On Quotient s-images of Metric Spaces

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Abstract

In this thesis, we show some results on quotient s-images of metric spaces.

Firstly we prove that if a Hausdorff space Y is a quotient s-image of a metric space, then Y has a point-countable base if and only if Y contains no closed copy of S_ω and no S_2. This gives a positive answer to a question of Y.Tanaka [35].

Then we prove that if a regular T_1 space Y has a point-countable closed k-network $\mathcal{B}$ which is closed for finite intersections, then there is a metric space M and a continuous onto map $f : M \to Y$ such that f is a compact-covering s-map.

Secondly we prove the following theorem.

Let Y have a point-countable closed k-network. If each metric closed subset of Y is locally compact, then Y has a point-countable countably compact k-network.

Then we construct the following example to explain "countably compact "in the theorem above can not be strengthened to "compact ". It also gives a negative answer to Question 37 of B of Problem Section in [15].

There is a regular T_1 countably compact space Y such that Y has a point-countable closed k-network and every first countable closed subspace of Y is compact, but Y has no any point-countable compact k-network.

Thirdly we study the products of sequential spaces and prove the following theorem:

Let Y be a quotient s-image of a metric space. If $S_\omega \times Y$ is sequential, then there exists a subcollection $\mathcal{P}_y$ of $\mathcal{P}$ such that $|\mathcal{P}_y| < \mathfrak{b}$ and $\cup\mathcal{P}_y$ is a neighborhood of y for each $y \in Y$.

Recall $\mathfrak{b} = \min\{|B| : B$ is an unbounded subset of ${}^\omega\omega\}$ in [6]. Here a subset of ${}^\omega\omega$ is called *unbounded* if it is unbounded in $< {}^\omega\omega, \leq >$. Then we have the following corollary:

Let regular T_1 space Y be a quotient s-image of a metric space. If $S_\omega \times Y$ is a sequential space, then Y is a locally $k_\mathfrak{c}$-space.

Finally we introduce a definition of weak neighborhoods and prove some basic properties of weak neighborhoods. Then using the results above, we construct a Hausdorff space Y such that Y is a quotient s-image of a metric space and is not any compact-covering quotient s-image of any metric space. It gives a negative answer to the 25 year-old Michael-Nagami's Problem.

Contents

1. Introduction

We assume all maps are continuous and onto.

Quotient s-images of metric spaces have been interesting topics for long years. E. Michael in [25] characterized quotient images of separable metric spaces as k-spaces with countable k-networks and gave many important properties. A.V.Arhangelskiǐ in [2] called it a very interesting result and raised the following problem.

Problem 2.1. How does one characterize, in intrinsic, quotient s-images of metric spaces?

T. Hoshina [13] firstly characterized quotient s-images of metric spaces. Recall that an onto map $f : X \to Y$ is an *s-map* if $f^{-1}(y)$ has a countable base for each $y \in Y$ and a *compact-covering* if every compact $K \subset Y$ is the image of some compact $C \subset X$. Then the following problem was raised in E. Michael and K. Nagami [20].

Problem. If a Hausdorff space Y is a quotient s-image of a metric space, must Y also be a compact-covering quotient s-image of a (possibly different) metric space?

Also the question was mentioned in M. E. Rudin [28] and studied in [16, 17, 18, 19] and so on. An interesting theorem concerning the question was given in G. Gruenhage, E. Michael and Y. Tanaka [9, Theorem 6.1]. It was proved that the answer to the question is "yes" if "compact-covering" is weakened to "sequence-covering". Here we call f a *sequence-covering* if every convergent sequence (including its limit point) is the image of some compact set $C \subset X$. E. Michael [22] posed the question again, analyzed the origin of the question and supplied references about the question.

It was asked in [9, Question 10.1] that suppose Y is a quotient s-image of a metric space, does Y have a point-countable closed k-network? Here a cover $\mathcal{P}$ of Y is a *k-network* for Y if, whenever $K \subset U$ with K compact and U open in Y, then $K \subset \cup\mathcal{F} \subset U$ for some finite $\mathcal{F} \subset \mathcal{P}$. Z. Yun found out a succinct counterexample in [36] and L. Foged constructed a strong counterexample in [7] to the question 10.1. The two counterexamples above enable us to consider Michael-Nagami's question from another angle. Professor Takao HOSHINA told me that "It is possible that there is a counterexample to Michael-Nagami's Problem." in a seminar of himself in June, 1995. Following this idea, we in [4] proved several propositions concerning weak neighborhoods and constructed the following counterexample which gave a negative answer to Michael-Nagami's Problem.

There is a Hausdorff space Y such that Y is a quotient s-image of a metric space and is no compact-covering quotient s-image of any metric space.

Before answering Michael-Nagami's Problem, we in [4] proved the following theorem which gave a positive answer to a **question** of Y.Tanaka [35].

Let Hausdorff space Y be a quotient s-image of a metric space. Then Y has a point-countable base if and only if Y contains no closed copy of S_ω and no S_2.

After A. V. Arkhangelskiĭ's paper [2], Maps as an important object is studied. G. Gruenhage, E. Michael and Y. Tanaka in [9] deeply discussed quotient maps and also showed relations between k-networks and quotient s-images of metric spaces. Also we in [5] used maps as a tool to study the relations among closed k-networks, countably compact k-networks and compact k-networks, also to study the following question which was raised by S. Lin [15] and was arranged as Question 37 of B of Problem Section in [27]:

Question 37. Suppose a space X has a point-countable closed k-network. Is X a space with a point-countable compact k-network if every first countable closed subspace of X is locally compact?

Recall that if $\mathcal{C}$ is a k-network for Y, then $\mathcal{C}$ is a closed (compact) k-network if each $C \in \mathcal{C}$ is closed (compact) in Y.

Firstly we proved the following theorem in [5].

If a regular T_1 space Y has a point-countable closed k-network $\mathcal{B}$ which is closed for finite intersections, then there is a metric space M and a continuous onto map $f : M \to Y$ such that f is a compact-covering s-map. Then we in [5] proved the following theorem.

Let Y have a point-countable closed k-network. If each metric closed subset of Y is locally compact, then Y has a point-countable countably compact k-network.

Then we in [5] constructed the following example to explain "countably compact "in the theorem above was strengthened to "compact "was impossible. It also gave a negative answer to Question 37 of B of Problem Section in [15].

There is a regular T_1 countably compact space Y such that Y has a point-countable closed k-network and every first countable closed subspace of Y is compact, but Y has no any point-countable compact k-network.

At the same time, the products quotient s-images of metric spaces is studied. It is well known that there are many interesting results about products of k-spaces. One of them is well known Cohen-Michael's Theorem which proved that for a regular T_1 space X, $X \times Y$ is a k-space for each k-space Y iff X is locally compact in [24]. Y. Tanaka, assuming the Continuum Hypothesis (CH), gave a characterization for the products of two closed images of metric spaces to be k-spaces in [32] and [34]. G. Gruenhage [10] showed that Y.Tanaka's the characterization above for the products of two closed images of metric spaces to be k-spaces is independent of the usual axioms of set theory.

We in [5] showd some results of the products of quotient s-images of metric spaces with $\mathfrak{b} = \aleph_1$. Firstly we proved the following theorem:

Let Y be a quotient s-image of a metric space. If $S_\omega \times Y$ is sequential, then there exists a subcollection $\mathcal{P}_y$ of $\mathcal{P}$ such that $|\mathcal{P}_y| < \mathfrak{b}$ and $\cup\mathcal{P}_y$ is a neighborhood of y for each $y \in Y$.

Recall $\mathfrak{b} = \min\{|B| : B$ is an unbounded subset of ${}^\omega\omega\}$ in [6]. Here a subset of ${}^\omega\omega$ is called *unbounded* if it is unbounded in $< {}^\omega\omega, \leq >$. Then we had the following corollary:

Let regular T_1 space Y be a quotient s-image of a metric space. If $S_\omega \times Y$ is a sequential space, then Y is a locally k_c-space.

Finally we raised the following question.

Question. Is it possible to find a counterexample to Michael-Nagami's question among regular T_1 spaces (or parecompact spaces)?

2. Theorems on Quotient s-images of Metric Spaces

For a cardinal number α, let the *sequential fan* S_α be the quotient space obtained from the topological sum of α convergent sequences by identifying all the limit points to a single point.

Y.Tanaka [35] raised a question about quotient s-images of metric spaces. We positively answer the question in the following Theorem 2.1 and use it in Section 5. S.Li has the same result in [14]. But we can see the relation between a metric space and its quotient s-image from our proof. This proof was announced in October, 1994 in a seminar.

Recall a canonical example S_2. That is,

$$S_2 = (N \times N) \cup N \cup \{0\},$$

where N is the set of all positive integers, with each point of $N \times N$ an isolated point. A basis of neighborhoods of $n \in N$ consists of all sets of the form $\{n\} \cup \{(m, n) : m_0 \leq m\}$. U is a neighborhood of 0 if and only if $0 \in U$ and U is a neighborhood of all but finitely many $n \in N$. S_2 is also called the *Arens' space.*

Recall that a space Y is *Frechét* iff whenever $B \subset Y$ and $y \in Cl_Y(B)$, there is a convergent sequence $S \subset B$ such that $S \longrightarrow y$.

Theorem 2.1. *Let Hausdorff space Y be a quotient s-image of a metric space. Then Y has a point-countable base if and only if Y contains no closed copy of S_ω and none of S_2.*

Proof. Let M be a metric space and $f : M \rightarrow Y$ be a sequence-covering quotient s-map by Theorem 6.1 in [9]. Let $\mathcal{B} = \cup_n \mathcal{B}_n$ be a σ-locally finite base of M.

Suppose that Y is not Frechét.

Following the proof of Proposition 7.3 of [8], we can choose a subset

$$Y' = \{y'\} \cup \{y_n : n \in omega\} \cup \{y_{nm} : n, m \in \omega\}$$

of Y such that:

(a). For each $n \in \omega$, there is an open neighborhood G_n of y_n in Y such that

$$G_n \cap G_m = \emptyset \qquad \text{if and only if} \qquad y_n \neq y_m.$$

(b). y_n converges to y' and y_{nm} converges to y_n for each $n \in \omega$.

(c). There is no sequence in

$$D = \{y_{nm} : n, m \in \omega\}$$

which converges to y'.

We may prove that for the $y' \in Y'$, there is an open set O_1 of M such that O_1 contains $f^{-1}(y')$ and $O_1 \bigcap f^{-1}(D) = \emptyset$.

In fact, for each $x \in f^{-1}(y')$, there is an

$$O_{n(x)} \in \mathcal{O}_x = \{O \in \mathcal{B} : x \in O\} = \{O_n : n \in N\}$$

satisfying $O_{n(x)} \cap f^{-1}(D) = \emptyset$.

Otherwise, there is an $x \in f^{-1}(y')$ satisfying $O_n \cap f^{-1}(D) \neq \emptyset$ for each $O_n \in \mathcal{O}_x$. Pick an $x_n \in O_n \cap f^{-1}(D)$ for each $n \in \omega$. Then $\{x_n : n \in \omega\}$ converges to x and $x \in f^{-1}(y')$. So $\{f(x_n) : n \in \omega\} \subset D$ and $\{f(x_n) : n \in \omega\}$ converges to $f(x) = y'$. This is a contradiction to (c).

Let

$$O_1 = \bigcup\{O_{n(x)} : x \in f^{-1}(y')\}.$$

Then O_1 contains $f^{-1}(y')$ and $O_1 \cap f^{-1}(D) = \emptyset$.

Let

$$S_0 = \{y_n : n \in \omega\}.$$

Then $S_0 \cup \{y'\}$ is compact. For each $y \notin S_0 \cup \{y'\}$, we take an open neighborhood U of y and an open neighborhood U_0 of $S_0 \cup \{y'\}$ satisfying $U \cap U_0 = \emptyset$. Then there is a $g \in {}^{\omega}\omega$ satisfying

$$U_g = \{y_{ij} : j \geq g(i)\} \subset U_0$$

by (b). If $x \in f^{-1}(U)$, then there is an $O_{n(x)} \in \mathcal{O}_x$ such that $O_{n(x)}$ meets only finitely many $f^{-1}(y_{ij})$'s for each $i \in \omega$ by

$$f^{-1}(U_g) = \bigcup\{f^{-1}(y_{ij}) : j \geq g(i)\} \subset f^{-1}(U_0)$$

and

$$f^{-1}(U_0) \cap f^{-1}(U) = \emptyset.$$

Let

$$\mathcal{B}_1 = \{B \in \mathcal{B} : B \cap f^{-1}(y') \neq \emptyset \text{ and } B \subset O_1\}.$$

Let

$$\mathcal{B}_x = \{B \in \mathcal{B} : x \in B \text{ and } B \subset O_{n(x)}\}$$

for each

$$x \in M - (f^{-1}(y') \cup f^{-1}(S_0)).$$

Note that $O_{n(x)} \in \mathcal{O}_x$ meets only finitely many $f^{-1}(y_{ij})$ for each $i \in \omega$. Let

$$\mathcal{B}_2 = \bigcup\{\mathcal{B}_x : x \in M - (f^{-1}(y') \cup f^{-1}(S_0))\}.$$

Let

$$\mathcal{B}_3 = \{B \in \mathcal{B} : B \cap f^{-1}(S_0) \neq \emptyset \text{ and } B \subset f^{-1}(G_i) \text{ for some } G_i\},$$

where G_i is given in (a). Let

$$\mathcal{B}' = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3.$$

Then $\mathcal{B}'$ is a σ-locally finite base of M also. Let

$$\mathcal{B}'' = \{B \in \mathcal{B}' : B \cap f^{-1}(Y') \neq \emptyset\}.$$

Since $f^{-1}(Y')$ is separable, then $\mathcal{B}''$ is countable. Let

$$\mathcal{B}''_2 = \mathcal{B}'' \cap \mathcal{B}_2 = \{B_n : n \in N\}.$$

Let

$$B_0 = \emptyset,\ X_0 = f^{-1}(S_0 \cup \{y'\})$$

and

$$X_n = f^{-1}(S_n \cup \{y_n\}) - f^{-1}(f(\bigcup_{i<n} B_i)) \text{ for each } n \geq 1,$$

where $S_n = \{y_{nj} : j < \omega\}$. Then $y_n \in f(X_n)$ and S_n is eventually in $f(X_n)$ by the definition of $\mathcal{B}_x$.

Define

$$Y_0 = \bigcup_{n<\omega} f(X_n).$$

If we prove that $A \cap Y_0$ is closed in A for each

$$A \in f(\mathcal{B}') = \{f(B) : B \in \mathcal{B}'\},$$

then Y_0 is a closed subset of Y by Lemma 1.7 of [9]. To see this, let $A = f(B)$ with $B \in \mathcal{B}'$. Then the following three cases arise:

Case 1. If $B \in \mathcal{B}_2$ and $B \cap f^{-1}(D) \neq \emptyset$, then $B \in \mathcal{B}''_2$ and $B = B_i$. So

$$f(B_i) \cap f(X_n) = \emptyset \text{ for } n > i.$$

But

$$f(B_i) \cap Y_0 \subset \bigcup_{n\leq i} f(X_n).$$

Then $f(B_i) \cap (\bigcup_{n\leq i} f(X_n))$ is closed in $f(B_i)$ by $\bigcup_{n\leq i} f(X_n)$ being closed in Y. So $f(B_i) \cap Y_0$ is closed in $f(B_i)$.

Case 2. If $B \in \mathcal{B}_3$, then $f(B) \subset G_i$ for some G_i. Then $y_i \in f(B) \cap Y_0$ by

$$B \cap f^{-1}(S_0) \neq \emptyset.$$

So $f(B) \cap Y_0$ is closed in $f(B)$.

Case 3. If $B \in \mathcal{B}_1$, then $B \subset O_1$ and $B \cap f^{-1}(D) = \emptyset$. Then

$$y' \in f(B) \cap Y_0 = f(B) \cap (S_0 \cup \{y'\}).$$

So $f(B) \cap Y_0$ is closed in $f(B)$.

Hence Y_0 is closed in Y. Since Y is a sequential space, Y_0 is a sequential space. So Y_0 is a closed copy of S_2. This is a contradiction. Therefore Y must be Frechét.

If Y is a Frechét space and Y contains no closed copy of S_ω, then Y is strongly Frechét by 16(b) of [31]. (E. Michael [21] called it countably bi-sequential). Hence $f : M \to Y$ is a countably bi-quotient s-map by Theorem 4.4 of [29]. Finally we prove that $f : M \to Y$ is a bi-quotient s-map. Then Y has a point-countable base.

To see it, let $\mathcal{O}$ be an open cover of $f^{-1}(y)$. Since $f^{-1}(y)$ is separable, there is a countable subcollection $\mathcal{O}' = \{O_1, O_2, \ldots\}$ of $\mathcal{O}$ which covers $f^{-1}(y)$. Hence there is an n such that $y \in \mathrm{Int}(\bigcup_{i \leq n} f(O_i))$ by $f : M \to Y$ being a countably bi-quotient map. Then $f : M \to Y$ is a bi-quotient s-map. □

Recall that "*Y is determined by $\mathcal{P}$*", or "*$\mathcal{P}$ determines Y*", if $U \subset Y$ is open (closed) in Y if and only if $U \cap P$ is relatively open (relatively closed) in P for each $P \in \mathcal{P}$. This terminology is used by [9]. Recall that a space Y is *sequential*, if it is determined by the cover consisting of all compact metric subsets of Y.

Proposition 2.2. *Let regular T_1 space Y be a quotient s-image of a metric space and $S_\omega \times Y$ be a sequential space. If a closed subspace B of Y contains no closed copy of S_ω and no S_2, then B is locally compact.*

Proof. Let B be the closed subspace. Then B is regular and first countable by Theorem 2.1. Suppose B is not locally compact. Then $S_\omega \times B$ is not sequential by Lemma 3 and Lemma 4 of [10]. It is a contradiction. □

Proposition 2.3. *Let M be a metric space and $f : M \to Y$ be a continuous map. If each metric closed subspace of regular T_1 space Y is locally compact, then for each point-countable base $\mathcal{B}$ of M, there is a base $\mathcal{B}' \subset \mathcal{B}$ such that $\overline{f(B)}$ is countably compact for each $B \in \mathcal{B}'$.*

Proof. Let $\mathcal{B}$ be a point-countable base of M. Let

$$\mathcal{B}(x) = \{B \in \mathcal{B} : x \in B\} = \{B_n : n \in \omega\} \subset \mathcal{B}.$$

Then:

1. $\cap_{n\in\omega} f(B_n) = \{f(x)\}$.
2. There is a $B_n \in \mathcal{B}(x)$ such that $\overline{f(B_n)}$ is countably compact.

Indeed. Let $y = f(x)$. Suppose $\cap_{i\leq n}\overline{f(B_i)}$ is not countably compact for each $n \in \omega$.

If $\overline{f(B_1)}$ is not countably compact, then there is a discrete closed subset

$$D_1 = \{y_{1m} : m \in \omega\} \subset \overline{f(B_1)}$$

such that $|D_1| = \aleph_0$ and y is not in D_1. Take an open neighborhood O_1 of y in Y such that $\overline{O_1} \cap D_1 = \emptyset$ since Y is a regular T_1 space. Then $f^{-1}(O_1)$ is open in M with $x \in f^{-1}(O_1)$. So there is a $B_{n_1} \in \mathcal{B}(x)$ with

$$x \in B_{n_1} \subset f^{-1}(O_1) \text{ and } 1 < n_1.$$

Then $\cap_{i\leq n_1}\overline{f(B_i)}$ is not countably compact by supposition. So there is a discrete closed subset

$$D_2 = \{y_{2m} : m \in \omega\} \subset \cap_{i \leq n_1} \overline{f(B_i)}$$

by

$$|\cap_{i \leq n_1} \overline{f(B_1)}| \geq \aleph_0$$

such that

$$|D_2| = \aleph_0, \; D_2 \cap D_1 = \emptyset$$

and y is not in D_2.

Then, by induction, there is an $n_i \in \omega$ with $n_i < n_{i+1}$ and a discrete closed subset

$$D_i = \{y_{im} : m \in \omega\} \subset \cap_{j \leq n_i} \overline{f(B_j)}$$

for each $i < \omega$ such that y is not in D_i,

$$|D_i| = \aleph_0 \text{ and } \; D_i \cap D_j = \emptyset \; (i \neq j).$$

Let

$$Y_1 = (\cup_{n \in \omega} D_n) \cup \{y\}.$$

Then we can prove that Y_1 is a closed metric subspace of Y and is not locally compact. It is a contradiction. Hence $\cap_{i \leq n} \overline{f(B_i)}$ is countably compact for some $n \in \omega$.

Since $\mathcal{B}(x)$ is a neighborhoods base of x in M, then there is a $B_m \in \mathcal{B}(x)$ with

$$B_m \subset \cap_{i \leq n} B_i.$$

Since

$$f(\cap_{i \leq n} B_i) \subset \cap_{i \leq n} f(B_i),$$

then

$$\overline{f(B_m)} \subset \overline{f(\cap_{i \leq n} B_i)}$$

is countably compact. Let

$$\mathcal{B}^*(x) = \{B \in \mathcal{B}(x) : B \subset B_m\}$$

and

$$\mathcal{B}' = \cup\{\mathcal{B}^*(x) : x \in X\}.$$

Then $\mathcal{B}'$ is a base of M such that $\overline{f(B)}$ is countably compact for each $B \in \mathcal{B}'$. □

Recall that a cover $\mathcal{F} = \{B_n : n \in F\}$ of Y is called *irreducible*, if $\cup\{B_n : n \in F_0\} \neq Y$ for each proper subset F_0 of F.

Theorem 2.4. *Let regular T_1 space Y have a point-countable closed k-network $\mathcal{B}$ which is closed for finite intersections. Then there is a metric space M and a continuous onto map $f : M \to Y$ such that f is a compact-covering s-map.*

Proof. Let $\mathcal{B}$ be a point-countable closed k-network of Y such that $\mathcal{B}$ is closed for finite intersections. Given $\mathcal{B}$ the discrete topology, the countable product $\Pi_{n>0}\mathcal{B}$ is a metric space. Pick

$$x' = (B_i') \quad \text{and} \quad x'' = (B_i'')$$

from $\Pi_{n>0}\mathcal{B}$. Let

$$d^*(x', x'') = 1/n,$$

where

$$n = min\{i : \ B_i' \neq B_i'', \ x' = (B_i') \ \text{ and } \ x'' = (B_i'')\}.$$

Then d^* is a metric of $\Pi_{n>0}\mathcal{B}$. Let $M \subset \Pi_{n>0}\mathcal{B}$ be all (B_n) such that there is a $y \in Y$ with $\cap_{n>0}B_n = \{y\}$ and every neighborhood of y contains some B_n. Let $f : M \to Y$ such that $f((B_n)) = y$ if $\cap_{n>0}B_n = \{y\}$ for each $(B_n) \in M$. We may show that f is an onto continuous s-map just as the proof of Theorem 6.1 of [9]. Let

$$\mathcal{C}_n = \{C \subset M : \text{ for each } i \leq n \text{ and } B_i \in \mathcal{B}, \quad C = (\{B_1\} \times \{B_2\} \times \ldots \times \{B_n\} \times \Pi_{j>n}\mathcal{B}) \cap M\}$$

for each $n > 0$. Let

$$\mathcal{C} = \cup_{n>0}\mathcal{C}_n.$$

Then $\mathcal{C}$ is a σ-discrete base of M. In the following proof, we show that $f : M \to Y$ is a compact-covering map.

Let K be a compact subset of Y. Then K is a metric subset of Y by Theorem 3.3 in [9]. If K is a finite subset of Y, then there is a finite subset C of M with $f(C) = K$. So we assume that K is infinite in the following proof. Let $\mathcal{F}$ be a finite subcollection of $\mathcal{B}$ which is an irreducible cover of K, and let

$$\mathcal{F}(y) = \{F \in \mathcal{F} : y \in F\}.$$

Claim 2.5. *If $y \in K$ and O is an open neighborhood of y in Y, then there is a finite subcollection $\mathcal{F}$ of $\mathcal{B}$ which is an irreducible cover of K such that $\cup\mathcal{F}(y) \subset O$.*

Proof. Case 1. y is an isolated point of K. Let O_1 be an open subset of O with $O_1 \cap K = \{y\}$. Then there is an $F_0 \in \mathcal{B}$ with $y \in F_0 \subset O_1$. Because the open set $Y - \{y\}$ contains the compact set $K - O_1$, there is a finite $\mathcal{F}' \subset \mathcal{B}$ which is an irreducible cover of $K - O_1$ such that

$$K - O_1 \subset \cup\mathcal{F}' \subset Y - \{y\}.$$

Let

$$\mathcal{F} = \mathcal{F}' \cup \{F_0\}.$$

Then $\mathcal{F}$ is an irreducible finite cover of K such that

$$\mathcal{F}(y) = \{F_0\}.$$

Case 2. Pick a cluster point y of K. Let d be a metric of K. Let

$$U_n = \{y' \in K : d(y, y') < 1/n\}$$

for each $n > 0$ with $\overline{U_{n(0)}} \subset O \cap K$. Then there is a finite subcollection $\mathcal{F}_1$ of $\mathcal{B}$ such that

$$\overline{U_{n(0)}} \subset \cup\mathcal{F}_1 \subset O.$$

Let

$$\mathcal{F}_1 = \{F_i : i \leq n\}.$$

We may assume that $\mathcal{F}_1$ is an irreducible finite cover of $\overline{U_{n(0)}}$.

Firstly we prove that there is an $\mathcal{F}_j \subset \mathcal{F}_1$ and an $U_{n(j)}$ such that for each $F_i \in \mathcal{F}_j$ and each U_m, $\cup(\mathcal{F}_j - \{F_i\})$ can not contain U_m and $U_{n(j)} \subset \cup\mathcal{F}_j \subset O$.

Indeed: If for each $i \leq n$, $\cup(\mathcal{F}_1 - \{F_i\})$ can not contain any U_m ($m \in \omega$), then pick an

$$x_{im} \in U_m - \cup(\mathcal{F}_1 - \{F_i\}) \subset U_{n(0)}$$

for each $m > n(0)$. Let

$$S_i = \{x_{im} : m \in \omega\}.$$

Then S_i converges to y and

$$S_i \cap (\cup(\mathcal{F}_1 - \{F_i\})) = \emptyset.$$

Since

$$S_i \subset \overline{U_{n(0)}} \subset \cup\mathcal{F}_1$$

and

$$S_i \cap (\cup(\mathcal{F}_1 - \{F_i\})) = \emptyset,$$

then $S_i \subset F_i$ for each $F_i \in \mathcal{F}_1$. Let

$$\mathcal{F}(y) = \mathcal{F}_1.$$

If there is an $i(1) \leq n$ and an $U_{n(1)}$ with

$$U_{n(1)} \subset \cup(\mathcal{F}_1 - \{F_{i(1)}\}).$$

Let

$$\mathcal{F}_2 = \mathcal{F}_1 - \{F_{i(1)}\}.$$

Suppose that for each $j \leq n$, there is an $i(j) \leq n$ and an

$$U_{n(j)} \subset \cup(\mathcal{F}_j - \{F_{i(j)}\})$$

here

$$\mathcal{F}_j = \mathcal{F}_{j-1} - \{F_{i(j-1)}\}.$$

Then we have

$$y \in U_{n(n)} = \cap_{j \leq n} U_{n(j)} \subset \cup(\mathcal{F}_1 - \{F_{i(1)}, ..., F_{i(n)}\}) = \emptyset$$

since we only arrange $\{F_1, ..., F_n\}$ into $\{F_{i(1)}, ..., F_{i(n)}\}$ again. It is a contradiction. So there is a $j < n$ such that for each $F_i \in \mathcal{F}_j$ and each U_m, $\cup(\mathcal{F}_j - \{F_i\})$ can not contain U_m and

$$U_{n(j)} \subset \cup\mathcal{F}_j \subset O.$$

Secondly, let $\mathcal{F}(y) = \mathcal{F}_j$. Then $S_i \subset F_i$, $S_i \cap (\cup(\mathcal{F}(y) - \{F_i\})) = \emptyset$ and $S_i \subset \overline{U_{n(j)+1}}$ for each $F_i \in \mathcal{F}(y)$.

Indeed: Pick an

$$x_{im} \in U_m - \cup(\mathcal{F}_j - \{F_i\}) \subset U_{n(j)+1}$$

for each $m > n(j) + 1$. Let

$$S_i = \{x_{im} : m \in \omega\}.$$

Then S_i converges to y and

$$S_i \cap (\cup(\mathcal{F}_j - \{F_i\})) = \emptyset.$$

Since

$$S_i \subset \overline{U_{n(j)+1}} \subset \cup \mathcal{F}_j$$

and

$$S_i \cap (\cup(\mathcal{F}_j - \{F_i\})) = \emptyset,$$

then $S_i \subset F_i$ for each $F_i \in \mathcal{F}_j$. Let

$$\mathcal{F}(y) = \mathcal{F}_j.$$

Then

$$S_i \subset F_i,\ S_i \cap (\cup(\mathcal{F}(y) - \{F_i\})) = \emptyset$$

and $S_i \subset \overline{U_{n(j)+1}}$ for each $F_i \in \mathcal{F}(y)$.

Now we construct the $\mathcal{F}$.

Indeed: Because the open set $Y - \overline{U_{n(j)+1}}$ contains the compact set $K - U_{n(j)}$, there is a finite $\mathcal{F}' \subset \mathcal{B}$ which is an irreducible cover of $K - U_{n(j)}$ such that

$$K - U_{n(j)} \subset \cup \mathcal{F}' \subset Y - \overline{U_{n(j)+1}}.$$

So

$$K \subset (K - U_{n(j)}) \cup \overline{U_{n(j)}} \subset \cup(\mathcal{F}' \cup \mathcal{F}(y)).$$

Then there is an irreducible finite cover $\mathcal{F}$ of K such that

$$\mathcal{F} \subset \mathcal{F}(y) \cup \mathcal{F}'.$$

Suppose there is an

$$F_i \in \mathcal{F}(y) - \mathcal{F}.$$

Then

$$\mathcal{F} \subset \mathcal{F}' \cup (\mathcal{F}(y) - \{F_i\}).$$

Since

$$S_i \subset F_i \cap U_{n(j)+1} \text{ and } \cup\mathcal{F}' \subset Y - \overline{U_{n(j)+1}},$$

then

$$S_i \cap (\cup\mathcal{F}') = \emptyset.$$

Notice

$$S_i \cap [\cup(\mathcal{F}(y) - \{F_i\})] = \emptyset.$$

So

$$S_i \cap (\cup\mathcal{F}) = \emptyset.$$

Then $\mathcal{F}$ is not a cover of K. It is a contradiction. So

$$\mathcal{F}(y) \subset \mathcal{F}.$$

Since $y \in \overline{S_i} \subset F_i$, then $y \in F_i$ for each $F_i \in \mathcal{F}(y)$. If y is not in F_i, then $U_n - F_i$ is an open neighborhood of y in K for each $n > 0$ since $F_i \in \mathcal{B}$ is closed. So

$$F_i \in \mathcal{F}(y) = \mathcal{F}_j \text{ if and only if } y \in F_i$$

as the definition before Claim 2.5. □

Claim 2.6. $|\{\mathcal{F} \subset \mathcal{B} : \mathcal{F}$ *is an irreducible finite cover of* $K\}| = \aleph_0$.

Proof. A. Mišcenko [26] proved that if $\mathcal{B}$ is a point-countable cover of K, then there are *at most* countably many finite subcollections of $\mathcal{B}$ which are irreducible covers of K.

Because K is infinite, there is a cluster point $y \in K$. Let d be a metric of K. Let

$$U_n = \{y' \in K : d(y, y') < 1/n\}$$

for each $n > 0$. Let O_n be an open set of Y with $K \cap O_n = U_n$ for each $n > 0$. Then there is a finite subcollection $\mathcal{F}_n$ of $\mathcal{B}$ which is an irreducible cover of K such that $\cup\mathcal{F}_n(y) \subset O_n$ by Claim 2.5 for each $n > 0$. So there must be *at least* infinitely countably many finite subcollections of $\mathcal{B}$ which are irreducible covers of K. □

Let $(\mathcal{F}_n)$ enumerates the all finite subcollections of $\mathcal{B}$ which are irreducible covers of K. Then $\Pi_{n>0}\mathcal{F}_n$ is a compact subset of $\Pi_{n>0}\mathcal{B}$. Let

$$D = (\Pi_{n>0}\mathcal{F}_n) \cap M.$$

Claim 2.7. $f(D) = K$.

Proof. Pick an $x = (B_n) \in D$. Suppose $f(x)$ is not in K. Then $Y - K$ is an open neighborhood of $f(x)$. So there is a B_n in (B_n) with $f(x) \in B_n \subset Y - K$ by the definition of the subspace M. Assume $B_n \in \mathcal{F}_n$. Then $\mathcal{F}_n - \{B_n\}$ is still a cover of K. It is a contradiction.

Pick a $y \in K$. Then for each $n > 0$, there is a $B'_n \in \mathcal{F}_n$ with $y \in B'_n$. Let $x = (B'_n)$. Then $x \in \Pi_{n>0}\mathcal{F}_n$ and $y \in \cap_n B'_n$. Pick an open set $O \subset Y$ with $y \in O$. Then, by Claim 2.5, there is an irreducible finite cover $\mathcal{F}$ of K such that $\cup\mathcal{F}(y) \subset O$. Since $(\mathcal{F}_n)$ enumerates the all finite subcollections of $\mathcal{B}$ which are irreducible covers of K, then there is an $\mathcal{F}_n = \mathcal{F}$. So B'_n is in $\mathcal{F}$. Then $B'_n \in \mathcal{F}(y)$ by $y \in B'_n$. Then

$$B'_n \subset \cup\mathcal{F}(y) \subset O.$$

This implies $x \in M$ and $f(x) = y \in f(D)$. □

Claim 2.8. *D is a compact subset of* $\Pi_{n>0}\mathcal{F}_n$.

Proof. Pick an $x = (B_n) \in \overline{D}$. Pick an open ball

$$O(x, 1/n) = (\{B_1\} \times \{B_2\} \times ... \times \{B_n\} \times \Pi_{i>n}\mathcal{B}) \cap M.$$

Then there is an $x_n \in D \cap O(x, 1/n)$ for each $n > 0$. Notice

$$x_n = (B_1, B_2, ..., B_n, B_{n+1}^n, B_{n+2}^n, ...)$$

since $x_n \in O(x, 1/n)$.

So sequence $S = \{x_n = (B_m^n) : n > 0\} \subset D$ converges to $x = (B_m)$. Then we may assume, without loss of generality, that

$$B_m^n = B_m \quad \text{for } n \geq m.$$

Then $B_n \in \mathcal{F}_n$ for each $\mathcal{F}_n$ in $(\mathcal{F}_n)$ since $x_n \in D$. So $f(x_{n+i}) \in f(B_n)$ for each $i > 0$. Since $f(S) = \{f(x_n) : n > 0\} \subset K$ and K is compact metric, then there is a subsequence $S' \subset S$ such that $f(S')$ converges to some $y \in K$. Since B_n is closed in Y, then $y \in B_n$ for each $n > 0$. So $y \in \cap_{n>0} B_n$. Let $O \subset Y$ be an open set with $y \in O$. Then, by Claim 2.5, there is an irreducible finite cover $\mathcal{F}$ of K such that

$$\cup\mathcal{F}(y) \subset O.$$

Since $(\mathcal{F}_n)$ enumerates the all finite subcollections of $\mathcal{B}$ which are irreducible covers of K, then there is an $\mathcal{F}_n = \mathcal{F}$. So $B_n \in \mathcal{F}_n = \mathcal{F}$. Then $B_n \in \mathcal{F}(y)$ since $y \in B_n$. Then

$$y \in B_n \subset \cup\mathcal{F}(y) \subset O.$$

This implies

$$x = (B_n) \in D.$$

So D is a closed subset of compact metric set $\Pi_{n>0}\mathcal{F}_n$. □

Proof. proof of Theorem 2.4. (continued)

If K is an infinite compact subset of Y, then there must be countably infinitely many finite subcollections of $\mathcal{B}$ which are irreducible covers of K by Claim 2.6. If $(\mathcal{F}_n)$ enumerates the all finite subcollections of $\mathcal{B}$ which are irreducible covers of K, then $D = (\Pi_{n>0}\mathcal{F}_n) \cap M$ is a compact subset of M by Claim 2.8. Then $f(D) = K$ by Claim 2.7. So $f : M \to Y$ is a compact-covering map. $\square$

3. Compact-covering s-images and Compact k-networks

Theorem 3.1. *Let regular T_1 space Y have a point-countable closed k-network. If each metric closed subset of Y is locally compact, then Y has a point-countable countably compact k-network.*

Proof. Let $\mathcal{B}$ be a point-countable closed k-network. Let $\mathcal{B}_1$ be the collection of all finite intersections of $\mathcal{B}$. Then $\mathcal{B}_1$ is a point-countable closed k-network which is closed for finite intersections. So we may assume that Y has a point-countable closed k-network $\mathcal{B}$ which is closed for finite intersections.

Let

$$M \subset \Pi_{n>0}\mathcal{B}$$

be the metric space,

$$\mathcal{C} = \cup_{n>0}\mathcal{C}_n$$

be the σ-discrete base of M and $f : M \to Y$ be the onto continuous compact-covering s-map as the Theorem 2.4 above. Then there is a subcollection $\mathcal{C}' \subset \mathcal{C}$ such that $\mathcal{C}'$ is a base of M and $\overline{f(C)}$ is countably compact for each $C \in \mathcal{C}'$ by Proposition 2.3. If

$$C = (\{B_1\} \times \{B_2\} \times \ldots \times \{B_n\} \times \Pi_{j>n}\mathcal{B}) \cap M \in \mathcal{C}',$$

then $f(C) = \cap_{i\leq n}B_i$ is closed. So $f(C) = \overline{f(C)}$ is countably compact. Then $\{f(C) : C \in \mathcal{C}'\}$ is a point-countable collection of countably compact subsets of Y. We have proved that $f : M \to Y$ is a compact-covering map by Theorem 2.4. Then $\{f(C) : C \in \mathcal{C}'\}$ is a point-countable countably compact k-network since $\mathcal{C}'$ is a base of M. Notice that $\mathcal{B}$ is closed for finite intersections. Then

$$f(C) = \cap_{i \leq n} B_i \in \mathcal{B}$$

for each $C \in \mathcal{C}'$. Then there is a countably compact k-network

$$\mathcal{B}' = \{f(C) : C \in \mathcal{C}'\}$$

which is a subcollection of $\mathcal{B}$. Then Y has a point-countable countably compact k-network. □

We would like to give a proposition about Question 37.

Proposition 3.2. *Let Y be a regular T_1 space. Then the following are equivalent.*

1. *Every metric closed subspace of Y is locally countably compact.*
2. *Every first countable closed subspace of Y is locally countably compact.*
3. *Every metric closed subspace of Y is locally compact.*
4. *Every first countable closed subspace of Y is locally compact.*

Proof. $1 \Rightarrow 2$.

Suppose that there is a first countable closed subset B of Y which is not locally countably compact. Then there is a point $y \in B$ such that y has no any countably compact neighborhood. Let

$$B = O_1 \supset \overline{O_2} \supset O_2 \supset \overline{O_3} \supset O_3 \supset \dots$$

such that

$$\mathcal{O} = \{O_n : n \in N\}$$

is an open decreasing neighborhoods base of y in B. Then each $\overline{O_n}$ is not countably compact.

If $\overline{O_1}$ is not countably compact, then there is a discrete closed subset

$$D_1 = \{y_{1m} : m \in N\} \subset \overline{O_1}$$

such that $|D_1| = \aleph_0$ and y is not in D_1. Take an open neighborhood $O_{n(1)}$ of y in B such that $\overline{O_{n(1)}} \cap D_1 = \emptyset$ since Y is a regular T_1 space. Then $1 < n(1)$. Then, by supposition, $\overline{O_{n(1)}}$ is not countably compact. So there is a discrete closed subset

$$D_2 = \{y_{2m} : m \in N\} \subset \overline{O_{n(1)}}$$

such that

$$|D_2| = \aleph_0, \, D_2 \cap D_1 = \emptyset$$

and y is not in D_2. Then, by induction, there is an $n(i) \in N$ with $n(i-1) < n(i)$ and a discrete closed subset

$$D_i = \{y_{im} : m \in N\} \subset \overline{O_{n(i)}}$$

for each $i \in N$ such that

$$|D_i| = \aleph_0, \; D_i \cap D_j = \emptyset \; (i \neq j)$$

and y is not in D_i. Let

$$Y_1 = (\cup_{n \in N} D_n) \cup \{y\}.$$

Then we can prove that Y_1 is a closed metric subspace of Y and is not locally countably compact. It is a contradiction. Hence each first countable closed subset B of Y is locally countably compact.

$2 \Rightarrow 1$.

Every metric space is a first countable space. If every first countable closed subspace of Y is locally countably compact, then every metric closed subspace of Y is locally countably compact.

$3 \Leftrightarrow 1$ and $4 \Leftrightarrow 2$.

Each compact subset is countably compact and each countably compact metric subset is compact. □

Proposition 3.3. *Let regular T_1 space Y have a point-countable closed k-network. If each first countable closed subset of Y is locally compact and each countably compact subset is compact, then Y has a point-countable compact k-network.*

Proof. By Theorem 3.1 and Proposition 3.2, Y has a point-countable countably compact k-network. Since each countably compact subset is compact, then Y has a point-countable compact k-network. □

In the following, we give an example which explains "countably compact "in Theorem 3.1 can not be strengthened to "compact "and the condition "each countably compact subset is compact "in Proposition 3.3 can not be omitted. It also give a negative answer to Question 37 of B of Problem Section in [15].

Example 3.4. *There is a regular T_1 countably compact space Y such that Y has a point-countable closed k-network and every first countable closed subspace of Y is compact, but Y has no any point-countable compact k-network.*

Claim 3.5. *To construct a regular T_1 space Y.*

Recall Example 9.1 in [9]. There is an infinite, completely regular countably compact space X, all of whose compact subsets are finite. Also X has a point-countable closed k-network.

Then X is uncountable since each countable completely regular countably compact space is compact. Then $\{\{x\} : x \in X\}$ is a compact point-countable k-network since all compact subsets of the space X are finite. Let $(X_n, \mathcal{T}_n)$ be a copy of the X above and $\mathcal{P}_n$ be the compact point-countable k-network of $(X_n, \mathcal{T}_n)$ for each $n \in \omega$. Let

$$X^* = \{\infty\} \cup \bigcup_{n\in\omega} X'_n.$$

Here

$$X'_n = \{x' : x \in X\}$$

with discrete topology and

$$\{U_n = \{\infty\} \cup \bigcup_{i\geq n} X'_i : n \in \omega\}$$

be a neighborhoods base of point ∞ in X^*. Let $\mathcal{T}^*$ denote the defined topology above of X^*. Then $(X^*, \mathcal{T}^*)$ is a metric space.

Let

$$Z = (\sum_{n\in\omega} X_n) \bigoplus X^*.$$

Let $g : Z \to Y$ be the obvious map. That is

$$f(\{x, x'\}) = \{x\} \quad \text{and} \quad f(\infty) = \infty.$$

Let $O \subset Y$ be an open set if and only if $f^{-1}(O)$ is open in Z. Then g is a continuous quotient s-map and Y is a regular T_1 space.

Claim 3.6. *Y is a countably compact space.*

Since

$$g|X_n : X_n \to X_n$$

is a homeomorphic map, then X_n is countably compact for each $n \in \omega$.

Let

$$\mathcal{P}' = \{g(U_n) : n \in \omega\}.$$

Notice that

$$Y = g(U_0) = g(Z).$$

We prove that each

$$g(U_n) \in \mathcal{P}'$$

is countably compact.

Pick a $g(U_n) \in \mathcal{P}'$. Let $A \subset g(U_n)$ is a countably infinite set.

Case 1. If there is an $n \in \omega$ such that $A \subset \cup_{i \leq n} X_i$, then there is an $i \leq n$ such that $A \cap X_i$ is countably infinite. So A has a cluster point in X_i since X_i is countably compact.

Case 2. If there are countably infinitely many X_i's such that $A \cap X_i$ is not empty in Y, then there are countably infinitely many X_i''s such that $g^{-1}(A) \cap X_i'$ is not empty in Z. Then there is a convergence sequence $S \subset g^{-1}(A)$ such that S converges to ∞ in Z. So there is a convergence sequence $g(S) \subset A$ such that $g(S)$ converges to ∞ in Y. So $g(U_n)$ is countably compact for each $n \in \omega$.

Claim 3.7. *Y has a point-countable countably compact closed k-network.*

Since

$$g|X_n : X_n \to X_n$$

is a homeomorphic map and $\mathcal{P}_n$ is a point-countable compact k-network of $(X_n, \mathcal{T}_n)$, then also $\mathcal{P}_n$ is a point-countable compact k-network of $g(X_n) = X_n$ for each $n \in \omega$. Notice that g is a quotient map. Then $g(U_n)$ is closed open since $g^{-1}(g(U_n))$ is closed open for each $n \in \omega$. So $\mathcal{P}'$ is also a closed open neighborhoods base of ∞ in Y.

Let $\mathcal{P} = \mathcal{P}' \cup \bigcup_{n \in \omega} \mathcal{P}_n$. We may prove that $\mathcal{P}$ is a point-countable countably compact closed k-network of Y.

In fact.

A. $\mathcal{P}$ is a point-countable countably compact closed collection.

B. $\mathcal{P}$ is a k-network.

Let $K \subset O \subset Y$ with K compact and O open in Y.

Case 1. If ∞ is not in K, then there is an open set $g(U_n)$ with $\infty \in g(U_n)$ and $g(U_n) \cap K = \emptyset$. So there is an $n \in \omega$ with $K \subset \cup_{i \leq n} X_i$. Then K is a finite subset of Y. Notice that $g|X_n : X_n \to X_n$ is a homeomorphic map and $\mathcal{P}_n$ is a point-countable compact k-network of $(X_n, \mathcal{T}_n)$ for each $n \in \omega$. Then there is a finite subcollection

$$\mathcal{F} \subset \bigcup_{i \leq n} \mathcal{P}_i$$

with

$$K \subset \cup \mathcal{F} \subset O.$$

Case 2. $\infty \in K$. Take a $g(U_n)$ with

$$\infty \in g(U_n) \subset O.$$

Then

$$K - g(U_n) \subset \cup_{i \leq n} X_i$$

is finite. So there is a finite subcollection

$$\mathcal{F} \subset \bigcup_{i \leq n} \mathcal{P}_i$$

with

$$K - g(U_n) \subset \cup \mathcal{F} \subset (\cup_{i \leq n} X_i) \cap O.$$

Let

$$\mathcal{F}' = \{g(U_n)\} \cup \mathcal{F}.$$

Then

$$\mathcal{F}' \subset \mathcal{P}$$

and

$$K \subset \cup \mathcal{F}' \subset O.$$

Claim 3.8. *Each first countable closed subspace of Y is locally compact.*

We prove only that each closed metric subspace of Y is locally compact by Proposition 3.1. Take a metric closed subspace B of Y. Then B is countably compact in Y. So B is compact.

Claim 3.9. *Y has no any point-countable compact k-network.*

Suppose that $\mathcal{C}$ is a point-countable compact k-network. Let

$$\mathcal{C}(\infty) = \{C \in \mathcal{C} : \infty \in C\} = \{C_n : n \in \omega\}$$

by $\mathcal{C}$ point-countable. Then $C_n \cap X_m$ is finite for each $m, n \in \omega$. So $\cup\mathcal{C}(\infty)$ is a countable subset of Y. Pick an $x_n \in X_n - \cup\mathcal{C}(\infty)$ since X_n is uncountable for each $n \in \omega$. Let

$$S = \{x_n : n \in \omega\}.$$

Then $S \cup \{\infty\}$ is compact. So there is a finite subcollection $\mathcal{F} \subset \mathcal{C}(\infty)$ and an $n' \in \omega$ with

$$\{x_n : n \geq n'\} \subset \cup\mathcal{F} \subset \cup\mathcal{C}(\infty).$$

It is a contradiction to

$$S \cap (\cup\mathcal{C}(\infty)) = \emptyset.$$

So Y has no any point-countable compact k-network.

4. Products of Quotient s-images of Metric Spaces

For two functions f and g from ω to ω, we define $f \leq g$ iff the set $\{n \in \omega : f(n) > g(n)\}$ is finite.

Recall $\mathfrak{b} = \min\{|B| : B$ is an unbounded subset of ${}^{\omega}\omega\}$ in [6]. Here a subset of ${}^{\omega}\omega$ is called *unbounded* if it is unbounded in $< {}^{\omega}\omega, \leq>$.

Recall that Y has *countable tightness* , $t(Y) \leq \omega$, if, whenever $x \in \overline{A}$ in X, then $x \in \overline{C}$ for some countable $C \subset A$. Sequential spaces and hereditarily separable spaces have countable tightness.

For an infinite cardinal number α, a space is a *k_α-space* , if it is determined by a cover $\mathcal{C}$ of compact subsets with $|\mathcal{C}| \leq \alpha$. A space Y is locally k_α, if each point $y \in Y$ has a neighborhood whose closure is a k_α-space. E. Michael [23] considered the concept of k_ω-space. He pointed out that each product of two k_ω-spaces is a k_ω-space. So is each product of two locally k_ω-spaces.

We can generalize Lemma 5 of [3] into the following theorem. It seems to be useful when discussing the relation between $S_\omega \times Y$ and another axioms. Let M be a metric space, $\mathcal{B}$ be a σ-locally finite base of M and $f : M \to Y$ be a quotient s-map. Let

$$\mathcal{P} = f(\mathcal{B}) = \{f(B) : B \in \mathcal{B}\}$$

such that $\overline{f(B)}$ is compact for each $B \in \mathcal{B}$ by Proposition 2.3. Then $\mathcal{P}$ is point-countable and determines Y by Theorem 6.1 of [9]. So $t(Y) \leq \omega$.

Theorem 4.1. *Let regular T_1 space Y be a quotient s-image of a metric space. If $S_\omega \times Y$ is sequential, then there exists a subcollection $\mathcal{P}_y$ of $\mathcal{P}$ such that $|\mathcal{P}_y| < \mathfrak{b}$ and $\cup\mathcal{P}_y$ is a neighborhood of y for each $y \in Y$.*

Proof. Suppose that there exists a point y_0 of Y such that if we take any subcollection $\mathcal{P}'$ of $\mathcal{P}$ with $|\mathcal{P}'| < \mathfrak{b}$, then $\cup\mathcal{P}'$ is not a neighborhood of y_0.

A. Let N_0 be a Moore-Smith net such that N_0 converges to y_0, y_0 is not in N_0 and $|N_0| = \aleph_0$ by $t(Y) \leq \omega$. Let

$$\mathcal{P}_0 = \{P \in \mathcal{P} : P \cap N_0 \neq \emptyset\}.$$

Then $|\mathcal{P}_0| \leq \aleph_0$ since $\mathcal{P}$ is point-countable. Let $\alpha < \mathfrak{b}$. Assume that we have defined a Moore-Smith net N_β for each $\beta < \alpha$ such that:

(1) N_β converges to y_0,

(2) y_0 is not in N_β,

(3) N_β is countable and

(4) $\mathcal{P}_\beta = \{P \in \mathcal{P} : P \cap N_\beta \neq \emptyset\}$ is countable.

Then $\cup(\cup_{\beta<\alpha}\mathcal{P}_\beta)$ is not a neighborhood of y_0 since

$$|\bigcup_{\beta<\alpha} \mathcal{P}_\beta| \leq \aleph_0 \cdot |\alpha| < \mathfrak{b}.$$

So we can take a Moore-Smith net N_α satisfying the (1)-(4) above and

$$N_\alpha \cap (\bigcup_{\beta<\alpha} N_\beta) = \emptyset.$$

Let

$$\mathcal{P}_\alpha = \{P \in \mathcal{P} : P \cap N_\alpha \neq \emptyset\}.$$

Then $\mathcal{P}_\alpha$ is countable by $\mathcal{P}$ point-countable. Thus, by induction, there exists a collection

$$\mathcal{N} = \{N_\alpha : \alpha < \mathfrak{b}\}$$

such that:

(a) N_α converges to y_0, y_0 is not in N_α and N_α is countable for each $N_\alpha \in \mathcal{N}$ and

(b) P meets only one N_α for each $P \in \mathcal{P}$.

B. Let

$$N_\alpha = \{x_{\alpha 1}, x_{\alpha 2}, \ldots, x_{\alpha n}, \ldots\} = \{1_\alpha, 2_\alpha, \ldots, n_\alpha, \ldots\}$$

for $\alpha < \mathfrak{b}$. Let

$$B = \{f_\alpha : \alpha < \mathfrak{b}\}$$

be an unbounded subset of ${}^\omega\omega$ by the definition of $\mathfrak{b}$. Let

$$H_\alpha = \bigcup_{n<\omega}(\{1_n, 2_n, \ldots, f_\alpha(n)_n\} \times \{1_\alpha, 2_\alpha, \ldots, n_\alpha\}) \subset S_\omega \times Y$$

for each $f_\alpha \in B$. Let

$$H = \bigcup_{\alpha<\mathfrak{b}} H_\alpha.$$

Here $S_n = \{1_n, 2_n, \ldots\}$ is a convergent sequence of S_ω.

a. $H \cap (S_n \times P)$ is closed in $S_n \times P$ for each $S_n \in \mathcal{S}$ and each $P \in \mathcal{P}$.

In fact. P meets only one N_α by (b) of A for each $P \in \mathcal{P}$. Then

$$\begin{aligned}(S_n \times P) \cap H &= (S_n \times P) \cap H_\alpha \\ &= (S_n \times P) \cap (\bigcup_{i \le n}\{1_i, 2_i, ..., f_\alpha(i)_i\} \times \{1_\alpha, 2_\alpha, ..., i_\alpha\})\end{aligned}$$

has only finitely many points. So $(S_n \times P) \cap H$ is closed in $(S_n \times P)$. If $S_\omega \times Y$ is sequential, then $S_\omega \times Y$ is determined by

$$\{S_n \times P : S_n \in \mathcal{S} \text{ and } P \in \mathcal{P}\}.$$

So H is closed in $S_\omega \times Y$.

b. H is not closed in $S_\omega \times Y$.

We prove

$$(\infty, y_0) \in Cl(H) - H.$$

Here "∞" denotes the nonisolated point in S_ω. Given an $f \in {}^\omega\omega$, let

$$U_f = \{\infty\} \cup \{n_m : n \geq f(m)\}.$$

Let U be a neighborhood of point y_0 in Y. Then $N_\alpha \cap U \neq \emptyset$ for each $N_\alpha \in \mathcal{N}$. Then there exists an

$$n(\alpha)_\alpha \in N_\alpha \cap U.$$

Let

$$g(\alpha) = n(\alpha).$$

Then $g \in {}^\mathfrak{b}\omega$. Since $f \in {}^\omega\omega$ and B is unbounded in ${}^\omega\omega$, there exists a function $f_{\alpha_0} \in B$ such that

$$A = \{n \in \omega : f_{\alpha_0}(n) > f(n)\}$$

is infinite. Because of

$$g(\alpha_0) = n(\alpha_0),$$

there exists an $n' \in A$ with $n' > n(\alpha_0)$. Then

$$(f_{\alpha_0}(n')_{n'}, n(\alpha_0)_{\alpha_0}) \in \{1_{n'}, 2_{n'}, \ldots, f_{\alpha_0}(n')_{n'}\}$$
$$\times \{1_{\alpha_0}, 2_{\alpha_0}, \ldots, n(\alpha_0)_{\alpha_0}, \ldots, n'_{\alpha_0}\} \subset H_{\alpha_0} \subset H.$$

On the other hand, the $n' \in A$ gives

$$f_{\alpha_0}(n') > f(n').$$

So

$$f_{\alpha_0}(n')_{n'} \in \{n_m : n \geq f(m)\} \subset U_f$$

and

$$n(\alpha_0)_{\alpha_0} \in N_{\alpha_0} \cap U \subset U.$$

Then

$$(f_{\alpha_0}(n')_{n'}, n(\alpha_0)_{\alpha_0}) \in (U_f \times U).$$

So

$$(f_{\alpha_0}(n')_{n'}, n(\alpha_0)_{\alpha_0}) \in (U_f \times U) \cap H_{\alpha_0} \subset (U_f \times U) \cap H.$$

Then H is not closed. This contradicts (a) of B. □

This technique of the proof of Theorem 4.1 is due to G. Gruenhage [10].

Remark 4.2. Assume $\mathfrak{b} = \aleph_1$. Let

$$\mathcal{U} = \{U_y : y \in Y\},$$

here U_y is a closed neighborhood of y in Y which is the union of countably many compact metric subsets of Y by Theorem 4.1. Then Y is determined by $\mathcal{U}$. In fact. Let $O \subset Y$. If $O \cap U_y$ is open in U_y for each $U_y \in \mathcal{U}$, then $O \cap Int_Y(U_y)$ is open in $Int_Y(U_y)$. So

$$O = \cup\{O \cap Int_Y(U_y) : y \in O\}$$

is open in Y. This implies that Y is determined by

$$\mathcal{U}' = \{Int_Y(U_y) : y \in Y\}.$$

So Y is determined by $\mathcal{U}$ since

$$Int_Y(U_y) \subset U_y.$$

Notice that $S_\omega \times Y$ is determined by

$$\{S_\omega \times U_y : y \in Y\}.$$

So $S_\omega \times Y$ is a sequential subspace if and only if $S_\omega \times U_y$ is a sequential space for each $U_y \in \mathcal{U}$. Then we can assume, without loss of generality under $\mathfrak{b} = \aleph_1$, that Y is the union of countably many compact metric subsets of Y.

Corollary 4.3. *Let regular T_1 space Y be a quotient s-image of a metric space. If $S_\omega \times Y$ is a sequential space, then Y is a locally $k_\mathfrak{c}$-space.*

Proof. Let $\mathfrak{d} < \mathfrak{b} \leq \mathfrak{c}$ and $Y = \cup_{\alpha<\mathfrak{d}} K_\alpha$ by Theorem 4.1 and Remark 4.2. Here K_α is a compact metric subset of Y for each $\alpha < \mathfrak{d}$. Then cardinality of Y is at most $\mathfrak{c}$. If Y is a quotient s-image of a metric space and $S_\omega \times Y$ is a sequential space, then there is a collection $\mathcal{P}$ such that $\mathcal{P}$ determines Y, $\mathcal{P}$ is a point-countable collection and $\overline{P}$ is a compact metric set for each $P \in \mathcal{P}$ by Proposition 2.3 and Theorem 6.1 of [9]. Because cardinality of Y is $\mathfrak{c}$ and $\mathcal{P}$ is a point-countable collection, cardinality of $\mathcal{P}$ is at most $\mathfrak{c}$. So $\mathcal{P}' = \{\overline{P} : P \in \mathcal{P}\}$ determines Y and cardinality of $\mathcal{P}'$ is at most $\mathfrak{c}$. This implies that Y is a locally $k_\mathfrak{c}$-space.

□

Question 4.4. *Let regular T_1 space Y be a quotient s-image of a metric space and $S_\omega \times Y$ be a sequential space. Has Y a point-countable compact k-network (even if we assume the continuum hypothesis)?*

Remark 4.5. According to A. V. Arkhangel'skiĭ [2], a cover $\mathcal{C}$ of Y is called a *k-system* if C is compact for each $C \in \mathcal{C}$ and $B \subset Y$ is closed in Y whenever $B \cap C$ is closed in C for each $C \in \mathcal{C}$.

If Y is a k-space and $\mathcal{C}$ is a compact k-network, then $\mathcal{C}$ is also a k-system by the definition of k-network. Notice that Remark 4.2 and Theorem 4.3. We may assume the space Y in Question 4.4 has a k-network $\mathcal{C}$ such that $|\mathcal{C}| = \mathfrak{c}$ ($|\mathcal{C}| = \aleph_1$ if we assume the Continuum Hypothesis).

5. Weak Neighborhoods and Quotient Maps

A. V. Arhangel'skiĭ [2] introduced definitions of *collections of weak neighborhoods*. It is more convenient to use the form of F. Siwiec [30] and Z. Gao [12] as follows.

Definition 5.1. *For a topological space Y and a point y in Y, a collection $\mathcal{W}_y$ of subsets of Y is called a collection of weak neighborhoods of y, if the following are satisfied:*

(1) *Each member of $\mathcal{W}_y$ contains y.*

(2) *For any two members W_1 and W_2 of $\mathcal{W}_y$, there is a W_3 in $\mathcal{W}_y$, such that $W_3 \subset W_1 \cap W_2$.*

(3) *A subset U of X is open iff for every point y in U there exists a W in $\mathcal{W}_y$ such that $W \subset U$.*

For the above condition (3), we have equivalent condition (3)′ as follows:

(3)′ *A subset B of X is closed iff for every point $x \notin B$ there exists a W in $\mathcal{W}_y$ such that $B \cap W = \emptyset$.*

Definition 5.2. *For a topological space Y and a point y in Y, a subset H of Y is called a weak neighborhood of y in Y, if there is a collection $\mathcal{W}_y$ of weak neighborhoods of y in Y and a $W_y \in \mathcal{W}_y$ such that $W_y \subset H$.*

Firstly we confirm some straightforward results derived from the definitions.

Proposition 5.3. *Let Y be a topological space. Then:*

1. *Every neighborhood of y is a weak neighborhood of y in Y.*

2. $O \subset Y$ *is open if and only if for each* $y \in O$*, there is a weak neighborhood* W_y *of* y *in* Y *with* $W_y \subset O$.

Proof. Let

$$\mathcal{O}_y = \{O : O \text{ is open in } Y \text{ and } y \in O\}.$$

Then $\mathcal{O}_y$ is a collection of weak neighborhoods of y in Y by Definition 5.1. So every neighborhood of y is a weak neighborhood of y in Y by Definition 5.2. So $O \subset Y$ is open if and only if for each $y \in O$, there is a weak neighborhood W_y of y in Y with $W_y \subset O$ by the condition 3 in Definition 5.1. □

Proposition 5.4. *Let* Y *be a sequential Hausdorff space and* y *be a point of* Y*. Then* W_y *is a weak neighborhood of* y *in* Y *if and only if for each compact metric subset (or convergent sequence containing the limit point* y*)* K *of* Y *with* $y \in K$*,* W_y *contains a neighborhood of* y *in* K *(or* W_y *contains* K *eventually).*

Proof. "only if". Suppose that there is a compact metric subset K of Y and a weak neighborhood W'_y which can not contain any neighborhood of y in K. Then there is a sequence $S \subset K$ such that S converges to y and $S \cap W'_y = \emptyset$.

Pick an open set O with $y \in O$. Then $O - S$ is not open.

On the other hand, we pick an $x \in O - S$. If $x \neq y$, then there is an open set $O_1 \subset O - S$ with $x \in O_1$. So there is a weak neighborhood W_x with $W_x \subset O_1$. When $x = y$, let $\mathcal{W}_y$ be the collection of weak neighborhoods of y such that there is a $W_y \in \mathcal{W}_y$ and $W_y \subset W'_y$ by Definition 5.2. Since O is an open set with $y \in O$, there is a $V_y \in \mathcal{W}_y$

such that $V_y \subset O$. Then there is a weak neighborhood $V'_y \in \mathcal{W}_y$ such that

$$V'_y \subset V_y \cap W_y \subset V_y \cap W'_y \subset O - S.$$

So $O - S$ is open by Proposition 5.3. It is a contradiction.

"if". Let $\mathcal{W}_y$ be a collection of subsets of Y such that each member of $\mathcal{W}_y$ contains a neighborhood of y in K for each compact metric subset K of Y with $y \in K$. We can check that $\mathcal{W}_y$ is a collection of weak neighborhoods of y. In fact:

1. Each member of $\mathcal{W}_y$ contains y.

2. Pick two members W_1 and W_2 from $\mathcal{W}_y$. Let K be a compact metric subset of Y with $y \in K$. Then both W_1 and W_2 contain neighborhoods of y in K. Then $W_1 \cap W_2$ contains a neighborhood of y in K. So $W_1 \cap W_2 \in \mathcal{W}_y$.

3. "if". Suppose that $A \subset Y$ is not open in Y. Then there is a point y and a sequence S such that S converges to y and $(S \cup \{y\}) \cap A$ is not open in $S \cup \{y\}$. So $y \in A$. Then S is eventually in W for each $W \in \mathcal{W}_y$. So $W \cap A$ is not open in W. Then $W - A \neq \emptyset$. "only if". If $A \subset Y$ is open, then for each $y \in A$, A is an open neighborhood of y in Y. So A is a weak neighborhood of y with $x \in A \subset A$ by Proposition 5.3.

So $\mathcal{W}_y$ is a collection of weak neighborhoods by Definition 5.1. Then each member of $\mathcal{W}_y$ is a weak neighborhood of y in Y by Definition 5.2. □

Corollary 5.5. *Let Y be a sequential Hausdorff space. Then W is a weak neighborhood of y in Y iff whenever $B \subset Y$ with $y \in Cl_Y(B-\{y\})$, $y \in Cl_Y((Cl_Y(B) - \{y\}) \cap W)$.*

Proof. Let $B \subset Y$ with $y \in Cl_Y(B - \{y\})$. Then $Cl_Y(B) - \{y\}$ is not closed in Y. So there is a converging sequence $S \cup \{y'\}$ such that

$$(S \cup \{y'\}) \cap (Cl_Y(B) - \{y\})$$

is not closed in $S \cup \{y'\}$. Let

$$S_1 = S \cap (Cl_Y(B) - \{y\}).$$

Then S_1 converges to $y' \notin Cl_Y(B) - \{y\}$. But

$$y' \in Cl_Y(S_1) \subset Cl_Y(B).$$

Then $y' = y$. Since each weak neighborhood W of y contains a neighborhood of y in $S \cup \{y\}$ by Proposition 5.4,

$$y \in Cl_Y((Cl_Y(B) - \{y\}) \cap W).$$

If W is not a weak neighborhood of y in Y, then there is a sequence S such that S converges to y and $S \cap W = \emptyset$ by Proposition 5.4. Then

$$y \in Cl_Y(S - \{y\}).$$

But

$$y \notin Cl_Y((Cl_Y(S) - \{y\}) \cap W).$$

□

Recall that a *k-sequence* (B_n) in a space Y (due to E. Michael [21]) is a decreasing sequence $\{B_n : n \in N\}$ of subsets of Y such that the set $\cap_{n=1}^{\infty} B_n = C$ is compact and each neighborhood of C contains some B_n. In particular, when the compact set C is a singleton $\{y\}$, we denote the k-sequence (B_n) by $(B_n) \downarrow y$.

Corollary 5.6. *Let Y be a sequential Hausdorff space. If $(B_n) \downarrow y$ is a k-sequence and W is a weak neighborhood of y in Y, then W contains some B_n.*

Proof. Suppose that there is a weak neighborhood W of y and a k-sequence $(B_n) \downarrow y$ such that $B_n - W \neq \emptyset$ for each $n \in \omega$. Let $y_n \in B_n - W$ for each $n \in \omega$. Then $\{y_n : n \in \omega\}$ converges to y. So $\{y_n : n \in \omega\}$ is eventually in W by Proposition 5.4. It is a contradiction.

□

Proposition 5.7. *Let Y be a sequential Hausdorff space, M be an arbitrary metric space and $f : M \to Y$ be an arbitrary quotient map. If U is a neighborhood of $f^{-1}(y)$ in M, then $f(U)$ is a weak neighborhood of y in Y for each $y \in Y$.*

Proof. Suppose that there is a sequence S such that S converges to y and $S \cap f(U) = \emptyset$ by Proposition 5.4. Let

$$Z = f^{-1}(S \cup \{y\}).$$

Then $f|_Z : Z \to S \cup \{y\}$ is a quotient map. So there is a sequence $S_1 \subset f^{-1}(S)$ such that S_1 converges to x and $x \in f^{-1}(y)$.

Because U is a neighborhood of $f^{-1}(y)$ in M, we may assume that $S_1 \subset U$. This implies that

$$\emptyset \neq f(S_1) \subset f(U) \cap S.$$

It is a contradiction. □

6. Michael-Nagami's Problem.

We construct a Hausdorff space Y such that Y is a quotient s-image of a metric space and is no compact-covering quotient s-image of any metric space.

Let N denote the set of all positive integers and i, j, l, m and n be members of N. Let

$$[a\ \ b) = \{r : r \text{ is a real number with } a \leq r < b\}$$

and

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Let Q be rational numbers in $[0\ \ 1]$ and $R_1 \cup R_2 = [0\ \ 1] - Q$ such that $R_1 \cap R_2 = \emptyset$ and $R_i (i = 1, 2)$ is $\mathfrak{c}$-dense in $[0\ \ 1]$. Let

$$M_1 = (R_1 \times \{1,\ 1/2,\ 1/3, ...\}) \cup ((R_1 \cup R_2) \times \{0\})$$

be a subspace of $[0\ \ 1] \times [0\ \ 1]$. Let

$$S_r = \{r\} \times \{1,\ 1/2,\ 1/3, ...\}$$

converge to $(r, 0)$ for each $r \in R_2$. Then

$$M_2 = \sum\{S_r \cup \{(r, 0)\} : r \in R_2\}$$

is locally compact metric. Let

$$M_3 = \sum\{[0\ \ 1] \times \{1/n\} : n \in N\}.$$

Then M_3 is locally compact metric. Let

$$M = M_1 \bigoplus M_2 \bigoplus M_3.$$

Then M is metric. Let $\mathcal{B} = \cup_n \mathcal{B}_n$ be a σ-locally finite base of M such that:

1. If $B \in \mathcal{B}$, then $B \subset M_l$ for some $l \in \{1, 2, 3\}$.
2. If $B \subset M_2$, then $B \subset S_r \cup \{(r, 0)\}$ for some $r \in R_2$.
3. If $B \subset M_3$, then $B \subset [0\ \ 1] \times \{1/n\}$ for some $n \in N$.

Let $g : M \to Y$ be the obvious map ($g(x) = x$ for each $x \in M$) and

$$\mathcal{P} = g(\mathcal{B}) = \{g(B) : B \in \mathcal{B}\}.$$

Let $U \subset Y$ be open in Y if and only if $U \cap P$ is relatively open in P for each $p \in \mathcal{P}$. Let

$$\mathcal{T} = \{O : O \cap P \text{ is relatively open in } P \text{ for every } p \in \mathcal{P}\}.$$

(i.e., Y is determined by a cover $\{M_1, M_2, M_3\}$.) Then g is a two to one quotient map and $(Y, \mathcal{T})$ is a sequential space.

Claim 6.1. *$(Y, \mathcal{T})$ is Hausdorff but not regular.*

Proof. Pick $y_1 = (r_1, 0)$ and $y_2 = (r_2, 0)$ in $(R_1 \cup R_2) \times \{0\}$ such that $r_1 < r_2$. Let

$$O_1 = \cup\{[0\ r_3) \times \{1/n\} : n \in N\} \cup (([0\ r_3) \cap (R_1 \cup R_2)) \times \{0\})$$

and

$$O_2 = \cup\{(r_3\ 1]) \times \{1/n\} : n \in N\} \cup (((r_3\ 1]) \cap (R_1 \cup R_2)) \times \{0\}).$$

Here $r_3 = (r_1 + r_2)/2$. Then O_1 and O_2 are disjoint open neighborhoods of y_1 and y_2 in Y respectively. So Y is Hausdorff.

Let

$$Q = \{q_n : n \in N\}$$

and

$$Y_1 = \cup\{\{q_i : i \leq n\} \times \{1/n\} : n \in N\}.$$

Then Y_1 is a discrete closed subset Y.

To see it, pick a $P \in \mathcal{P}$. If $P \subset M_1 \cup M_2$, then $P \cap Y_1 = \emptyset$. If $P \subset M_3$, then $P \cap Y_1$ is finite. So $P \cap Y_1$ is closed in P for each $P \in \mathcal{P}$. Y_1 is discrete closed since Y is determined by $\mathcal{P}$ (also see footnote 2 of page 303 in [9]).

Let

$$O = Y - Y_1.$$

Then O is open. Let

$$y \in (R_1 \cup R_2) \times \{0\} \subset O.$$

Suppose that B is a closed neighborhood of y in Y such that $B \subset O$. Then there is an open set O_y with $y \in O_y \subset B$. Then $O_y \cap M_1$ is an open neighborhood of y in M_1. So there is an open ball

$$B(y) = \{y' \in M_1 : d(y\ y') < 1/n\}$$

of M_1 with

$$y \in B(y) \subset O_y \cap M_1.$$

Because M_1 is dense in Y, there is an $m \in N$ and a $[r_1\ \ r_2]$ such that

$$\cup_{i>m}([r_1\ \ r_2] \times \{1/i\}) \subset Cl_Y(B(y)).$$

Take a convergent sequence

$$S = \{q_{i(n)} : n \in N\} \subset Q \cap [r_1\ \ r_2]$$

such that S converges to r ($y = (r,\ 0)$). Then

$$(q_{i(n)}, 1/i(n)) \in Cl_Y(B(y)) \cap Y_1$$

for each $i(n) > m$. We have

$$Cl_Y(B(y)) \cap Y_1 \neq \emptyset \quad \text{and} \quad Cl_Y(B(y)) \subset B \subset O.$$

It is a contradiction. So Y is not regular. □

Claim 6.2. *$M_1 \cup S_r$ is a metric subspace of Y for each $(r, 0) = y \in R_2 \times \{0\}$.*

Proof. Given an $r \in R_2$, then let

$$y = (r, 0)$$

and

$$M_r = M_1 \bigoplus (S_r \cup \{(r,0)\})$$
$$\bigoplus \sum \{(R_1 \cup \{r\}) \times \{1/n\} : n \in N\} \subset M.$$

Let

$$g|M_r \ = g_r$$

and

$$\mathcal{T}_1 = \{O \cap (M_1 \cup S_r) : O \in \mathcal{T}\}.$$

1. g_r is a two to one continuous onto map from M_r to $M_1 \cup S_r$.

In fact, g is a two to one continuous onto map.

2. $(M_1 \cup S_r, \mathcal{T}_1)$ is a sequential subspace of $(Y, \mathcal{T})$.

To see it, let $B \subset M_1 \cup S_r$ be not closed in $M_1 \cup S_r$.

If there is a

$$y' \in Cl_{M_1 \cup S_r}(B) - B$$

with $y' \neq (r,0)$, then there is an open neighborhood $O_{y'}$ of y' in $M_1 \cup S_r$ such that $Cl_{M_1 \cup S_r}(O_{y'})$ is metric in $M_1 \cup S_r$. Because of

$$y' \in Cl_{M_1 \cup S_r}(O_{y'} \cap B),$$

there is a sequence S of B which converges to y'. So $B \cap (S \cup \{y'\})$ is not closed in $S \cup \{y'\}$.

If

$$(Cl_Y(B) - B) \cap (M_1 \cup S_r) = \{(r,0)\} = \{y\},$$

then there is a sequence $S \subset Cl_Y(B) - \{y\}$ which converges to y by $(Y, \mathcal{T})$ being sequential. Because $g : M \to Y$ is a two to one quotient map,

$$g^{-1}(y) \subset M_1 \bigoplus (S_r \cup \{(r,0)\})$$

and

$$M_1 \bigoplus (S_r \cup \{(r,0)\})$$

is open in M, $M_1 \cup S_r$ is a weak neighborhood of $y = (r,0)$ in Y by Proposition 5.7. Then S is eventually in $M_1 \cup S_r$ by Proposition 5.4. So S is eventually in

$$(Cl_Y(B) - \{y\}) \cap (M_1 \cup S_r) = Cl_{M_1 \cup S_r}(B) - \{y\}.$$

Since

$$Cl_{M_1 \cup S_r}(B) - B = (Cl_Y(B) - B) \cap (M_1 \cup S_r) = \{y\},$$

S is eventually in

$$Cl_{M_1 \cup S_r}(B) - \{y\} = B.$$

Then $B \cap (S \cup \{y\})$ is not closed in $S \cup \{y\}$. This implies that $(M_1 \cup S_r, \mathcal{T}_1)$ is sequential.

3. $g_r : M_r \longrightarrow M_1 \cup S_r$ is a sequence-covering map. So it is a quotient map.

To see it, let $S \subset M_1 \cup S_r$ converge to $y = (r,0)$. If

$$S \subset M_1 \quad \text{or} \quad S \subset S_r,$$

then it is simple. If

$$|S \cap M_1| = |S \cap S_r| = \omega,$$

then

$$S = (S \cap M_1) \cup (S \cap S_r).$$

So there are two compact sets $C_1 \subset M_1$ and $C_2 \subset S_r \cup \{(r,0)\}$ in M such that

$$g_r(C_1) = (S \cap M_1) \cup \{y\}$$

and

$$g_r(C_2) = (S \cap S_r) \cup \{y\}.$$

4. $M_1 \cup S_r$ is regular.

In fact. Let $y' \in M_1 \cup S_y$. We notice that M_1 is a metric subspace of Y.

If $y' \neq (r, 0)$, then there is an open metric subset $O_{y'}$ of $M_1 \cup S_r$ such that $y' \in O_{y'}$ and $Cl_{M_1 \cup S_r}(O_{y'})$ is metric in $M_1 \cup S_r$.

If $y' = (r, 0) = y$, then whenever open set $O \subset M_1 \cup S_r$ with $y \in O$, there is an open set O_1 of M_1 such that

$$y \in O_1 \subset Cl_{M_1}(O_1) \subset O \cap M_1.$$

Let

$$S_{r,l} = \{(r, 1/n) : (r, 1/n) \in S_r \text{ and } n > l\}.$$

Because S_r is eventually in O, we have $S_{r,l} \cup Cl_{M_1}(O_1) \subset O$ for some $l \in N$. Let

$$M_l = (M_1 \cup S_r) - (\cup\{(R_1 \cup \{r\}) \times \{1/n\} : n = 1, 2, \dots, l\}).$$

We may assume $O_1 \subset M_l$. Then

$$\begin{aligned} Cl_{M_1 \cup S_r}(O_1) &= Cl_{M_l}(O_1) \\ &= Cl_Y(O_1) \cap M_l \\ &\subset Cl_Y(O_1) \cap (M_1 \cup S_{r,l}) \\ &\subset (Cl_Y(O_1) \cap M_1) \cup (Cl_Y(O_1) \cap S_{r,l}) \\ &\subset Cl_{M_1}(O_1) \cup S_{r,l} \subset O. \end{aligned}$$

Let

$$B = (M_1 \cup S_r) - Cl_{M_1 \cup S_r}(O_1).$$

Suppose

$$y \in Cl_{M_1 \cup S_r}(B).$$

Because

$$Cl_{M_1 \cup S_r}(B) - \{y\}$$

is not closed in $M_1 \cup S_r$, there is a sequence

$$S \subset Cl_{M_1 \cup S_r}(B) - \{y\}$$

which converges to y by 2 of Claim 6.2. Let

$$S = \{y_n : n \in \omega\}.$$

a. If for each $n \in \omega$, there is an $i(n) > n$ with $y_{i(n)} \notin S_r$, then we may assume

$$S \cap S_r = \emptyset \text{ and } S \subset M_1.$$

Because $y \in O_1$ and O_1 is open in M_1, S is eventually in O_1. If arbitrary $S_n \subset M_1$ converges to $y_n \in S$, then S_n is eventually in O_1. On the other hand, if

$$y_n \in S \quad \text{and} \quad S \cap S_r = \emptyset,$$

then $y_n \notin S_r \cup \{y\}$. Because $S_r \cup \{y\}$ is compact and $M_1 \cup S_r$ is Hausdorff, there are two open sets U_1 and U_2 of $M_1 \cup S_r$ such that

$$U_1 \cap U_2 = \emptyset,$$
$$y_n \in U_1 \subset M_1 \quad \text{and} \quad S_r \cup \{y\} \subset U_2.$$

Because of

$$y_n \in Cl_{M_1 \cup S_r}(B) \cap U_1,$$

there is a sequence

$$S_n \subset U_1 \cap B \subset U_1 \subset M_1$$

such that S_n converges to y_n by U_1 being metric. Because of

$$U_1 \cap B = U_1 \cap ((M_1 \cup S_r) - Cl_{M_1 \cup S_r}(O_1)),$$

we have

$$(\cup_n S_n) \cap O_1 = \emptyset.$$

It is a contradiction.

b. If there is an l such that $y_n \in S_r$ for each $n > l$, then we may assume

$$S = \{y_{l(n)} : n \in \omega\} \subset S_r.$$

So there is a clopen metric set

$$U_n = (R_1 \cup \{r\}) \times \{1/l(n)\}$$

for each $y_{l(n)} \in S_r$. Since

$$U_n \cap ((M_1 \cup S_r) - Cl_{M_1 \cup S_r}(O_1))$$

is an open neighborhood of $y_{l(n)}$ in $M_1 \cup S_r$, there is a sequence

$$S_n \subset U_n \cap ((M_1 \cup S_r) - Cl_{M_1 \cup S_r}(O_1))$$

such that $S_n \subset M_1$ and S_n converges to $y_{l(n)}$. Then

$$(\cup_n S_n) \subset M_1 \quad \text{and} \quad (\cup_n S_n) \cap O_1 = \emptyset.$$

On the other hand, we have

$$y \in Cl_{M_1 \cup S_r}(\cup_n S_n).$$

If $O \subset M_1 \cup S_r$ is open with $y \in O$, then

$$(\cup_n S_n) \cap O \neq \emptyset.$$

Let O be open in $M_1 \cup S_r$ with $O_1 = O \cap M_1$. Then

$$O_1 \cap (\cup_n S_n) = (O \cap M_1) \cap (\cup_n S_n)$$
$$= O \cap (M_1 \cap (\cup_n S_n)) = O \cap (\cup_n S_n) \neq \emptyset.$$

It is a contradiction.

So $y \notin Cl_{M_1 \cup S_r}(B)$ and $M_1 \cup S_r$ is regular.

5. $M_1 \cup S_r$ contains no closed copy of S_ω and none of S_2.

Because $g_r^{-1}(y)$ is compact and $M_1 \cup S_r$ is regular, $M_1 \cup S_r$ contains no closed copy of S_ω by Theorem 1.7 of [33]. Suppose that $M_1 \cup S_r$ contains a copy of S_2. Let

$$(\cup_n S_n') \cup S' \cup \{y\}$$

be the copy. Then $y = (r, 0)$ and

$$(\cup_n (S_n' - S_r)) \cup S' \cup \{y\}$$

is a copy of S_2 also. On the other hand, let

$$M_1 = U_1 \supset U_2 \supset ...$$

be a countable base of neighborhoods of y in M_1. Then, in $M_1 \cup S_r$, there is a decreasing open sets sequence

$$M_1 \cup S_r = O_1 \supset O_2 \supset ...$$

such that $O_n \cap M_1 = U_n$ for each $n > 0$. Since

$$y \in Cl_{M_1 \cup S_r}(\cup_n (S_n' - S_r)),$$

then

$$O_n \cap (\cup_n (S_n' - S_r)) \neq \emptyset.$$

So

$$(O_n - S_r) \cap (\cup_n (S_n' - S_r)) \neq \emptyset.$$

Because of

$$O_n - S_r = O_n \cap M_1 = U_n,$$

we have

$$U_n \cap (\cup_n (S_n' - S_r)) \neq \emptyset$$

for each $n > 0$. Pick a

$$z_n \in U_n \cap (\cup_n (S_n' - S_r))$$

for each $n > 0$. Then the sequence

$$S = \{z_n : n \in N\} \subset \cup_n(S'_n - S_r)$$

converges to y. It is a contradiction.

So $M_1 \cup S_r$ has a point-countable base by Theorem 2.1. Because $M_1 \cup S_r$ is a $\aleph_0$-space, $M_1 \cup S_r$ is a metric space by Theorem 11.4 of [11]. □

Claim 6.3. *Y is no compact-covering quotient s-image of any metric space.*

Proof. Let X be an arbitrary metric space and $\mathcal{B} = \cup_n \mathcal{B}_n$ be a σ-locally finite base of X. Suppose that f is a compact-covering quotient s-map from the space X to the space $(Y, \mathcal{T})$ and

$$\mathcal{P} = \{f(B) : B \in \mathcal{B}\}.$$

Then $\mathcal{P}$ is point-countable and $(Y, \mathcal{T})$ is determined by cover $\mathcal{P}$ (see also [9, footnote 2]). Let

$$\mathcal{B}'_1 = \{B \in \mathcal{B} : B \cap f^{-1}((r, 0)) \neq \emptyset \text{ for some } r \in R_2\}$$

and

$$\mathcal{B}'_2 = \{B \in \mathcal{B}'_1 : f(B) \subset M_1 \cup S_r \text{ for some } r \in R_2\}.$$

Since we have proved that $g : M \to Y$ is a two to one quotient map and for

$$y = (r, 0) \ (r \in R_2),$$
$$g^{-1}(y) \subset M_1 \bigoplus (S_r \cup \{(r, 0)\}),$$

then $M_1 \cup S_r$ is a weak neighborhood of y in Y by Proposition 5.7. Let $B \in \mathcal{B}'_1$ such that

$$B \cap f^{-1}((r, 0)) \neq \emptyset.$$

If

$$x \in B \cap f^{-1}((r,0))$$

and

$$B = U_1 \supset U_2 \supset \dots$$

is a neighborhoods base of x in X, then $(f(U_n)) \downarrow f(x)$ is a k-sequence. So there is an n such that $f(U_n) \subset M_1 \cup S_r$ by Corollary 5.6. Then $(\mathcal{B} - \mathcal{B}_1') \cup \mathcal{B}_2'$ is a σ-locally finite base of X also. So we may assume that, without loss of generality, if

$$B \in \mathcal{B} \quad \text{and} \quad B \cap f^{-1}((r,0)) \neq \emptyset$$

for some $r \in R_2$, then

$$f(B) \subset M_1 \cup S_r.$$

All $\mathcal{B}$ in the following proof refer to the σ-locally finite base and

$$\mathcal{P} = \{f(B) : B \in \mathcal{B}\}.$$

(*Continued after Subclaim 6.4.*)

Subclaim 6.4. *Given an arbitrary $r \in R_2$ and $y = (r,0)$, let*

$$S_r = \{y_l : y_l = (r, 1/l) \text{ and } l \in N\},$$

$$\mathcal{P}'(y) = \{P \in \mathcal{P} : y = (r,0) \in P \text{ and } |P \cap S_r| = \omega\} = \{P_n : n \in N\}$$

and $\mathcal{F}_n = \{P_i \in \mathcal{P}'(y) : i \leq n\}$ for each $n \in \omega$. Then there is a finite $\mathcal{F}_n \subset \mathcal{P}'(y)$ such that for each $l > n$, $\cup\{P \in \mathcal{F}_n : y_l \in P\}$ contains a neighborhood U_l of y_l in $M_1 \cup S_r$.

Proof. Suppose that for each

$$\mathcal{F}_n = \{P_i \in \mathcal{P}'(y) : i \leq n\},$$

there is an $l > n$ such that

$$\cup\{P \in \mathcal{F}_n : y_l \in P\}$$

can not contain any neighborhood of y_l in $M_1 \cup S_r$. Let $\{O_n : n \in N\}$ be a decreasing open countable base of neighborhoods of y in $M_1 \cup S_r$ such that $y_n \in O_n$ for each $n \in N$.

Let

$$\mathcal{F}_1 = \{P_1\} \subset \mathcal{P}'(y).$$

Then there is an $l_1 > 1$ such that $y_{l_1} \in P_1$ and P_1 can not contain any neighborhood of y_{l_1} in O_1. Let S_1 converge to y_{l_1} such that

$$S_1 \cap P_1 = \emptyset \text{ and } S_1 \cup \{y_{l_1}\} \subset O_1.$$

Assume that $S_n \cup \{y_{l_n}\}$ have been defined such that

$$l_n > l_{n-1},$$

$$S_n \cap (\cup\{P \in \mathcal{F}_n : y_{l_n} \in P\}) = \emptyset$$

and

$$S_n \cup \{y_{l_n}\} \subset O_n.$$

Let

$$\mathcal{F}_{n+1} = \{P_1, P_2, ..., P_{n+1}\} \subset \mathcal{P}'(y).$$

Then there is an $l_{n+1} > l_n$ such that $\cup\{P \in \mathcal{F}_{n+1} : y_{l_{n+1}} \in P\}$ can not contain any neighborhood of $y_{l_{n+1}}$ in $M_1 \cup S_r$. Let

$$\mathcal{F}' = \{P \in \mathcal{F}_{n+1} : y_{l_{n+1}} \in P\}.$$

Then we can define S_{n+1} such that S_{n+1} converges to $y_{l_{n+1}}$,

$$S_{n+1} \cup \{y_{l_{n+1}}\} \subset O_{n+1} \quad \text{and} \quad S_{n+1} \cap (\cup\mathcal{F}') = \emptyset.$$

Let

$$S = \{y_{l_n} : n \in N\}$$

and

$$E_2 = (\cup_n S_n) \cup S \cup \{y\}.$$

Since f is a compact-covering map and E_2 is a compact subset of Y, then there is a compact subset C of X with $f(C) = E_2$. For each $y_{ij} \in E_2$, pick an

$$x_{ij} \in f^{-1}(y_{ij}).$$

Let

$$S_i'' = \{x_{ij} : j \in N\}$$

for each $i \in N$. Then, for each $i \in N$, there is a subsequence

$$S_i' = \{x_{ij(n)} : n \in N\}$$

of S_i'' which converges to some $x_i \in f^{-1}(y_{l_i}) \cap C$ since

$$f^{-1}(S_i \cup \{y_{l_i}\}) \cap C$$

is compact and

$$S_i'' \subset f^{-1}(S_i \cup \{y_{l_i}\}) \cap C$$

is infinite. Let

$$S'' = \{x_i : i \in N\}.$$

Then there is a subsequence

$$S' = \{x_{i(n)} : n \in N\}$$

of S'' which converges to some $x_0 \in f^{-1}(y) \cap C$ since

$$f^{-1}(S \cup \{y\}) \cap C$$

is compact and

$$S'' \subset f^{-1}(S \cup \{y\}) \cap C$$

is infinite. Let $\{O_n' : n \in N\}$ be a decreasing open neighborhoods base of x_0 in C. Then we may assume, without loss of generality,

$$S_{i(n)}' \cup \{x_{i(n)}\} \subset O_n'$$

for each $n \in N$ (Otherwise we may take infinite subsequences of S' and S'_i $(i \in N)$). Let

$$K = (\cup_n S'_{i(n)}) \cup S' \cup \{x_0\}.$$

Then K is compact. Pick a $B \in \mathcal{B}$ with $x_0 \in B$. Then there is an n_0 with $O'_{n_0} \subset B$. So

$$f(S'_{i(m)} \cup \{x_{i(m)}\}) \subset f(B)$$

for each $m > n_0$ by

$$S'_{i(m)} \cup \{x_{i(m)}\} \subset O'_m \subset O'_{n_0}.$$

Notice that $f(S'_{i(m)})$ is an infinite subsequence of $S_{i(m)}$. Then $S_{i(m)} \cap f(B)$ is infinite for each $m > n_0$. On the other hand, since $O'_n \subset B$ for each $n > n_0$, then $S \cap f(B)$ is infinite. So $f(B) = P_{n'} \in \mathcal{P}'(y)$ for some n'. Then, for each $i > n'$,

$$S_i \cap (\cup\{P \in \mathcal{F}_{n'} : y_{l_i} \in P\}) = \emptyset$$

since

$$S_i \cap (\cup\{P \in \mathcal{F}_i : y_{l_i} \in P\}) = \emptyset$$

and

$$\mathcal{F}_{n'} \subset \mathcal{F}_i.$$

Then, whenever $i(m) > m > n_0 + n'$, $S_{i(m)} \cap f(B)$ is infinite. But $S_{i(m)} \cap P_{n'} = \emptyset$ since $P_{n'} \in \mathcal{F}_{n'} \subset \mathcal{F}_{i(m)}$. It is a contradiction to $f(B) = P_{n'}$. □

Proof. *proof of Claim 6.3.* (*continued*)

Let

$$\mathcal{P}(y) = \{P_{y1}, P_{y2}, ..., P_{yn(y)}\}$$

be a finite collection which satisfies Subclaim 6.4 for each $y = (r, 0) \in R_2 \times \{0\}$. Let

$$\mathcal{P}' = \cup\{\mathcal{P}(y) : y = (r, 0) \in R_2 \times \{0\}\}.$$

Then $\mathcal{P}' \subset \mathcal{P}$ is point-countable. We can prove that $\mathcal{P}'$ satisfies the following condition $*$.

$*$. If y and y' are distinct points of $R_2 \times \{0\}$, then

$$\mathcal{P}(y) \cap \mathcal{P}(y') = \emptyset.$$

Here $y = (r, 0)$, $y' = (r', 0)$ and $r \neq r'$.

In fact, if there is a $P \in \mathcal{P}(y) \cap \mathcal{P}(y')$, then

$$|P \cap S_r| = |P \cap S_{r'}| = \aleph_0$$

and

$$P \subset (M_1 \cup S_r) \cap (M_1 \cup S_{r'}) = M_1 \cup (S_r \cap S_{r'}) = M_1.$$

It is a contradiction to

$$M_1 \cap (S_r \cup S_{r'}) = \emptyset.$$

Let

$$\mathcal{P}'' = \{\cup\mathcal{P}(y) : y = (r, 0) \in R_2 \times \{0\}\}.$$

Then $\mathcal{P}''$ is point-countable and uncountable by the condition $*$. Let

$$\mathcal{O} = \{Int_{M_1 \cup S_r}(\cup\mathcal{P}(y)) : y = (r, 0) \in R_2 \times \{0\}\}.$$

If

$$y \neq y',$$

then

$$Int_{M_1 \cup S_r}(\cup\mathcal{P}(y)) \cap S_{r'} = \emptyset$$

by

$$\cup\mathcal{P}(y) \subset M_1 \cup S_r.$$

Then S_r is eventually in $Int_{M_1 \cup S_r}(\cup \mathcal{P}(y))$ by Subclaim 6.4. So

$$Int_{M_1 \cup S_r}(\cup \mathcal{P}(y)) \neq Int_{M_1 \cup S_{r'}}(\cup \mathcal{P}(y')).$$

Then $\mathcal{O}$ is point-countable and uncountable. Because of Subclaim 6.4, there is an $n \in N$ such that whenever $l > n$, $\cup \mathcal{P}(y)$ contains an open neighborhood of $(r, 1/l)$ in $M_1 \cup S_r$ for each $y = (r, 0) \in R_2 \times \{0\}$. So

$$Int_{M_1 \cup S_r}(\cup \mathcal{P}(y)) - (S_r \cup \{y\}) \neq \emptyset.$$

Then

$$Int_{M_1 \cup S_r}(Int_{M_1 \cup S_r}(\cup \mathcal{P}(y)) - (S_r \cup \{y\})) \neq \emptyset.$$

This implies

$$O_y = Int_{M_1 \cup S_r}(Int_{M_1 \cup S_r}(\cup \mathcal{P}(y)) - S_r) \neq \emptyset$$

and O_y is open in $M_1 \cup S_r$. Let

$$U_y = M_1 \cap O_y.$$

Then U_y is open in M_1. Let

$$\mathcal{U} = \{U_y : y = (r, 0) \in R_2 \times \{0\}\}.$$

Then $\mathcal{U}$ is a collection of point-countable open sets of M_1 by $U_y \subset \cup \mathcal{P}(y)$ for each $y = (r, 0) \in R_2 \times \{0\}$. Because M_1 is separable metric, $\mathcal{U}$ is countable.

On the other hand, we notice that:

$$\begin{aligned} M_1 &\supset Int_{M_1 \cup S_r}(\cup \mathcal{P}(y)) - (S_r \cup \{y\}) \\ &= Int_{M_1 \cup S_r}(Int_{M_1 \cup S_r}(\cup \mathcal{P}(y)) - (S_r \cup \{y\})) \\ &= Int_{M_1 \cup S_r}(Int_{M_1 \cup S_r}(\cup \mathcal{P}(y)) - S_r) \\ &= O_y = M_1 \cap O_y = U_y \end{aligned}$$

Then

$$\{O_y : y = (r, 0) \in R_2 \times \{0\}\}$$

is countable. So there is an uncountable subset $A \subset R_2$ such that whenever we pick $r \neq r'$ in A,

$$Int_{M_1 \cup S_r}(\cup \mathcal{P}(y)) - (S_r \cup \{y\})$$
$$= Int_{M_1 \cup S_{r'}}(\cup \mathcal{P}(y')) - (S_{r'} \cup \{y'\}) \neq \emptyset.$$

Then $\mathcal{O}$ is not point-countable. It is a contradiction. This implies Claim 6.3. □

Question 6.5. *Is it possible to find a counterexample to Michael-Nagami's question among regular T_1 spaces (or paracompact spaces)?*

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