The Hydrodynamical Formulation of Quantum Mechanics and the Two-Dimensional Parabolic Potential Barrier

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Abstract

The dissertation deals with the two-dimensional isotropic parabolic potential barrier as a solvable model of a two-dimensional unstable system in non-relativistic quantum mechanics. The time-independent Schrödinger equation for this model is set up as the eigenvalue problem in Gel'fand triplet and its exact solutions are expressed by generalized eigenfunctions belonging to complex energy eigenvalues. There are stationary states with a real energy eigenvalue involved in those solutions, and they are infinitely degenerate. A physical picture of the generalized eigenstates of the unstable system is obtained from the hydrodynamical point of view. For the first few stationary states of the two-dimensional parabolic potential barrier, the complex velocity potentials in the hydrodynamical formulation of quantum mechanics express the two-dimensional irrotational flows round a right angle. We find that this result can be deduced from the general connection between complex velocity potentials and singular potentials in two-dimensional quantum systems.

Contents

Chapter 1

Introduction

QUANTUM mechanics has been developed continuously from the last century[†] and applied to an ever-widening range of dynamical systems, including the gauge fields in interaction with matter [1–3]. The formulation of the underlying ideas and the laws requires the use of the mathematics of Hilbert spaces [4]. In fact, the closed states of a stable system form a Hilbert space and have real discrete eigenvalues. However, we see the need for extending this space when we are dealing with the eigenstates belonging to continuous spectrum. For example, the Dirac δ function [1] that plays an important part in the normalization of these eigenstates, is not a quantity which can be well defined in the theory of Hilbert spaces, but its definition requires the theory of distributions [5] or generalized functions [6]. If we did not extend the Hilbert spaces, the phenomenon of continuous spectrum could not occur and our formulation would be too weak for most practical problems. On these lines it has been found possible to set up a new formulation [7–12], called a *rigged Hilbert space* or a *Gel'fand triplet* [6], which is a more general space and which is more suitable for the description of collision processes than a Hilbert space. In this new formulation any Hilbert space $\mathcal{H}$ can be extended to a Gel'fand triplet $\Phi \subset \mathcal{H} \subset \Phi^\times$, where Φ is a nuclear space whose properties are analogous to a finite-dimensional space, and $\Phi^\times$ is the conjugate space of Φ, i.e. the set of bounded anti-linear functionals on Φ. With this extension, the plane waves that are required to represent stationary states of the whole system of scatterer plus particle, are not in a Hilbert space, but in the conjugate space of a Gel'fand triplet. Further, Bohm and Gadella [11] suggested by the analysis

[†]This year marks the 100th anniversary of Dirac's birth.

of S-matrices that the states belonging to continuous spectrum or complex eigenvalues are expressible in terms of the *generalized functions*, and their time evolution associates with a one-parameter semigroup. These possibilities are due to the change of boundary conditions which are determined by the choice of the nuclear space of a Gel'fand triplet.

The idea of the complex energy eigenvalue $E \in \mathbb{C}$ arises when our dynamical system is an unstable system, that is, when the potential energy $V(x)$ does not have a stable stationary point. However, since the Hamiltonian $\hat{H}$ is a self-adjoint operator on a Hilbert space $\mathcal{H}$, the eigenfunction $u(x)$ of $\hat{H}$ for the unstable system is not expressible in terms of the vectors of $\mathcal{H}$. The physical reason why $u \notin \mathcal{H}$ is brought out clearly if we consider the resonance scattering in which the complex energy eigenvalues play an important part. Since a state of resonance is only approximately stationary, its mean lifetime will be finite. This means the probability of the particle being inside the system tends to zero after a sufficient lapse of time, and hence the eigenfuction $u(x)$ remains finite at $|x| \to \infty$. Such a state is called a *non-stationary state* [1, 3] or a *quasi-stationary state* [2]. It is therefore necessary to use the mathematics of Gel'fand triplets in order to describe the generalized eigenstates for the unstable system adequately. But since the generalized eigenstates form a over-complete system and are not normalized, there is a serious difficulty to abandon the interpretation in terms of probabilities based on the states and observables [1]. The question we must now consider is: How are we to acquire a physical interpretation of these generalized eigenstates?

The main object of the present dissertation is to give a physical picture of the generalized eigenstates for a two-dimensional unstable system from the point of view of the *Hydrodynamical Formulation of Quantum Mechanics*. Classical hydrodynamics enables one to account for the natural phenomena on the macroscopic scale [13–15]; in the case of meteorological phenomena, for instance, typhoons developed from the vortices of the atmosphere, ocean currents driven by the circulations of the oceans and so on. For the microscopic scale, on the other hand, some kind of *velocity* was introduced in quantum mechanics by Madelung [16], and the connection between hydrodynamics and quantum mechanics was vigorously investigated in the earlier stage of the development of quantum mechanics [17–23]. A description of the dynamical properties of superconductors and superfluids was given by the hydrodynamical formulation on second quantization [14, 24].

From the chemical side the connection between the nodes of the wave function and *quantized vortices* was carefully examined by Hirschfelder [25–28]. A review article including these works is given by [29]. It is also noticed that such a velocity and a vortex are still useful in present-day quantum physics [30–34].

Important hydrodynamical variables in usual quantum theory are the probability density and the probability current [1–3]. For quantum problems, in particular, connected with the time-independent wave function which is not normalized, such as the free particle or the stationary states of collision problems, one can obtain physical results in terms of the ratios of the probability densities or the probability currents in the initial and final states. In picturing a generalized eigenstate of an unstable system from the hydrodynamical point of view, however, it is more convenient to work with the velocity instead of the probability current. This is because the velocity has the following two advantages over the probability current, namely, (i) it has no ambiguity arising from the normalization of the state, while the probability current cannot avoid it, and (ii) it does not depend on time even for a non-stationary state, whereas the probability current generally depends on time. These fundamental properties mean that we cannot speak of the probability current having an absolute meaning for a generalized eigenstate of an unstable system, but we can speak of the velocity having it for the state. Thus the velocity can be a good quantity in the unstable system. For the problem of a two-dimensional unstable system, in order to display a stream line (its tangent line at each point being given by the velocity) we can go further and introduce the *complex velocity potential* into non-relativistic quantum mechanics [32, 33]. The complex velocity potential $W(z)$ is a mathematical tool which expresses an irrotational flow of an incompressible perfect fluid in the two-dimensional xy-plane in hydrodynamics [13–15]. $W(z)$ is a regular function of the complex variable $z = x + iy$ and its singularities correspond to the hydrodynamical concepts such as two-dimensional source, sink, vortex, etc. Any two-dimensional irrotational flow is formed by *superposition of complex velocity potentials*. The stream lines having a complex boundary condition can be mapped on those having a simple boundary condition under *conformal transformations*. Thus the problem of two-dimensional irrotational flow of an incompressible perfect fluid is in accordance with the theory of functions of a complex variable. For this reason the problems of two-dimensional unstable system, if fit in with the hydrodynamical formulation of quantum mechanics, will have mathematical beauty.

As a model Hamiltonian of the two-dimensional unstable system in non-relativistic quantum mechanics, we take the 2D PPB (two-dimensional parabolic potential barrier) defined by the potential energy $V(x,y) = V_0 - m\gamma^2(x^2+y^2)/2$. We will investigate this model on the same lines as the 2D HO (two-dimensional harmonic oscillator) [32]. It is well known that the 2D HO is equivalent to the dynamical system consisting of two independent 1D HOs—the energy eigenvalues of the 2D HO are given by the sum of the energy eigenvalues of the 1D HOs and the eigenstates of the system are given by the product of the eigenstates of the 1D HOs. When degenerate eigenstates of the 2D HO are superposed with suitable weights, the new states will be the eigenstates of orbital angular momentum. These results were studied a long time ago by Dirac [35].

The 2D PPB is also equivalent to the dynamical system consisting of two independent 1D PPBs. The collision problem in the 1D PPB was first solved by Kemble [36] by the method of the WKB approximation. Various authors [37–45] have from time to time studied the model of the 1D PPB for revealing the motion of non-stationary states. The eigenvalue problem of the 1D PPB can be solved exactly by a operator method [41, 43, 44, 46] and can be adapted to the concept of supersymmetry which gives a cornerstone in the supersymmetric quantum mechanics of scattering [47]. Discussions similar to the present model were performed by the method of complex scaling [48, 49] and also in terms of the complex HO [50, 51]. In the 1D PPB defined by the potential energy $V(x) = V_0 - m\gamma^2 x^2/2$, we will show that the energy eigenvalues are complex numbers and the corresponding eigenfunctions are expressible in terms of the generalized functions of a Gel'fand triplet [44, 45]. It should be noticed that all energy eigenvalues appear in the pairs of conjugate complexes $V_0 \mp i(n+1/2)\hbar\gamma$ (n a non-negative integer), which, respectively, correspond to the resonance states of decay and growth. Under the assumption that the time factor $T(t) = Ae^{-iEt/\hbar}$ (A a complex number) of an unstable system is square integrable, we provide a probabilistic interpretation of it [45]. This assumption leads to the separation of the domain of the time evolution, namely all the time factors belonging to the complex energy eigenvalues in the lower half-plane [$V_0 - i(n+1/2)\hbar\gamma$, as the 1D PPB case] exist on the future part and all those belonging to the complex energy eigenvalues in the upper half-plane [$V_0 + i(n+1/2)\hbar\gamma$, as the same case] exist on the past part. In this scheme the physical energy distributions worked out from these time factors are found to be the Breit-Wigner resonance formulas. The half-widths of these resonance formulas are determined by the imaginary parts of complex energy eigenvalues.

In two dimensions, however, the exact solutions of the eigenvalue problem have much more variety. We will see that they are separated into four types [32]. Two of the four types, that are expressed by the generalized eigenfunctions belonging to complex (not real) energy eigenvalues, represent diverging and converging flows. In these two types the solutions will be the eigenstates of orbital angular momentum as same as the 2D HO. In the other two of the four types all the solutions are *infinitely degenerate* and involve the special solutions with the real energy eigenvalue V_0, which are *stationary* and do not grow or decay with time. It is a striking result that the eigenstates of the 2D PPB can involve stationary states, while no such state exists in the 1D PPB. One would expect to be able to get a more direct solution of the eigenvalue problem of the 2D PPB by working all the time in the two-dimensional polar coordinates, instead of working in the Cartesian coordinates and transforming at the end to the two-dimensional polar coordinates, as will be done in this dissertation, but under suitable boundary conditions in the two-dimensional polar coordinates one would obtain only the former two of the four types, i.e. not the latter two. It is also pointed out that one can get only the former two types from the analytical continuation of the solutions of the 2D HO. In dealing with the eigenvalue problem of the unstable system in non-relativistic quantum mechanics, it is important to remember that coordinate systems impose a restriction on the symmetry of boundary conditions. The source of this feature lies in the existence of a very large class of solutions for the unstable system.

The situation giving rise to stationary states in the 2D PPB may be well understood in comparison with the results of classical mechanics for this system. That is to say, for the 1D PPB, the Newton equation of motion is $\ddot{x}(t) = \gamma^2 x(t)$ and then the fundamental solutions give $x^{\pm}(t) = x^{\pm} e^{\pm\gamma t}$, where $x^{\pm}$ are real numbers. These solutions, $x^{+}(t)$ and $x^{-}(t)$, respectively, represent the diverging solution (tends to infinity) and the converging one (tends to zero). In two dimensions we find two different orbits expressed by $x^{\pm}(t)/y^{\pm}(t)$ = constant and $x^{\pm}(t)y^{\mp}(t)$ = constant. Now it is transparent that the former two of the four types of quantum solution of the 2D PPB correspond to the linear orbits $x^{\pm}(t)/y^{\pm}(t)$ = constant and the latter two involving the stationary states to the hyperbolic orbits $x^{\pm}(t)y^{\mp}(t)$ = constant. We may say that linear orbits such that $x^{\pm}(t)$ and $y^{\pm}(t)$ go toward the origin as $t \to \mp\infty$ correspond to the states of resonance in quantum mechanics, whereas the hyperbolic orbits which cannot pass through the origin correspond to the scattering states in collision problems.

The appearance of infinite degeneracy in the 2D PPB is another difference. They enable us to write the states in many various expressions in terms of superposition of the infinitely degenerate states. We can actually see that quantized vortices, which are connected with the nodal singularities of the superposed wave function, appear [32–34]. Further, the existence of the infinite degeneracy can possibly play an interesting role in statistical mechanics on Gel'fand triplets [52–54]. These considerations indicate that the analysis of the 2D PPB will give us an interesting insight for understanding of the quantum theory of two-dimensional unstable systems.

The organization of this dissertation is as follows: in Chapter 2 we build up the hydrodynamical formulation of quantum mechanics. Elementary applications of this formulation are given at the latter half of the chapter. In Chapter 3 we consider the problem of the 1D PPB. The eigenvalue problems of the 1D PPB are worked out in § 3.1 both in the coordinate representation and in the momentum representation. We show that the energy eigenvalues are complex numbers and the corresponding eigenfunctions are expressible in terms of the generalized functions of a Gel'fand triplet. The motion in the 1D PPB is considered in § 3.2. We obtain the connection between the complex energy eigenvalues and the Breit-Wigner resonance formulas. In Chapter 4 we study the problem of the 2D PPB. The eigenvalue problems of the 2D PPB are worked out in § 4.1 with the help of the work on the 1D PPB. We find that the generalized eigenstates belonging to a real energy eigenvalue exist in the solutions of the 2D PPB and these stationary states are infinitely degenerate. The flows in the 2D PPB are studied in § 4.2 based on the hydrodynamical formulation of quantum mechanics. We find that, for the first few stationary flows, the corresponding complex velocity potentials describe the two-dimensional irrotational flows round a right angle. In Chapter 5, using the techniques of conformal transformations, we show the connection between complex velocity potentials and singular potentials in two dimensions. We find that the result for the stationary flows in the 2D PPB is a special case of this connection. Finally, Chapter 6 is devoted to our conclusion.

Chapter 2

The Hydrodynamical Formulation of Quantum Mechanics

WE shall begin to build up the hydrodynamical formulation of quantum mechanics by dealing with the probability density and the probability current. Let us call the probability density $\rho(t, \boldsymbol{r})$ and the probability current $\boldsymbol{j}(t, \boldsymbol{r})$ of a state $\psi(t, \boldsymbol{r})$ in non-relativistic quantum mechanics, which are defined by [1–3]

$$\rho(t, \boldsymbol{r}) \equiv |\psi(t, \boldsymbol{r})|^2 , \tag{2.1}$$

$$\boldsymbol{j}(t, \boldsymbol{r}) \equiv \Re\left[\psi(t, \boldsymbol{r})^* \left(-i\hbar\nabla\psi\right)(t, \boldsymbol{r})\right]/\, m, \tag{2.2}$$

where m is the mass of the particle. They satisfy the equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0. \tag{2.3}$$

However, these ρ and $\boldsymbol{j}$ are independent of the time t when the considered state is a closed state of the stable system (e.g. the HO, the hydrogen atom) or a stationary state (e.g. the free particle). For such closed or stationary states the probability currents will be at rest or uniform, and equation (2.3) reduces to

$$\nabla \cdot \boldsymbol{j} = 0.$$

Thus the probability currents are simple for the closed or stationary states of the stable system. In this chapter we shall deal with some kind of velocity, which is a more important concept in classical hydrodynamics and which is a stepping-stone to a precise description in quantum mechanics.

Velocities

We start out with the equations of hydrodynamics, consisting of Euler's equation of continuity for the density and velocity of a fluid and so on. Let us try to introduce a velocity which will be the analogue of the hydrodynamical one. In hydrodynamics [13–15], the fluid at one time can be represented by the density ρ and the fluid velocity $\boldsymbol{v}$. They satisfy Euler's equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0. \tag{2.4}$$

Comparing this equation with (2.3), we are thus led to the following definition for the quantum velocity of a state $\psi(t, \boldsymbol{r})$:

$$\boldsymbol{v} \equiv \frac{\boldsymbol{j}(t, \boldsymbol{r})}{|\psi(t, \boldsymbol{r})|^2}, \tag{2.5}$$

in which $\boldsymbol{j}(t, \boldsymbol{r})$ is given by (2.2). It is stressed that this velocity is different from the eigenvalue of the velocity operator $\hat{\boldsymbol{v}} = -i\hbar\nabla/m$, except for the cases where $\psi(t, \boldsymbol{r})$ is an eigenfunction of the momentum operator $\hat{\boldsymbol{p}} = -i\hbar\nabla$. Note that, if we can separate variables of $\psi(t, \boldsymbol{r})$, then $\boldsymbol{v}$ does not contain the time t explicitly.

Equation (2.5) is justified from the following point of view. In semiclassical cases the time-dependent wave function can be written [1–3]

$$\psi(t, \boldsymbol{r}) = \sqrt{\rho}\, e^{iS/\hbar}, \tag{2.6}$$

where ρ is the probability density (2.1) and S is the quantum analogue of the classical action, which is a real function of t and $\boldsymbol{r}$. The velocity (2.5) gives

$$\boldsymbol{v} = \nabla S/m. \tag{2.7}$$

The right-hand side is known just as the velocity in classical dynamics. Madelung [16] introduced the velocity by this relation (2.7).

Vortices

We shall now consider the vorticity in quantum mechanics. It is defined, as in hydrodynamics, by

$$\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{v}. \tag{2.8}$$

With the above definition we have

$$\boldsymbol{\omega} = 0 \tag{2.9}$$

for the domain in which the wave function does not have nodes. Using standard formulas of vector analysis, equation (2.9) is verified with the help of (2.7), in semiclassical cases. However, equation (2.9) does not hold when the velocity has singularities. From the definition (2.5), in the nodal region where the wave function vanishes, a vortex may exist [26, 28]. The strength of the vortex is characterized by the circulation round a closed contour C encircling the nodal singularity

$$\Gamma \equiv \oint_C \boldsymbol{v} \cdot d\boldsymbol{s}. \tag{2.10}$$

We make use of Stokes' theorem,

$$\Gamma = \iint_S \boldsymbol{\omega} \cdot d\boldsymbol{S}, \tag{2.11}$$

where S is a two-dimensional surface whose boundary is the closed contour C. The wave function (2.6) is required to have a single value at each point, so the circulation (2.10) becomes, with the help of (2.7) [26, 28, 55–57],

$$\Gamma = 2\pi n\hbar/m, \tag{2.12}$$

where n is an integer. This result informs us that *the circulations are quantized in units of* $2\pi\hbar/m$ *for the state* (2.6) *in semiclassical cases.* Equation (2.12) is known as *Onsager's Quantization of Circulations* [58]. Examples of this quantized vortex will be dealt with in Example 3 and § 4.2.1.

In hydrodynamics [13–15], the flow satisfying $\boldsymbol{\omega} = 0$ is called potential flow or irrotational flow. Thus the result (2.9) asserts that *the quantum flow is irrotational flow except for the nodal singularities.* The velocity in irrotational flow satisfying (2.9) may be described by the gradient of the velocity potential Φ,

$$\boldsymbol{v} = \nabla\Phi. \tag{2.13}$$

Comparing this with (2.7), we see that

$$\Phi = S/m, \tag{2.14}$$

in semiclassical cases. The right-hand side here is undefined to the extent of an arbitrary additive real constant. On substituting (2.13) in (2.10), the circulation becomes

$$\mathit{\Gamma} = [\mathit{\Phi}]_C\,, \tag{2.15}$$

$[\mathit{\Phi}]_C$ being the change in $\mathit{\Phi}$ round the closed contour C. Applied to the velocity potential (2.14) it gives (2.12) again.

Complex velocity potentials

We now proceed to study only the two-dimensional or plane flow. Let us consider the velocity (2.5) which is solenoidal, namely

$$\nabla \cdot \boldsymbol{v} \equiv \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \tag{2.16}$$

The velocity in two-dimensional flow satisfying (2.16) may be described by the rotation of the stream function $\mathit{\Psi}$,

$$v_x = \frac{\partial \mathit{\Psi}}{\partial y}, \quad v_y = -\frac{\partial \mathit{\Psi}}{\partial x}. \tag{2.17}$$

The magnitude of the vorticity $\boldsymbol{\omega}$ satisfies, with the help of (2.17),

$$\omega = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = -\nabla^2 \mathit{\Psi}, \tag{2.18}$$

where ∇^2 is written for the two-dimensional Laplacian operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$. The solenoidal condition (2.16) holds for incompressible fluids in hydrodynamics, since ρ is a constant. In quantum mechanics, however, the probability density (2.1) depends generally on $\boldsymbol{r}$, so the velocity (2.5) is not always solenoidal.

Further, in the two-dimensional irrotational flow, by combining (2.17) with (2.13) we get

$$v_x = \frac{\partial \mathit{\Phi}}{\partial x} = \frac{\partial \mathit{\Psi}}{\partial y},$$
$$v_y = \frac{\partial \mathit{\Phi}}{\partial y} = -\frac{\partial \mathit{\Psi}}{\partial x}.$$

These are known as the Cauchy-Riemann equations between the velocity potential and the stream function. We can therefore take the complex velocity potential

$$W(z) = \mathit{\Phi}(x, y) + i\mathit{\Psi}(x, y), \tag{2.19}$$

which is a regular function of the complex variable $z = x + iy$. The complex velocity potential then expresses the two-dimensional irrotational flow, with the lines $\Phi(x, y) =$ constant and $\Psi(x, y) =$ constant determining the equipotential lines and stream lines, respectively. The differentiation of $W(z)$ gives us the complex velocity

$$\frac{dW}{dz} = v_x - iv_y. \tag{2.20}$$

In this way we know that in the two-dimensional irrotational flow it is advantageous to use the theory of functions of a complex variable [13–15]. As a simple example of $W(z)$, we may consider a power

$$W = Az^a, \tag{2.21}$$

where A is a number and a is an integer. According to hydrodynamics [13–15], this expresses the flow round the angle π/a for positive integer a, e.g. the uniform flow for $a = 1$, the flow round a right angle for $a = 2$, and so on. On the other hand, this expresses the two-dimensional $2^{|a|}$-pole or the two-dimensional multipole for negative integer a, e.g. the two-dimensional doublet for $a = -1$, the two-dimensional quadrupole for $a = -2$, and so on. The form (2.21) plays an important part in the quantum theory of two-dimensional singular potentials (see Chapter 5).

We shall now illustrate some elementary examples of complex velocity potentials which express the two-dimensional flows in quantum mechanics.

Example 1. The free particle

Let us first consider the free particle in two dimensions as an example of a stationary state. The plane wave is of the form

$$U_{k_x k_y}(x, y) = A_{k_x k_y} e^{i(k_x x + k_y y)}, \tag{2.22}$$

where $A_{k_x k_y}$ is independent of t, x and y. The probability current of the plane wave is

$$j_x(x, y) = |A_{k_x k_y}|^2 \hbar k_x/m, \quad j_y(x, y) = |A_{k_x k_y}|^2 \hbar k_y/m,$$

and hence their velocity is

$$v_x = \hbar k_x/m, \quad v_y = \hbar k_y/m.$$

Equations (2.9) and (2.16) are easily seen to hold for the plane wave. Therefore the velocity potential satisfying (2.13) or (2.14) of the plane wave is

$$\Phi = \hbar(k_x x + k_y y)/m, \tag{2.23}$$

and the stream function satisfying (2.17) is

$$\Psi = \hbar(k_x y - k_y x)/m. \tag{2.24}$$

For the state represented by (2.22), the complex velocity potential (2.19) gives, from (2.23) and (2.24),

$$\begin{aligned} W &= \hbar(k_x x + k_y y)/m + i\hbar(k_x y - k_y x)/m \\ &= \hbar(k_x - ik_y)z/m. \end{aligned} \tag{2.25}$$

Equation (2.25) is of the form (2.21) with $a = 1$, and it shows that *the complex velocity potential of the plane wave just expresses uniform flow.*

Example 2. The harmonic oscillator

Let us consider the eigenstate of the 2D HO as an example of a closed state of the stable system. The eigenfunction is, in terms of the Cartesian coordinates x, y,

$$U_{n_x n_y}(x, y) = N_{n_x} N_{n_y} e^{-\alpha^2(x^2+y^2)/2} H_{n_x}(\alpha x) H_{n_y}(\alpha y) \tag{2.26}$$

from the result (A.7) of Appendix A. Since the Hermite polynomials $H_{n_x}(\alpha x)$, $H_{n_y}(\alpha y)$ are real functions of x, y, respectively, the probability current of the HO vanishes and their velocity also vanishes. Therefore the velocity potential and the stream function all vanish. For the state represented by (2.26), the complex velocity potential gives

$$W = 0, \tag{2.27}$$

which expresses *fluid at rest* in hydrodynamics. This fluid at rest, however, is not the only one that is physically permissible for a closed state in quantum mechanics, as we can also have flows which are *vortical*. For these flows the vorticity may contain singularities in the xy-plane. Such flows will be dealt with in Example 3.

Example 3. Flows in a central field of force

We shall now consider the bound state in a certain central field of force. The eigenfunction is, in terms of the polar coordinates r, θ, φ,

$$U_{nlm_l}(r,\theta,\varphi) = R_{nl}(r)Y_{lm_l}(\theta,\varphi), \tag{2.28}$$

where the spherical harmonics $Y_{lm_l}(\theta,\varphi)$ are of the form

$$Y_{lm_l}(\theta,\varphi) = C_{lm_l}P_l^{|m_l|}(\cos\theta)e^{im_l\varphi}, \tag{2.29}$$

and C_{lm_l} are the normalizing constants. Since $R_{nl}(r)$ for the bound state and the associated Legendre polynomials $P_l^{|m_l|}(\cos\theta)$ are real functions, the polar coordinates j_r, j_θ, j_φ of (2.2) in a central field of force are

$$j_r(r,\theta,\varphi) = j_\theta(r,\theta,\varphi) = 0, \;\; j_\varphi(r,\theta,\varphi) = |u_{nlm_l}(r,\theta,\varphi)|^2\frac{m_l\hbar}{mr\sin\theta}.$$

In consequence, a simple treatment becomes possible, namely, we may consider the velocity for a definite direction θ and then we can introduce the radius $\rho = r\sin\theta$ in the above equations and get a problem in two degrees of freedom ρ, φ. The two-dimensional polar coordinates v_ρ, v_φ of (2.5) give

$$v_\rho = 0, \;\; v_\varphi = \frac{m_l\hbar}{m\rho}.$$

Their divergence readily vanishes. If we transform to two-dimensional polar coordinates ρ, φ, equations (2.17) become

$$v_\rho = \frac{1}{\rho}\frac{\partial\Psi}{\partial\varphi}, \;\; v_\varphi = -\frac{\partial\Psi}{\partial\rho}, \tag{2.30}$$

and the stream function in a central field of force is thus

$$\Psi = -\frac{m_l\hbar}{m}\log\rho. \tag{2.31}$$

On substituting (2.31) in (2.18) we obtain

$$\omega = \frac{m_l\hbar}{m}\nabla^2\log\rho = 2\pi\frac{m_l\hbar}{m}\delta(\boldsymbol{\rho}), \tag{2.32}$$

where $\delta(\boldsymbol{\rho})$ is the two-dimensional Dirac δ function. Thus the vorticity in a central field of force vanishes everywhere except the origin $\rho = 0$. This singularity will lie along the quantization axis $\theta = 0$ and π in three-dimensional space. The velocity potential satisfying (2.13) or (2.14) in a central field of force is

$$\varPhi = \frac{m_l \hbar}{m} \varphi. \tag{2.33}$$

For the state represented by (2.28), the complex velocity potential (2.19) gives, from (2.33) and (2.31),

$$\begin{aligned} W &= \frac{m_l \hbar}{m} \varphi - i \frac{m_l \hbar}{m} \log \rho \\ &= -i \frac{m_l \hbar}{m} \log z, \end{aligned} \tag{2.34}$$

since $z = \rho e^{i\varphi}$. According to hydrodynamics [15], this expresses the *vortex filament*. The strength of the vortex filament is defined by the circulation (2.10) round a closed contour C encircling the singularity at the origin $\rho = 0$. On substituting (2.32) in (2.11) we obtain

$$\varGamma = 2\pi \frac{m_l \hbar}{m}, \tag{2.35}$$

where m_l is an integer. This result is to be expected from (2.33) and (2.15). This is of exactly the same form as (2.12), and it shows that *the circulations are quantized for the state* (2.28) *moving in a central field of force*.

Example 4. Flows in the $-1/\rho^2$ potential

As a final example we shall consider the approximate solutions of the two-dimensional inverse square potential $V(\rho) = -g/\rho^2$ $(g > 0)$. The approximate solutions for the neighborhood of the center $\rho = 0$ are, according to (B.3) and (B.8) of Appendix B,

$$U_l^{\pm}(\rho, \varphi) \approx \rho^{\pm i\sqrt{\gamma}} e^{il\varphi} / \sqrt{2\pi} \tag{2.36}$$

with γ, defined by (B.6), is positive. The probability currents for small ρ are,

$$j_\rho^{\pm}(\rho, \varphi) \approx \pm \frac{\sqrt{\gamma}\hbar}{2\pi m\rho}, \quad j_\varphi^{\pm}(\rho, \varphi) \approx \frac{l\hbar}{2\pi m\rho},$$

and hence their velocities are

$$v_\rho^{\pm} \approx \pm \frac{\sqrt{\gamma}\hbar}{m\rho}, \quad v_\varphi^{\pm} \approx \frac{l\hbar}{m\rho}.$$

If $l = 0$, $v_\varphi^\pm$ vanish, so the velocity potentials satisfying (2.13) become

$$\Phi_0^\pm \approx \pm\frac{\sqrt{\gamma}\hbar}{m}\log\rho, \tag{2.37}$$

and the stream functions satisfying (2.30) become

$$\Psi_0^\pm \approx \pm\frac{\sqrt{\gamma}\hbar}{m}\varphi, \tag{2.38}$$

as discussed in the above examples. So the complex velocity potentials give

$$\begin{aligned} W_0^\pm &\approx \pm\frac{\sqrt{\gamma}\hbar}{m}\log\rho \pm i\frac{\sqrt{\gamma}\hbar}{m}\varphi \\ &\approx \pm\frac{\sqrt{\gamma}\hbar}{m}\log z. \end{aligned} \tag{2.39}$$

According to hydrodynamics [13, 15], W_0^+ and W_0^- express the *two-dimensional source and sink*, respectively. The strengths of these two-dimensional source and sink are defined by the flux across a closed contour C encircling the singularity at the origin $\rho = 0$

$$\begin{aligned} Q_0^\pm &\equiv \oint_C v_\rho^\pm ds \\ &\approx \pm 2\pi\frac{\sqrt{\gamma}\hbar}{m}. \end{aligned} \tag{2.40}$$

Now if $l \neq 0$, $v_\varphi^\pm$ are not zero and will then contribute to the vortex filament having a strength (2.35) with l written for m_l. For the states represented by (2.36), the complex velocity potentials are given by superposition of (2.39) and (2.34), thus

$$W^\pm \approx \left(\pm\frac{\sqrt{\gamma}\hbar}{m} - i\frac{l\hbar}{m}\right)\log z. \tag{2.41}$$

They express *logarithmic spiral vortices*.

The above-mentioned method will be applied to the flows in the 2D PPB, as an example of an unstable system, in § 4.2. A general application of the hydrodynamical formulation of quantum mechanics will be discussed in Chapter 5 under the heading of the connection between complex velocity potentials and singular potentials in two dimensions.

Chapter 3

The Parabolic Potential Barrier in One Dimension

3.1 Eigenvalue problems of the parabolic potential barrier

3.1.1 The coordinate representation

A SIMPLE and interesting model of an unstable system in quantum mechanics is the 1D PPB. The Hamiltonian in quantum mechanics is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_0 - \frac{1}{2}m\gamma^2 x^2, \tag{3.1}$$

where $V_0 \in \mathbb{R}$ is the maximum potential energy, $m > 0$ is the mass and $\gamma > 0$ is the Liapunov exponent in the classical theory [43]. This Hamiltonian can be obtained from the Hamiltonian of the 1D HO (A.1) with the replacement

$$\omega \to \mp i\gamma. \tag{3.2}$$

$\hat{H}$ is essentially self-adjoint on a Schwartz space $\mathcal{S}(\mathbb{R}_x)$ [5]. Also, the following two conditions are satisfied (see Appendix A):

(i) $\mathcal{S}(\mathbb{R}_x)$ is an invariant subspace of $\hat{H}$.

(ii) $\hat{H}$ is continuous on $\mathcal{S}(\mathbb{R}_x)$.

The time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}u(x) + \left(V_0 - \frac{1}{2}m\gamma^2 x^2\right)u(x) = Eu(x). \tag{3.3}$$

It is convenient to introduce the dimensionless variables

$$\xi \equiv \beta x, \quad \beta \equiv \sqrt{\frac{m\gamma}{\hbar}}, \tag{3.4}$$

$$\lambda \equiv \frac{2(E - V_0)}{\hbar\gamma}. \tag{3.5}$$

Equation (3.3) now becomes

$$\frac{d^2u}{d\xi^2} + \left(\lambda + \xi^2\right) u = 0. \tag{3.6}$$

Step 1. The asymptotic solutions

We shall now obtain the asymptotic solutions of equation (3.6). If we neglect λ in (3.6) altogether, approximate solutions for large $|\xi|$ are

$$u^{\pm}(\xi) \approx \xi^n e^{\pm i\xi^2/2}, \tag{3.7}$$

where n is an arbitrary number. Substituting (3.7) into (3.6), we see that *the values* $E^{\pm}$ *of* (3.5) *associated with* $u^{\pm}$ *are complex numbers*:

$$E^+ \in \mathbb{C}_-, \quad E^- \in \mathbb{C}_+,$$

where the symbols $\mathbb{C}_{\pm}$ have the meaning

$$\mathbb{C}_{\pm} \equiv \{z = x + iy \,|\, x \in \mathbb{R}, \;\; y \in \mathbb{R}_{\pm}\},$$
$$\mathbb{R}_+ \equiv [0, +\infty), \;\; \mathbb{R}_- \equiv (-\infty, 0].$$

Since $\hat{H}$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R}_x)$, $u^{\pm}$ do not belong to a Lebesgue space $L^2(\mathbb{R}_x)$. For such eigenfunctions of $\hat{H}$ belonging to the complex energy eigenvalues, one can use the Gel'fand triplet which is formed by the Schwartz space $\mathcal{S}(\mathbb{R}_x)$ as a nuclear space [10, 11],

$$\mathcal{S}(\mathbb{R}_x) \subset L^2(\mathbb{R}_x) \subset \mathcal{S}(\mathbb{R}_x)^{\times}. \tag{3.8}$$

Let us treat the extension $\hat{H}^{\times}$ of the Hamiltonian to the conjugate space $\mathcal{S}(\mathbb{R}_x)^{\times}$. We should be able to apply it to a generalized function $u \in \mathcal{S}(\mathbb{R}_x)^{\times}$, the product $\hat{H}^{\times}u$ being defined by [10, 11]

$$\langle v \,|\, \hat{H}^{\times}u \rangle = \langle \hat{H}v \,|\, u \rangle$$

for all functions $v \in \mathcal{S}(\mathbb{R}_x)$. Taking the x-representatives, we get

$$\int_{-\infty}^{\infty} v(x)^* \big(\hat{H}^\times u\big)(x)dx = \int_{-\infty}^{\infty} \big(\hat{H}v\big)(x)^* u(x)dx.$$

We can transform the right-hand side by partial integration and get

$$\int_{-\infty}^{\infty} v(x)^* \big(\hat{H}^\times u\big)(x)dx = \int_{-\infty}^{\infty} v(x)^* \big(\hat{H}u\big)(x)dx,$$

since v is a rapidly decreasing function and then the contributions from the limits of integration vanish. This gives

$$\big\langle v \,\big|\, \hat{H}^\times u \big\rangle = \big\langle v \,\big|\, \hat{H}u \big\rangle,$$

showing that

$$\hat{H}^\times u = \hat{H}u.$$

Thus $\hat{H}^\times$ operating to a generalized function has the meaning of $\hat{H}$ operating.

The asymptotic solutions (3.7) will then be generalized functions or tempered distributions in $\mathcal{S}(\mathbb{R}_x)^\times$. We may verify that $u^\pm \in \mathcal{S}(\mathbb{R}_x)^\times$ by the Gauss-Fresnel integral, e.g., for any $x^m e^{-\alpha^2 x^2/2} \in \mathcal{S}(\mathbb{R}_x)$ $(m \in \mathbb{Z}_+, \alpha \in \mathbb{R})$,

$$\int_{-\infty}^{\infty} \left(x^m e^{-\alpha^2 x^2/2}\right)^* x^n e^{\pm i\beta^2 x^2/2} dx = (m+n-1)!! \sqrt{\frac{2\pi}{(\alpha^2 \mp i\beta^2)^{m+n+1}}}$$

when $m+n$ is even and zero otherwise.

Let us examine physical properties of the asymptotic states. The result of the momentum operator $\hat{p} = -i\hbar d/dx$ applied to asymptotic solutions (3.7) is

$$-i\hbar \frac{d}{dx} u^\pm(x) \approx \pm \hbar \beta^2 x u^\pm(x). \tag{3.9}$$

Thus $u^+(x)$ represents particles moving outward to the infinity $|x| = \infty$, and $u^-(x)$ represents particles moving inward to the origin $x = 0$. We shall see in § 3.2.1 that these asymptotic behaviors are justified by those of the probability currents.

Step 2. The method of power series expansion

We shall work out the whole solutions of equation (3.6) by the method of power series expansion from (3.7). Put

$$u^\pm(\xi) = e^{\pm i\xi^2/2} H^\pm(\xi), \tag{3.10}$$

introducing two new functions $H^{\pm}(\xi)$. Equation (3.6) becomes

$$\frac{d^2H^{\pm}}{d\xi^2} \pm 2i\xi\frac{dH^{\pm}}{d\xi} + \left(\lambda^{\pm} \pm i\right) H^{\pm} = 0. \tag{3.11}$$

We now look for a solution of these equations in the form of power series†

$$H^{\pm}(\xi) = \xi^s \sum_{n=0}^{\infty} c_n^{\pm}\xi^n \quad \left(c_0^{\pm} \neq 0\right), \tag{3.12}$$

in which values of the leading order s need not be integers. Substituting (3.12) in (3.11) and picking out coefficients of ξ^n, we obtain

$$\begin{cases} s(s-1)c_0^{\pm} = 0, \\ (s+1)sc_1^{\pm} = 0, \\ (s+n+2)(s+n+1)c_{n+2}^{\pm} = \left[\mp i(2s+2n+1) - \lambda^{\pm}\right] c_n^{\pm}. \end{cases} \tag{3.13}$$

From the indicial equations of (3.13)

$$s = 0 \ \text{ or } \ 1. \tag{3.14}$$

We can investigate the convergence of the series (3.12) on the same lines as the 1D HO. The third of equations (3.13) give approximately, when n is large,

$$\frac{c_{n+2}^{\pm}}{c_n^{\pm}} \approx \mp i\frac{2}{n}.$$

The series (3.12) will therefore converge like

$$H^{\pm}(\xi) \approx e^{\mp i\xi^2}$$

for large values of $|\xi|$. Thus, from (3.10), we have

$$u^{\pm}(\xi) \approx e^{\mp i\xi^2/2}.$$

These results change the asymptotic behaviors in which $u^+(\xi)$ goes off to infinity and $u^-(\xi)$ arrives from infinity. There are therefore in general no permissible solutions of (3.3) for which the series (3.12) extend to infinity on the side of large n. The permissible

†One cannot infer that $u^{\pm}(\xi)$ have a definite parity in this step, even though the potential energy of (3.1) is a symmetrical of ξ, since $u^{\pm}(\xi)$ are generally degenerate [2, 3].

solutions, which have the asymptotic behaviors mentioned in (3.9), arise when the series (3.12) terminate on the side of large n. The conditions for this termination of the series are

$$c_1^{\pm} = 0 \tag{3.15}$$

and

$$\lambda^{\pm} = \mp\, i(2s + 2m + 1) \tag{3.16}$$

for some even m. In this case we get, from the recurrence formulas of (3.13),

$$c_1^{\pm} = c_3^{\pm} = \cdots = c_{m-1}^{\pm} = c_{m+1}^{\pm} = c_{m+2}^{\pm} = \cdots = 0.$$

Thus $H^{\pm}(\xi)$ are the polynomials of degree $s + m$. Since the only possible values of s are 0 or 1, we obtain from (3.16), by putting $s + m = n$,

$$E_n^{\pm} = V_0 \mp i\left(n + \frac{1}{2}\right)\hbar\gamma \in \mathbb{C}_{\mp} \quad (n = 0, 1, 2, \dots)\,. \tag{3.17}$$

This formulas give the *complex energy eigenvalues* for the 1D PPB. For generalized eigenstates of $\hat{H}$ belonging to the generalized eigenvalues $E_n^{\pm}$, the time-independent wave functions are

$$u_n^{\pm}(x) = B_n^{\pm} e^{\pm i\beta^2 x^2/2} H_n^{\pm}(\beta x) \in \mathcal{S}(\mathbb{R}_x)^{\times}, \tag{3.18}$$

where $H_n^{\pm}(\beta x)$ are the polynomials of degree n (see Appendix C), and $B_n^{\pm}$ are the numerical coefficients. These numerical coefficients cannot be determined by normalizing condition, since a scalar product of generalized functions is not defined. In general, a generalized function is not necessarily normalized [10]. However, we can determine $B_n^{\pm}$ with the help of the following argument. Comparing above results with (A.6) and (A.7), we see that $E_n^{\pm}$ and $u_n^{\pm}(x)$ are the same form as in the 1D HO from the analytical continuation (3.2) (see Appendix A). From (A.8), we therefore determine that

$$B_n^{\pm} = \left(\frac{\beta}{(\pm 2i)^n n! \sqrt{\pm i\pi}}\right)^{\frac{1}{2}} \tag{3.19}$$

with the correction from our definition of $H_n^{\pm}(\beta x)$ of Appendix C.

Let us now see the properties of time-independent wave functions under a space inversion. Property (C.4) of Appendix C shows that $u_n^{\pm}(x)$ are eigenstates of the parity,

$$u_n^{\pm}(-x) = (-)^n u_n^{\pm}(x).$$

The validity of this result depends on the behaviors of the asymptotic states.

The above results suggest that the eigenfunctions of the 1D PPB are *non-stationary states*. It is obvious physically that the observational energy spectrum of this system is real and continuous. But the energy eigenvalues (3.17) are complex and discrete. The connection between the real continuous spectrum of observed energies and the complex discrete eigenvalues of mathematical eigenvalue problem will be dealt with in § 3.2.2.

3.1.2 The momentum representation

The coordinate and the momentum representatives of a good function $f \in \mathcal{S}(\mathbb{R}_x)$ are connected by the Fourier transformations [1]

$$\begin{cases} \tilde{f}(p) = \big(\hat{\mathcal{F}} f\big)(p) \equiv \dfrac{1}{\sqrt{2\pi\hbar}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{-ipx/\square} dx, \\ f(x) = \big(\hat{\mathcal{F}}^{-1} \tilde{f}\big)(x) \equiv \dfrac{1}{\sqrt{2\pi\hbar}} \displaystyle\int_{-\infty}^{\infty} \tilde{f}(p) e^{ipx/\square} dp. \end{cases} \tag{3.20}$$

Let us treat the extension $\hat{\mathcal{F}}^{\times}$ of the Fourier transformation to the conjugate space $\mathcal{S}(\mathbb{R}_x)^{\times}$. We should be able to apply it to a generalized function $u \in \mathcal{S}(\mathbb{R}_x)^{\times}$, the product $\hat{\mathcal{F}}^{\times} u$ being defined by [5, 10, 11]

$$\big\langle v \,\big|\, \hat{\mathcal{F}}^{\times} u \big\rangle = \big\langle \hat{\mathcal{F}}^{-1} v \,\big|\, u \big\rangle$$

for all functions $v \in \mathcal{S}(\mathbb{R}_x)$. Taking the representatives, we get

$$\int_{-\infty}^{\infty} \tilde{v}(p)^* \big(\hat{\mathcal{F}}^{\times} u\big)(p) dp = \int_{-\infty}^{\infty} \big(\hat{\mathcal{F}}^{-1} \tilde{v}\big)(x)^* u(x) dx.$$

We can transform the right-hand side by the second of equations (3.20) and get

$$\int_{-\infty}^{\infty} \tilde{v}(p)^* \big(\hat{\mathcal{F}}^{\times} u\big)(p) dp = \int_{-\infty}^{\infty} \tilde{v}(p)^* \big(\hat{\mathcal{F}} u\big)(p) dp,$$

with the exchange of the order of the integrals by Fubini's theorem. This gives

$$\big\langle v \,\big|\, \hat{\mathcal{F}}^{\times} u \big\rangle = \big\langle v \,\big|\, \hat{\mathcal{F}} u \big\rangle,$$

showing that

$$\hat{\mathcal{F}}^{\times} u = \hat{\mathcal{F}} u.$$

Thus $\hat{\mathcal{F}}^{\times}$ operating to a generalized function has the meaning of $\hat{\mathcal{F}}$ operating.

We note that the Fourier transformation and its extension applied to (3.8) give a new Gel'fand triplet

$$\begin{array}{ccccc} \mathcal{S}(\mathbb{R}_x) & \subset & L^2(\mathbb{R}_x) & \subset & \mathcal{S}(\mathbb{R}_x)^{\times} \\ {\scriptstyle \hat{\mathcal{F}}}\big\downarrow & & {\scriptstyle \hat{\mathcal{F}}}\big\downarrow & & \big\downarrow{\scriptstyle \hat{\mathcal{F}}^{\times}} \\ \mathcal{S}(\mathbb{R}_p) & \subset & L^2(\mathbb{R}_p) & \subset & \mathcal{S}(\mathbb{R}_p)^{\times}, \end{array} \tag{3.21}$$

which is a unitary equivalent to (3.8) [5, 10, 11]. Written in terms of p-representatives, the Hamiltonian (3.1) gives

$$\hat{\mathcal{F}}\hat{H}\hat{\mathcal{F}}^{-1} = \frac{p^2}{2m} + V_0 + \frac{1}{2}m\gamma^2\hbar^2\frac{d^2}{dp^2}, \tag{3.22}$$

essentially self-adjoint on $\mathcal{S}(\mathbb{R}_p)$, and also the two conditions (i) and (ii) of § 3.1.1 are satisfied. For generalized eigenstates of (3.22) belonging to the generalized eigenvalues $E_n^{\pm}$, the time-independent wave functions are

$$\tilde{u}_n^{\pm}(p) = \tilde{B}_n^{\pm} e^{\mp i\tilde{\beta}^2 p^2/2} H_n^{\mp}(\tilde{\beta}p) \in \mathcal{S}(\mathbb{R}_p)^{\times}, \tag{3.23}$$

$$\tilde{B}_n^{\pm} = (-i)^n \left(\frac{\tilde{\beta}}{(\mp 2i)^n n! \sqrt{\mp i\pi}} \right)^{\frac{1}{2}}, \quad \tilde{\beta} \equiv \frac{1}{\sqrt{m\gamma\hbar}}. \tag{3.24}$$

Note that the phase factor $(-i)^n$ appears as a result of the Fourier transformations of $H_n^{\pm}$.

3.2 Motion in the parabolic potential barrier

3.2.1 The time evolution

Our work in § 3.1 has been concerned with one instant of time. The present section will be devoted to the variation with time of the non-stationary states.

We can get the time factors $T_n^{\pm}(t)$ corresponding to the complex energy eigenvalues $E_n^{\pm}$ by substituting (3.17) into the form $e^{-iE_n^{\pm}t/\hbar}$. They give

$$T_n^{\pm}(t) = A_n^{\pm} e^{-iV_0 t/\hbar} e^{\mp(n+1/2)\gamma t}, \tag{3.25}$$

where $A_n^{\pm}$ are the numerical coefficients. The time-dependent wave functions $\psi_n^{\pm}(t,x)$ representing non-stationary states of complex energy eigenvalues $E_n^{\pm}$ are

$$\begin{aligned}\psi_n^{\pm}(t,x) =& T_n^{\pm}(t)u_n^{\pm}(x)\\ =& A_n^{\pm}B_n^{\pm}e^{-iV_0t/\hbar}e^{\mp(n+1/2)\gamma t}e^{\pm i\beta^2x^2/2}H_n^{\pm}(\beta x).\end{aligned} \tag{3.26}$$

Before proceeding to discuss the physical meanings of the time factors (3.25), we shall first verify the equation of continuity (2.3). From (3.26) we see now that the probability densities (2.1) are

$$\begin{aligned}\rho_n^{\pm}(t,x) &\equiv \left|\psi_n^{\pm}(t,x)\right|^2\\ &= \left|A_n^{\pm}\right|^2\left|B_n^{\pm}\right|^2 e^{\mp(2n+1)\gamma t}H_n^{\mp}(\beta x)H_n^{\pm}(\beta x)\end{aligned} \tag{3.27}$$

and the probability currents (2.2) are

$$\begin{aligned}j_n^{\pm}(t,x) &\equiv \Re\left[\psi_n^{\pm}(t,x)^*\left(-i\hbar\partial\psi_n^{\pm}/\partial x\right)(t,x)\right]/\,m\\ &= \pm\left|A_n^{\pm}\right|^2\left|B_n^{\pm}\right|^2 e^{\mp(2n+1)\gamma t}\times\\ &\quad\times\gamma\left\{xH_n^{\mp}(\beta x)H_n^{\pm}(\beta x)\pm 2n\beta^{-1}\Im\left[H_n^{\mp}(\beta x)H_{n-1}^{\pm}(\beta x)\right]\right\}.\end{aligned} \tag{3.28}$$

Equations (3.27) and (3.28) depend on the time t which satisfy the equation of continuity

$$\frac{\partial}{\partial t}\rho_n^{\pm}(t,x)+\frac{\partial}{\partial x}j_n^{\pm}(t,x)=0, \tag{3.29}$$

with the help of formulas (C.7) and (C.8) of Appendix C. Note that the numerical coefficients $A_n^{\pm}$ and $B_n^{\pm}$ do not affect the validity of (3.29).

We must examine how the probability currents $j_n^{\pm}(t,x)$ behave for large values of $|x|$. The second term in the $\{\}$ brackets in (3.28) can be neglected as $|x|\to\infty$. We are left with

$$\begin{aligned}j_n^{\pm}(t,x) &\approx \pm\, 2^{2n}\left|A_n^{\pm}\right|^2\left|B_n^{\pm}\right|^2 e^{\mp(2n+1)\gamma t}\gamma\beta^{2n}x^{2n+1}\\ &\approx \pm\, e^{\mp(2n+1)\gamma t}x^{2n+1}.\end{aligned}$$

These asymptotic behaviors of probability currents show that the index + (ψ_n^+, u_n^+, etc.) means only outward moving particles and the index − (ψ_n^-, u_n^-, etc.) means only inward moving particles. These asymptotic behaviors are the one expected in § 3.1.1 for dealing with the asymptotic solutions.

Let us now examine how the probability densities $\rho_n^{\pm}(t,x)$ vary with time. From (3.27), $\rho_n^{\pm}(t,x)$ will tend to infinity as $t \to \mp\infty$. The result is a necessary consequence of the form of time-dependent wave functions (3.26). In any case the time-dependent wave functions $\psi_n^{\pm}(t,x)$ must not tend to infinity as $t \to \mp\infty$, or they will represent states that have no physical meaning. The way of escape from the difficulty lies in the domains of the time factors (3.25). We assume that $T_n^+(t)$ exists only on the future part, for which $t > 0$, and $T_n^-(t)$ exists only on the past part, for which $t < 0$, i.e.

$$\begin{aligned} T_n^+(t) =& A_n^+ e^{-iV_0 t/\Box} e^{-(n+1/2)\gamma t} \quad \text{when } \ t >0, \\ T_n^-(t) =& A_n^- e^{-iV_0 t/\Box} e^{(n+1/2)\gamma t} \quad \ \text{when } \ t <0. \end{aligned}$$

These equations can be combined into the single equations

$$T_n^{\pm}(t) = A_n^{\pm} e^{-iV_0 t/\Box} e^{\mp(n+1/2)\gamma t}\theta(\pm t), \tag{3.30}$$

where $\theta(t)$ is the Heaviside step function. We take the squares of the moduli of the time factors (3.30)

$$\left|T_n^{\pm}(t)\right|^2 = \left|A_n^{\pm}\right|^2 e^{\mp(2n+1)\gamma t}\theta(\pm t). \tag{3.31}$$

To interpret the result (3.31), we may suppose that the index + (ψ_n^+, T_n^+, etc.) refer to the *states of decay* when $t > 0$, while the index − (ψ_n^-, T_n^-, etc.) refer to the *states of growth* when $t < 0$. This condition expresses mathematically that

$$T_n^+(t) \in L_+^2(\mathbb{R}_t), \ \ T_n^-(t) \in L_-^2(\mathbb{R}_t), \tag{3.32}$$

where spaces $L_{\pm}^2(\mathbb{R}_t)$ are defined by

$$L_{\pm}^2(\mathbb{R}_t) \equiv \left\{ f \in L^2(\mathbb{R}_t) \;\middle|\; f(t) = 0 \ \text{ a.e. } \ t \in \mathbb{R}_{\mp} \right\}. \tag{3.33}$$

This condition will be referred to as the *time boundary condition*.

The asymptotic behaviors and this boundary condition require that $\psi_n^-(t,x)$ represents the growing state lying on the past part together with only inward moving particle and $\psi_m^+(t,x)$ represents the decaying state lying on the future part together with only outward moving particle. We must now obtain the probability of a transition taking place from state ψ_n^- to state ψ_m^+ at time $t = 0$. This gives us the "selection rule" at time $t = 0$, and can be worked out from the following transition matrix elements

$$S_{mn} = \left\langle u_m^+ \,\middle|\, u_n^- \right\rangle \equiv \int_{-\infty}^{\infty} u_m^+(x)^* u_n^-(x) dx, \tag{3.34}$$

the last definition expresses $\left\langle u_m^+ \,\middle|\, u_n^- \right\rangle$ in terms of x-representatives. When we substitute for $u_n^\pm(x)$ their value given by (3.18), equation (3.34) gives

$$S_{mn} = \beta^{-1} B_m^{+*} B_n^- \int_{-\infty}^{\infty} H_m^+(\xi)^* H_n^-(\xi) e^{-i\xi^2} d\xi$$

with the help of (3.4). The right-hand side becomes, from an application of formula (C.9) of Appendix C,

$$S_{mn} = \delta_{mn} \tag{3.35}$$

with the help of (3.19). The transition matrix is then a unit matrix. This result expresses that the state ψ_n^- $(n = 0, 1, 2, \dots)$ on the past part is connected with the state ψ_n^+ having the same label n on the future part. Thus our selection rule at time $t = 0$ is that only those transitions can take place in which the complex energy changes from E_n^- to E_n^+.

Our problem now is to obtain the numerical coefficients $A_n^\pm$ of (3.30). We shall distinguish between the growing state ψ_n^- and the decaying state ψ_n^+ of the unstable system, with the growth and decay processes governed independently each by its own probability law [1]. We shall require that *the total number of growing particles is equal to the total number of decaying particles*, i.e.

$$\int_{-\infty}^{0} \rho_n^-(t, x) dt = \int_{0}^{\infty} \rho_n^+(t, x) dt. \tag{3.36}$$

By substituting for $\rho_n^\pm$ here their values given by (3.27), we now see that this condition (3.36) is independent of x. Thus (3.36) provides the condition for the time factor. From (3.36) we can require *the normalizing condition for the time factor*:

$$\int_{-\infty}^{0} \left|T_n^-(t)\right|^2 dt = \int_{0}^{\infty} \left|T_n^+(t)\right|^2 dt = 1. \tag{3.37}$$

These conditions will be seen later (see equations (3.46)) to be connected with the fact that the Breit-Wigner resonance formulas are normalized. If we now apply the conditions

(3.37) to the time factors (3.30), we have

$$A_n^{\pm} = \sqrt{(2n+1)\gamma}, \tag{3.38}$$

where the phase factors are chosen unity.†

We can now proceed to introduce a physical interpretation of the time factors $T_n^{\pm}(t)$ such that the absolute probabilities of the particles being during the time interval t to $t + \varDelta t$ are given by [45]

$$\left| T_n^{\pm}(t) \right|^2 \varDelta t$$

corresponding to Born's probabilistic interpretation. This interpretation provides a physical description of a process of time for a non-stationary state, say from growth to decay. In this case the expectation values of the time t for the states $\psi_n^{\pm}$ are

$$\begin{aligned} \langle t \rangle_n^{\pm} &\equiv \int_{-\infty}^{\infty} t \left| T_n^{\pm}(t) \right|^2 dt \\ &= \left| A_n^{\pm} \right|^2 \int_{-\infty}^{\infty} t e^{\mp(2n+1)\gamma t} \theta(\pm t) dt \\ &= \pm \frac{1}{(2n+1)\gamma}, \end{aligned} \tag{3.39}$$

respectively. Again, we find

$$\begin{aligned} \langle t^2 \rangle_n^{\pm} &\equiv \int_{-\infty}^{\infty} t^2 \left| T_n^{\pm}(t) \right|^2 dt \\ &= \left| A_n^{\pm} \right|^2 \int_{-\infty}^{\infty} t^2 e^{\mp(2n+1)\gamma t} \theta(\pm t) dt \\ &= \frac{2}{[(2n+1)\gamma]^2}. \end{aligned}$$

Hence the positive square roots of the variance of t for the states $\psi_n^{\pm}$ are

$$\begin{aligned} (\varDelta t)_n^{\pm} &\equiv \sqrt{\left\langle \left(t - \langle t \rangle_n^{\pm} \right)^2 \right\rangle_n^{\pm}} \\ &= \sqrt{\langle t^2 \rangle_n^{\pm} - \left(\langle t \rangle_n^{\pm} \right)^2} \\ &= \frac{1}{(2n+1)\gamma}. \end{aligned} \tag{3.40}$$

†We choose these phase factors so that the time-dependent wave functions are simple forms under a time reversal (cf. references [46, 47]).

Thus $(\varDelta t)_n^+$ is equal to $(\varDelta t)_n^-$, since the probability densities $|T_n^+(t)|^2$ and $|T_n^-(t)|^2$ are symmetrical with respect to the center $t = 0$. We may call these quantities $(\varDelta t)_n^\pm$ the *mean lifetimes*. Equations (3.40) inform us that *the mean lifetimes are quantized in the unstable system with the Hamiltonian* (3.1). These results are closely connected with the uncertainty relation between time and energy. This matter will be dealt with in the next section.

Let us return to the time-dependent wave functions (3.26). We shall verify finally a time reversal. By application of property (C.5) of Appendix C, we get

$$\psi_n^\pm(-t, x)^* = \psi_n^\mp(t, x),$$

with the help of (3.38) and also of equation (3.19). Similarly we have in the p-representation

$$\tilde{\psi}_n^\pm(-t, -p)^* = \tilde{\psi}_n^\mp(t, p),$$

since the phase factor in equation (3.24) and the parity of $H_n^\pm(\tilde{\beta} p)$ cancel out. We see in this way that a time reversal occurs resulting in the interchange of states on the future part and on the past part. It is known as the *reciprocity theorem* [2].

3.2.2 The energy distribution

The time factor and the *physical* energy dependence of a function $f \in L^2_\pm(\mathbb{R}_t)$ are connected by the inverse Fourier transformations

$$\begin{cases} \tilde{f}(E) = \big(\hat{\mathcal{F}}^{-1} f\big)(E) \equiv \dfrac{1}{\sqrt{2\pi\hbar}} \displaystyle\int_{-\infty}^{\infty} f(t) e^{iEt/\hbar} dt, \\ f(t) = \big(\hat{\mathcal{F}} \tilde{f}\big)(t) \equiv \dfrac{1}{\sqrt{2\pi\hbar}} \displaystyle\int_{-\infty}^{\infty} \tilde{f}(E) e^{-iEt/\hbar} dE, \end{cases} \tag{3.41}$$

since the theory of relativity puts physical energy in the same relation to time as momentum to coordinate (see equations (3.20)).

We must now evaluate the physical energy dependence $\tilde{T}_n^\pm(E)$ for the time factors (3.30), and obtain the probability density of them. We make use of the Paley-Wiener theorem, which states that the inverse Fourier transformation is a unitary mapping from $L^2_\pm(\mathbb{R}_t)$ onto $H^2_\pm$, i.e.

$$\hat{\mathcal{F}}^{-1} L^2_\pm(\mathbb{R}_t) = H^2_\pm, \tag{3.42}$$

H^2_+ denoting a Hardy space on the upper half-plane and H^2_- denoting a Hardy space on the lower half-plane. If we apply the first of equations (3.41) to the time factors (3.30) and (3.32), we then have

$$\begin{aligned}\tilde{T}^{\pm}_n(E) =& \frac{A^{\pm}_n}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{i(E-V_0)t/\square} e^{\mp(n+1/2)\gamma t}\theta(\pm t)dt,\\ =& \frac{\pm i}{\sqrt{\pi}}\frac{\sqrt{(n+1/2)\hbar\gamma}}{E-V_0 \pm i(n+1/2)\hbar\gamma} \in H^2_{\pm},\end{aligned} \tag{3.43}$$

with the help of (3.38). Thus $\tilde{T}^{+}_n(E)$ has a simple pole at the point

$$V_0 - i(n+1/2)\hbar\gamma \in \mathbb{C}_-,$$

and $\tilde{T}^{-}_n(E)$ has a simple pole at the symmetrical point

$$V_0 + i(n+1/2)\hbar\gamma \in \mathbb{C}_+$$

with respect to the real axis. To obtain the original equations (3.30) we note that, from an application of the second of equations (3.41) to (3.43), the integral being taken along a large semicircle in the lower half-plane when $t > 0$ and in the upper half-plane when $t < 0$.

By taking the squares of the moduli of (3.43), respectively, we obtain

$$\left|\tilde{T}^{-}_n(E)\right|^2 = \left|\tilde{T}^{+}_n(E)\right|^2 = \frac{1}{\pi}\frac{(n+1/2)\hbar\gamma}{(E-V_0)^2 + [(n+1/2)\hbar\gamma]^2}. \tag{3.44}$$

These physical energy distributions are the *Breit-Wigner resonance formulas*, having the same resonance energy V_0 and the half-widths

$$\mathit{\Gamma}^{\pm}_n = \left(n + \frac{1}{2}\right)\hbar\gamma. \tag{3.45}$$

Specially, the quantity $\mathit{\Gamma}^{+}_n$ is what is sometimes called the *decay width*. Equations (3.45) inform us that *the half-widths are quantized in the unstable system with the Hamiltonian* (3.1). The integrals of (3.44) with respect to E are

$$\int_{-\infty}^{\infty}\left|\tilde{T}^{-}_n(E)\right|^2 dE = \int_{-\infty}^{\infty}\left|\tilde{T}^{+}_n(E)\right|^2 dE = 1, \tag{3.46}$$

corresponding to (3.37), so $\tilde{T}^{\pm}_n(E)$ *are normalized*.

Let us take the products of the mean lifetimes $(\Delta t)_n^{\pm}$ and the half-widths $\Gamma_n^{\pm}$. From (3.40) and (3.45) these products read

$$(\Delta t)_n^{\pm} \Gamma_n^{\pm} = \frac{\hbar}{2}. \tag{3.47}$$

These results are independent of n. *There are thus Heisenberg's Uncertainty Relations between the mean lifetimes and the half-widths*.

Chapter 4

The Parabolic Potential Barrier in Two Dimensions

4.1 Eigenvalue problems of the two-dimensional parabolic potential barrier

LET us now suppose the PPB of the preceding chapter is also located in a second direction, at right angles to the first, with the same Liapunov exponent γ. We shall then have a 2D isotropic PPB, whose Hamiltonian is

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + V_0 - \frac{1}{2}m\gamma^2\left(x^2 + y^2\right), \tag{4.1}$$

where x and y are the two Cartesian coordinates.

A state is represented by a wave function $U(x, y)$ satisfying the Schrödinger equation, which now reads, with $\hat{H}$ given by (4.1),

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)U(x, y) + \left[V_0 - \frac{1}{2}m\gamma^2\left(x^2 + y^2\right)\right]U(x, y) = EU(x, y). \tag{4.2}$$

The energy eigenvalues of (4.2) will be the sum of the energy eigenvalues of the 1D PPB in the x- and y-direction, respectively, i.e.

$$E_{n_x n_y} = E_{n_x} + E_{n_y} \tag{4.3}$$

and the eigenfunctions belonging to these energy eigenvalues will be the product of their corresponding eigenfunctions

$$U_{n_x n_y}(x, y) = u_{n_x}(x)u_{n_y}(y). \tag{4.4}$$

With the notation of the preceding chapter, the results (4.3) and (4.4) of the 2D PPB separate into four types:

$$\begin{aligned}
&\textit{Type } 1. \qquad && E^{++}_{n_x n_y} = E^{+}_{n_x} + E^{+}_{n_y} = V_0 - i(n_x + n_y + 1)\hbar\gamma, \\
& && U^{++}_{n_x n_y}(x, y) = u^{+}_{n_x}(x) u^{+}_{n_y}(y) = e^{i\beta^2(x^2+y^2)/2} H^{+}_{n_x}(\beta x) H^{+}_{n_y}(\beta y). \\
&\textit{Type } 2. \qquad && E^{+-}_{n_x n_y} = E^{+}_{n_x} + E^{-}_{n_y} = V_0 - i(n_x - n_y)\hbar\gamma, \\
& && U^{+-}_{n_x n_y}(x, y) = u^{+}_{n_x}(x) u^{-}_{n_y}(y) = e^{i\beta^2(x^2-y^2)/2} H^{+}_{n_x}(\beta x) H^{-}_{n_y}(\beta y). \\
&\textit{Type } 3. \qquad && E^{-+}_{n_x n_y} = E^{-}_{n_x} + E^{+}_{n_y} = V_0 + i(n_x - n_y)\hbar\gamma, \\
& && U^{-+}_{n_x n_y}(x, y) = u^{-}_{n_x}(x) u^{+}_{n_y}(y) = e^{-i\beta^2(x^2-y^2)/2} H^{-}_{n_x}(\beta x) H^{+}_{n_y}(\beta y). \\
&\textit{Type } 4. \qquad && E^{--}_{n_x n_y} = E^{-}_{n_x} + E^{-}_{n_y} = V_0 + i(n_x + n_y + 1)\hbar\gamma, \\
& && U^{--}_{n_x n_y}(x, y) = u^{-}_{n_x}(x) u^{-}_{n_y}(y) = e^{-i\beta^2(x^2+y^2)/2} H^{-}_{n_x}(\beta x) H^{-}_{n_y}(\beta y).
\end{aligned}$$

The numerical coefficients $B^{\pm}_{n_x}$ and $B^{\pm}_{n_y}$ here are discarded. These eigenfunctions $U^{++}_{n_x n_y}$, $U^{+-}_{n_x n_y}$, $U^{-+}_{n_x n_y}$ and $U^{--}_{n_x n_y}$ are also generalized functions in $\mathcal{S}(\mathbb{R}^2)^{\times}$ of the Gel'fand triplet

$$\mathcal{S}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) \subset \mathcal{S}(\mathbb{R}^2)^{\times} \tag{4.5}$$

instead of (3.8). Note that the eigenfunctions of types 4 and 3 are conjugate complex functions of types 1 and 2, respectively, i.e.

$$U^{\pm\pm}_{n_x n_y}(x, y)^* = U^{\mp\mp}_{n_x n_y}(x, y)$$

and

$$U^{\pm\mp}_{n_x n_y}(x, y)^* = U^{\mp\pm}_{n_x n_y}(x, y).$$

4.1.1 Diverging and converging flows

Let us consider first types 1 and 4. In this case the energy eigenvalues $E^{\pm\pm}_{n_x n_y}$ are always complex numbers and the time factors corresponding to them are

$$e^{-iE^{\pm\pm}_{n_x n_y} t/\hbar} = e^{-iV_0 t/\hbar} e^{\mp(n_x+n_y+1)\gamma t}.$$

Thus the solutions of type 1 are well defined when $t > 0$, and those of type 4 are well defined when $t < 0$, according to the time boundary condition that time factors of an unstable system are square integrable (see the bottom of p. 25). Also, $U^{++}_{n_x n_y}(x, y)$ represent particles moving outward from the center as in Fig. 4.1, and $U^{--}_{n_x n_y}(x, y)$ represent particles moving inward to the center as in Fig. 4.2. Thus we shall call these types *diverging* and *converging flows*, respectively. Note that a time reversal occurs, resulting in the interchange of the diverging and converging flows.

For $n_x = n_y = 0$, we get the energy eigenvalue

$$E^{\pm\pm}_{00} = V_0 \mp i\hbar\gamma \tag{4.6}$$

and only one eigenfunction

$$U^{\pm\pm}_{00}(x, y) = e^{\pm i\beta^2(x^2+y^2)/2}, \tag{4.7}$$

respectively. For $n_x + n_y = 1$, namely $n_x = 1, n_y = 0$ and $n_x = 0, n_y = 1$, we get

$$E^{\pm\pm}_{10} = E^{\pm\pm}_{01} = V_0 \mp 2i\hbar\gamma \tag{4.8}$$

and two eigenfunctions

$$\left.\begin{aligned} U^{\pm\pm}_{10}(x, y) &= 2\beta x e^{\pm i\beta^2(x^2+y^2)/2}, \\ U^{\pm\pm}_{01}(x, y) &= 2\beta y e^{\pm i\beta^2(x^2+y^2)/2}. \end{aligned}\right\} \tag{4.9}$$

There is a twofold degenerate state of types 1 and 4 with $n_x + n_y = 1$. For $n_x + n_y = 2$, namely $n_x = 2, n_y = 0$; $n_x = 1, n_y = 1$; $n_x = 0, n_y = 2$, we get

$$E^{\pm\pm}_{20} = E^{\pm\pm}_{11} = E^{\pm\pm}_{02} = V_0 \mp 3i\hbar\gamma \tag{4.10}$$

and three eigenfunctions

$$\left.\begin{aligned} U^{\pm\pm}_{20}(x, y) &= (4\beta^2 x^2 \mp 2i) e^{\pm i\beta^2(x^2+y^2)/2}, \\ U^{\pm\pm}_{11}(x, y) &= 4\beta^2 xy e^{\pm i\beta^2(x^2+y^2)/2}, \\ U^{\pm\pm}_{02}(x, y) &= (4\beta^2 y^2 \mp 2i) e^{\pm i\beta^2(x^2+y^2)/2}. \end{aligned}\right\} \tag{4.11}$$

There is a threefold degenerate state of types 1 and 4 with $n_x + n_y = 2$. Generally, there is an $(n + 1)$-fold degenerate state of types 1 and 4 with $n_x + n_y = n$. This result is just the same degree of degeneracy as the 2D HO.

For the further discussion of the state of types 1 and 4, we now pass from the Cartesian coordinates x, y to the two-dimensional polar coordinates ρ, φ by means of the equations

$$\left.\begin{aligned} x &= \rho\cos\varphi, \\ y &= \rho\sin\varphi. \end{aligned}\right\} \tag{4.12}$$

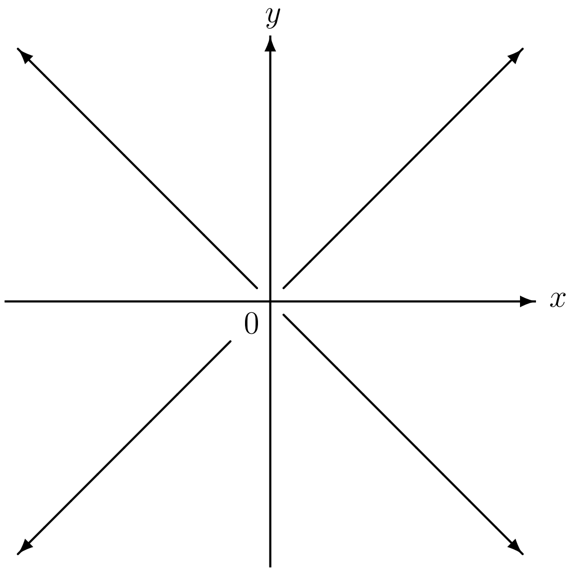

Fig. 4.1: Diverging flows.

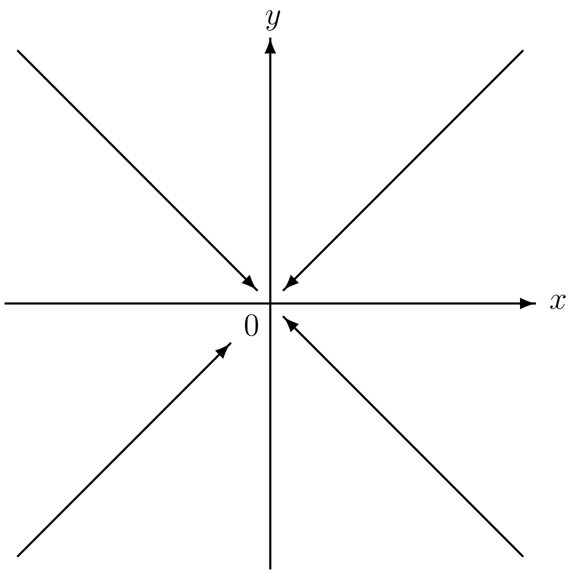

Fig. 4.2: Converging flows.

If in the new coordinates we superpose the above-mentioned eigenstates with suitable weights, the result will be the eigenstates of orbital angular momentum $\hat{L}$ defined by

$$\hat{L} = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\hbar\frac{\partial}{\partial\varphi}. \tag{4.13}$$

For $n \equiv n_x + n_y = 0$, the eigenfunction (4.7) will be

$$U_{0_0}^{\pm\pm}(\rho,\varphi) \equiv U_{00}^{\pm\pm}(\rho,\varphi) = e^{\pm i\beta^2\rho^2/2}, \tag{4.14}$$

the suffix l_n of $U_{l_n}^{\pm\pm}$ being the eigenvalue of $\hat{L}/\hbar$. Thus $U_{0_0}^{\pm\pm}(\rho,\varphi)$ is independent of φ and has zero orbital angular momentum. For $n = 1$, a linear combination of the eigenfunctions (4.9) gives

$$\left.\begin{aligned} U_{1_1}^{\pm\pm}(\rho,\varphi) &\equiv U_{10}^{\pm\pm}(\rho,\varphi) + iU_{01}^{\pm\pm}(\rho,\varphi) = 2\beta\rho e^{\pm i\beta^2\rho^2/2}e^{i\varphi}, \\ U_{-1_1}^{\pm\pm}(\rho,\varphi) &\equiv U_{10}^{\pm\pm}(\rho,\varphi) - iU_{01}^{\pm\pm}(\rho,\varphi) = 2\beta\rho e^{\pm i\beta^2\rho^2/2}e^{-i\varphi}. \end{aligned}\right\} \tag{4.15}$$

These states are eigenstates of $\hat{L}$ with eigenvalues $\hbar$ and $-\hbar$, respectively. For $n = 2$, (4.11) give

$$\left.\begin{aligned} U_{2_2}^{\pm\pm}(\rho,\varphi) &\equiv U_{20}^{\pm\pm}(\rho,\varphi) + 2iU_{11}^{\pm\pm}(\rho,\varphi) - U_{02}^{\pm\pm}(\rho,\varphi) = 4\beta^2\rho^2 e^{\pm i\beta^2\rho^2/2}e^{2i\varphi}, \\ U_{0_2}^{\pm\pm}(\rho,\varphi) &\equiv U_{20}^{\pm\pm}(\rho,\varphi) + U_{02}^{\pm\pm}(\rho,\varphi) = 4(\beta^2\rho^2 \mp i)e^{\pm i\beta^2\rho^2/2}, \\ U_{-2_2}^{\pm\pm}(\rho,\varphi) &\equiv U_{20}^{\pm\pm}(\rho,\varphi) - 2iU_{11}^{\pm\pm}(\rho,\varphi) - U_{02}^{\pm\pm}(\rho,\varphi) = 4\beta^2\rho^2 e^{\pm i\beta^2\rho^2/2}e^{-2i\varphi}. \end{aligned}\right\} \tag{4.16}$$

These states are also eigenstates of $\hat{L}$ with eigenvalues $2\hbar$, 0 and $-2\hbar$. Substituting these eigenfunctions in (2.2), we obtain the two-dimensional polar coordinates $j_{l_n\rho}^{\pm\pm}$, $j_{l_n\varphi}^{\pm\pm}$ of $\boldsymbol{j}_{l_n}^{\pm\pm}$, which are the probability currents of the states $U_{l_n}^{\pm\pm}$. The result is

$$j_{0_0\rho}^{\pm\pm}(t,\rho,\varphi) = \pm e^{\mp 2\gamma t}\gamma\rho, \;\; j_{0_0\varphi}^{\pm\pm}(t,\rho,\varphi) = 0, \tag{4.17}$$

$$\left.\begin{aligned} j_{1_1\rho}^{\pm\pm}(t,\rho,\varphi) &= \pm 4e^{\mp 4\gamma t}\gamma\beta^2\rho^3, \;\; j_{1_1\varphi}^{\pm\pm}(t,\rho,\varphi) = 4e^{\mp 4\gamma t}\gamma\rho, \\ j_{-1_1\rho}^{\pm\pm}(t,\rho,\varphi) &= \pm 4e^{\mp 4\gamma t}\gamma\beta^2\rho^3, \;\; j_{-1_1\varphi}^{\pm\pm}(t,\rho,\varphi) = -4e^{\mp 4\gamma t}\gamma\rho, \end{aligned}\right\} \tag{4.18}$$

$$\left.\begin{aligned} j_{2_2\rho}^{\pm\pm}(t,\rho,\varphi) &= \pm 16e^{\mp 6\gamma t}\gamma\beta^4\rho^5, \;\; j_{2_2\varphi}^{\pm\pm}(t,\rho,\varphi) = 32e^{\mp 6\gamma t}\gamma\beta^2\rho^3, \\ j_{0_2\rho}^{\pm\pm}(t,\rho,\varphi) &= \pm 16e^{\mp 6\gamma t}\gamma\rho(\beta^4\rho^4 + 3), \;\; j_{0_2\varphi}^{\pm\pm}(t,\rho,\varphi) = 0, \\ j_{-2_2\rho}^{\pm\pm}(t,\rho,\varphi) &= \pm 16e^{\mp 6\gamma t}\gamma\beta^4\rho^5, \;\; j_{-2_2\varphi}^{\pm\pm}(t,\rho,\varphi) = -32e^{\mp 6\gamma t}\gamma\beta^2\rho^3. \end{aligned}\right\} \tag{4.19}$$

Thus $\boldsymbol{j}_{l_n}^{\pm\pm}$ depend on the time t. There is a similar procedure for large values of n.

4.1.2 Corner flows

Let us now study types 2 and 3. In this case the energy eigenvalues $E^{\pm\mp}_{n_x n_y}$ are also in general complex numbers, but with the striking difference that all the eigenstates belonging to each energy eigenvalue are infinitely degenerate. The corresponding time factors are

$$e^{-iE^{\pm\mp}_{n_x n_y}t/\square} = e^{-iV_0 t/\square} e^{\mp(n_x - n_y)\gamma t}.$$

Thus the solutions of type 2 are well defined when $t > 0$, and those of type 3 are well defined when $t < 0$, for the case of $n_x > n_y$, and vice versa. Also, $U^{+-}_{n_x n_y}(x, y)$ represent particles which, coming from the y-direction, round the center and go off to the x-direction as in Fig. 4.3, and $U^{-+}_{n_x n_y}(x, y)$ represent particles which, coming from the x-direction, round the center and go off to the y-direction as in Fig. 4.4. Note that a time reversal occurs, resulting in the interchange of these corner flows.

Stationary flows

The above time factors now show that for the case of $n_x = n_y$, there are *stationary flows*. For $n_x = n_y \equiv n = 0, 1, 2, \ldots$, the energy eigenvalues associated with stationary flows are the same real number:

$$E^{\pm\mp}_{nn} = V_0. \tag{4.20}$$

The first few infinitely degenerate eigenfunctions belonging to this energy eigenvalue (4.20) are

$$\left.\begin{aligned}
U^{\pm\mp}_{00}(x, y) &= e^{\pm i\beta^2(x^2-y^2)/2}, \\
U^{\pm\mp}_{11}(x, y) &= 4\beta^2 xy e^{\pm i\beta^2(x^2-y^2)/2}, \\
U^{\pm\mp}_{22}(x, y) &= 4\left[4\beta^4 x^2 y^2 + 1 \pm 2i\beta^2(x^2 - y^2)\right] e^{\pm i\beta^2(x^2-y^2)/2}, \\
&\ldots.
\end{aligned}\right\} \tag{4.21}$$

For the further study of the stationary flows in the 2D PPB with the Hamiltonian (4.1), it is convenient to make a transformation to the rectangular hyperbolic coordinates u, v, given by

$$\left.\begin{aligned}
u &= x^2 - y^2, \\
v &= 2xy.
\end{aligned}\right\} \tag{4.22}$$

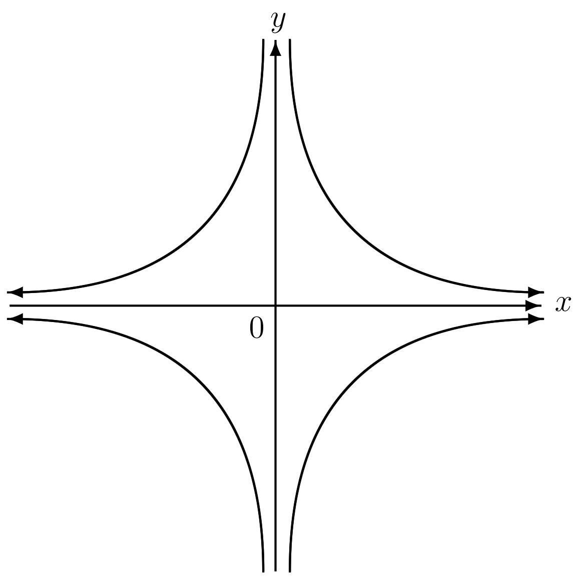

FIG. 4.3: Corner flows moving from the y- to the x-direction.

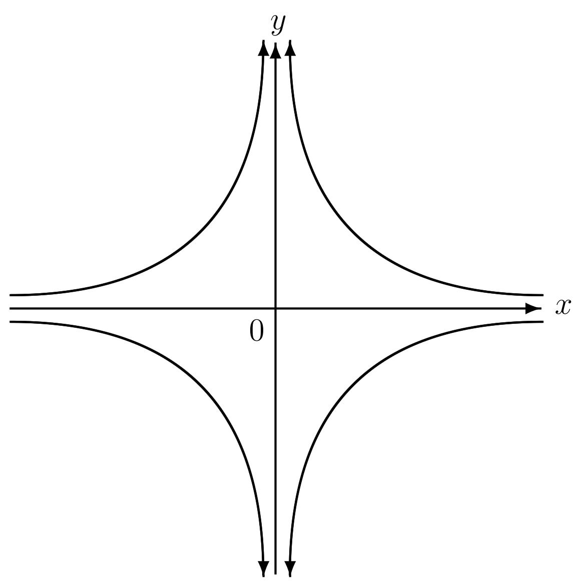

FIG. 4.4: Corner flows moving from the x- to the y-direction.

The eigenfunctions (4.21) will become in the new representation

$$\left.\begin{aligned} U_{00}^{\pm\mp}(u,v) &= e^{\pm i\beta^2 u/2}, \\ U_{11}^{\pm\mp}(u,v) &= 2\beta^2 v e^{\pm i\beta^2 u/2}, \\ U_{22}^{\pm\mp}(u,v) &= 4\left(\beta^4 v^2 + 1 \pm 2i\beta^2 u\right) e^{\pm i\beta^2 u/2}, \\ &\cdots\cdot \end{aligned}\right\} \tag{4.23}$$

The factors $e^{\pm i\beta^2 u/2}$ occurring in (4.23) describe plane waves in the uv-plane, i.e. the motion of the wave $e^{i\beta^2 u/2}$ is in the direction specified by Fig. 4.3 and that of the wave $e^{-i\beta^2 u/2}$ is in the direction specified by Fig. 4.4. These eigenfunctions substituted in (2.2) give the rectangular hyperbolic coordinates $j_{nnu}^{\pm\mp}$, $j_{nnv}^{\pm\mp}$ of $\boldsymbol{j}_{nn}^{\pm\mp}$, which are the probability currents of the states $U_{nn}^{\pm\mp}$. They give

$$\left.\begin{aligned} j_{00u}^{\pm\mp}(u,v) &= \pm\gamma h_u/2, \;\; j_{00v}^{\pm\mp}(u,v) = 0, \\ j_{11u}^{\pm\mp}(u,v) &= \pm 2\gamma\beta^4 v^2 h_u, \;\; j_{11v}^{\pm\mp}(u,v) = 0, \\ j_{22u}^{\pm\mp}(u,v) &= \pm 8\gamma\left[(\beta^4 v^2 + 5)(\beta^4 v^2 + 1) + 4\beta^4 u^2\right] h_u, \\ j_{22v}^{\pm\mp}(u,v) &= \mp 64\gamma\beta^4 uv h_v, \\ &\cdots, \end{aligned}\right\} \tag{4.24}$$

where the scale factors $h_u = h_v = 2\sqrt[4]{u^2 + v^2}$. Thus $\boldsymbol{j}_{nn}^{\pm\mp}$ can never depend on the time t. We see from this result the suitability of the term "stationary flows".

4.2 Hydrodynamical formulation of the two-dimensional parabolic potential barrier

In the present section we shall apply the hydrodynamical formulation of quantum mechanics given in Chapter 2 to the 2D PPB flows dealt with in § 4.1, in § 4.2.1 the velocities and the vortices of diverging and converging flows studied in § 4.1.1 and in § 4.2.2 the velocities, the vortices and the complex velocity potentials of stationary flows as special cases of corner flows studied in § 4.1.2.

4.2.1 Vortices of diverging and converging flows

The non-vanishing φ-components of the probability currents (4.18), the first and third of equations (4.19), show that for the eigenstates of orbital angular momentum there are vortices around the origin $\rho = 0$. Note that this point $\rho = 0$ is the node of the corresponding wave functions (4.15), (4.16). To get an understanding of the physical features of these vortices it is better to work with the velocity defined by (2.5) and the circulation defined by (2.10). For the states (4.14), (4.15) and (4.16), we get the following velocities:

$$v^{\pm\pm}_{0_0\rho} = \pm\gamma\rho, \;\; v^{\pm\pm}_{0_0\varphi} = 0, \tag{4.25}$$

$$\left.\begin{aligned} v^{\pm\pm}_{1_1\rho} &= \pm\gamma\rho, \;\; v^{\pm\pm}_{1_1\varphi} = \frac{\hbar}{m\rho}, \\ v^{\pm\pm}_{-1_1\rho} &= \pm\gamma\rho, \;\; v^{\pm\pm}_{-1_1\varphi} = -\frac{\hbar}{m\rho}, \end{aligned}\right\} \tag{4.26}$$

$$\left.\begin{aligned} v^{\pm\pm}_{2_2\rho} &= \pm\gamma\rho, \;\; v^{\pm\pm}_{2_2\varphi} = \frac{2\hbar}{m\rho}, \\ v^{\pm\pm}_{0_2\rho} &= \pm\gamma\rho\frac{\beta^4\rho^4+3}{\beta^4\rho^4+1}, \;\; v^{\pm\pm}_{0_2\varphi} = 0, \\ v^{\pm\pm}_{-2_2\rho} &= \pm\gamma\rho, \;\; v^{\pm\pm}_{-2_2\varphi} = -\frac{2\hbar}{m\rho}. \end{aligned}\right\} \tag{4.27}$$

Note that the nodes of (4.14), (4.15) and (4.16) go over into the singularities of the corresponding φ-components of the above velocities. Now the circulation (2.10), round a closed contour C encircling the singularity at the origin $\rho = 0$, reads

$$\Gamma^{\pm\pm}_{l_n} = \oint_C v^{\pm\pm}_{l_n\varphi} ds. \tag{4.28}$$

On substituting (4.25), (4.26) and (4.27) in (4.28) we get the following formula:

$$\Gamma^{\pm\pm}_{l_n} = 2\pi\frac{l_n\hbar}{m}, \tag{4.29}$$

where l_n on the right-hand side of (4.29) is one of the values $n, n-2, n-4, \ldots, -n$. These are in agreement with expression (2.12) of Chapter 2, and they show that, *for the eigenstates* (4.14), (4.15) *and* (4.16) *of orbital angular momentum, the circulations of the 2D PPB are quantized*. One could make calculations for large values of n and one would be led to the same results.

Comparing (4.29) with (2.35) of Chapter 2, we see that the vorticities of these states must be of the form (2.32). This leads to the result that the velocity potentials of the form (2.33) can exist except the singularity $\rho = 0$. But the stream functions of these states do not exist, since the velocities (4.25), (4.26) and (4.27) cannot be solenoidal. As an example we try to calculate the divergence (2.16) for (4.25). The result is

$$\nabla \cdot \boldsymbol{v}_{0_0}^{\pm\pm} \equiv \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho v_{0_0\rho}^{\pm\pm}\right) + \frac{1}{\rho}\frac{\partial}{\partial \varphi} v_{0_0\varphi}^{\pm\pm} = \pm 2\gamma \neq 0.$$

Now the $\pm$ sign shows that $U_{0_0}^{++}$ is connected with the diverging flow and $U_{0_0}^{--}$ is connected with the converging one. This result follows from the fact that the probability densities of these flows depend on t in (2.4). Therefore we cannot obtain the complex velocity potentials defined by (2.19). This result also holds for large values of n.

4.2.2 Vortices and complex velocity potentials of stationary flows

Taking into account that a node of the wave function $\psi(t, \boldsymbol{r})$ derives a singularity of the velocity (2.5) of Chapter 2 which is divided by $|\psi(t, \boldsymbol{r})|^2$, one may consider that vortices exist at the nodal singularities of the wave function [26, 28]. As noted in § 4.1.2, the solutions involving the stationary states are infinitely degenerate. One can therefore make up the wave function which has a countable number of nodes at any points in terms of the superposition of the infinitely degenerate states [33, 34].

Complex velocity potentials of stationary flows

The probability currents (4.24) now show that, for the case of $n = 0$ and 1, there are stationary flows which move along the hyperbolas (each line with v constant). To get an understanding of the physical features of these flows it is better to work with the velocity defined by (2.5) and the complex velocity potential defined by (2.19). For $n = 0$ and 1, the velocities give the same result

$$v_u^{\pm\mp} = \pm\frac{1}{2}\gamma h_u, \quad v_v^{\pm\mp} = 0. \tag{4.30}$$

Note that the nodal singularity $v = 0$ of the second of equations (4.23) does not appear in these velocities. Thus we can take their rotation, so that we obtain the vorticity (2.8)

$$\omega^{\pm\mp} \equiv h_u h_v \left[\frac{\partial}{\partial u}\left(\frac{v_v^{\pm\mp}}{h_v}\right) - \frac{\partial}{\partial v}\left(\frac{v_u^{\pm\mp}}{h_u}\right)\right] = 0.$$

These equations are the general result (2.9), and therefore the velocity potentials defined by (2.13) must exist. If we transform to rectangular hyperbolic coordinates u, v, equations

(2.13) become

$$v_u = h_u \frac{\partial \Phi}{\partial u}, \;\; v_v = h_v \frac{\partial \Phi}{\partial v}, \tag{4.31}$$

and the velocity potentials for $n = 0$ and 1 are thus

$$\Phi^{\pm\mp} = \pm\frac{1}{2}\gamma u. \tag{4.32}$$

Note that they are proportional to the phase factors of (4.23). Further, the divergence (2.16) gives

$$\nabla \cdot \boldsymbol{v}^{\pm\mp} \equiv h_u h_v \left[\frac{\partial}{\partial u}\left(\frac{v_u^{\pm\mp}}{h_v}\right) + \frac{\partial}{\partial v}\left(\frac{v_v^{\pm\mp}}{h_u}\right)\right] = 0.$$

Thus the velocities (4.30) are solenoidal, so we can obtain the stream functions defined by (2.17). The equations (2.17) are also expressed, as in equations (4.31),

$$v_u = h_v \frac{\partial \Psi}{\partial v}, \;\; v_v = -h_u \frac{\partial \Psi}{\partial u}, \tag{4.33}$$

and the stream functions for $n = 0$ and 1 are thus

$$\Psi^{\pm\mp} = \pm\frac{1}{2}\gamma v. \tag{4.34}$$

Substituting (4.34) into (2.18) we get $\omega^{\pm\mp} = 0$ again. For the states represented by the first and second of equations (4.23), the complex velocity potential (2.19) gives, from (4.32) and (4.34),

$$\begin{aligned} W^{\pm\mp} &= \pm\frac{1}{2}\gamma u \pm \frac{i}{2}\gamma v \\ &= \pm\frac{1}{2}\gamma z^2, \end{aligned} \tag{4.35}$$

since $z^2 = u + iv$. Equations (4.35) are of the form (2.21) with $a = 2$, and they show that, *for the stationary states* (4.23) *with* $n = 0$ *and* 1, *the complex velocity potentials of the 2D PPB express the flows round a right angle.* One could make calculations in terms of Cartesian coordinates and one would be led to the same conclusion.

Chapter 5

The Connection between Complex Velocity Potentials and Singular Potentials in Two Dimensions

At the end of § 4.2, we saw that the first few stationary states of the 2D PPB $V(x,y) = -m\gamma^2(x^2+y^2)/2$ (with neglect of the constant term V_0 in (4.1)) correspond to the complex velocity potentials $W^{\pm\mp} = \pm\gamma z^2/2$ (i.e. the flows round a right angle). This connection, however, does not stem from the mathematical fact that the 2D PPB can be solved exactly. In this chapter we shall investigate complex velocity potentials connected with the asymptotic flows in the two-dimensional singular potentials†

$$V_a(\rho) = -a^2 g_a \rho^{2(a-1)} \quad (a = \pm 1, \pm 2, \pm 3, \dots), \tag{5.1}$$

where $\rho = \sqrt{x^2+y^2}$ is the radius in the xy-plane and $g_a > 0$ are the coupling constants. Note here that $V_a(\rho)$ describe repulsive forces for $a > 1$ and attractive forces for $a < 1$. As a consequence of the following argument, we can deduce that $V_a(\rho)$ refer to the complex velocity potentials

$$W_a^{\pm} \approx \pm\sqrt{\frac{2g_a}{m}}z^a, \tag{5.2}$$

which are of the form (2.21) of Chapter 2. The result (4.35) of the 2D PPB is the special case of (5.2) with $a = 2$ and $g_2 = m\gamma^2/8$.

†In the three-dimensional case, see references [59, 60].

Conformal transformations of the Schrödinger equations

We shall first investigate the general structure of the Schrödinger equations written in terms of the Cartesian coordinates x, y, by applying conformal transformations to them. The time-independent Schrödinger equation reads, with the form (5.1) for the potential energies,

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)U_a(x,y)-a^2 g_a \rho^{2(a-1)}U_a(x,y)=E_a U_a(x,y). \tag{5.3}$$

Since we solve equations (5.3) on the Gel'fand triplet (4.5) as in the preceding two chapters, the energy eigenvalues E_a are generally complex numbers.

Let us consider that the complex variables $z = x + iy$ and $w_a = u_a + iv_a$ are connected by the following conformal transformations [13–15]:

$$w_a = z^a \quad (a = \pm 1, \pm 2, \pm 3, \dots). \tag{5.4}$$

Note that these transformations are *single-valued functions* for any integers a, and have a *pole of order* $|a|$ *at* $z = 0$ for a negative integer a. Introducing the two-dimensional polar coordinates ρ, φ, we have

$$\left.\begin{aligned} u_a &= \rho^a \cos a\varphi, \\ v_a &= \rho^a \sin a\varphi. \end{aligned}\right\} \tag{5.5}$$

They show that the transformations (5.4) map the region $0 < \varphi < \pi/|a|$ in the z-plane on the upper half-plane in the w_a-plane for a positive integer a and on the lower half-plane for a negative integer a.

We now use the general formula, if the two complex variables z and w are connected by the conformal transformation $w = f(z)$,

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=|f(z)'|^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right),$$

since the Jacobian can be written

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}=h_u h_v,$$

and the scale factors are now given by

$$h_u = h_v = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2}=\sqrt{\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2}=|f(z)'|$$

with the help of the Cauchy-Riemann equations between u and v. Taking (5.4), we find, since

$$h_{u_a} = h_{v_a} = a\sqrt{u_{a-1}^2 + v_{a-1}^2} = a\rho^{a-1}$$

with the help of (5.5), that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = a^2\rho^{2(a-1)}\left(\frac{\partial^2}{\partial u_a^2} + \frac{\partial^2}{\partial v_a^2}\right). \tag{5.6}$$

Thus (5.3) become, on dividing by $a^2\rho^{2(a-1)}$,

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial u_a^2} + \frac{\partial^2}{\partial v_a^2}\right)U_a(u_a, v_a) - g_a U_a(u_a, v_a) = \frac{E_a}{a^2}\rho^{2(1-a)}U_a(u_a, v_a). \tag{5.7}$$

These give our Schrödinger equations expressed in the (u_a, v_a)-representations. They are more useful than the (ρ, φ)-representation such as (B.2) of Appendix B because, while the (ρ, φ)-representation enables one to express the states having the rotational symmetry such as the eigenstates of orbital angular momentum (see § 4.1.1), the (u_a, v_a)-representations enable one so to express the states having the specific direction such as the scattering states of collision problems (see § 4.1.2), and the asymptotic states in the following discussions have specific directions connecting the potential energies $V_a(\rho)$.

The asymptotic solutions of the two-dimensional singular potentials

We shall now obtain the approximate solutions of equations (5.7) for large ρ in the case when $a > 1$, and for large $1/\rho$ or small ρ in the case when $a < 1$. The behavior of these approximate solutions in the two cases will here be called "asymptotic". Now the equations for the asymptotic states U_a are, with neglect of the right-hand sides of (5.7),

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 U_a}{\partial u_a^2} + \frac{\partial^2 U_a}{\partial v_a^2}\right) - g_a U_a \approx 0. \tag{5.8}$$

Equations (5.8) are of exactly the same simple form for all a. It is interesting to note that they are just the Schrödinger equations in the w_a-planes for the free particles with the energy eigenvalues g_a. In the special case when the potential energy becomes constant $V_1(\rho) = -g_1$ for $a = 1$, equation (5.7) can be solved exactly to give (2.22) in Example 1 of Chapter 2 with $k_1 \equiv \sqrt{k_x^2 + k_y^2} = \sqrt{2m(E_1 + g_1)}/\hbar$ and the possible energy eigenvalues for E_1 are all numbers from $-g_1$ to ∞, because the right-hand side of (5.7) has no ρ-dependence. In the serious case when equation (5.7) is not well defined for $a = 0$, we must obtain the asymptotic solutions with (ρ, φ)-representation in Appendix B.

For $a \neq 0$, we can write down solutions of (5.8) immediately, namely

$$U_a(u_a, v_a) \approx e^{i(k_{u_a} u_a + k_{v_a} v_a)}, \tag{5.9}$$

where k_{u_a}, k_{v_a} are numbers satisfying

$$k_a \equiv \sqrt{k_{u_a}^2 + k_{v_a}^2} = \sqrt{2mg_a}/\hbar > 0 \tag{5.10}$$

and the numerical coefficients are discarded. The asymptotic solutions of (5.7) are generalized functions of Gel'fand triplet of the type (4.5), since plane waves can be expressed as generalized functions of Gel'fand triplet of which nuclear space is given by a Schwartz space [10, 11]. It should be noted that the asymptotic solutions (5.9) are infinitely degenerate, on account of the different directions of the vectors (k_{u_a}, k_{v_a}) with the same magnitudes (5.10).

The infinite degeneracy of the two-dimensional parabolic potential barrier

The effect of rotations on (u_a, v_a)-representatives should be noted. Take the asymptotic solutions $U_a(u_a, v_a)$ and apply to them rotations through angles $-\alpha$ and $\pi - \alpha$ about the origin $\rho = 0$ in the w_a-planes, satisfying the condition $\tan\alpha = k_{v_a}/k_{u_a}$. They will change into [33, 34]

$$U_a^{\pm}(u_a, v_a) \approx e^{\pm i k_a u_a} \tag{5.11}$$

with k_a defined by (5.10). Putting $a = 2$, we find that

$$U_2^{\pm}(u_2, v_2) \approx e^{\pm i\beta^2 u_2/2},$$

since $k_2 = \sqrt{2mg_2}/\hbar = m\gamma/2\hbar = \beta^2/2$. They agree the first of equations (4.23) of § 4.1.2 and show that the stationary states $U_{00}^{\pm\mp}$ of the 2D PPB corresponding to the energy eigenvalues $E_{00}^{\pm\mp} = 0$ are interpreted as the asymptotic states $U_2^{\pm}$ for large ρ. Further, we see that two types of states U_{00}^{+-} and U_{00}^{-+} (in general, types 2 and 3 of § 4.1) are degenerate by the meaning of the foregoing discussion, that is, on account of the opposite directions of the vectors $(k_2, 0)$ and $(-k_2, 0)$, respectively.

However each U_{nn}^{+-} and U_{nn}^{-+} $(n = 0, 1, 2, \dots)$ of (4.23) have another kind of infinite degeneracy. The equations (4.23) for $a = 2$ can readily be extended to give the solutions of (5.8) for $a = \pm1, \pm2, \pm3, \dots$ in terms of (u_a, v_a)-representatives. If we denote such solutions by $U_{an}^{\pm}$ $(n = 0, 1, 2, \dots)$, we get [33, 34]

$$\left.\begin{aligned} U_{a0}^{\pm}(u_a, v_a) &\approx e^{\pm ik_a u_a}, \\ U_{a1}^{\pm}(u_a, v_a) &\approx 4k_a v_a e^{\pm ik_a u_a}, \\ U_{a2}^{\pm}(u_a, v_a) &\approx 4\left(4k_a^2 v_a^2 + 1 \pm 4ik_a u_a\right) e^{\pm ik_a u_a}, \\ &\dots . \end{aligned}\right\} \tag{5.12}$$

It should be noted that, for $a = 1$, equations (5.12) give exactly the solutions in the two-dimensional constant potential $V_1(\rho) = -g_1$ with $k_1 = \sqrt{2m(E_1 + g_1)}/\hbar$. They are also the exact solutions in the two-dimensional singular potentials (5.1) belonging to the zero energy eigenvalue.

Complex velocity potentials of asymptotic flows

We shall conclude this chapter with a discussion of complex velocity potentials which refer to the asymptotic states in the potential energies $V_a(\rho)$ given by (5.1). The asymptotic solutions (5.9) are of the form (2.22) with u_a, v_a for x, y and k_{u_a}, k_{v_a} for k_x, k_y, and it shows that we can get the complex velocity potentials on the same lines as the derivation of (2.25) given in Example 1 of Chapter 2. The complex velocity potentials for the asymptotic solutions (5.9) are therefore

$$\begin{aligned} W_a &\approx \hbar\,(k_{u_a} - ik_{v_a})\, w_a/m \\ &\approx \hbar\,(k_{u_a} - ik_{v_a})\, z^a/m \end{aligned} \tag{5.13}$$

with the help of (5.4), or for (5.11),

$$W_a^{\pm} \approx \pm\frac{\hbar k_a}{m} z^a. \tag{5.14}$$

If in (5.14) we put (5.10), we get formulas (5.2) mentioned at the beginning of the present chapter. One should note that equations (5.13) and (5.14) are the accurate formulas for $a = 1$ with $k_1 = \sqrt{2m(E_1 + g_1)}/\hbar$. With the help of the hydrodynamical formulation of quantum mechanics of Chapter 2, we can summarize our results of the present chapter as follows:

(i) *In the two-dimensional central potentials* $V_a(\rho) = -a^2 g_a \rho^{2(a-1)}$ $(a = 1, 2, 3, \dots)$*, the asymptotic flows for large* ρ *are the flows round the angle* π/a*; in particular, in the two-dimensional constant potential* $V_1(\rho) = -g_1$*, the flows at any point in space are the uniform flows.*

(ii) *In the two-dimensional singular potentials* $V_a(\rho) = -a^2 g_a/\rho^{2(1-a)}$ $(a = -1, -2, -3, \dots)$*, the approximate flows for small* ρ *are the two-dimensional multipole.*

As $a = 0$, the complex velocity potentials (2.41) obtained in Example 4 of Chapter 2 deduce to:

(iii) *In the two-dimensional inverse square potential* $V(\rho) = -g/\rho^2$*, the approximate flows for the neighborhood of the center* $\rho = 0$ *are the logarithmic spiral vortices.*

Chapter 6

Conclusion

In this dissertation we have studied the problem of the 2D PPB. Particularly, we have described the generalized eigenstates of this unstable system according to the hydrodynamical formulation of quantum mechanics.

In Chapter 2 we built up the hydrodynamical formulation of quantum mechanics involving the three hydrodynamical variables, namely the velocity (2.5), the vorticity (2.8) and the complex velocity potential (2.19), and we dealt with some simple dynamical systems in terms of them. The complex velocity potentials and their analytical properties are sufficient to display the two-dimensional stream lines in these usual dynamical systems.

In Chapter 3 we obtained some insight into the main features of the unstable system by making a study of the 1D PPB. The solutions of the time-independent Schrödinger equation for this system, which have complex energy eigenvalues, are generalized eigenfunctions in the conjugate space of the Gel'fand triplet $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})^{\times}$. These generalized eigenfunctions are non-stationary states. We assumed that the time factor corresponding to a non-stationary state is a square-integrable function. This assumption requires that the states of complex energy eigenvalues in the lower half-plane exist only on the future part and those of complex energy eigenvalues in the upper half-plane exist only on the past part. They correspond to the states of decay and growth, having finite lifetimes, respectively. The distributions of physical energy of these decaying and growing states are given by the Breit-Wigner resonance formulas whose half-widths are determined by the complex energy eigenvalues. We saw that these half-widths are connected with the mean lifetimes by the uncertainty relations.

In Chapter 4 we obtained first the exact solutions of the 2D PPB, using the results of Chapter 3. One class of the solutions is diverging and converging flows. These solutions always have complex energy eigenvalues and are expressed by generalized eigenfunctions, which mean that the diverging and converging flows are not stationary as same as the 1D PPB. These generalized eigenfunctions can be superposed to give the eigenstates of orbital angular momentum. The other class of the solutions is corner flows. All the solutions are infinitely degenerate and involve stationary flows with a real energy eigenvalue. It should be noted that there are no stationary flows in the 1D or 3D *isotropic* PPB. We proceeded to calculate the three hydrodynamical variables given in Chapter 2 for both these classes. Comparing the velocities of (4.25)–(4.27) and (4.30) with the probability currents of (4.17)–(4.19) and (4.24), we see that the velocities satisfy the fundamental properties (i), (ii) discussed in Chapter 1. Further, for the eigenstates of orbital angular momentum, the circulations are quantized as (4.29). From this we see, bearing in mind the argument of Chapter 2, that the vorticity has just the same properties as the angular momentum for all dynamical systems in quantum theory, including semiclassical cases. Furthermore, for the stationary states (4.23) with $n = 0$ and 1, we found the flows round a right angle that are expressed by the complex velocity potentials (4.35), but for $n \geqslant 2$ the complex velocity potentials do not exist, because the imaginary parts of the polynomials $H^{\pm}_{n_x}(\beta x)$ and $H^{\pm}_{n_y}(\beta y)$ cause the stream lines to depart from hyperbolas. This means the states for which the complex velocity potentials exist are characterized by the asymptotic solutions of the 2D PPB in the uv-plane.

In Chapter 5, by extending the above characterization connected with the complex velocity potentials of the 2D PPB to the asymptotic solutions of general two-dimensional unstable systems, we showed that the complex velocity potential has just the power law z^a for the asymptotic states with the radial power law $-\rho^{2(a-1)}$ $(a \neq 0)$ of potential energy in the w_a-plane. Now the more accurate formulas (5.2) do not contain any order of $\hbar$ at all, on account of the physical property that the asymptotic solutions of the two-dimensional singular potentials (5.1) are semiclassical. The complex velocity potentials are also sufficient to get the semiclassical property of two-dimensional unstable systems. We see in this way how the generalized eigenstates for those general unstable systems are describable from the hydrodynamical point of view.

There is some problem in connection with the infinite degeneracy of the stationary states in two-dimensional unstable systems. One expects that this infinite degeneracy is caused by certain special symmetry in Nature. We shall leave the problem of finding the unknown symmetry to the future.

The problem of comparison with observation is a more important one. Hydrodynamical variables, introduced to describe the generalized eigenstates for the 2D PPB or more general unstable systems, are semiclassical. This would mean that the stream lines in the unstable systems correspond to the motion of the dynamical system composed of many particles having the same wave function. Experiments have shown that the closed stream line which would be the picture of an eigenstate in a Hilbert space is observed (for example, the quantized vortex in superfluidity). On the other hand the open stream line which would be the picture of a generalized eigenstate in the conjugate space of a Gel'fand triplet is not known experimentally at the present time (except the free particle in collision problems). It appears that all flows occurring in classical hydrodynamics are both closed and open stream lines. To acquire a closer description of Nature, we have been taking the philosophical point of view that hydrodynamics is good enough to describe phenomena on all scales, therefore, the open stream lines in the unstable systems should be realizable in some phenomenon that occurs in quantum mechanics.

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Appendix A

The Harmonic Oscillator

THE main features of the HO will be briefly reviewed here. Details may be found in Dirac's book [35], §§ 41 and 42.

The Hamiltonian of the 1D HO is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_0 + \frac{1}{2}m\omega^2x^2, \tag{A.1}$$

where $V_0 \in \mathbb{R}$ is the minimum potential energy, $m > 0$ is the mass and $\omega > 0$ is the angular frequency. This Hamiltonian is essentially self-adjoint on a Schwartz space $\mathcal{S}(\mathbb{R}_x)$ [5]. Further, the following two conditions are satisfied [10, 11]:

(i) $\mathcal{S}(\mathbb{R}_x)$ is an invariant subspace of $\hat{H}$.

(ii) $\hat{H}$ is continuous on $\mathcal{S}(\mathbb{R}_x)$.

The time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}u(x) + \left(V_0 + \frac{1}{2}m\omega^2x^2\right)u(x) = Eu(x). \tag{A.2}$$

It is convenient to introduce the dimensionless variables

$$\xi \equiv \alpha x, \quad \alpha \equiv \sqrt{\frac{m\omega}{\hbar}}, \tag{A.3}$$

$$\lambda \equiv \frac{2(E - V_0)}{\hbar\omega}. \tag{A.4}$$

Equation (A.2) now becomes

$$\frac{d^2u}{d\xi^2} + \left(\lambda - \xi^2\right)u = 0. \tag{A.5}$$

Under the boundary condition $u \in \mathcal{S}(\mathbb{R}_x)$, this equation can be solved by the method of power series expansion. The energy eigenvalues are

$$E_n = V_0 + \left(n + \frac{1}{2}\right)\hbar\omega \in \mathbb{R} \quad (n = 0, 1, 2, \dots) \tag{A.6}$$

and the eigenfunctions are

$$u_n(x) = N_n e^{-\alpha^2 x^2/2} H_n(\alpha x) \in \mathcal{S}(\mathbb{R}_x), \tag{A.7}$$

$$N_n = \left(\frac{\alpha}{2^n n! \sqrt{\pi}}\right)^{\frac{1}{2}}. \tag{A.8}$$

Here $H_n(\alpha x)$ are the Hermite polynomials of degree n, and the constants N_n are determined by normalizing condition. The set of eigenfunctions $\{u_n\}_{n=0}^{\infty}$, given by (A.7), forms a complete orthonormal system on a Lebesgue space $L^2(\mathbb{R}_x)$ [5].

The energy eigenvalues (A.6) show that the probability densities (2.1) of the 1D HO are time-independent, so the time-differentials of them vanish in the equation of continuity (2.3).

Appendix B

The $-1/\rho^2$ Potential

We shall here examine the motion of a particle in the two-dimensional inverse square potential†

$$V(\rho) = -\frac{g}{\rho^2}, \tag{B.1}$$

which replaces (5.1) with $a = 0$.

The time-independent Schrödinger equation is, in terms of the two-dimensional polar coordinates ρ, φ,

$$-\frac{\hbar^2}{2m}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right]U(\rho,\varphi) - \frac{g}{\rho^2}U(\rho,\varphi) = EU(\rho,\varphi). \tag{B.2}$$

For an eigenstate of orbital angular momentum $\hat{L}$, given by (4.13), belonging to the eigenvalue $l\hbar$ $(l = 0, \pm1, \pm2, \dots)$ the eigenfunction will be of the form

$$U_l(\rho,\varphi) = R_l(\rho)e^{il\varphi}/\sqrt{2\pi}. \tag{B.3}$$

The factor $1/\sqrt{2\pi}$ is inserted in (B.3) for the normalization in φ. Substituting (B.3) into (B.2), we get as the equation for R_l,

$$\frac{d^2R_l}{d\rho^2} + \frac{1}{\rho}\frac{dR_l}{d\rho} + \left[\frac{2mE}{\hbar^2} + \left(\frac{2mg}{\hbar^2} - l^2\right)\frac{1}{\rho^2}\right]R_l = 0. \tag{B.4}$$

We must now examine how our solution $R_l(\rho)$ behaves for the neighborhood of the center $\rho = 0$. For small values of ρ equation (B.4) is

$$\frac{d^2R_l}{d\rho^2} + \frac{1}{\rho}\frac{dR_l}{d\rho} + \frac{\gamma}{\rho^2}R_l \approx 0, \tag{B.5}$$

†To reveal certain properties of quantum motion, Landau and Lifshitz examined the three-dimensional case, see reference [2], § 35.

where

$$\gamma \equiv \frac{2mg}{\hbar^2} - l^2. \tag{B.6}$$

We must distinguish between the two cases of γ positive and γ negative. For γ negative, the solutions of (B.5) will be $\rho^{\sqrt{-\gamma}}$ and $\rho^{-\sqrt{-\gamma}}$. However one solution $\rho^{-\sqrt{-\gamma}}$ will tend to infinity as $\rho \to 0$ and will not represent a physically possible state. Thus, for negative values of γ, the permissible solution of (B.5) is

$$R_l(\rho) \approx \rho^{\sqrt{-\gamma}}. \tag{B.7}$$

It should be noted that this approximate solution does not give the whole solution of (B.4) with $E = 0$, since the solution (B.7) will tend to infinity as $\rho \to \infty$.

For any positive values of γ, the solutions of (B.5) will be

$$R_l^{\pm}(\rho) \approx \rho^{\pm i\sqrt{\gamma}}. \tag{B.8}$$

These approximate solutions are physically permissible, since, although the solutions (B.8) remain finite as $\rho \to 0$ or ∞, they will not tend to infinity. According to Example 4 of Chapter 2, $R_l^{+}(\rho)$ represents particles rising from the center, and $R_l^{-}(\rho)$ represents particles falling to the center.

Appendix C

Properties of the Polynomials $H_n^{\pm}(\xi)$

OUR work in § 3.1.1 led us to introduce the polynomials of degree n, namely $H_n^{\pm}(\xi)$, satisfying the differential equations (3.11) with (3.16)

$$\frac{d^2 H_n^{\pm}}{d\xi^2} \pm 2i\xi \frac{dH_n^{\pm}}{d\xi} \mp 2inH_n^{\pm} = 0. \tag{C.1}$$

If we use the analytical continuation [44], the solutions of these equations will be written

$$H_n^{\pm}(\xi) = e^{\pm in\pi/4} H_n(e^{\mp i\pi/4}\xi), \tag{C.2}$$

$H_n(e^{\mp i\pi/4}\xi)$ are the Hermite polynomials of degree n. The phase factors $e^{\pm in\pi/4}$ must be inserted in (C.2) so that the highest power of ξ appears with the coefficient 2^n. The successive polynomials calculated from (3.13) or (C.2) are

$$\begin{aligned} H_0^{\pm}(\xi) =& 1, \;\; H_1^{\pm}(\xi) = 2\xi, \;\; H_2^{\pm}(\xi) = 4\xi^2 \mp 2i, \\ H_3^{\pm}(\xi) =& 8\xi^3 \mp 12i\xi, \;\; H_4^{\pm}(\xi) = 16\xi^4 \mp 48i\xi^2 - 12, \ldots . \end{aligned}$$

These polynomials in general can be expressed as

$$H_n^{\pm}(\xi) = (\mp i)^n \, e^{\mp i\xi^2} \frac{d^n}{d\xi^n} e^{\pm i\xi^2}. \tag{C.3}$$

Thus the parity of $H_n^{\pm}(\xi)$ is

$$H_n^{\pm}(-\xi) = (-)^n H_n^{\pm}(\xi), \tag{C.4}$$

and the conjugate complex of $H_n^{\pm}(\xi)$ is

$$H_n^{\pm}(\xi)^* = H_n^{\mp}(\xi). \tag{C.5}$$

An alternative way of defining the polynomials $H_n^{\pm}(\xi)$ is as the generating functions $S^{\pm}(\xi, s)$ given by

$$S^{\pm}(\xi, s) \equiv e^{\mp i(2s\xi - s^2)} = \sum_{n=0}^{\infty} \frac{H_n^{\pm}(\xi)}{n!}(\mp is)^n. \tag{C.6}$$

From this definition we obtain the recurrence formulas

$$H_{n+1}^{\pm} - 2\xi H_n^{\pm} \pm 2inH_{n-1}^{\pm} = 0, \tag{C.7}$$

$$dH_n^{\pm}/d\xi = 2nH_{n-1}^{\pm}. \tag{C.8}$$

Corresponding to the orthogonality relation of the Hermite polynomials, we now have, from (C.6)

$$\int_{-\infty}^{\infty} H_m^{\mp}(\xi)^* H_n^{\pm}(\xi) e^{\pm i\xi^2} d\xi = (\pm 2i)^n n! \sqrt{\pm i\pi}\,\delta_{mn}, \tag{C.9}$$

where δ_{mn} is the Kronecker delta symbol.

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