On Convergence Conditions of the Order Dependent Mapping

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Abstract

We investigate convergence conditions for a special class of order dependent mappings which is well known to work for the zero dimensional and quantum mechanical (quartic) anharmonic oscillators. We estimate large order behavior of the resummed series directly and determine under what conditions the series converge or diverge. From the comparison of the present results with the previously established sufficient conditions by other authors, we clarify the origin of the each condition and of the differences between the results. To examine a controversial problem, we investgate the 0-dimensional model in view of the realization of the coefficients of the strong coupling expansion.

Contents

Chapter 1

Introduction

Perturbation theory is one of the most widely used methods to calculate physical quantities in quantum systems. However, validity of perturbation theory has been yet a long standing problem since the theory wasestablished. In general, framework of a physical theory is constructed, at least in the beginning, at the cost of mathematical rigor. However, a criticism on the validity of perturbation theory already appeared for a realistic physical problem only a few years after perturbation theory in quantum mechanics [1] had been established in 1926. For the Stark effect in Hydrogen atom, Oppenheimer claimed, in 1928 [2], that because an electron in the atomic orbit could escape outside the potential barrier due to the quantum tunnelling, the system could not possess any stationary states and thus the eigenstates calculated by perturbation theory [1, 3] do not describe the correct effect of the external field. In addition to this kind of stability problems, question on convergence of formal power series in perturbation theory has been one of the main mathematical subjects on perturbation theory for linear operators [4,5]. First important results on the convergence was established by Rellich in 1930's [6]. Among the results, the following is directly connected with what the Rayleigh-Schrödinger perturbation theory anticipated [4].

Let $T(g)$ be a bounded selfadjoint operator in a Hilbert space $\mathtt{H}$*, depending on a real parameter g as a convergent power series,*

$$T(g) = T^{[0]} + gT^{[1]} + g^2T^{[2]} + \cdots .$$

Suppose that the unperturbed operator $T^{[0]}$ has an isolated eigenvalues λ with a finite multiplicity m. Then, $T(g)$ has exactly m eigenvalues $\lambda_j(g)$ $(j = 1 \sim m)$ in the neighborhood of λ for sufficiently small $|g|$, and these eigenvalues can be expanded into convergent series,

$$\lambda_j(g) = \lambda + g\lambda_j^{[1]} + g^2\lambda_j^{[2]} + \cdots , \qquad (j = 1 \sim m).$$

The associated eigenvectors $|\varphi_j\rangle$ of $T(g)$ can also be chosen as convergent series,

$$|\varphi_j(g)\rangle = |\varphi_j^{[0]}\rangle + g|\varphi_j^{[1]}\rangle + g^2|\varphi_j^{[0]}\rangle + \cdots , \qquad (j = 1 \sim m),$$

satisfying the orthonormality conditions

$$\langle\varphi_j(g)|\varphi_k(g)\rangle = \delta_{jk},$$

and where the $|\varphi_j^{[0]}\rangle$ form an orthonormal family of eigenvectors of $T^{[0]}$ for the eigenvalue λ.

As a simple example, let us consider spin-states represented by finite-dimensional hermitian matrices such as Pauli matrices. In this case, perturbation series for spin-states and corresponding eigenvalues have non-zero convergent radius, as far as any matrix elements of the total Hamiltonian are expressed by regular functions near the origin in the expansion parameter. However, operators in quantum theory, which are defined in infinite dimensional Hilbert space, are unbounded in general. In fact, the appearance of unbounded operators in quantum theory is easily expected by the following [7].

Let P and Q be operators in a Hilbert space H and have an invariant subspace $\mathcal{D}$ in H. If P and Q satisfy canonical commutation relation

$$[P, Q] = 1$$

in $\mathcal{D}$, then either P or Q must be unbounded.

For the general case including unbounded operators, the notion of *analytic families of operators* plays a central role for analyticity of the eigenvalues. Here we omit the rigorous definitions for the analytic families of operators but only exhibit a crucial resultant property from the analytic family, which is known as Kato-Rellich theorem [4, 5].

Let $T(g)$ be an analytic family and E_0 be a nondegenerate discrete eigenvalue of $T(g_0)$. Then, for g near g_0, there is exactly one isolated nondegenerate eigenvalue $E(g)$ of $T(g)$, and $E(g)$ is an analytic function of g near g_0.

Therefore, once an operator $T(g)$ under consideration is known to be an analytic family, above theorem assures the existence of a nondegenerate discrete eigenvalue near that of the *free* operator $T(0)$ and perturbation series for this eigenvalue in powers of g have non-zero convergence radius. One of the useful criteria for $T(g)$ to be an analytic family, especially in the analyses of perturbation series in quantum theory, is the following [4,5];

Let H_0 be a closed operator with nonempty resolvent set. Define $H(g) = H_0 + gV$ on $\mathcal{D}(H_0) \cap \mathcal{D}(V)$. Then, $H(g)$ is an analytic family (of type A) near $g = 0$ if and only if;

(i) $\mathcal{D}(V) \supset \mathcal{D}(H_0)$.

(ii) *For some a and b and for all $\psi \in \mathcal{D}(H_0)$,*

$$\|V\psi\| \leq a\|H_0\psi\| + b\|\psi\|.$$

That is, if V is H_0-*bounded*, there exists a unique eigenvalue $E(g)$ of $H(g)$ analytic near $g = 0$ and hence the perturbation series in powers of g for $E(g)$ have non-zero convergence radius. If an operator is an analytic family, perturbation is said to be *regular*.

On the other hand, a conjecture that perturbation series might be divergent arose by making use of purely physical intuition. In 1952, Dyson considered the vacuum stability in quantum electrodynamics under the analytic continuation on electric charge e-plane [8]. In the world $e^2 < 0$ where like charges attract. the vacuum would be unstable under decay into electron-positron pairs and therefore certain singularity exists at $e^2 = 0$, which means the zero radius of convergence. Nowadays, this kind of consideration, which is sometimes called *the sign-change argument*, is known not to be true in general and some counter examples have been also proposed [9–12]. However, after this striking conjecture, perturbation series indeed diverge had been shown by Jaffe in 1965 for two-demensional boson field theories with polynomial self-interactions [13]. In 1968, Bender and Wu investigated the analytic structure of the energy eigenvalues of quantum mechanical anharmonic oscillator (AHO) by WKB analysis [14] and found that the origin $g = 0$ is really singular (and surprisingly not an isolated singularity). They also calculated the perturbation coefficients for the ground state energy up to 75th order and explicitly showed a strongly divergent behavior [15]. Then, Simon rigorously proved [9] that most of the conjectures made by Bender and Wu [14,15] are correct. One of the most powerful tools established in these studies to investigate large order behavior of perturbation series is the following, which (Eq.(1.1) or (1.2)) we will hereafter call the *Bender-Wu relation* [16,17];

Suppose $S(g)$ be a function obeying,

(i) *analytic in $\{g|0 < |g| < R; |\arg g| < \pi\}$ and continuous in the closure of this region,*

(ii) *$S(g)$ has $\sum_{p=0}^{\infty} c^{[p]} g^p$ as asymptotic series for $g \downarrow 0$.*

One then finds,

$$c^{[p]} = \frac{(-1)^{p+1}}{2\pi i} \int_{1/R}^{\infty} d\mu \mu^{p-1} \mathrm{Disc} S(-\mu^{-1}) + S_p, \tag{1.1}$$

with

$$|S_p| \leq C R^{-p},$$

where C is a constant and

$$\mathrm{Disc} S(-\mu^{-1}) = \lim_{\epsilon \downarrow 0} \Big[S(-\mu^{-1} + i\epsilon) - S(-\mu^{-1} - i\epsilon) \Big].$$

Especially, if $R = \infty$, that is, $S(g)$ is analytic in the cut plane,

$$c^{[p]} = \frac{(-1)^{p+1}}{2\pi i} \int_{0}^{\infty} d\mu \mu^{p-1} \mathrm{Disc} S(-\mu^{-1}). \tag{1.2}$$

With these relations, one can obtain large order behavior by calculating asymptotic behavior for $g \uparrow 0$ of some nonperturbative quantity related to quntum tunnelling. This method was widely applied to variants of AHO [16–19,21], to the Stark effect [20,21] and so on, and clarified the general feature of factorial divergence of the perturbation series. It also turned out that, in addition to the AHO, many examples of interest in quantum physics, such as the Zeeman and Stark effect in hydrogen atom, don't fit into the scheme of regular perturbation theory mentioned earlier. Therefore, the analyses of the nature of perturbation series, e.g., asymptoticity, large order behavior and so on, have been done for each specific model Hamiltonian often without abstract operator theories. For references, see Ref. [22].

The analyses of analytic structure and large order behavior provide another important direction on research, that is, summability of the perturbation series. The research of this direction is based on the belief that one can well define the *sum* of these divergent perturbation series as in the case of conditionally convergent series. Indeed, with the aid of the knowledge obtained by the analyses on the quartic AHO, two methods are proved to be summable for the perturbation series for the quartic AHO; Padé approximation [23] in 1969, and Borel summation [24] in 1970. These methods had been already employed to calculate physical quantities [25] and some sufficient conditions for uniquely defining the sum have been established; Stieltjes theorem for convergence and Carleman's condition for uniquness in Padé [25], Watson-Nevanlinna theorem for Borel summability [26]. Once one knows that one of the sufficient conditions is satisfied for the problem concerned, the resummation provides a more reliable tool for calculating physical quantities than any nonperturbative methods; the latter ones are often known to work quite well for a specific problem but one cannot rigorously judge, in general, their validity, correctness and limitation of applicability in both qualitative and quantitative aspects. Of course, to get the information on the summability for each case is itself non-trivial problems, but apart from this kind of difficulty the two methods above have their common disadvantages; since the methods are essentially the summation at the neighborhood of the origin on the complex expansion-parameter (coupling constant) plane, they are only valid for weak coupling regime, or provide so slowly convergent approximants in strong coupling regime that one must calculate hundreds of the perturbative coefficients to achieve satisfactory results.

In 1990's, there have appeared several papers which prove the convergence of the new resummation methods for some specific problems [27–33]. The methods are called the δ expansion (DE)[1] and order dependent mappings (ODM). As far as we know, these methods were explored to obtain accurate results for the eigenvalues of quantum mechanical (quartic) AHO around the end of 1970's by several authors [25,34]. One of the significant features is that, in spite of simplicity of the method, it provides quite accurate results for wide range of the coupling constant even at the lowest order perturbation calculation.

[1] There are many equivallent methods to the DE but have different names such as variational perturbation, renormalized perturbation, optimized perturbation and so on.

However, the DE methods have sufficient reasons that they do not look like more than an art. One of the reason will be too wide arbitrariness that the method possesses. In terms of the DE, one will choose a solvable free Hamiltonian with artificial parameters such that this describes the full Hamiltonian under consideration as close as possible. However, there is no criterion that can judge the closeness of two Hamiltonians. In addition to the arbitrariness of the selection of a free Hamiltonian, there is another freedom in the method, that is, the way to fix parameters introduced artificially. Since there is no definite principle such as a variational principle in the DE, some *plausible* criterions such as the fastest apparent convergence (FAC) [35,36] and the principle of minimal sensitivity (PMS) [37] are often employed to determine the values of the artificial parameters introduced (see also, Appendix A). Hence, it will be inevitable to arise such a criticism that there may always exist a set of particular Hamiltonian and specific values of parameters that can lead us to obtain accurate approximations even at the lowest perturbation calculation. Therefore, the lack of guiding principles to delete the arbitrariness has been one of the crucial defects of the DE. Indeed, few examples are known to succeed as greatly as the (quartic) AHO although the method has been widely applied to various models. For recent references, see, e.g., Ref. [38, 39].

The ODM, first proposed by Seznec and Zinn-Justin [35] in 1979, on the other hand, provided the mechanism of why the DE for AHO in 0-dimension and in quantum mechanics succeed. Actually, they first began with the DE and constructed the ODM as a generalization of the DE by detecting the fact that the essence of the DE consists in the change of the expansion variable via a certain transformation. In view of the ODM, the selection of a free Hamiltonian in the DE corresponds to the choice of a transformation from a original expansion variable to a new one, and the success of the method is found to rely on the compatibility of the mapping and the analytic property of a quantity concerned as a function of the expansion parameter. Although their analyses were not rigorous, their conjectures are almost supported by the recent convergence proofs [27,31–33].

Another important consequence of the convergence proofs is about the choice of the artificial parameters. In the analyses of Ref. [35] and [27,28], the authors rely on a specific criterion (the FAC in the former and the PMS in the latters). However, it was eventually recognized in Ref. [29,30] and [32,33] that the essential point for the convergence is not on a special condition for determining the parameters but on the *order dependence* of the parameters adjusted. One can see in Ref. [32.33] that the convergence can be achieved in a certain region of the parameter space while a condition like the FAC and the PMS picks up only finite points for the criterion on the choice of the parameters. It was also argued that the FAC criterion (with the largest module) leads us to just the boundary of the convergence region [32] and the both of the solutions of the FAC and the PMS would be in the convergence region as far as the region exists [33]. These features are preferable in both theoretical and practical points of view. Theoretically, if the method would prefer a specific condition strongly depending on a physical quantity and a model Hamiltonian concerned, the method might lose its universality in applications and would be nothing more than parameter fitting. Practically, it becomes more difficult and cumbersome to calculate a condition and solutions of the condition increases in number, as the order of the perturbation increases. The above features of the method suggest the following strategy; for the lower order calculations (at most 3rd order) one would invoke either the FAC or the PMS condition; for the larger order, one might only assume the order dependence with an initial condition suggested by the calculated lower order results and need not solve optimization conditions any more.

In this way, the series of the convergence proofs have compensated the defects of the method to great extent. However, one would like to know about general convergence conditions independent of specific models or quantities rather than a convergence proof itself for each special case. In this regard, Guida et al. [33] have established sufficient conditions for convergence of a special class of ODM (which we will call *type AHO*, see chapter 2). The main result is stated as follows.

Theorem 1. *Let a function $S(g)$ be given such that;*

(I) *$S(g)$ is analytic in the complex g-plane cut along the negative axis, continuous on the two edges of the cut and behaves as $S(g) \sim g^{\varsigma}$ for $|g| \to \infty$ in the cut plane.*

(II) *$S(g)$ has $\sum_{p=0}^{\infty} c^{[p]} g^p$ as asymptotic series for $g \downarrow 0$,*

(III) *The discontinuity of $S(g)$ along the cut behaves as*

$$\mathrm{Disc}S(-g) \sim 2iS_D g^{-c} \exp\left(-\frac{a}{g^{1/b}}\right) \qquad \textit{for } g \downarrow 0, \tag{1.3}$$

with S_D a constant, $a > 0$ and $0 < b < 2$.

Then, the sequence $\{S^{[N]}(\rho_N)\}_N$ with $S^{[N]}(\rho_N)$, constructed by the ODM of type AHO with restriction

$$1 < \alpha \leq 2, \tag{1.4}$$

converges to $S(g)$ as $N \to \infty$, uniformly in each compact subset of the region

$$\Re \frac{1}{g^{1/\alpha}} > 0, \tag{1.5}$$

if the positive parameter ρ is scaled as

$$\rho_N \sim \rho_1 N^{-\gamma} \tag{1.6}$$

with

$$b < \gamma < \alpha, \tag{1.7}$$

where ρ_1 is independent of g.

This theorem is indeed preferable in that it does not invoke any specific models but only needs some information about the analytic properties in the cut plane of a function under consideration. The establishment of a general theorem on summability like the above indicates that the ODM stands not only as one of the useful technique for calculations but also as one of the summation methods comparable with Padé approximation and Borel summation. But the applicability of the above theorem is rather restricted. One of the reason of the limitations comes from the upper bound 2 for α, Eq.(1.4). Fulfillment of the condition requires $b < \alpha$, which results in the restriction on b, that is, $b < 2$. On the other hand, under the assumptions (I)-(III) above, large order behavior of perturbation series reads

$$\left|c^{[N]}\right| \sim \Gamma(bN), \tag{1.8}$$

with the aid of the Bender-Wu relation Eq.(1.2). Therefore, the restriction $b < 2$ means that the theorem above can be applied only if the series grow strictly slower than $(2n)!$. Thus, one cannot judge if the ODM of type AHO can correctly sum perturbation series of, e.g., sextic AHO (b=2). So, if the restriction $\alpha \leq 2$ is not only a sufficient but also a necessary condition, it should be said that the validity of the ODM, at least of type AHO, is too restrictive.

In this thesis, motivated by the above considerations, we carefully investigate the convergence conditions, especially with concentration on the α-dependence. For this purpose, we take a completely different approach from that in Ref. [33]. In the latter, the sufficient conditions have been obtained by the use of an integral representation and its estimate. In this thesis, we directly investigate the resummed series and examine whether the series converges or diverges in a certain condition. Although the convergence of the series does not always mean the convergence to the *true* answer, this approach can clarify the structure of the resummed series and the mechanism of the convergence or the divergence more apparently than the one based on an integral representation. Furthermore, from the examination of both the convergence and the divergence, we can establish the necessary and sufficient conditions for the convergence, which play complementary roles to the sufficient conditions for the convergence to the *true* answer.

The rest of the thesis is organized as follows. We first summarize the method of ODM in chapter 2. Not only the direct resummed series expressions but also some integral representations which were employed for convergence proofs are exhibited both for general case and for the type AHO. The relation between the representations is clarified. The analyses on the resummed series are made in chapter 3, which constitutes the main part of the thesis. In chapter 4, the analyses in chapter 3 are reexamined in the case where a convergent strong coupling expansion exists. To confirm our analyses, illustrations on zero-dimensional models for which analytic controll is almost available are performed in chapter 5. In chapter 6, we mention the δ-expansion method of mass-renormalized type and disscuss the validity and limitations of it in view of the ODM of type AHO. Discussion and summary is given in chapter 7. In appendix A, the optimization conditions are summarized. In appendix B, the result of Ref. [33] for the boundary of the convergence region $\gamma = b$ is represented.

Chapter 2

Order dependent mappings

The order dependent mapping method [35,40,41] is based on a change of the expansion parameter. One starts from a conventional perturbation series in powers of a variable g for a physical quantity $S(\tilde{g})$

$$\tilde{S}(\tilde{g}) = \sum_{p=0}^{\infty} c^{[p]} \tilde{g}^{p}, \tag{2.1}$$

where and hereafter the symbol tilde is employed if we would like to stress that a quantity in an expression must be scaled to be dimensionless. Otherwise, whether a quantity is dimensionless or not is not so important or trivial.

One then transforms the expansion variable g to a new variable λ by a conformal transformation

$$\tilde{g} = \rho F(\lambda), \tag{2.2}$$

where ρ is an adjustable positive real parameter and $F(\lambda)$ satisfies

$$F(\lambda) \sim \lambda \quad \text{for} \quad \lambda \to 0. \tag{2.3}$$

One then expand in λ aside from a suitable function $f(\lambda)$,

$$\tilde{S}(\rho) = f(\lambda) \sum_{s=0}^{\infty} P^{[s]}(\rho) \lambda^{s}, \tag{2.4}$$

where $P^{[s]}(\rho)$ are s-th order polynomials in ρ, as far as $f(\lambda)$ is regular at the origin $\lambda = 0$ and the condition Eq.(2.3) is satisfied. Indeed, this can be shown as follows. The procedure above immediately reads

$$\tilde{S}(\rho) = f(\lambda)\sum_{p=0}^{\infty} c^{[p]}\rho^p \frac{F(\lambda)^p}{f(\lambda)}. \tag{2.5}$$

Applying the Cauchy's integral formula, one gets

$$\tilde{S}(\rho) = f(\lambda)\sum_{s=0}^{\infty}\lambda^s \sum_{p=0}^{\infty} c^{[p]}\rho^p \oint_{C_0} \frac{dw}{2\pi i}\frac{F(w)^p}{w^{s+1}f(w)}, \tag{2.6}$$

where the integration countour C_0 encloses the origin such that any other singularities involving in $F(w)$ and $1/f(w)$ are outside the countour. Then, if $f(\lambda)$ is regular at the origin and the condition Eq.(2.3) is satisfied, the pole at $w = 0$ of the integrand is of order $s + 1 - p$ and thus the summation with respect to p terminates up to $p = s$. As a result, $P^{[s]}(\rho)$ becomes s-th order polynomial given by

$$\begin{aligned} P^{[s]}(\rho) &= \sum_{p=0}^{s}\rho^p c^{[p]} \oint_{C_0}\frac{dw}{2\pi i}\frac{F(w)^p}{w^{s+1}f(w)} && (2.7)\\ &= \sum_{p=0}^{s}\rho^p c^{[p]} \frac{1}{(s-p)!}\frac{d^{s-p}}{dw^{s-p}}\frac{F(w)^p}{w^p f(w)}\bigg|_{w=0}. && (2.8)\end{aligned}$$

The N-th order approximation $S^{[N]}$ in ODM is obtained by truncating the power series in λ of Eq.(2.4) at the order N and the parameter ρ is fixed order by order

$$\tilde{S}^{[N]}(\rho_N) = f(\lambda_N)\sum_{s=0}^{N} P^{[s]}(\rho_N)\lambda_N^s. \tag{2.9}$$

Note that for a given g, the value of λ becomes also order dependent through Eq.(2.2) in accordance with the order dependence of ρ, see Eq.(3.8). The essential and difficult problems in ODM are the choice of the mapping $F(\lambda)$ and prefactor $f(\lambda)$ which should be compatible with the analytic structure of $S(g)$. For the quartic AHO in 0-dimension and in quantum mechanics, it was proved [33] that the choice

$$F(\lambda) = \frac{\lambda}{(1-\lambda)^\alpha} \qquad f(\lambda) = \frac{1}{(1-\lambda)^\sigma} \tag{2.10}$$

with suitable choice of the indicies $\alpha(>1)$ and σ can lead to convergent sequence $S^{[N]}(\rho_N)$ to the exact answers. In the case of the above choice Eq.(2.10), which we may hereafter call *type AHO*, N-th order approximation can be calculated directly and is expressed as

$$\tilde{S}^{[N]}(\rho) = \frac{1}{(1-\lambda)^{\sigma}}\sum_{p=0}^{N} c^{[p]}(\rho\lambda)^p \sum_{r=0}^{N-p}\frac{(\alpha p-\sigma)_r}{r!}\lambda^r \tag{2.11}$$

$$= \frac{1}{(1-\lambda)^{\sigma}}\sum_{s=0}^{N}\lambda^s\sum_{p=0}^{s} c^{[p]}\frac{(\alpha p-\sigma)_{s-p}}{(s-p)!}\rho^p, \tag{2.12}$$

where $(a)_n$ is the Pochhammer symbol for *shifted* factorial

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{2.13}$$

If $S(g)$ satisfies the analyticity condition of (I) and (II), the Bender-Wu relation Eq.(1.2) can be substituted and one obtains

$$\tilde{S}^{[N]}(\rho) = \frac{1}{(1-\lambda)^{\sigma}}\sum_{s=0}^{N}\lambda^s\left[c^{[0]}\frac{(-\sigma)_s}{s!} + \sum_{p=1}^{s}\rho^p\frac{(\alpha p-\sigma)_{s-p}}{(s-p)!}\int_0^{\infty}\frac{d\mu}{2\pi i}(-\mu)^{s-1}\mathrm{Disc}\tilde{S}(-\mu^{-1})\right]. \tag{2.14}$$

This expression can be also obtained with recourse to the Cauchy's integral formula. Under the analyticity conditions of (I) and (II), the following once-subtracted dispersion relation holds [9]

$$\tilde{S}(\tilde{g}) = c^{[0]} + \tilde{g}\int_{-\infty}^{0}\frac{dg'}{2\pi i}\frac{\mathrm{Disc}\tilde{S}(g')}{g'(g'-\tilde{g})} \tag{2.15}$$

$$= c^{[0]} + \int_0^{\infty}\frac{d\mu}{2\pi i}\mathrm{Disc}\tilde{S}(-\mu^{-1})\frac{\tilde{g}}{1+\mu\tilde{g}}. \tag{2.16}$$

One then substitutes the transformation Eq.(2.2) in the variable $\tilde{g}$ in the integrand and applies the Cauchy's integral formula, yielding

$$\begin{aligned}\tilde{S}(\tilde{g}) &= f(\lambda)\left[\frac{c^{[0]}}{f(\lambda)} + \frac{1}{f(\lambda)}\int_0^{\infty}\frac{d\mu}{2\pi i}\mathrm{Disc}\tilde{S}(-\mu^{-1})\frac{\rho F(\lambda)}{1+\mu\rho F(\lambda)}\right] \\ &= f(\lambda)\left[c^{[0]}\sum_{s=0}^{\infty}\lambda^s\oint_{C_0}\frac{dw}{2\pi i}\frac{f_1(w)}{w^{s+1}}\right. \\ &\quad\left.+\int_0^{\infty}\frac{d\mu}{2\pi i}\mathrm{Disc}\tilde{S}(-\mu^{-1})\sum_{s=0}^{\infty}\lambda^s\oint_{C_0}\frac{dw}{2\pi i}\frac{f_2(w)}{w^{s+1}}\right],\end{aligned} \tag{2.17}$$

where

$$f_1(w) = \frac{1}{f(w)}, \qquad f_2(w) = \frac{1}{f(w)}\frac{\rho F(w)}{1+\mu\rho F(w)}, \tag{2.18}$$

and the integration contour C_0 encloses the origin and not any singurarities of $f_1(w)$ and $f_2(w)$. The N-th approximation is given by terminating the power series in λ at the N-th order in Eq.(2.17),

$$\begin{aligned}\tilde{S}^{[N]}(\rho) &= f(\lambda)\sum_{s=0}^{N}\lambda^s\left[c^{[0]}\oint_{C_0}\frac{dw}{2\pi i}\frac{f_1(w)}{w^{s+1}}\right.\\ &\quad \left.+\int_0^\infty\frac{d\mu}{2\pi i}\mathrm{Disc}\tilde{S}(-\mu^{-1})\oint_{C_0}\frac{dw}{2\pi i}\frac{f_2(w)}{w^{s+1}}\right]. \end{aligned}\tag{2.19}$$

For the ODM of type AHO, one obtains

$$\begin{aligned}\tilde{S}^{[N]}(\rho) &= \frac{1}{(1-\lambda)^\sigma}\sum_{s=0}^{N}\lambda^s\left[c^{[0]}\oint_{C_0}\frac{dw}{2\pi i}\frac{(1-w)^\sigma}{w^{s+1}}\right.\\ &\quad \left.+\int_0^\infty\frac{d\mu}{2\pi i}\mathrm{Disc}\tilde{S}(-\mu^{-1})\oint_{C_0}\frac{1}{w^s}\frac{\rho(1-w)^\sigma}{(1-w)^\alpha+\mu\rho w}\right] \end{aligned}\tag{2.20}$$

$$\begin{aligned} &= \frac{1}{(1-\lambda)^\sigma}\left[c^{[0]}\oint_{C_0}\frac{dw}{2\pi i}\frac{1-(\lambda/w)^{N+1}}{w-\lambda}(1-w)^\sigma\right.\\ &\quad \left.+\int_0^\infty\frac{d\mu}{2\pi i}\mathrm{Disc}\tilde{S}(-\mu^{-1})\oint_{C_0}\frac{dw}{2\pi i}\frac{1-(\lambda/w)^{N+1}}{w-\lambda}\frac{\rho w(1-w)^\sigma}{(1-w)^\alpha+\mu\rho w}\right]. \end{aligned}\tag{2.21}$$

This integral representation is the ODM version of the equation, which Guida et al. analysed as the starting point the convergence of the δ expansion for the eigenvalues of quartic AHO ($\alpha = 3/2$, $\sigma = 1/2$, see chapter 6) in Ref. [32]. If one further expands the integrand of the second term in powers of ρ and formally interchanges the order of integration and summation, the series terminates at finite order and finally one obtains

$$\begin{aligned}\tilde{S}^{[N]}(\rho) &= \frac{1}{(1-\lambda)^\sigma}\sum_{s=0}^{N}\lambda^s\left[c^{[0]}\oint_{C_0}\frac{dw}{2\pi i}\frac{(1-w)^\sigma}{w^{s+1}}\right.\\ &\quad \left.+\sum_{p=1}^{s}\rho^p\int_0^\infty\frac{d\mu}{2\pi i}(-\mu)^{p-1}\mathrm{Disc}\tilde{S}(-\mu^{-1})\oint_{C_0}\frac{dw}{2\pi i}\frac{(1-w)^{\sigma-\alpha p}}{w^{s+1-p}}\right]. \end{aligned}\tag{2.22}$$

Calculus of residues yields

$$\oint_{C_0}\frac{dw}{2\pi i}\frac{(1-w)^{\sigma-\alpha p}}{w^{s+1-p}} = \frac{1}{(s-p)!}\frac{d^{s-p}}{dw^{s-p}}(1-w)^{\sigma-\alpha p}\bigg|_{w=0} = \frac{(\alpha p-\sigma)_{s-p}}{(s-p)!}, \tag{2.23}$$

then the expression Eq.(2.12) can be obtained again.

Chapter 3

Analyses of convergence conditions

In this chapter, we analyze the structure of the sequence $\{S^{[N]}(\rho_N)\}_N$ constructed by the ODM of type AHO Eq.(2.12). To investigate the convergence of a sequence $\{S_N\}$, it is sometimes convenient to study in terms of the series,

$$S_N = \sum_{n=0}^{N} a_n.$$

So, the quantity to be estimated here is the following,

$$a_N = S^{[N]}(\rho_N) - S^{[N-1]}(\rho_{N-1}). \tag{3.1}$$

If the arguments of the two terms in the right are the same, this quantity is simply obtained by Eq.(2.12) as

$$a_N = \frac{\lambda^N}{(1-\lambda)^\sigma} \sum_{p=0}^{N} c^{[p]} \frac{(\alpha p - \sigma)_{N-p}}{(N-p)!} \rho^p. \tag{3.2}$$

It turns out that the effect of the difference on the arguments is so small that Eq.(3.2) with $\rho = \rho_N$ well approximates Eq.(3.1), as we will see below. If $S^{[N]}(\rho_{N-1})$ is subtracted from and added to a_N and one applies mean value theorem (under the assumption that ρ_N is monotone with respect to N), a_N reads

$$\begin{aligned} a_N &= S^{[N]}(\rho_N) - S^{[N]}(\rho_{N-1}) + S^{[N]}(\rho_{N-1}) - S^{[N-1]}(\rho_{N-1}) \\ &= (\rho_N - \rho_{N-1}) \frac{\partial}{\partial \rho} S^{[N]}(\rho_{N-\theta}) + S^{[N]}(\rho_{N-1}) - S^{[N-1]}(\rho_{N-1}) \quad (0 < \theta < 1). \end{aligned} \tag{3.3}$$

The each term of the r.h.s. of Eq.(3.3) can be calculated as

$$\begin{aligned}
\frac{\partial}{\partial\rho}S^{[N]}(\rho) &= \left(\frac{d\rho}{d\lambda}\right)^{-1}\frac{\partial}{\partial\lambda}S^{[N]}(\rho) \\
&= \rho^{-1}\frac{\lambda}{1+(\alpha-1)\lambda}\frac{\lambda^N}{(1-\lambda)^\sigma}\sum_{p=0}^{N}c^{[p]}\frac{(\alpha p-\sigma)_{N-p+1}}{(N-p)!}\rho^p, \qquad (3.4)
\end{aligned}$$

$$S^{[N]}(\rho)-S^{[N-1]}(\rho)=\frac{\lambda^N}{(1-\lambda)^\sigma}\sum_{p=0}^{N}c^{[p]}\frac{(\alpha p-\sigma)_{N-p}}{(N-p)!}\rho^p, \qquad (3.5)$$

where Eq.(2.2) with (2.10) has been used. Therefore, one gets

$$\begin{aligned}
a_N &= \frac{\lambda_{N-1}^N}{(1-\lambda_{N-1})^\sigma}\sum_{p=0}^{N}c^{[p]}\frac{(\alpha p-\sigma)_{N-p}}{(N-p)!}\rho_{N-1}^p \\
&+\frac{\rho_N-\rho_{N-1}}{\rho_{N-\theta}}\frac{\lambda_{N-\theta}}{1+(\alpha-1)\lambda_{N-\theta}}\frac{\lambda_{N-\theta}^N}{(1-\lambda_{N-\theta})^\sigma}\sum_{p=0}^{N}c^{[p]}\frac{(\alpha p-\sigma)_{N-p+1}}{(N-p)!}\rho_{N-\theta}^p, \qquad (3.6)
\end{aligned}$$

which involves only one summation symbol in each term.

Next, we assume the order dependence of ρ_N for large N. If ρ_N is scaled as

$$\rho_N \sim \rho_1 N^{-\gamma} \qquad (\gamma>0, \quad \rho_1>0), \qquad (3.7)$$

where ρ_1 is a positive real constant independent of N and g, the corresponding order dependence of λ follows, from Eq.(2.2) with (2.10), as

$$\lambda_N \sim 1-\left(\frac{\rho_1}{g}\right)^{1/\alpha}N^{-\gamma/\alpha}. \qquad (3.8)$$

By the use of Eqs.(3.7) and (3.8), one can estimate the leading behavior of the prefactors outside the summation symbols in Eq.(3.6);

$$\frac{\rho_N-\rho_{N-1}}{\rho_{N-\theta}} \sim \frac{1-(1-1/N)^{-\gamma}}{(1-\theta/N)^{-\gamma}} = -\frac{\gamma}{N}\left[1+O(N^{-1})\right], \qquad (3.9)$$

$$\begin{aligned}
\frac{\lambda_{N-\theta}}{1+(\alpha-1)\lambda_{N-\theta}} &\sim \frac{1}{\alpha}\left[1-\left(\frac{\rho_1}{g}\right)^{1/\alpha}N^{-\gamma/\alpha}+\cdots\right]\left[1+\frac{\alpha-1}{\alpha}\left(\frac{\rho_1}{g}\right)^{1/\alpha}N^{-\gamma/\alpha}+\cdots\right] \\
&= \frac{1}{\alpha}\left[1+O(N^{-\gamma/\alpha})\right], \qquad (3.10)
\end{aligned}$$

$$\frac{1}{(1-\lambda_{N-\bar{\theta}})^{\sigma}} \sim \left[\left(\frac{\rho_1}{g}\right)^{1/\alpha}(N-\bar{\theta})^{-\gamma/\alpha}\right]^{-\sigma} = \left(\frac{g}{\rho_1}\right)^{\sigma/\alpha} N^{\sigma\gamma/\alpha}\Big[1+O(N^{-1})\Big], \qquad (3.11)$$

$$\begin{aligned} \ln \lambda_{N-\bar{\theta}}^{N} &\sim N\left[-\left(\frac{\rho_1}{g}\right)^{1/\alpha} N^{-\gamma/\alpha}\left(1-\frac{\bar{\theta}}{N}\right)^{-\gamma/\alpha} + O(N^{-2\gamma/\alpha})\right] \\ &= -\left(\frac{\rho_1}{g}\right)^{1/\alpha} N^{1-\gamma/\alpha} + O(N^{-\gamma/\alpha}), \end{aligned}$$

hence,

$$\lambda_{N-\bar{\theta}}^{N} \sim \exp\left[-\left(\frac{\rho_1}{g}\right)^{1/\alpha} N^{1-\gamma/\alpha}\right], \qquad (3.12)$$

where $\bar{\theta}$ above denotes either θ or 1. From these results above, the estimate for a_N now reads to

$$\begin{aligned} a_N &\sim \left(\frac{g}{\rho_1}\right)^{\sigma/\alpha} N^{\sigma\gamma/\alpha} \exp\left[-\left(\frac{\rho_1}{g}\right)^{1/\alpha} N^{1-\gamma/\alpha}\right] \\ &\quad \times \sum_{p=0}^{N} \frac{c^{[p]}(\rho_1 N^{-\gamma})^p}{(N-p)!}\left[(\alpha p-\sigma)_{N-p} - \frac{\gamma}{\alpha}N^{-1}(\alpha p-\sigma)_{N-p+1}\right]. \end{aligned} \qquad (3.13)$$

Furthermore, since

$$\begin{aligned} &(\alpha p-\sigma)_{N-p} - \frac{\gamma}{\alpha}N^{-1}(\alpha p-\sigma)_{N-p+1} \\ &= \left(1-\frac{\gamma}{\alpha}-\gamma\frac{\alpha-1}{\alpha}\Delta_p + \frac{\gamma\sigma}{\alpha}N^{-1}\right)(\alpha p-\sigma)_{N-p} \qquad \Delta_p = \frac{p}{N}, \end{aligned}$$

the summation in Eq.(3.13) can be approximated as

$$\begin{aligned} &\sum_{p=0}^{N} \frac{c^{[p]}(\rho_1 N^{-\gamma})^p}{(N-p)!}\left[(\alpha p-\sigma)_{N-p} - \frac{\gamma}{\alpha}N^{-1}(\alpha p-\sigma)_{N-p+1}\right] \\ &\sim \left(1-\frac{\gamma}{\alpha}-\gamma\frac{\alpha-1}{\alpha}\Delta_Q\right)\sum_{p=0}^{N} c^{[p]}(\rho_1 N^{-\gamma})^p \frac{(\alpha p-\sigma)_{N-p}}{(N-p)!} \qquad (0 < Q < N). \end{aligned}$$

Then, one yields

$$a_N \sim \text{Const.} \times g^{\sigma/\alpha} N^{\sigma\gamma/\alpha} \exp\left[-\left(\frac{\rho_1}{g}\right)^{1/\alpha} N^{1-\gamma/\alpha}\right] \times P^{[N]} \qquad (3.14)$$

where

$$P^{[N]} = \sum_{p=0}^{N} c^{[p]}(\rho_1 N^{-\gamma})^p \frac{(\alpha p-\sigma)_{N-p}}{(N-p)!}, \qquad (3.15)$$

and 'Const.' above and hereafter denotes a quantity independent of N and g. Note that $P^{[N]}$ is independent of g. Therefore, one can judge the convergence of the sequence $\sum a_n$ in the region

$$\Re\frac{1}{g^{1/\alpha}} > 0. \tag{3.16}$$

If $\gamma < \alpha$, the exponential term in Eq.(3.14) tends to 0 faster than any finite power of N in the region Eq.(3.16). Hence, for $\gamma < \alpha$, the sequence $\sum a_n$ is convergent in the region Eq.(3.16) if $P^{[N]}$ behaves at most as a finite power of N for large N. On the other hand, if $\gamma \geq \alpha$, the exponential term in Eq.(3.14) tends to (or equals) 1 for any complex value of g. Hence, for $\gamma \geq \alpha$, the sequence $\sum a_n$ diverges for any complex g unless $P^{[N]}$ tends to 0 faster than $N^{-\sigma\gamma/\alpha}$ at worst.

Next, let us look at α-dependence of a_N. Apart from the exponential term in Eq.(3.14), the only essential dependence on α consists in the Pochhammer symbol in the summand. So, to clarify the α-dependence, we would like to represent asymptotic behavior of the summand by the proper use of the Stirling formula.

(a) For $p \ll N$, the factorial term in the summand behaves as

$$\frac{(\alpha p-\sigma)_{N-p}}{(N-p)!} = \frac{1}{\Gamma(\alpha p-\sigma)}\frac{\Gamma\Big(N+(\alpha-1)p-\sigma\Big)}{\Gamma(N-p+1)} \sim \frac{N^{\alpha p-\sigma-1}}{\Gamma(\alpha p-\sigma)}. \tag{3.17}$$

(b) For $|q| = |p-N/A| \ll N$; $1 < A \sim O(1)$,

$$\begin{aligned}\frac{(\alpha p-\sigma)_{N-p}}{(N-p)!} &= \frac{\Gamma\left(\frac{A+\alpha-1}{A}N+(\alpha-1)q-\sigma\right)}{\Gamma\left(\frac{\alpha}{A}N+\alpha q-\sigma\right)\Gamma\left(\frac{A-1}{A}N-q+1\right)}\\ &\sim \text{Const.}\times N^{-1}B_1(\alpha)^{N/A}B_2(\alpha)^q,\end{aligned} \tag{3.18}$$

with

$$B_1(\alpha) = \left(\frac{\alpha+A-1}{\alpha}\right)^{\alpha}\left(\frac{\alpha+A-1}{A-1}\right)^{A-1}, B_2(\alpha) = \left(\frac{\alpha+A-1}{\alpha}\right)^{\alpha}\left(\frac{\alpha+A-1}{A-1}\right)^{-1}.$$

(c) For $q = N - p \ll N$,

$$\frac{(\alpha p-\sigma)_{N-p}}{(N-p)!} = \frac{1}{q!}\frac{\Gamma\left(\alpha N-(\alpha-1)q-\sigma\right)}{\Gamma(\alpha N-\alpha q-\sigma)} \sim \frac{(\alpha N)^q}{q!}. \tag{3.19}$$

From the analyses above, one can see that strong dependence on α for large N would arise from lower p of the summand, (a).

Let us come back to estimate the $P^{[N]}$. The assumption (III) tempts us to decompose the discontinuity of $S(g)$ as,

$$\mathrm{Disc}S(-\mu^{-1}) = 2iS_D\mu^c\exp\left(-a\mu^{1/b}\right) + D(\mu), \tag{3.20}$$

where

$$D(\mu) \sim \mu^c\exp\left(-a\mu^{1/b}\right) \times O(\mu^{-1}) \quad \text{for } \mu \to 0_+.$$

Then, the application of the Bender-Wu relation Eq.(1.2) reads $c^{[p]} = c_1^{[p]} + c_2^{[p]}$ with

$$c_1^{[p]} = -\frac{S_D b}{\pi a^{bc}}\left(-\frac{1}{a^b}\right)^p\Gamma(bp+bc), \tag{3.21}$$

$$c_2^{[p]} = \int_0^\infty \frac{d\mu}{2\pi i}(-\mu)^{p-1}D(\mu), \tag{3.22}$$

where

$$\left|c_2^{[p]}\right| = \left|c_1^{[p]}\right| \times O(p^{-1}) \quad \text{for large } p.$$

That is, c_1 denotes the leading asymptotic part of the perturbation coefficients while c_2 denotes the remaining *moderate* part. Thus, the contribution of $c_2^{[p]}$ for large p need not be considered or, alternatively, $c_2^{[p]}$ can be assumed to be bounded, that is to say,

$$\sup_p\left|c_2^{[p]}\right| = C_2 < \infty. \tag{3.23}$$

and hence one obtains $P^{[N]} = P_1^{[N]} + P_2^{[N]}$ with

$$\begin{aligned} P_1^{[N]} &= \text{Const.} \times \sum_{p=0}^{N}\left(-\frac{\rho_1}{a^b}N^{-\gamma}\right)^p\Gamma(bp+bc)\frac{(\alpha p-\sigma)_{N-p}}{(N-p)!} \\ &= \text{Const.} \times \left(-\frac{\rho_1}{a^b}N^{-\gamma}\right)^N\sum_{q=0}^{N}\frac{J_q}{q!}\left(-\frac{a^b}{\rho_1}N^{\gamma}\right)^q \end{aligned} \tag{3.24}$$

where

$$J_q = \Gamma(bN + bc - bq)(\alpha N - \sigma - \alpha q)_q, \tag{3.25}$$

and

$$\begin{aligned} P_2^{[N]} &= \sum_{p=0}^{N} c_2^{[p]} (\rho_1 N^{-\gamma})^p \frac{(\alpha p - \sigma)_{N-p}}{(N-p)!} \\ &= (\rho_1 N^{-\gamma})^N \sum_{q=0}^{N} \frac{c_2^{[N-q]}}{q!} (\alpha N - \sigma - \alpha q)_q \left(\frac{N^\gamma}{\rho_1}\right)^q. \end{aligned} \tag{3.26}$$

To facilitate the estimate of these quantities, we approximate the factorial terms with more tractable functions. First, we write

$$(\alpha N - \sigma - \alpha q)_q = (\alpha N)^q \prod_{k=0}^{q-1} \left(1 - \frac{\sigma}{\alpha N} - \Delta_q + \frac{k}{\alpha N}\right). \tag{3.27}$$

The following inequality holds by considering quadrature by parts,

$$\int_{x_0 - 1/(\alpha N)}^{x_1 - 1/(\alpha N)} dx \ln x < \sum_{k=0}^{q-1} \frac{1}{\alpha N} \ln\left(1 - \frac{\sigma}{\alpha N} - \Delta_q + \frac{k}{\alpha N}\right) < \int_{x_0}^{x_1} dx \ln x, \tag{3.28}$$

with

$$x_0 = 1 - \Delta_q - \frac{\sigma}{\alpha N}, \quad x_1 = 1 - \frac{\alpha - 1}{\alpha}\Delta_q - \frac{\sigma}{\alpha N}. \tag{3.29}$$

Then, one can obtain the following approximations,

$$\begin{aligned} &\frac{1}{\alpha N} \sum_{k=0}^{q-1} \ln\left(1 - \frac{\sigma}{\alpha N} - \Delta_q + \frac{k}{\alpha N}\right) \\ &\quad \sim \left[1 - \frac{\alpha - 1}{\alpha}\Delta_q\right] \ln\left(1 - \frac{\alpha - 1}{\alpha}\Delta_q\right) - (1 - \Delta_q)\ln(1 - \Delta_q) - q, \end{aligned} \tag{3.30}$$

or

$$\prod_{k=0}^{q-1} \left(1 - \frac{\sigma}{\alpha N} - \Delta_q + \frac{k}{\alpha N}\right) \sim \left(1 - \frac{\alpha - 1}{\alpha}\Delta_q\right)^{\alpha N - (\alpha - 1)q} (1 - \Delta_q)^{-\alpha N + \alpha q} e^{-q} \tag{3.31}$$

$$\sim \left(1 - \frac{\alpha - 1}{\alpha}\Delta_q\right)^{-(\alpha - 1)q} (1 - \Delta_q)^{\alpha q}. \tag{3.32}$$

Thus, combining Eqs.(3.27) and Eq.(3.32) reads

$$(\alpha N - \sigma - \alpha q)_q \sim (\alpha N)^q \left(1 - \frac{\alpha - 1}{\alpha}\Delta_q\right)^{-(\alpha - 1)q} (1 - \Delta_q)^{\alpha q}. \tag{3.33}$$

The similar procedure for $\Gamma(bN + bc - bq)$ reads

$$\Gamma(bN + bc - bq) \sim \Gamma(bN + bc)(bN)^{-bq}(1 - \Delta_q)^{-bq}. \tag{3.34}$$

Substituting Eqs.(3.33) and (3.34) into Eq.(3.25), one obtains

$$P_1^{[N]} \sim \text{Const.} \times \Gamma(bN + bc)\left(-\frac{\rho_1}{a^b}N^{-\gamma}\right)^N \sum_{q=0}^{N} \frac{I_q}{q!}\left(-\frac{\alpha a^b}{\rho_1 b^b}N^{1+\gamma-b}\right)^q, \tag{3.35}$$

where I_q is a moderate part in the change of q,

$$I_q \sim \left(1 - \frac{\alpha - 1}{\alpha}\Delta_q\right)^{-(\alpha-1)q}(1 - \Delta_q)^{(\alpha-b)q}. \tag{3.36}$$

Alternatively, more rough estimate with more tractable form can be obtained as follows. Since $1 + x < e^x$ for arbitrary $x \neq 0$, an upper bound for the right hand side of Eq.(3.30) is yielded as,

$$\begin{aligned} \text{r.h.s.} &< (1 - \Delta_q)\frac{\Delta_q/\alpha}{1 - \Delta_q} - \frac{\alpha - 1}{\alpha^2}\Delta_q^2 - \frac{\Delta_q}{\alpha} \\ &= -\frac{\alpha - 1}{\alpha^2}\frac{q^2}{N^2}. \end{aligned} \tag{3.37}$$

On the other hand, the use of the inequality $1 + x > e^{x/(1+x)}$ for any $x > -1$ gives an lower bound for the right hand side of Eq.(3.30);

$$\begin{aligned} \text{r.h.s.} &> \left(1 - \frac{\alpha - 1}{\alpha}\Delta_q\right)\frac{-\frac{\alpha-1}{\alpha}\Delta_q}{1 - \frac{\alpha-1}{\alpha}\Delta_q} - (1 - \Delta_q)(-\Delta_q) - \frac{\Delta_q}{\alpha} \\ &= -\frac{q^2}{N^2}. \end{aligned} \tag{3.38}$$

Thus, combining the resultant Eqs.(3.37) and (3.38) reads to the following estimate;

$$\exp\left(-\alpha\frac{q^2}{N}\right) < \prod_{k=0}^{q-1}\left(1 - \frac{\sigma}{\alpha N} - \Delta_q + \frac{k}{\alpha N}\right) < \exp\left(-\frac{\alpha - 1}{\alpha}\frac{q^2}{N}\right). \tag{3.39}$$

Similarly, for $\Gamma(bN + bc - bq)$ one obtains

$$1 < \frac{\Gamma(bN + bc - bq)(bN)^{bq}}{\Gamma(bN + bc)} < \exp\left(b\frac{q^2}{N}\right). \tag{3.40}$$

Finally, the estimate for J_q becomes by Eqs.(3.39) and (3.40)

$$e^{-\alpha q} < e^{-\alpha q^2/N} < \frac{J_q (bN)^{bq}}{\Gamma(bN+bc)(\alpha N)^q} < e^{(b+1/\alpha-1)q^2/N} < e^{(b+1/\alpha-1)q}, \tag{3.41}$$

from which one yields a rough approximation

$$\frac{J_q (bN)^{bq}}{\Gamma(bN+bc)(\alpha N)^q} = e^{\Theta_q q}, \quad \left(-\alpha < \Theta_q < b + \frac{1}{\alpha} - 1\right). \tag{3.42}$$

In this case, one obtains instead of Eq.(3.35).

$$P_1^{[N]} \sim \text{Const.} \times \Gamma(bN+bc)\left(-\frac{\rho_1}{a^b} N^{-\gamma}\right)^N \sum_{q=0}^{N} A_q, \tag{3.43}$$

with

$$A_q = \frac{1}{q!}\left(-\frac{\alpha a^b e^{\Theta_q}}{\rho_1 b^b} N^{1+\gamma-b}\right)^q. \tag{3.44}$$

From the above, one sees

$$\left|\frac{A_{q+1}}{A_q}\right| = \text{Const.} \times \frac{N^{1+\gamma-b}}{q+1} \le \text{Const.} \times N^{\gamma-b}. \tag{3.45}$$

Thus, if $\gamma > b$,

$$\begin{aligned}\left|\sum_{q=0}^{N} A_q\right| - |A_N| \;&\le\; \sum_{q=0}^{N-1} |A_q| \le |A_N| \sum_{p=1}^{N} N^{-p(\gamma-b)} \\ &=\; |A_N|\, N^{-(\gamma-b)} \frac{1-N^{-N(\gamma-b)}}{1-N^{-(\gamma-b)}},\end{aligned}$$

that is to say,

$$\left|\sum_{q=0}^{N} A_q\right| \sim |A_N| \times \left[1 + O(N^{-(\gamma-b)})\right].$$

Therefore, if $\gamma > b$, the contribution to $P_1^{[N]}$ is domimated by the summand with $q = N$ and from Eq.(3.24), one obtains

$$P_1^{[N]} \sim \frac{(-\sigma)_N}{N!} \sim N^{-\sigma-1}. \tag{3.46}$$

As has been mentioned earlier below Eq.(3.16), the convergence can be achieved if $P^{[N]}$ behaves at most as a finite power of N for large N, as far as $\gamma < \alpha$. Thus, the contribution from $P_1^{[N]}$ is convergent if $b < \gamma < \alpha$.

On the other hand, if $\gamma \leq b$, each term in the summand Eq.(3.44) can be comparable with each other. So, if one further approximates Eq.(3.26) by neglecting the q-dependence of Θ_q, say, $\Theta_q \to \Theta$ and notes the relation

$$\sum_{q=0}^{N} \frac{z^q}{q!} = e^z Q(N+1, z), \tag{3.47}$$

where $Q(a, z)$ is the normalized imcomplete gamma function defined by

$$Q(a, z) = \frac{\Gamma(a, z)}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_z^{\infty} dt e^{-t} t^{a-1}, \tag{3.48}$$

the estimate for $P_1^{[N]}$ eventually becomes

$$P_1^{[N]} \sim \text{Const.} \times N^{bc-1/2} \exp\left(-B_2 N^{1+\gamma-b}\right) B_1^N N^{(b-\gamma)N} Q\left(N+1, -B_2 N^{1+\gamma-b}\right), \tag{3.49}$$

with

$$B_1 = \frac{\rho_1 b^b}{a^b e^b}, \quad B_2 = \frac{\alpha a^b e^{\Theta}}{\rho_1 b^b}. \tag{3.50}$$

From Eq.(3.49), one immediately knows that if $\gamma < b$, the dominant behavior is $N^{(b-\gamma)N}$ and thus strongly divergent.

In the case of $\gamma = b$, Eq.(3.49) becomes

$$P_1^{[N]} \sim \text{Const.} \times N^{bc-1/2} Q(N+1. -B_2 N) \exp\Big[(\ln B_1 - B_2) N\Big]. \tag{3.51}$$

For the careful treatment of $Q(a, z)$ with $a \sim O(z)$ large in the case, we must invoke the uniform asymptotic expansion Eq.(5.25). which is also valid for $|\arg a| < \pi$ and $|\arg z/a| < 2\pi$ [46], see chapter 5. The resultant convergence condition is

$$\rho_1 \left(\ln \rho_1 + b \ln \frac{b}{ae}\right) \leq \frac{\alpha a^b e^{\Theta}}{b^b}. \tag{3.52}$$

Therefore, the convergence depends on ρ_1 if $\gamma = b$. The situation is similar to that of the result in Ref. [32,33], see Appendix B. However, our result involves a parameter Θ whose precise estimate is difficult in our approach, and thus the precise bound for ρ_1 cannot be obtained here.

The estimate of $P_2^{[N]}$ can be made by the same way. From the assumption Eq.(3.23) and the bounds Eq.(3.39) one obtains

$$\left|P_2^{[N]}\right| \leq C_2(\rho_1 N^{-\gamma})^N \sum_{q=0}^{N} \frac{1}{q!}\left(\frac{\alpha}{\rho_1 e^{\Theta'_q}} N^{1+\gamma}\right)^q, \quad \left(\frac{\alpha-1}{\alpha} < \Theta'_q < \alpha\right). \tag{3.53}$$

As has been seen in the analysis of $P_1^{[N]}$ for $\gamma > b$, the dominant contiribution of the right hand side of the above comes from the summand with $q = N$ for $\gamma > 0$. Thus, one yields

$$\left|P_2^{[N]}\right| \sim \left|c_2^{[0]}\right| \frac{(-\sigma)_N}{N!} \sim N^{-\sigma-1}, \tag{3.54}$$

which assures convergent contribution to a_N under $\gamma < \alpha$.

Summarizing the results in this chapter, the sequence $S^{[N]}(\rho_N)$ constructed by the ODM of type AHO is convergent in the region Eq.(3.16) (but not assures the covergence to the *true* answer) under the assumption (I)-(III) in Theorem 1 and $\gamma < \alpha$ if and only if either of the following condition is satisfied;

(A) $\gamma > b$.

(B) $\gamma = b$ and ρ_1 lies in the region like Eq.(3.52).

If one compares the result with those of Theorem 1, one immediately finds that the condition on γ is completely the same (see, Eq.(1.7)) while the restriction on α like Eq.(1.4) in Theorem 1 does not appear here. The situation is completely the same in the case of $\gamma = b$, see Appendix B. Therefore, in view of mere convergence of the resummed series, the restriction on α only comes from $0 < b < \alpha$, at least in the region Eq.(3.16). In the next section, we consider the convergence outside the region Eq.(3.16) in connection with the existence of a strong coupling expansion.

Chapter 4

Strong coupling expansions

In Ref. [33], Guida et al. proved that the convergence region in g-plane can be extended to wider region than Eq.(3.16) if there exists a strong coupling expansion for $S(g)$ with non-zero convergence radius. The result is stated as follows.

Theorem 2. *Let* $S(g)$ *be given which satisfies* (I)-(III) *in* Theorem 1 *and moreover:*

(IV) $S(g)$ *possesses a large* g *expansion*

$$S(g) = g^{\varsigma} \sum_{p=0}^{\infty} d^{[p]} (g^{-\tau})^p \tag{4.1}$$

which converges for $|g| \geq g_0$ *and there exists a positive integer* k *such that*

$$b\tau < k < 2\tau. \tag{4.2}$$

Then, the sequence $\{S^{[N]}(\rho_N)/g^{\varsigma}\}_N$ *with* $S^{[N]}(\rho_N)$, *constructed by the ODM of type AHO with* $\alpha = k/\tau$ *and* $\sigma = \alpha\varsigma$, *converges to* $S(g)/g^{\varsigma}$ *as* $N \to \infty$, *uniformly in each compact subset of the region (including* $g = \infty$*)*

$$\Re \frac{1}{g^{1/\alpha}} + \frac{1}{g_0^{1/\alpha}} \cos\left(\frac{\alpha - 1}{\alpha}\pi\right) > 0 \tag{4.3}$$

for any choice of scaling Eq.(1.7) of the positive parameter $\rho_N = \rho_1/N^\gamma$ with ρ_1 independent of g.

In this chapter, we consider the above theorem from a viewpoint of the present analysis. This might reveal further insight into the structure of ODM and perturbation series itself. Needless to say, the present result does not contradict with the above theorem since it states about convergence or divergence only in the region Eq.(3.16) and not at all outside the region Eq.(3.16). One of the significant points is that, in the expression of the behavior of a_N, g-dependence exists only in the prefactor outside the summation symbol. (Note that ρ is independent of g.) If we consider the scaled quantity $S(g)/g^\varsigma$ and $S^{[N]}(\rho)/g^\varsigma$ and choose $\sigma = \alpha\varsigma$, the g-dependence of the behavior of a_N only exists in the exponential term,

$$\exp\left[-\left(\frac{\rho_1}{g}\right)^{1/\alpha} N^{1-\gamma/\alpha}\right]. \tag{4.4}$$

From this fact, one can easily construct sufficient conditions for convergence of the series $\sum a_n$ outside the region Eq.(3.16), as follows:

(i) Convergence on the whole Riemann surface of g can be achieved if the large order behavior of $P^{[N]}$ in Eq.(3.14) is like,

$$P^{[N]} \sim A_0^{-N} \qquad A_0 > 1. \tag{4.5}$$

Note that in this case, the condition $\gamma < \alpha$ is not necessary at all. On the other hand, if A_0 depends on α, the condition $A_0 > 1$ may lead to the restriction on α. If this is the case, the convergence conditions involve a certain restriction on α in the whole Riemann surface of g including the region Eq.(3.16).

(ii) Convergence in a region like Eq.(4.3) can be achieved if the large order behavior of $P^{[N]}$ in Eq.(3.14) is like,

$$P^{[N]} \sim A_1^{-N^{1-\gamma/\alpha}} \qquad A_1 > 1 \tag{4.6}$$

$$\sim \exp\left(-A_2\rho_N^{1/\alpha}N\right) \sim \left(1 + A_2\rho_N^{1/\alpha}\right)^{-N} \quad A_2 > 0. \tag{4.7}$$

In this case, the convergence region for $\gamma < \alpha$ becomes,

$$\Re\frac{1}{g^{1/\alpha}} + A_2 > 0. \tag{4.8}$$

Contrary to the case (i) above, α-dependence of A_2 (or A_1) does not immediately lead to a restriction on the convergence conditions. If $A_2 < 0$ for some α, Eq.(4.8) only shows narrower convergence region than Eq.(3.16).

The resulting region Eq.(4.3) in Theorem 2 therefore means that if there exists a strong coupling expansion like Eq.(4.1) with convergence radius $g_0^{-\tau}$, $P^{[N]}$ decreases like or faster than Eq.(4.7) at large N with

$$A_2 = \frac{1}{g_0^{1/\alpha}}\cos\left(\frac{\alpha-1}{\alpha}\pi\right). \tag{4.9}$$

Note that A_2 in this case does depend on α and the condition $A_2 \geq 0$ leads to $\alpha \leq 2$, which is the same as the restriction in the necessary conditions of Theorem 1 and 2. However, as has been just stated in (ii) above, the violation of $\alpha \leq 2$ only results in narrower convergence region.

Now, let us come back to the analysis in chapter 3 and remember that contributions from large p in the sum to $P^{[N]}$ decreases rapidly for the convergent case $\gamma > b$. Then, it is apparent that one needs to have much informations on the lower order perturbation coefficients if one would like to obtain an estimate like Eqs.(4.5) and (4.7). On the contrary, one only needs to show that the leading behavior of $P^{[N]}$ is at most finite power of N (and indeed it is the case for $\gamma > b$) and does not need to know the decreasing behavior from the leading term like Eqs. (4.5) and (4.7), if one does not consider the convergence outside the region Eq.(3.16). The importance of lower order perturbation coefficients rather than higher order ones is also expected if one observes that Eq.(4.3) does not depend on the parameter b which is related to the large order behavior.

The above disscussion thus indicates a strong connection between existence of a strong coupling expansion and lower order behavior of perturbation coefficients. It seems a bit strange, if one is reminded of, for instance, the fact that the existence of a strong coupling expansion in the quantum mechanical AHO can be proved by the only use of Kato-Rellich theorem and Symanzik's scaling relation [9]. On the other hand, the Bender-Wu relation Eq.(1.2) tells us that the perturbation coefficients are uniquely determined under the conditions (I) and (II) if we know the function $S(g)$ on the cut completely. The full knowledge of a function on the cut means, in principle, that we also know the function on the whole Riemann surface by the uniqueness of analytic continuation and thus the existence of a convergent strong coupling expansion. The Bender-Wu relation also tells us that contribution from larger negative g becomes more important if one considers lower order coefficients. Threfore, the knowledge of lower order behavior of perturbation series for a function $S(g)$ is strongly connected with the knowledge of analyticity of the function for large g region.

The connection mentioned above is further confirmed by the fact that the strong coupling expansion can be represented solely by the original divergent perturbation coefficients. Although it was shown that representations can be obtained not only by ODM [33, 42] but also by the other methods [43, 44], the procedure by ODM seems simplest among them. Especially, the procedure in Ref. [33] with the choice stated in Theorem 2 with $k = 1$ is quite simple and here we follow it. Let start with a resummed expression Eq.(2.11) with $\alpha = 1/\tau$ and $\sigma = \varsigma/\tau$,

$$S^{[N]}(\lambda) = \frac{1}{(1-\lambda)^{\varsigma/\tau}} \sum_{p=0}^{N} c^{[p]} g^p (1-\lambda)^{p/\tau} \sum_{r=0}^{N-p} \frac{(p/\tau - \varsigma/\tau)_r}{r!} \lambda^r. \tag{4.10}$$

One then substitutes an order dependence for λ of the form

$$\lambda_N \sim 1 - \left(\frac{\varrho_N}{g}\right)^{\tau} \qquad (\varrho_N \sim \rho_N) \tag{4.11}$$

to obtain

$$S^{[N]}(\varrho_N) = \left(\frac{g}{\varrho_N}\right)^{\varsigma} \sum_{p=0}^{N} c^{[p]} \varrho_N^p \sum_{r=0}^{N-p} \frac{(p/\tau - \varsigma/\tau)_r}{r!} \sum_{t=0}^{r} \frac{r!(-1)^t}{(r-t)!t!} \left(\frac{\varrho_N}{g}\right)^{t\tau}. \tag{4.12}$$

Noting that the upper value of the summation with respect to t can formally extend to, say, infinity, one can arrange the above as

$$S^{[N]}(\varrho_N) = \left(\frac{g}{\varrho_N}\right)^{\varsigma} \sum_{t=0}^{N} \left(\frac{\varrho_N}{g}\right)^{t\tau} \frac{(-1)^t}{t!} \sum_{p=0}^{N-t} c^{[p]} \varrho_N^p \sum_{r=0}^{N-p-t} \frac{(p/\tau - \varsigma/\tau)_{r+t}}{r!}. \tag{4.13}$$

The summation with respect to r is summed up to

$$\sum_{r=0}^{N-p-t} \frac{(p/\tau - \varsigma/\tau)_{r+t}}{r!} = \frac{(p/\tau - \varsigma/\tau)_{N-p+1}}{(p/\tau - \varsigma/\tau + t)(N-p-t)!}, \tag{4.14}$$

and one gets the following expression

$$S^{[N]}(\varrho_N) = g^{\varsigma} \sum_{t=0}^{N} (g^{-\tau})^t \left[\frac{(-1)^t}{t!} \sum_{p=0}^{N-t} c^{[p]} \frac{(p/\tau - \varsigma/\tau)_{N-p+1}}{(p/\tau - \varsigma/\tau + t)(N-p-t)!} \varrho_N^{p-\varsigma+t\tau}\right], \tag{4.15}$$

and hence the representations for the coefficients of the strong coupling expansion are

$$d^{[t]} = \lim_{N\to\infty} \frac{(-1)^t}{t!} \sum_{p=0}^{N-t} c^{[p]} \frac{(p/\tau - \varsigma/\tau)_{N-p+1}}{(p/\tau - \varsigma/\tau + t)(N-p-t)!} \varrho_N^{p-\varsigma+t\tau}, \tag{4.16}$$

with a proper order dependence of ϱ_N. If Eq. (4.16) is correct, one can also calculate the convergence radius $g_0^{-\tau}$ of the strong coupling expansion in principle, for instance,

$$\begin{aligned} \frac{1}{g_0^{-\tau}} &= \overline{\lim_{t\to\infty}} \left|d^{[t]}\right|^{1/t} \\ &= \overline{\lim_{t\to\infty}} \left|\lim_{N\to\infty} \frac{(-1)^t}{t!} \sum_{p=0}^{N-t} c^{[p]} \frac{(p/\tau - \varsigma/\tau)_{N-p+1}}{(p/\tau - \varsigma/\tau + t)(N-p-t)!} \varrho_N^{p-\varsigma+t\tau}\right|^{1/t}. \end{aligned} \tag{4.17}$$

Therefore, for two physical quantities given, if the large order behavior of the perturbation coefficients for these quantities are the same, one may say that if there exists a convergent strong coupling expansion or not is determined by the behavior of the lower order coefficients.

Chapter 5

Illustrations on zero-dimensional models

Analyses on a model, of which the analytic structures could be well understood, enable us to get deeper understandings on validity of resummation methods. Even in a simple quantum mechanical model with one degree of freedom, however, structure of the Riemann surface, on which the analytic continuation of the eigenvalues are defined, is known to have quite complicated structure [9]. Therefore, a 0-dimensional model (simple integral with expansion parameters) is often employed. Let us consider the 0-dimensional AHO,

$$Z(g) = \int_{-\infty}^{\infty} dx \exp\left(-x^2 - g x^{2M}\right) \quad (M = 2, 3, 4, \cdots). \tag{5.1}$$

One can calculate the conventional perturbation coefficients explicitly,

$$\begin{aligned} Z(g) &= \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} g^p \int_{-\infty}^{\infty} dy \exp(-y^2) y^{2Mp} \\ &= \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \Gamma\left(Mp + \frac{1}{2}\right) g^p, \end{aligned} \tag{5.2}$$

that is,

$$c^{[p]} = \frac{(-1)^p}{p!} \Gamma\left(Mp + \frac{1}{2}\right) \tag{5.3}$$

$$\sim (-1)^p \Gamma\Big((M-1)p\Big) \quad \text{for large } p. \tag{5.4}$$

The analyticity conditions (I) and (II) is fulfilled with $\varsigma = -1/2M$ and hence, large order behavior Eq.(5.4) is equivallent to the asymptotic behavior of Disc$Z(g)$ for small negative g Eq.(1.3) with

$$b = M - 1. \tag{5.5}$$

On the other hand, one can also calculate the coefficients of the strong couping expansion explicitely,

$$\begin{aligned} Z(g) &= g^{-1/2M} \sum_{t=0}^{\infty} \frac{(-1)^t}{t!} \left(g^{-1/M}\right)^t \int_{-\infty}^{\infty} dy \exp(-y^{2M}) y^{2t} \\ &= g^{-1/2M} \sum_{t=0}^{\infty} \frac{(-1)^t}{Mt!} \Gamma\left(\frac{t}{M} + \frac{1}{2M}\right) \left(g^{-1/M}\right)^t, \end{aligned} \tag{5.6}$$

that is,

$$d^{[t]} = \frac{(-1)^t}{Mt!} \Gamma\left(\frac{t}{M} + \frac{1}{2M}\right) \tag{5.7}$$

$$\sim \frac{(-1)^t}{\Gamma\big((1 - 1/M)t\big)} \quad \text{for large } t. \tag{5.8}$$

The large order behavior Eq.(5.8) shows that the convergence radius of the strong coupling expansion Eq.(5.6) is ∞ for any M and thus Eq.(5.6) represents the exact $Z(g)$ for any $g \neq 0$. Therefore, one can investigate the convergence of the ODM for $Z(g)$ with recourse to the construction of the strong coupling expansion via the ODM in chapter 4. Let us substitute the coefficients Eq.(5.3), $\varsigma = -1/2M$ and $\alpha = 1/\tau = M$ into Eq.(4.15),

$$\begin{aligned} Z^{[N]}(\varrho_N) &= g^{-1/2M} \sum_{t=0}^{N} \left(g^{-1/M}\right)^t \\ &\times \left[\frac{(-1)^t}{t!} \sum_{p=0}^{N-t} \frac{(-1)^p}{p!} \frac{\Gamma\big(N + (M-1)p + 3/2\big)}{(Mp + 1/2 + t)(N - p - t)!} \varrho_N^{(Mp+1/2+t)/M}\right]. \end{aligned} \tag{5.9}$$

Now, the N-th approximation $d^{[t]}(\varrho_N)$ for $d^{[t]}$ reads to, by the use of the Euler integral representation,

$$d^{[t]}(\varrho_N) = \frac{(-1)^t}{t!} \sum_{p=0}^{N-t} \frac{(-1)^p}{p!(N-p-t)!} \frac{\varrho_N^{(Mp+1/2+t)/M}}{Mp + 1/2 + t} \int_0^{\infty} ds e^{-s} s^{N+(M-1)p+1/2}$$

$$
\begin{aligned}
&= \frac{(-1)^t}{t!}\int_0^\infty dse^{-s}s^{N+1/2}\sum_{p=0}^{N-t}\frac{(-s^{M-1})^p}{p!(N-p-t)!}\int_0^{\varrho_N}\frac{d\rho}{M}\rho^{(Mp+1/2+t)/M-1} \\
&= \frac{(-1)^t}{Mt!}\int_0^\infty dse^{-s}s^{N+1/2}\int_0^{\varrho_N}d\rho\rho^{(1/2+t)/M-1}\sum_{p=0}^{N-t}\frac{(-\rho s^{M-1})^p}{p!(N-p-t)!}.
\end{aligned}
$$

Applying the binomial theorem,

$$
\begin{aligned}
d^{[t]}(\varrho_N) &= \frac{(-1)^t}{Mt!(N-t)!}\int_0^\infty dse^{-s}s^{N+1/2}\int_0^{\varrho_N}d\rho\rho^{(1/2+t)/M-1}\left(1-\rho s^{M-1}\right)^{N-t} \\
&= \frac{(-1)^t}{Mt!(N-t)!}\int_0^\infty dse^{-s}s^{N-t+(1/2+t)/M}B_{\varrho_N s^{M-1}}\left(\frac{1/2+t}{M}, N-t+1\right),
\end{aligned}
$$

where $B_x(p,q)$ denotes the incomplete beta function. Using the reflection formula,

$$
B_x(p,q) = B(p,q) - B_{1-x}(q,p),
$$

one finally obtains,

$$
d^{[t]}(\varrho_N) = \frac{(-1)^t}{Mt!}\Gamma\left(\frac{t}{M}+\frac{1}{2M}\right) + R_N(\varrho_N) \tag{5.10}
$$

with

$$
R_N(\varrho_N) = -\frac{(-1)^t}{Mt!(N-t)!}\int_0^\infty dse^{-s}s^{N-t+(1/2+t)/M}B_{1-\varrho_N s^{M-1}}\left(N-t+1, \frac{1/2+t}{M}\right) \tag{5.11}
$$

Therefore, convergence of the ODM can be proved for $Z(g)$ on the whole Riemann surface, if one can choose ϱ_N such that $R_N(\varrho_N)$ vanishes as $N \to \infty$.

For the proof, we first divide $R_N(\varrho_N)$ as $R_N = R_N^{(1)} + R_N^{(2)}$ with,

$$
\begin{aligned}
R_N^{(1)}(\varrho_N) &= -\frac{(-1)^t}{Mt!(N-t)!} \\
&\quad\times\int_0^{\varrho_N^{-1/(M-1)}}dse^{-s}s^{N-t+(1/2+t)/M}B_{1-\varrho_N s^{M-1}}\left(N-t+1, \frac{1/2+t}{M}\right), \qquad (5.12)
\end{aligned}
$$

$$
\begin{aligned}
R_N^{(2)}(\varrho_N) &= -\frac{(-1)^t}{Mt!(N-t)!} \\
&\quad\times\int_{\varrho_N^{-1/(M-1)}}^\infty dse^{-s}s^{N-t+(1/2+t)/M}B_{1-\varrho_N s^{M-1}}\left(N-t+1, \frac{1/2+t}{M}\right). \qquad (5.13)
\end{aligned}
$$

(1) estimate of $R_N^{(1)}$

For $R_N^{(1)}$, the argument of the incomplete beta function in the integrand ranges bounded region

$$0 \leq 1 - \varrho_N s^{M-1} \leq 1.$$

So, for the estimate, we can invoke the *uniform* asymptotic expansion of the incomplete beta function $B_x(p,q)$ for large p, uniformly valid for both $x \in [0,1]$ and $q \geq 0$ [45]. The resulting form is

$$B_x(p,q) \sim B(p,q)Q\Big(q, \alpha(x)p\Big) + \frac{x^p(1-x)^q}{p}\sum_{s=0}^{\infty}\frac{B_s\big(\alpha(x)\big)}{p^s}, \qquad (5.14)$$

where $Q(a,z)$ is defined by Eq.(3.48) and $\alpha(x)$ is defined by the following implicit relation

$$-\ln x - \mu\ln(1-x) = \alpha(x) - \mu\ln\alpha(x) + (1+\mu)\ln(1+\mu) - \mu, \qquad \mu = q/p. \quad (5.15)$$

$\alpha(x)$ ranges from ∞ to 0 as x ranges from 0 to 1. For more details of the expansion Eq.(5.14), see Ref. [45]. With the aid of the expansion Eq.(5.14), we have, for $p = N - t + 1$ large and with $q = (1/2 + t)/M$,

$$\begin{aligned}
\left|R_N^{(1)}(\varrho_N)\right| &\sim \text{Const.} \times \frac{1}{\Gamma(p)}\int_0^{\varrho_N^{-1/(M-1)}} ds e^{-s}s^{p+q-1}B(p,q)Q\Big(q, \alpha(1-\varrho_N s^{M-1})p\Big) \\
&= \text{Const.} \times \frac{\Gamma(q)}{\Gamma(p+q)}Q(q.\alpha p)\int_0^{\varrho_N^{-1/(M-1)}} ds e^{-s}s^{p+q-1} \quad (0 < \alpha < \infty) \\
&= \text{Const.} \times \Gamma(q,\alpha p)P\left(p+q, \varrho_N^{-1/(M-1)}\right).
\end{aligned}$$

Since the normalized incomplete gamma function $P(a,z)$ defined by

$$P(a,z) = \frac{\gamma(a,z)}{\Gamma(a)} = \frac{1}{\Gamma(a)}\int_0^z dt e^{-t}t^{a-1} \qquad (5.16)$$

is bounded by 1 for any $0 \leq z < \infty$, one finally gets

$$\left|R_N^{(1)}(\varrho_N)\right| \leq \text{Const.} \times \Gamma\left(\frac{1/2+t}{M}, \alpha N\right) \to 0 \qquad (N \to \infty), \qquad (5.17)$$

irrespective of M and γ.

(2) estimate of $R_N^{(2)}$

For $R_N^{(2)}$, the argument of the incomplete beta function in the integrand takes any $1 - \varrho_N s^{M-1} < 0$. In this case, the following series representation [47]

$$B_x(p,q) = \frac{x^p(1-x)^{q-1}}{p} \sum_{n=0}^{\infty} \frac{(1-q)_n}{(1+p)_n} \left(\frac{x}{1-x}\right)^n, \tag{5.18}$$

which is convergent for $|x/(1-x)| < 1$, i.e., $x < 1/2$ is adequate[1]. Then, with $p = N - t + 1$, $q = (1/2 + t)/M$ and $x = 1 - \varrho_N s^{M-1}$,

$$\begin{aligned} R_N^{(2)}(\varrho_N) &= -\frac{(-1)^t}{Mt!\Gamma(p+1)} \\ &\times \int_{\varrho_N^{-1/(M-1)}}^{\infty} ds e^{-s} s^{p+q-1} x^p (1-x)^{q-1} \sum_{n=0}^{\infty} \frac{(1-q)_n}{(1+p)_n} \left(\frac{x}{1-x}\right)^n. \end{aligned} \tag{5.19}$$

Since the series in the integrand is uniformly convergent, we can safely interchange the order of integration and summation. After the interchange, applying the mean value theorem on the each term reads,

$$\begin{aligned} R_N^{(2)}(\varrho_N) &= -\frac{(-1)^t}{Mt!\Gamma(p+1)} \sum_{n=0}^{\infty} \frac{(1-q)_n}{(1+p)_n} (\theta_n)^n \\ &\times \int_{\varrho_N^{-1/(M-1)}}^{\infty} ds e^{-s} s^{p+q-1} x^p (1-x)^{q-1} \quad (|\theta_n| < 1) \\ &\sim -\frac{(-1)^{t+p}}{Mt!\Gamma(p+1)} \int_{\varrho_N^{-1/(M-1)}}^{\infty} ds e^{-s} s^{p+q-1} (1-x)^{p+q-1} \quad \text{for large } p \\ &= \frac{(-1)^{t+p+1} \varrho_N^{p+q-1}}{Mt!\Gamma(p+1)} \Gamma\left(M(p+q-1)+1, \varrho_N^{-1/(M-1)}\right). \end{aligned}$$

Finally, substituting $\varrho_N \sim \varrho_1 N^{-\gamma}$ and applying the Stirling formula yields,

$$\begin{aligned} \left|R_N^{(2)}(\varrho_N)\right| &= \frac{\varrho_N^{N-t+q-1}}{Mt!} \frac{\Gamma\left(MN - M(t-q+1)+1\right)}{\Gamma(N-t+2)} \\ &\times Q\left(MN - M(t-q+1)+1, \varrho_N^{-1/(M-1)}\right) \\ &\sim \text{Const.} \times N^{(\gamma-M)(t-q+1)+t-1} \left(\varrho_1 e^{1-M} M^M\right)^N \\ &\times N^{(M-1-\gamma)N} Q\left(MN, R_1 N^{\gamma/(M-1)}\right), \end{aligned} \tag{5.20}$$

[1]This series is nothing but one of the analytic continuation of the hypergeometric function outside the unit circle $|x| = 1$.

with

$$R_1 = \varrho_1^{-1/(M-1)}.$$

So, let us estimate the normalized incomplete gamma function in Eq.(5.20).

(i) $1 > \gamma/(M-1) = 1 - \delta$

In this case, the following series representation [47] is adequate;

$$Q(a, z) = 1 - \frac{e^{-z} z^a}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{z^n}{(a)_n} \sim 1 - O\Big((z/a)^a\Big). \tag{5.21}$$

For Eq.(5.20), $z/a \sim N^{-\delta}$ and hence the behavior of $R_N^{(2)}$ is

$$\left|R_N^{(2)}(\varrho_N)\right| \sim N^{(M-1-\gamma)N} = N^{(M-1)\delta N}, \tag{5.22}$$

that is, divergent.

(ii) $1 < \gamma/(M-1) = 1 + \delta$

In this case, it is sufficient to use the boundedness of the incomplete gamma ratio. The behavior of $R_N^{(2)}$ is,

$$\left|R_N^{(2)}(\varrho_N)\right| \sim N^{(M-1-\gamma)N} = N^{-(M-1)\delta N} \to 0 \quad (N \to \infty), \tag{5.23}$$

that is, convergent to zero.

(iii) $\gamma = M - 1$

In this case, since $a \sim O(z)$, neither the expansion Eq. (5.21) nor the well-known asymptotic expansion [47]

$$Q(a, z) \sim \frac{z^{a-1} e^{-z}}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{(-1)^n (1-a)_n}{z^n} \tag{5.24}$$

is inadequate. So, we invoke the *uniform* asymptotic expansion of the incomplete gamma function for large a, uniformly valid for $z \geq 0$ [46]. The resulting form is

$$Q(a, z) \sim \frac{1}{2}\mathrm{erfc}\left(\eta\sqrt{a/2}\right) + \frac{e^{-a\eta^2/2}}{\sqrt{2\pi a}} \sum_{k=0}^{\infty} \frac{c_k(\eta)}{a^k}, \tag{5.25}$$

where $\mathrm{erfc}(z)$ is the complementary error function defined by

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} dt e^{-t^2}, \tag{5.26}$$

and η is defined by

$$\eta = (\lambda - 1)\sqrt{2\frac{\lambda - 1 - \ln \lambda}{(\lambda - 1)^2}}, \qquad \lambda = \frac{z}{a}. \tag{5.27}$$

For more details of the expansion Eq.(5.25), see Ref. [46]. Thus, the leading behavior of $R_N^{(2)}$ reads

$$\left|R_N^{(2)}(\varrho_N)\right| \sim \mathrm{Const.} \times N^{q-2} \left(\varrho_1 e^{1-M} M^M\right)^N \mathrm{erfc}\left(\sqrt{\frac{M}{2}}\eta N^{1/2}\right), \tag{5.28}$$

with

$$\lambda \sim \frac{R_1}{M}.$$

Since the behavior of $\mathrm{erfc}(z)$ is critical at $z = 0$, the estimate should be done separately as;

(a) $R_1/M > 1 \quad \Leftrightarrow \quad \varrho_1 < 1/M^{M-1}$

In this case, $\eta > 0$ and the argument of the error function tends to infinity as $N \to \infty$. By the well-known asymptotic expansion for the complementary error function [47]

$$\mathrm{erfc}(z) \sim \frac{e^{-z^2}}{\sqrt{\pi} z} \sum_{m=0}^{\infty} \frac{(-1)^m (1/2)_m}{z^{2m}}, \tag{5.29}$$

we obtain

$$\mathrm{erfc}\left(\sqrt{\frac{M}{2}}\eta N^{1/2}\right) \sim N^{-1/2} e^{-M\eta^2 N/2}$$

and thus

$$\left|R_N^{(2)}(\varrho_N)\right| \sim \mathrm{Const.} \times N^{q-5/2} \left(\varrho_1 M^M e^{1-M-M\eta^2/2}\right)^N. \tag{5.30}$$

Therefore, the condition that $R_N^{(2)}$ tends to 0 becomes

$$\varrho_1 M^M e^{1-M-M\eta^2/2} = e^{1-R_1} R_1 < 1, \tag{5.31}$$

which is always satistfied for any $R_1 > M \geq 2$.

(b) $R_1/M = 1 \quad \Leftrightarrow \quad \varrho_1 = 1/M^{M-1}$

In this case, $\mathrm{erfc}(0) = 1$ and thus for $M \geq 2$,

$$\left| R_N^{(2)}(\varrho_N) \right| \sim \text{Const.} \times N^{q-2} \left(e^{1-M} M \right)^N \to 0 \quad (N \to \infty). \tag{5.32}$$

(c) $R_1/M < 1 \quad \Leftrightarrow \quad \varrho_1 > 1/M^{M-1}$

In this case, $\eta < 0$ and the argument of the error function tends to negative infinity as $N \to \infty$. By the reflection formula

$$\mathrm{erfc}(-z) = 2 - \mathrm{erfc}(z)$$

and the asymptotics Eq.(5.29), one obtains

$$\left| R_N^{(2)}(\varrho_N) \right| \sim \text{Const.} \times N^{q-2} \left(\varrho_1 e^{1-M} M^M \right)^N . \tag{5.33}$$

Therefore, $R_N^{(2)}$ tends to 0 as $N \to \infty$ (for any q and thus any t) if and only if

$$\varrho_1 < \frac{e^{M-1}}{M^M}. \tag{5.34}$$

Note that since $M < e^{M-1}$ for any $M > 1$, the set of ϱ_1 satisfying

$$\frac{1}{M^{M-1}} < \varrho_1 < \frac{e^{M-1}}{M^M}$$

is not empty.

Summarizing the above results, the remainder R_N which is given by Eq.(5.11) tends to zero for any t if and only if the either of the following conditions are satisfied;

(A') $\gamma > M - 1 = b$.

(B') $\gamma = M - 1 = b$ and $\varrho_1 < \varrho_1^{\mathrm{UB}}$ with the upper bound

$$\varrho_1^{\mathrm{UB}} = \frac{e^{M-1}}{M^M}. \tag{5.35}$$

As has been mentioned earlier, these conditions are equivalent to the conditions for the convergence of the ODM of type AHO for the 0-dimensional AHO Eq.(5.1). Note that the condition $\gamma < \alpha = M$ does not appear and thus the conjecture made in (i) of chapter 4 for the case where the convergence in the whole g-plane can be achieved is confirmed. The restriction on α, or equivallently on M, like Eq.(1.4) does not also appear in this case.

For the comparison, we also checked the initial value ρ_1 suggested by the first order optimization conditions. With $\alpha = M$, $\sigma = \alpha_\zeta = -1/2$ and $c^{[0]} = \Gamma(1/2)$, $c^{[1]} = -\Gamma(M + 1/2)$, the first order FAC condition Eq.(A.6) gives,

$$\rho_1^{\mathrm{FAC}} = \frac{1}{2}\frac{\Gamma(1/2)}{\Gamma(M + 1/2)}, \tag{5.36}$$

and the first order PMS condition Eq.(A.7) gives,

$$\rho_1^{\mathrm{PMS}} = \frac{3}{4}\frac{\Gamma(1/2)}{\Gamma(M + 3/2)}, \tag{5.37}$$

In Table 5.1, round numbers of ϱ_1^{UB}, ρ_1^{FAC} and ρ_1^{PMS} for $M = 2, 3, 4, 5$ are explicitly shown. The results almost confirm the assertion in Ref. [32] that the FAC criterion leads us to the boundary of the convergence region. Indeed, for any M,

$$\begin{aligned}
\rho_1^{\mathrm{FAC}} &\sim \frac{1}{2\sqrt{2}}\left(M + \frac{1}{2}\right)^{-M} e^{M+1/2}\left[1 - \frac{1}{12(M + 1/2)} + \cdots\right] \\
&\leq \left(\frac{e}{2}\right)^{3/2}\left(\frac{4}{5}\right)^{2}\frac{e^{M-1}}{M^M} \\
&\approx 1.01409 \times \varrho_1^{\mathrm{UB}},
\end{aligned}$$

and thus the optimal values by the first order FAC criterion will be a bit less than the upper bound Eq.(5.35) for any M in this case.

M	2	3	4	5
ϱ_1^{UB}	0.67957	0.27367	0.078459	0.017471
ρ_1^{FAC}	0.66667	0.26667	0.076190	0.016931
ρ_1^{PMS}	0.40000	0.11429	0.025397	0.004618

Table 5.1: Round values for the upper bound ϱ_1^{UB} and ρ_1 obtained by the optimization conditions for zero-dimensional AHO with anharmonicity M.

Chapter 6

The δ expansion and its limitations

The idea of the δ expansion [25, 34] is based on the arbitrariness in a way of splitting a Hamiltonian or Lagrangian into a free part and a perturbative part. In the δ expansion, we take a free part, which constitutes the basis of the perturbation, such that this free part involves artificial parameters. Then, one regards the remainder term as a perturbative part and carries out the conventional perturbation calculation on the parameter-dependent basis. The above procedure can be formally written as follows. One first defines a Hamiltonian of the form

$$
\begin{aligned}
H_\delta(\Omega) &= (1-\delta)H_0(\Omega) + \delta H(g) &(6.1)\\
&= H_0(\Omega) + \delta\left[H(g) - H_0(\Omega)\right], &(6.2)
\end{aligned}
$$

where $H(g)$ is the original Hamiltonian concerned, and $H_0(\Omega)$ is a Hamiltonian which is regarded as a free part and depends on artificial parameters Ω. One then performs the perturbation expansion for a physical quantity $S(g)$ in powers of δ on the basis of $H_0(\Omega)$,

$$S_\delta(g) = \sum_{p=0}^{\infty} d^{[p]}(g;\Omega)\delta^p. \tag{6.3}$$

The N-th order approximation in the δ expansion is obtained by truncating the series at the N-th order in δ, setting $\delta = 1$ and adjusting the parameters Ω order by order,

$$S^{[N]}(g;\Omega_N) = \sum_{p=0}^{N} d^{[p]}(g;\Omega_N). \tag{6.4}$$

Note that $S_\delta(g)$ is independent of the artificial parameters Ω at $\delta = 1$ since, at this point, $H_\delta(\Omega)$ reduces to the original Hamiltonian $H(g)$. On the other hand, any truncations of the series Eq.(6.3) causes Ω-dependence of the truncated quantity even at $\delta = 1$. Therefore, one can suitably adjust the values of Ω order by order. This is the cruicial point of the δ expansion. Needless to say, the essential and difficult problems in the δ expansion consist in the choice of $H_0(\Omega)$. For quartic AHO in 0-dimension and quantum mechanics, it has been proved that the convergence of the sequence Eq.(6.4) to the exact answer can be achieved with a proper order dependence if one employs the harmonic oscillator with trial frequency Ω as a free Hamiltonian $H_0(\Omega)$ [32] (see, Eq.(6.7). Convergence property of this type of the δ expansion (mass-renormalized perturbation) can be easily examined from the fact that the δ expansion with the above choice is realized as a special case of the ODM of type AHO [33,35], as we will see below. Suppose we have a following AHO Hamiltonian (density) in D dimensions,

$$\hat{\mathcal{H}}(g) = \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\omega^2\hat{\phi}^2 + \frac{g}{(2M)!}\hat{\phi}^{2M}, \quad (M = 2,3,4,\cdots). \tag{6.5}$$

Then, the conventional perturbation expansion in g for a quantity $S(g)$ which has mass-dimension κ takes, from dimensional analysis, the form

$$S(g) = \omega^\kappa \sum_{p=0}^{\infty} c^{[p]} \left(\frac{g}{\omega^{2M-(M-1)D}}\right)^p. \tag{6.6}$$

The δ expansion for Eq.(6.5) with the assignment

$$\hat{\mathcal{H}}_\delta(\Omega) = \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\Omega^2\hat{\phi}^2 + \delta\left[\frac{1}{2}(\omega^2 - \Omega^2)\hat{\phi}^2 + \frac{g}{(2M)!}\hat{\phi}^{2M}\right] \tag{6.7}$$

corresponds to substitutions in Eq.(6.6)

$$\omega \rightarrow \Omega\left(1+\delta\frac{\omega^2-\Omega^2}{\Omega^2}\right)^{1/2} = \omega v(1-\delta\lambda)^{1/2} \tag{6.8}$$

$$g \rightarrow \delta g \tag{6.9}$$

with

$$v = \frac{\Omega}{\omega} = (1-\lambda)^{-1/2}, \tag{6.10}$$

followed by the expansion in δ, truncating the series at the order N in powers of δ and setting $\delta = 1$ at the end. Eventually, one obtains

$$S(\rho) = (\omega v)^{\kappa}\sum_{p=0}^{N} c^{[p]}\rho^{p}\sum_{r=0}^{N-p}\frac{\left(Mp-(M-1)Dp/2-\kappa/2\right)_r}{r!}\lambda^{r+p}, \tag{6.11}$$

with

$$\rho = \frac{g}{\lambda(\omega v)^{2M-(M-1)D}}. \tag{6.12}$$

Comparison of Eq.(6.11) to Eq.(2.11) and Eq.(6.12) to Eq.(2.2) with (2.10) shows that the δ expansion Eq.(6.7) is realized as a special case of the ODM of type AHO with

$$\sigma = \frac{\kappa}{2}, \quad \alpha = M - \frac{(M-1)D}{2}. \tag{6.13}$$

In this way, one can construct a particular ODM starting from a DE [35]. Now, let us first consider the 0-dimensional case,

$$Z(g) = \int_{-\infty}^{\infty} dx \exp\left(-\frac{1}{2}\omega^2 x^2 - \frac{g}{(2M)!}x^{2M}\right). \tag{6.14}$$

In this case, $b = M-1$ (see, chapter 5), $\kappa = -1$ and hence

$$\sigma = -\frac{1}{2}, \quad \alpha = M. \tag{6.15}$$

Therefore, a necessary condition for convergence $b < \alpha$ is satistied for any anharmonicity M in $D=0$ and a sufficient condition $\alpha \leq 2$ requires $M \leq 2$. Note that the resulting choices Eq.(6.15) are just the ones stated in Theorem 2 with $k = 1$. The analyses in these choices have been already shown in chapter 5.

Next, let us consider the quantum mechanical case ($D = 1$)

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\omega^2 x^2 + \frac{g}{(2M)!}x^{2M}. \tag{6.16}$$

For eigenvalues $E(g)$, $\kappa = 1$ and hence,

$$\sigma = \frac{1}{2}, \quad \alpha = \frac{M+1}{2}. \tag{6.17}$$

The large order behavior of the perturbation coefficiets is known as [16]

$$c_n^{[p]} \sim (-1)^{p+1}\Gamma\Big((M-1)p + n + 1/2\Big) \tag{6.18}$$

for the n-th level and thus

$$b = M - 1. \tag{6.19}$$

As is well known [9], there exist strong coupling expansions for the quantum mechanical AHOs with $\varsigma = 1/(M+1)$ and $\tau = 2/(M+1)$. Hence, the resulting choices Eq.(6.17) are also exactly the ones stated in Theorem 2 with $k = 1$ as in the case of $D = 0$ and thus look preferable at first sight. However, the crucial difference of the quantum mechanical case from the 0-dimensional case is that a necessary condition for convergence $b < \alpha$ now leads to $M < 3$. That is to say, convergence is possible only for the quartic AHO and the sextic AHO ($M = 3$) is on the boundary. A sufficient condition $\alpha \leq 2$ also requires $M \leq 3$. Therefore, the δ expansion Eq.(6.7) can achieve convergence only for quartic AHO (and for sextic AHO for any chance) in quantum mechanics.

Finally, let us come back to consider the field theoretical case. A difficulty is immediately expected from the fact that for $D \geq 2$, one always has $\alpha \leq 1$ for any anharmonicity M from Eq.(6.13). On the other hand, the fulfillment of a necessary condition $b < \alpha \leq 1$

indicates that the large order behavior of perturbation coefficients must grow slower than factorial,

$$|c^{[N]}| < N! \quad \text{for large } N, \tag{6.20}$$

at least, under the analyticity conditions (I) and (II). However, as far as I know, such a slow divergent behavior has not yet observed in any quantum systems and otherwise the series would be convergent.

In addition to the above fact, it should be noted that property of the conformal mapping of Eq.(2.2) with (2.10) for $\alpha \leq 1$ is quite different from that for $\alpha > 1$. In the latter case, positive real axis in g plane is mapped into bounded interval $[0,1)$ in λ plane while, in the formar case, the inverse image of Eq.(2.2) for positive real axis in g plane is no longer bounded. For this reason, one cannot expect moderate property of asymptotic series in λ, especially for large g, in the case of $\alpha \leq 1$. Note also that since this bounded nature of the mapping for $\alpha > 1$ is fully taken into account in the proof of convergence in Ref. [33], Theorem 1 and 2 cannot apply in the case of $\alpha \leq 1$.

In the case where g has mass-dimension 0, that is, the theory Eq.(6.5) is renormalizable but not super renormalizable (e.g., $M = 3$ for $D = 3$, $M = 2$ for $D = 4$), the mapping Eq.(6.12) becomes trivial and thus the δ expansion may at most reguralize some singular behavior at $g = \infty$. So, the naive application of the δ expansion to a just renormalizable model may be of the least benefit.

Of course, validity of the above considerations is questionable since the theory in $D \geq 2$ needs renormalization and the above inspections are only based on the bare quantities ω and g. However, there seems no active reasoning that the δ expansion really improve the conventinal perturbation for $D \geq 2$.

Chapter 7

Discussion and summary

In this thesis, we investigated the convergence conditions for the ODM of type AHO in completely different way from those by other authors [27–33]. In chapter 3, we analyzed the resummed series directly under the same assumptions (I)-(III) of Theorem 1 and restrict the consideration in the region Eq.(3.16) and the choice $\gamma < \alpha$ (remember that these two are strongly connected with each other). Under these assumptions we obtained the neccessary and sufficient conditions for the convergence of the resummed series, (A) and (B) of chapter 3. Since the sufficient conditions for the convergence to the *true* answer consist of (A), (B) and the restriction Eq.(1.4), one can conclude that the neccessary and sufficient conditions for the convergence to the *true* answer lies between the two conditions; (A), (B) and the restriction on α between $0 < b < \alpha$ and $\max\{1, b\} < \alpha \leq 2$. The appearance of $\alpha \leq 2$ has an enough reason if we analyze the convergence by the analytic properties. The region Eq.(3.16) is equivallent to

$$|\arg g| < \frac{\alpha}{2}\pi. \tag{7.1}$$

Then, under the assumptions (I)-(III), one can only use the analyticity in the cut plane and hence may say about the convergence only in the cut plane, which means from Eq.(7.1) $\alpha \leq 2$. The present analyses in chapter 3 assure that if the analyticity region is known to be wider, the convergence to the *true* answer in the region Eq.(3.16) with $\alpha > 2$ is possible.

In chapter 4, we considered the case where the function under consideration is known to have a specific analyticity region in addition to the cut plane, that is, the case where the function possesses a convergent strong coupling expansion, in connection with Theorem 2 of Ref. [33]. By investigating the structure of the resummed series, there are two cases where wider convergence region than Eq.(3.16) is possible, (i) and (ii) in chapter 4. The former corresponds to the case where the convergence radius of the strong coupling expansion is infinity, or equivalently, the function under consideration is analytic in the whole Riemann surface except the origin. In this case, it was argued that the condition $\gamma < \alpha$ would not be necessary and the restriction on α (if there exists) immediately affects not only the convergence in the region Eq.(3.16) but also that in the whole Riemann surface. Therefore, if the restriction $\alpha \leq 2$ exists in the region Eq.(3.16), the restriction is also valid in the whole Riemann surface. On the contrary, as has been just stated above, the analyticity in the higher Riemann sheets has a possibility to surmount the restriction $\alpha > 2$.

To examine the possibility, we investigated in chapter 5 the 0-dimensional AHO as an extreme example which is analytic in the whole Riemann surface except the origin and where analytic study is almost available. We analyzed these models in a completely different way from that in chapter 3; realization of the coefficients of the strong coupling expansion. The conclusions are that both the condition $\gamma < \alpha$ and the restriction on α is indeed not necessary and therefore the possibility mentioned above is confirmed at least in one example.

If one comes back to the case (ii) of chapter 4, one finds that there still exists a reason which prevents one from surmounting the restriction $\alpha \leq 2$. The case (ii) would correspond to the situation where a strong coupling expansion exists but the convergence radius is *finite*. In this case, there is a possibility where one of the singularities, say, $g = g_0$ lies in the sector $\pi < |\arg g_0| < \pi + \epsilon$ for any $\epsilon > 0$. If this is the case, $\alpha > 2$ may be impossible in view of analyticity consideration.

In conclusion, we find that under the assumption (I)-(III) of Theorem 1, the convergence conditions such as $\gamma > b$, $\gamma < \alpha$ and the region Eq.(3.16) are consequence of the structure of the resummed series while the restriction $\min\{1, b\} < \alpha \leq 2$ comes from purely analytic properties of the quantity concerned. Therefore, although it is possible to achieve the convergence in the case of $b \geq 2$, as is the sextic AHO in quantum mechanics, more information on the analyticity region in the higher Riemann sheets or on the discontinuity on the cut is needed to establish the convergence to the *true* answer.

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Appendix A

The optimization conditions

In this appendix, we summarize and show the closed forms for the two optimization conditions, say, the fastest apparent convergence (FAC) and the principle of minimal sensitivity (PMS). The FAC condition is

$$S^{[N]}(\rho) - S^{[N-1]}(\rho) = 0. \quad \Leftrightarrow \quad P^{[N]}(\rho) = 0. \tag{A.1}$$

The spirit of the FAC is as follows. Since the series expansion in powers of λ is also asymptotic by the property Eq. (2.3), the dominant contribution comes from the last term in the truncated series in general. Therefore, the order by order applications of the FAC condition will render the dominant divergent behavior supressed in all orders calculated [35]. One of the advantages over the PMS is facility of the calculation. Representations for the FAC condition can be easily obtained by any representations for $S^{[N]}$ or $P^{[N]}$ in chapter 2. For example, one obtains from Eq.(2.9) with (2.7),

$$\sum_{p=0}^{N} c^{[p]} \rho^p \oint_{C_0} \frac{dw}{2\pi i} \frac{F(w)^p}{w^{N+1} f(w)} = 0. \tag{A.2}$$

On the other hand, the PMS condition is

$$\frac{\partial S^{[N]}(\rho)}{\partial \rho} = 0. \tag{A.3}$$

The spirit of the PMS is easy to understand intuitively. Since the original Hamiltonian does not involve any adjustable parameters and hence any physical quantities do not depend on them, the quantities calculated by the method should also be independent of them. Representations for the PMS condition are also obtained using the same results in chapter 2. From the most abstract form Eq.(2.9), the PMS condition reads

$$\sum_{s=0}^{N}\left[\frac{f'(\lambda)}{f(\lambda)}p^{[s]}(\rho)+\frac{d\rho}{d\lambda}p^{[s]\prime}(\rho)+\frac{s}{\lambda}\right]\lambda^{s}=0, \tag{A.4}$$

where the prime denotes the derivative with respect to each argument. If one substitutes Eq.(2.7) into the above, one yields

$$\sum_{s=0}^{N}\lambda^{s}\sum_{p=0}^{s}c^{[p]}\rho^{p}\left[\frac{f'(\lambda)}{f(\lambda)}-p\frac{F'(\lambda)}{F(\lambda)}+\frac{s}{\lambda}\right]\oint_{C_0}\frac{dw}{2\pi i}\frac{F(w)^{p}}{w^{s+1}f(w)}=0. \tag{A.5}$$

From the general representations Eqs.(A.2) and (A.5), one can obtain the closed forms for type AHO, which have been also reprersented in Ref. [33]. Using the Eq.(2.10) for $F(\lambda)$ and $f(\lambda)$, and noting that the integral in Eqs.(A.2) and Eq.(A.5) in the case of type AHO is just the same as Eq.(2.23), one immediately obtains for the FAC,

$$\sum_{p=0}^{N}c^{[p]}\frac{(\alpha p-\sigma)_{N-p}}{(N-p)!}\rho^{p}=0. \tag{A.6}$$

For the PMS, the square bracket in Eq.(A.5) reads

$$\left[\frac{f'(\lambda)}{f(\lambda)}-p\frac{F'(\lambda)}{F(\lambda)}+\frac{s}{\lambda}\right]=\frac{1}{1-\lambda}\Big[(\alpha p-\sigma+s-p)-(s-p)\lambda^{-1}\Big],$$

and thus Eq.(A.5) is equivallent to

$$\sum_{s=0}^{N}\lambda^{s}\sum_{p=0}^{s}c^{[p]}\frac{(\alpha p-\sigma)_{s-p+1}}{(s-p)!}\rho^{p}-\sum_{s=1}^{N}\lambda^{s-1}\sum_{p=0}^{s-1}c^{[p]}\frac{(\alpha p-\sigma)_{s-p}}{(s-p-1)!}\rho^{p}=0.$$

Arranging the above sum, one finally gets,

$$\sum_{p=0}^{N}c^{[p]}\frac{(\alpha p-\sigma)_{N-p+1}}{(N-p)!}\rho^{p}=0, \tag{A.7}$$

which has a strong resemblance to the FAC Eq.(A.6) in the form. Of course, the results Eq.(A.6) and Eq.(A.7) can be obtained directly from Eq. (2.12) as have been demonstrated in Ref. [33]. As has just be seen above, the final expression of the PMS condition of type AHO does not involve the summation symbol with respect to s and hence does not depend on g. This point was first pointed out by Janke and Kleinert for the special case $\alpha = 3/2$ and $\sigma = 1/2$ [48]. Under what conditions on the choice $F(\lambda)$ and $f(\lambda)$ the same phenomenon occurs will be an interesting problem.

Appendix B

The convergence condition at the boundary

Here we only represent the result of Ref. [33] for the boundary of the convergence region $\gamma = b$. As will be seen below, the precise estimate for the upper bound of ρ_1 involves quite complicated analyses.

Theorem 3. *Theorem 1 and Theorem 2 hold if (maintaining the other respective hypotheses) we scale the parameter ρ, instead of Eq.(1.7) as*

$$\rho_N = \rho_1 N^{-b}, \tag{B.1}$$

that is, $\gamma = b$, and

$$0 < \rho_1 < \left(\frac{a}{u_*}\right)^b, \tag{B.2}$$

where u_* is a solution of the following equation,

$$\Phi\Big(\lambda_*(u_*), u_*\Big) = 0 \tag{B.3}$$

with

$$\Phi(\lambda, u) = \ln \lambda + u\frac{(1+\lambda)^{a/b}}{\lambda^{1/b}} \tag{B.4}$$

and $\lambda_*(u)$ is a saddle point of Φ defined by

$$\frac{\partial}{\partial\lambda}\Phi\Big(\lambda_*(u), u\Big) = 0. \qquad \text{(B.5)}$$

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