right\}.$$

By Lemma 31, the inner sum is equal to

$$\#\left\{\ell:\ \begin{matrix}\ell<\ell' \\ \ell\equiv\ell'((d,d')) \\ (\ell,ad^*)=1\end{matrix}\right\} = \frac{\varphi(ad^*)}{ad^*}\ \frac{\ell'}{(d,d')} + O(\tau(ad^*))$$

$$= \frac{1}{\varphi((d,d'))} \sum_{\substack{\ell<\ell' \\ (\ell,ad)=1}} 1 + O(\tau(ad^*)),$$

whence

$$\sum_{\ell<\ell'} = \frac{1}{\varphi((d,d'))} \sum_{\substack{\ell'\leq L \\ (\ell',ad')=1}} \left(\min\left(2Q,\ \frac{x-a}{\ell'}\right) - Q\right) \sum_{\substack{\ell<\ell' \\ (\ell,ad)=1}} 1 + O(\tau(ad^*)LQ)$$

$$= \frac{1}{\varphi((d,d'))} \sum\sum_{\substack{\ell<\ell'\leq L \\ (\ell,ad)=(\ell',ad')=1}} \left(\min\left(2Q,\ \frac{x-a}{\ell},\ \frac{x-a}{\ell'}\right) - Q\right) + O(\tau(ad^*)x).$$

The same argument applies to the symmetric sum in ℓ and ℓ'. Hence we get

$$\sum_{\ell\neq\ell'} = \frac{1}{\varphi((d,d'))} \sum\sum_{\substack{\ell\neq\ell'\leq L \\ (\ell,ad)=(\ell',ad')=1}} (Q'-Q) + O(\tau(a)x(\tau(d^*)+\tau(d'^*)))$$

or

$$W_0 = \sum_{(dd',a)=1}\sum \frac{\lambda_d\lambda_{d'}}{dd'}\ \frac{\varphi(a)}{a}\ \frac{(d,d')}{\varphi((d,d'))} \sum\sum_{\substack{\ell\neq\ell'\leq L \\ (\ell,ad)=(\ell',ad')=1}} (Q'-Q) + O(\tau(a)x\log^4 x)$$

(4) $$= W_1 + O(\tau(a)x\log^4 x), \quad \text{say.}$$

We turn to V:

$$V = \sum_{\substack{Q<q\le 2Q\\(q,a)=1}} \sum_{\substack{a<n'\le x\\ n'\equiv a(q)\\(n',a)=1}} \Big(\sum_{\substack{d'|n'\\(d',q)=1}} \lambda_{d'} \Big) \Big(\sum_{(d,aq)=1} \frac{\lambda_d}{d} \Big) \sum_{\substack{a<n\le x\\ n\equiv a(q)\\(n,a)=1}} 1$$

$$= \sum_{\substack{Q<q\le 2Q\\(q,a)=1}} \sum_{(d'd,aq)=1}\sum \lambda_{d'} \frac{\lambda_{d'}}{d} \sum_{\substack{a<n'\le x\\ n'\equiv a(q)\\ n'\equiv 0(d')\\(n',a)=1}} 1 \sum_{\substack{a<n\le x\\ n\equiv a(q)\\(q,a)=1\\ n\ne n'}} 1 + O(\log x \cdot \sum_{a<n\le x} \tau(n)\tau(n-a)).$$

As before we write $n'= a+q\ell'$ and $n= a+q\ell$. Then the condition for ℓ' and ℓ is

$$\ell'\ne\ell\le \frac{x-a}{q}, \quad a+q\ell'\equiv 0 \ (d'), \quad \begin{cases}(\ell'\ell,a)=1\\(\ell',d')=1\end{cases} .$$

Changing the order of summaton we have

$$V = \sum_{(d'd,a)=1}\sum \lambda_{d'} \frac{\lambda_d}{d} \sum_{\substack{\ell'\ne\ell\le \frac{x-a}{Q}= L\\(\ell'\ell,a)=1\\(\ell',d')=1}}\sum \#\{q:\ \substack{Q<q\le Q'\\ q\equiv\ -a\overline{\ell'}(\mathrm{mod}.d')\\(q,ad'd)=1}\} + O(x\log^3 x).$$

By Lemma 31, The innermost sum is equal to

$$\#\{q:\ \substack{Q<q\le Q'\\ q\equiv\ -a\overline{\ell'}(\mathrm{mod}.d')\\(q,ad^*)=1}\} = \frac{\varphi(ad^*)}{ad^*}\,\frac{Q'-Q}{d'} + O(\tau(ad^*)),$$

whence

$$V = \sum_{(d'd,a)=1}\sum \frac{\lambda_{d'}\lambda_d}{d'd}\,\frac{\varphi(ad^*)}{ad^*} \sum_{\substack{\ell'\ne\ell\le L\\(\ell'\ell,a)=1\\(\ell',d')=1}}\sum (Q'-Q) + O(\ \tau(a)\tfrac{x}{Q}D\log^3 x+ x\log^3 x)$$

(5) $$= \sum_{(d'd,a)=1}\sum \frac{\lambda_{d'}\lambda_d}{d'd}\,\frac{\varphi(a)}{a}\,\mathcal{L}(d',d) + O(\ \tau(a)\tfrac{x}{Q}D\log^3 x+ x\log^3 x), \quad \text{say.}$$

We proceed to evaluate $\mathcal{L}$:

$$\mathcal{L} = \frac{\varphi(d^*)}{d^*} \sum_{\substack{\ell' \neq \ell \leq L \\ (\ell', ad')=1 \\ (\ell, a)=1}} \left(\min\left(2Q, \frac{x-a}{\ell'}, \frac{x-a}{\ell}\right) - Q\right)$$

$$= \sum_{\substack{\ell \leq L \\ (\ell, a)=1}} \left(\min\left(2Q, \frac{x-a}{\ell}\right) - Q\right) \frac{\varphi(d^*)}{d^*} \#\{\ell' : \substack{\ell' < \ell \\ (\ell', ad')=1}\}$$

$$+ \sum_{\substack{\ell' \leq L \\ (\ell', ad')=1}} \left(\min\left(2Q, \frac{x-a}{\ell'}\right) - Q\right) \frac{\varphi(d^*)}{d^*} \#\{\ell : \substack{\ell < \ell' \\ (\ell, a)=1}\}.$$

Lemma 31 yields

$$\frac{\varphi(d^*)}{d^*} \#\{\ell' : \substack{\ell' < \ell \\ (\ell', ad')=1}\} = \frac{\varphi([d', d])}{[d', d]} \#\{\ell' : \substack{\ell' < \ell \\ (\ell', a)=1}\} + O(\tau(ad')),$$

and

$$\frac{\varphi(d^*)}{d^*} \#\{\ell : \substack{\ell < \ell' \\ (\ell, a)=1}\} = \frac{(d', d)}{\varphi((d', d))} \#\{\ell : \substack{\ell < \ell' \\ (\ell, ad')=1}\} + O(\tau(ad')\log x).$$

Thus,

$$\mathcal{L} = \frac{\varphi([d', d])}{[d', d]} \sum_{\substack{\ell' < \ell \leq L \\ (\ell'\ell, a)=1}} (Q'-Q) + \frac{(d', d)}{\varphi((d', d))} \sum_{\substack{\ell < \ell' \leq L \\ (\ell', ad')=(\ell, ad)=1}} (Q'-Q) + O(\tau(ad')x\log x).$$

Combining this with (5) we have

$$V = \sum_{(d'd, a)=1} \sum \frac{\lambda_{d'}\lambda_d}{d'd} \frac{\varphi(a)}{a} \left\{ \frac{\varphi([d', d])}{[d', d]} \sum_{\substack{\ell' < \ell \leq L \\ (\ell'\ell, a)=1}} (Q'-Q) + \frac{(d', d)}{\varphi((d', d))} \sum_{\substack{\ell < \ell' \leq L \\ (\ell', ad')=(\ell, ad)=1}} (Q'-Q) \right\}$$

(6) $$+ O\left(\tau(a)x\log^4 x + \tau(a)\frac{x}{Q}D\log^3 x \right).$$

Interchanging the role of (d', ℓ') with that of (d, ℓ), we may obtain the corresponding expression to (6). Hence,

(7) $$2V = U_1 + W_1 + O\left(\tau(a)x\log^4 x + \tau(a)\frac{x}{Q}D\log^3 x \right)$$

where

$$U_1 = \sum_{(d'd,a)=1}\sum \frac{\lambda_{d'}\lambda_d}{d'd}\frac{\varphi(a[d',d])}{a[d',d]}\sum_{\substack{\ell'\neq\ell\leq L\\(\ell'\ell,a)=1}}\sum (Q'-Q).$$

Finaly we consider U. By the same argument as above, we have

$$U = \sum_{\substack{Q<q\leq 2Q\\(q,a)=1}}\Big(\sum_{(d',aq)=1}\frac{\lambda_{d'}}{d'}\Big)\Big(\sum_{(d,aq)=1}\frac{\lambda_d}{d}\Big)\sum_{\substack{a<n'\leq x\\n'\equiv a(q)\\(n',a)=1}}1\sum_{\substack{a<n\leq x\\n\equiv a(q)\\(n,a)=1}}1$$

$$= \sum_{\substack{Q<q\leq 2Q\\(q,a)=1}}\Big(\sum_{(d',aq)=1}\frac{\lambda_{d'}}{d'}\Big)\Big(\sum_{(d,aq)=1}\frac{\lambda_d}{d}\Big)\sum_{\substack{\ell'\neq\ell\leq\frac{x-a}{q}\\(\ell'\ell,a)=1}}\sum 1 + O(\log^2 x\sum_{a<n\leq x}\tau(n-a))$$

$$= \sum_{(d'd,a)=1}\sum\frac{\lambda_{d'}\lambda_d}{d'd}\sum_{\substack{\ell'\neq\ell\leq\frac{x-a}{Q}\\(\ell'\ell,a)=1}}\sum \#\Big\{q:\ \begin{matrix}Q<q\leq Q'\\(q,a[d',d])=1\end{matrix}\Big\} + O(x\log^3 x)$$

$$(8)\qquad = U_1 + O(\ \tau(a)(\tfrac{x}{Q})^2\log^4 x + x\log^3 x\ ).$$

From (1),(3),(4),(7) and (8) we infer the following lemma.

LEMMA 32.

$$\mathfrak{D} = \Gamma + O(\ \tau(a)x\log^4 x + \tau(a)(\tfrac{x}{Q})D\log^3 x\ ).$$

Here,

$$\Gamma = \sum_{(dd',a)=1}\sum \lambda_d\lambda_{d'}\sum_{\substack{\ell\neq\ell'\leq(x-a)/Q\\(\ell,ad)=(\ell',ad')=1\\\ell\equiv\ell'((d,d'))}}\sum\left(\sum_{\substack{Q<q\leq Q'\\q\equiv b([d,d'])\\(q,a)=1}}1 - \frac{\varphi(a)}{a}\frac{Q'-Q}{[d,d']}\right)$$

with b defined by (2).

Now, we proceed to Γ. ℓ is uniquely decomposed into the form $\ell = \ell^{**}\ell^{\#}$ for which

(9) $$(\ell^{**},\ \ell^{\#}) = 1$$

(10) $$(\ell',\ell^{**}) = 1$$

(11) $$p\,|\,\ell^{\#} \quad \text{implies} \quad p\,|\,(\ell,\ell').$$

Then, since $(d,\ell)=(d',\ell')=1$,

(12) $$(dd',\ell^{\#}) = 1.$$

We write $\ell^{*} = \ell/(\ell,\ell')$ and $\ell'^{*} = \ell'/(\ell,\ell')$.

LEMMA 33.

For b defined by (2) and $e\,|\,a$, we have

$$-\frac{b\bar{e}}{[d,d']} \equiv \frac{a/e}{[d,d']\ell} + \theta\left(\frac{d^{*}}{d'^{*}}\right) \quad (\mathrm{mod}.1)$$

where, with the notation in secton 7,

$$\theta\left(\frac{d^{*}}{d'^{*}}\right) = \theta\left(\frac{a}{e},\ \frac{\ell^{*}-\ell'^{*}}{(d,d')},\ \ell^{\#},\ (d,d')\ell^{**},\ \ell^{**},\ \ell^{\#}\ell'^{*}\right).$$

Proof.

By the reciprocity relation

(13) $$\frac{\bar{m}}{n}+\frac{\bar{n}}{m} = \frac{1}{mn} \quad (\mathrm{mod}.1) \qquad \text{for} \quad (m,n)=1,$$

and the definition of b,

$$-\frac{b\bar{e}}{[d,d']} \equiv -\frac{b\overline{d'^{*}e}}{d} - \frac{b\overline{de}}{d'^{*}}$$

$$\equiv \frac{a\ell\overline{d'^{*}e}}{d} + \frac{a\overline{\ell'de}}{d'^{*}}$$

$$\equiv \frac{a}{e}\left(\frac{\overline{d'^{*}\ell}}{d} + \frac{\overline{d\ell'}}{d'^{*}}\right)$$

$$\equiv \frac{a}{e}\left(\frac{1}{dd'^{*}\ell} - \frac{\bar{d}}{d'^{*}\ell} + \frac{\overline{d\ell'}}{d'^{*}}\right) \quad (\mathrm{mod}.1)$$

snce $\mu^2(d)=\mu^2(d')=1$ and $(d,\ell)=1$. Moreover, because of (12) and (9),

$$(15)\qquad \frac{\overline{d}}{d'^{*}\ell} \equiv \frac{\overline{d}}{d'^{*}\ell^{**}\ell^{\#}} \equiv \frac{\overline{d\ell^{\#}}}{d'^{*}\ell^{**}} + \frac{\overline{dd'^{*}\ell^{**}}}{\ell^{\#}} \quad (\mathrm{mod}.1)$$

by (13).

Here, let

$$(16)\qquad g= -(d\ell^{\#})^{\dagger\dagger}+ (d\ell')^{\dagger}\ell^{**}$$

where $m^{\dagger\dagger}m \equiv 1\ (\mathrm{mod}.d'^{*}\ell^{**})$ and $m^{\dagger}m \equiv 1\ (\mathrm{mod}.d'^{*})$. We then have, with a certain integer k,

$$\begin{aligned} d\ell^{\#}\ell'^{*}g &= -(d\ell^{\#})^{\dagger\dagger}d\ell^{\#}\ell'^{*}+ (d\ell')^{\dagger}d\ell'^{*}\ell^{\#}\ell^{**} \\ &\equiv -\ell'^{\#}+ (d\ell')^{\dagger}d\ell'^{*}\ell \qquad (\mathrm{mod}.d'^{*}\ell^{**}) \\ &= -\ell'^{*}+ (d\ell')^{\dagger}d\ell'\ell^{*} \\ &= -\ell'^{*}+ (1+ kd'^{*})\ell^{*} \\ &\equiv \ell^{*}- \ell'^{*} \qquad (\mathrm{mod}.d'^{*}\ell^{**}) \end{aligned}$$

since $\ell^{**}|\ell^{*}$ by (9) and (11). By (9),(10) and (12), we see $(d\ell^{\#}\ell'^{*}, d'^{*}\ell^{**})= 1$. Hence,

$$g \equiv \overline{d\ell^{\#}\ell'^{*}}(\ell^{*}-\ell'^{*}) \qquad (\mathrm{mod}.d'^{*}\ell^{**}).$$

Since $((d,d'), \ell\ell')=1$, the condition $(d,d')|\ell-\ell'$ implies $(d,d')|\ell^{*}-\ell'^{*}$. Therefore we have

$$(17)\qquad g \equiv \frac{\ell^{*}-\ell'^{*}}{(d,d')}\ \overline{d^{*}\ell^{\#}\ell'^{*}} \qquad (\mathrm{mod}.d'^{*}\ell^{**}).$$

In conjunction with (14),(15),(16) and (17), we get

$$- \frac{b\bar{e}}{[d,d']} \equiv \frac{a}{e}\left(\frac{1}{dd'^{*}\ell} - \frac{\overline{dd'^{*}\ell^{**}}}{\ell^{\#}} - \frac{\overline{d\ell^{\#}}}{d'^{*}\ell^{**}} + \frac{\overline{d\ell'}}{d'^{*}} \right)$$

$$\equiv \frac{a}{e}\left(\frac{1}{[d,d']\ell} - \frac{\overline{[d,d']\ell^{**}}}{\ell^{\#}} + \frac{g}{d'^{*}\ell^{**}} \right)$$

$$\equiv \frac{a}{e}\left(\frac{1}{[d,d']\ell} - \frac{\overline{d^{*}d'^{*}(d,d')\ell^{**}}}{\ell^{\#}} + \frac{\ell^{*}-\ell'^{*}}{(d,d')} \frac{\overline{d^{*}\ell^{\#}\ell'^{*}}}{d'^{*}\ell^{**}} \right) \qquad (\mathrm{mod}.1)$$

$$= \frac{a/e}{[d,d']\ell} + \theta(d^{*},\ d'^{*},\ \frac{a}{e},\ \frac{\ell^{*}-\ell'^{*}}{(d,d')},\ \ell^{\#},\ (d,d')\ell^{**},\ \ell^{**},\ \ell^{\#}\ell'^{*}).$$

LEMMA 34.

$$\Gamma = \sum_{f|a} \mu(\tfrac{a}{f}) \sum_{\substack{\ell \neq \ell' \leq L \\ (\ell\ell',a)=1}} \sum_{\substack{\delta | \ell-\ell' \\ (\delta,a\ell\ell')=1}} \Gamma_1(f,\ell,\ell',\delta)$$

where

$$\Gamma_1 = \Gamma_1(f,\ell,\ell',\delta) = \sum \sum_{\substack{(\nu,\nu')=1 \\ (\nu,a\ell)=1 \\ (\nu',a\ell')=1}} \lambda_{\delta\nu}\lambda_{\delta\nu'} \sum_{j=1,2} (-1)^{j} \psi\left(\frac{t_j}{\nu\nu'} + \theta(\tfrac{\nu}{\nu'}) \right)$$

with

$$t_2 = \frac{fQ'(\ell,\ell')}{a\delta} + \frac{f}{\delta\ell}, \qquad t_1 = \frac{fQ}{a\delta} + \frac{f}{\delta\ell},$$

and $\theta(\frac{\nu}{\nu'}) = \theta(\nu,\ \nu',\ f,\ \frac{\ell^{*}-\ell'^{*}}{\delta},\ \ell^{\#},\ \delta\ell^{**},\ \ell^{**},\ \ell^{\#}\ell'^{*})$.

Proof. By Lemma 32,

$$\Gamma = \sum_{\substack{\ell \neq \ell' \leq L \\ (\ell\ell',a)=1}} \sum_{\substack{(d,d') | \ell-\ell' \\ (d,a\ell)=1 \\ (d',a\ell')=1}} \sum \lambda_d \lambda_{d'} \sum_{e|a} \mu(e) \left(\sum_{\substack{Q<q\leq Q' \\ q \equiv b([d,d']) \\ e|q}} 1 - \frac{Q'-Q}{e[d,d']} \right)$$

$$= \sum_{e|a} \mu(e) \sum_{\substack{\ell \neq \ell' \le L \\ (\ell\ell', a)=1}} \sum_{\substack{(d,d')|\ell-\ell' \\ (d,a\ell)=1 \\ (d',a\ell')=1}} \lambda_d \lambda_{d'} \left\{ \psi\left(\frac{Q'}{e[d,d']} - \frac{b\bar{e}}{[d,d']}\right) - \psi\left(\frac{Q}{e[d,d']} - \frac{b\bar{e}}{[d,d']}\right) \right\}$$

$$= \sum_{e} \mu(e) \sum_{\ell} \sum_{\ell'} \Gamma_2 , \quad \text{say.}$$

Now, we write

$$a = ef, \quad (d,d') = \delta, \quad d^* = \nu, \quad d'^* = \nu'.$$

Then,

$$\Gamma_2 = \sum_{\substack{\delta | \ell-\ell' \\ (\delta, a\ell\ell')=1}} \sum_{\substack{(\nu,\nu')=1 \\ (\nu,a\ell)=1 \\ (\nu',a\ell')=1}} \lambda_{\delta\nu} \lambda_{\delta\nu'} \sum_{j=1,2} (-1)^j \psi\left(\frac{t_j}{\nu\nu'} - \frac{f}{\delta\nu\nu'\ell} - \frac{b\bar{e}}{[d,d']}\right) .$$

By Lemma 33, we see

$$- \frac{f}{\delta\nu\nu'\ell} - \frac{b\bar{e}}{[d,d']} \equiv \theta\left(\frac{\nu}{\nu'}\right) \quad (\mathrm{mod}.1).$$

12. Proof of Theorem 5.

Let $\mathcal{A}= \{n: a<n\le x,\ n\equiv a(\mathrm{mod}.q),\ (n,a)=1\ \}$,

$$\mathcal{P}= (p: p|q\ \},$$

$$\omega(d)= \begin{cases} 1, & (d,a)=1 \\ 0, & (d,a)>1\ , \end{cases}$$

$$r_d(\mathcal{A})= \#\mathcal{A}_d - \frac{\varphi(a)}{a}\frac{x-a}{q}\frac{\omega(d)}{d}.$$

Then, by Lemma 30. we have

(1) $$\#\{P_2: P_2\le x,\quad P_2\equiv a\ (\mathrm{mod}.q)\} > \frac{Cx}{q\log x} - R(q)$$

where

$$R(q)= R(\ x;\ q,\ a)= \sum_d \lambda_d r_d(\mathcal{A}) + \sum_{x^{4\varepsilon}<p<x^{1/2}} \#\{m:\ p^2m\in\mathcal{A}\ \}$$

$$= R_1(q) + R_2(q),\qquad \text{say.}$$

Let $\mathcal{E}$ denote the exceptional set of moduli, namely,

$$\mathcal{E} = \{q:\ Q<q\le 2Q,\ (q,a)=1,\ \#\{P_2: \begin{matrix} P_2\equiv 0\ (\mathrm{mod}.q) \\ P_2\le \tau(a)q\log^7 q \end{matrix}\}= 0\ \}.$$

Put $x= \tau(a)Q\log^7 Q$ in (1). We see

(2) $$R(q)> \frac{Cx}{q\ \log\ x}\qquad \text{for all } q\in\mathcal{E},$$

since the left hand side of (1) is zero. Suppose

(3) $$\sum_{\substack{Q<q\le 2Q \\ (q,a)=1}} R(q)^2 << \tau(a)x\log^4 x + \frac{x^2}{Q}\log^{-3} x + \tau(a)^2 Q\log^2 x.$$

Then, from this claim and (2) we infer that

$$\#\mathcal{E}\left(\frac{x}{Q\log\ x}\right)^2 < \sum_{q\in\mathcal{E}} R(q)^2$$

$$\leq \sum_{\substack{Q<q\leq 2Q \\ (q,a)=1}} R(q)^2$$

$$\ll \tau(a)x\log^4 x + x^2Q^{-1}\log^{-3}x + \tau(a)^2Q\log^2 x,$$

or

$$\#\mathcal{E} \ll \tau(a)Q^2x^{-1}\log^6 x + Q\log^{-1}x + \tau(a)^2Q^3x^{-2}\log^4 x$$

$$\ll Q\log^{-1}Q,$$

as required.

It remains to verify (13). Since

$$R_2(q) \ll x^{\varepsilon}\frac{x}{q},$$

we have

$$\sum_{\substack{Q<q\leq 2Q \\ (q,a)=1}} R_2(q)^2 \ll x^{\varepsilon}\frac{x}{Q} \sum_{x^{4\varepsilon}<p<x^{1/2}} \sum_{\substack{a<n\leq x \\ p^2|n}} \tau(n-a)$$

$$\ll x^{2\varepsilon}\frac{x}{Q} \sum_{p>x^{4\varepsilon}} \frac{x}{p^2}$$

(4) $$\ll x^{2-2\varepsilon}Q^{-1}.$$

We proceed to R_1. By the definition of r_d and Lemma 31,

$$R_1(q) = \sum_{\substack{a<n\leq x \\ (n,a)=1}} \Big(\sum_{\substack{d|n \\ (d,q)=1}} \lambda_d\Big) - \Big(\sum_{(d,q)=1} \frac{\lambda_d}{d}\Big) \sum_{\substack{a<n\leq x \\ n\equiv a(q) \\ (n,a)=1}} 1 + O(\tau(a)\log D).$$

Also, with the notation in section 11,

(5) $$\sum_{\substack{Q<q\leq 2Q \\ (q,a)=1}} R_1(q)^2 \ll \mathfrak{D} + O(\tau(a)^2Q\log^2 x).$$

By Lemma 32 and 34, we have

$$\mathfrak{D} = \Gamma + O\big(\tau(a)x\log^4 x + \tau(a)(\tfrac{x}{Q})x^{1-2\varepsilon}\big)$$

$$(6) \qquad \ll \tau(a)\left(\frac{x}{Q}\right)^{2} \sum_{\delta\le L} \delta^{-1} \sup_{f,\ell,\ell'} |\Gamma_1| + \tau(a)x\log^4 x + x^{2-\varepsilon}Q^{-1}.$$

Now we decompose (λ_d) in the same manner as in section 4, getting

$$\Gamma_1 \ll \tau(\delta)^2\log^8 x \left\{K(3Q, Q^{\frac{1}{2}-4\varepsilon}, Q^{\frac{1}{20}-\varepsilon}; f, \frac{\ell^*-\ell'^*}{\delta}, \ell^\#, \delta\ell^{**}, \frac{a}{f}, \ell^{**}, \ell^\#\ell'^*) + Q^{1-2\varepsilon}\right\}.$$

Lemma 26 yields

$$\Gamma_1 \ll Q^{1-\varepsilon},$$

since $\omega\upsilon = \ell^\#\ell^{**} = \ell \ll Q^{\varepsilon}$. Combining this with (4),(5) and (6), we obtain the bound (3).

This completes the proof of Theorem 5.