Chapter 2

Poincaré formulas

2.1 Coarea formula

Before the main discussion of this chapter, we shall introduce a fundamental theorem called "*coarea formula*", which plays an important part in this thesis.

Definition 2.1.1. Let V and W be inner product spaces of dimension m and n respectively, and assume that $m \geq n$. Let $F : V \to W$ be a linear mapping. Then we define the *Jacobian* JF of F by

$$JF = \sup\{\|F(u_1) \wedge \cdots \wedge F(u_n)\| \mid u_1, \cdots, u_n \text{ is an orthonormal system of } V\}.$$

Proposition 2.1.2. *If F is not an onto mapping then $JF = 0$. If F is an onto mapping then*

$$JF = \frac{\|F(v_1) \wedge \cdots \wedge F(v_n)\|}{\|v_1 \wedge \cdots \wedge v_n\|}$$

for any basis $v_1, \cdots, v_n$ of $(\mathrm{Ker}F)^{\perp}$.

Definition 2.1.3. Let M and N be Riemannian manifolds of dimension m and n respectively, and assume that $m \geq n$. Let $f : M \to N$ be a smooth mapping. Then $x \in M$ is a *regular point* of f if and only if $df_x : T_xM \to T_{f(x)}N$ is an onto mapping, and is a *critical point* otherwise. And $y \in N$ is a *regular value* of f if and only if every point of $f^{-1}(y)$ is a regular point of f, and is a *critical value* otherwise.

Theorem 2.1.4. (Sard's theorem)
The set of critical values of f has measure zero.

If $y \in N$ is a regular value of f then $f^{-1}(y)$ is empty or a closed embedded $(m-n)$-dimensional submanifold of M, by the implicit function theorem. Therefore if ϕ is a measurable function on M, then the function

$$y \mapsto \int_{f^{-1}(y)} \phi(x) d\mu_{f^{-1}(y)}(x)$$

is defined for all regular values of f and thus almost everywhere on N. So this is measurable function on N.

Theorem 2.1.5. (Coarea formula) *If $\phi J f$ is integrable on M or $\phi \geq 0$ then*

$$\int_N \left(\int_{f^{-1}(y)} \phi(x) d\mu_{f^{-1}(y)}(x) \right) d\mu_N(y) = \int_M \phi(x) J f(x) d\mu_M(x),$$

where $Jf(x)$ is the Jacobian of the linear mapping $df_x : T_x M \to T_{f(x)} N$.

2.2 Generalized Poincaré formula

In this section, we shall review the generalized Poincaré formula for Riemannian homogeneous spaces obtained by Howard [13]. We begin with the definition of the angle between subspaces in an inner product space.

Definition 2.2.1. Let E be a finite dimensional real vector space with an inner product $\langle \, , \, \rangle$. For two vector subspaces V and W of dimensions p and q in E, we take orthonormal bases $v_1, \ldots, v_p$ and $w_1, \ldots, w_q$ of V and W respectively. And we define

$$\sigma(V, W) = \|v_1 \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge w_q\|$$

where

$$\|x_1 \wedge \cdots \wedge x_k\|^2 = \det(\langle x_i, x_j \rangle).$$

This definition is independent of the choice of orthonormal bases. Under this definition the following proposition is clear.

Proposition 2.2.2. $0 \leq \sigma(V, W) \leq 1$ *with* $\sigma(V, W) = 0$ *if and only if* $V \cap W \neq \{0\}$ *and* $\sigma(V, W) = 1$ *if and only if V is perpendicular to W. Also if ρ is a linear isometry of E, then*

$$\sigma(\rho V, \rho W) = \sigma(V, W) \tag{2.2.1}$$

Furthermore if $p + q = \dim E$, then

$$\sigma(V, W) = \sigma(V^{\perp}, W^{\perp}). \tag{2.2.2}$$

Let G be a Lie group and K a compact subgroup of G. We denote by π the natural projection from G to G/K. Then $\pi(e)$ is called the origin of a homogeneous space G/K and denoted by o, where e is the identity element of G. We assume that G has a left invariant Riemannian metric that is also invariant under the right action of K. This metric induces an invariant Riemannian metric on G/K. And then π becomes a Riemannian submersion.

We shall define the angle of subspaces tangent to homogeneous space G/K at different points. If V is a subspace of $T_x(G/K)$ and W is a subspace of $T_y(G/K)$ then there are g_x and g_y in G with $g_x o = x$ and $g_y o = y$. We translate V and W to the subspaces of $T_o(G/K)$ by the differential mapping of translations g_x and g_y, and try to define the angle between V and W as the angle between $(g_x)_*^{-1}V$ and $(g_y)_*^{-1}W$. But the choice of g_x and g_y is not unique, so it is not well-defined. We can overcome this problem by averaging over all possible choice of g_y.

Definition 2.2.3. For x and y in G/K and vector subspaces V in $T_x(G/K)$ and W in $T_y(G/K)$, we define $\sigma_K(V, W)$ the angle between V and W by

$$\sigma_K(V, W) = \int_K \sigma((g_x)_*^{-1}V, k_*^{-1}(g_y)_*^{-1}W)dk$$

where g_x and g_y are in G with $g_x o = x$ and $g_y o = y$.

K is a unimodular Lie group because K is compact, this implies that the measure of K is invariant under the changes of variable $k \mapsto k^{-1}$, left and right translations. If g'_x and g'_y are any other elements of G with $g'_x o = x$ and $g'_y o = y$, then $g'_x = g_x a$ and $g'_y = g_y b$ for some a and b in K. Therefore

$$\begin{aligned}
& \int_K \sigma((g'_x)_*^{-1}V, k_*^{-1}(g'_y)_*^{-1}W)dk \\
= & \int_K \sigma(a_*^{-1}(g_x)_*^{-1}V, k_*^{-1}b_*^{-1}(g_y)_*^{-1}W)dk \\
= & \int_K \sigma((g_x)_*^{-1}V, a_* k_*^{-1}b_*^{-1}(g_y)_*^{-1}W)dk \qquad \text{(by (2.2.1))} \\
= & \int_K \sigma((g_x)_*^{-1}V, k_*^{-1}(g_y)_*^{-1}W)dk.
\end{aligned}$$

Thus Definition 2.2.3 is independent of the choice of g_x and g_y.

Proposition 2.2.4. *For all g in G*

(1) $\sigma_K(V,W) = \sigma_K(g_*V,W) = \sigma_K(V,g_*W)$,

(2) $0 \le \sigma_K(V,W) \le \mathrm{vol}(K)$.

Furthermore if $\dim V + \dim W = \dim(G/K)$ *then*

(3) $\sigma_K(V,W) = \sigma_K(V^\perp, W^\perp)$.

Proof. (1) Put $z = gx$ then $g_*V \in T_z(G/K)$ and $gg_xo = z$. From Definition 2.2.3

$$\begin{aligned}\sigma_K(g_*V,W) &= \int_K \sigma((gg_x)_*^{-1}g_*^{-1}V,\ k_*^{-1}(g_y)_*^{-1}W)dk \\ &= \int_K \sigma((g_x)_*^{-1}V,\ k_*^{-1}(g_y)_*^{-1}W)dk \\ &= \sigma_K(V,W).\end{aligned}$$

By the same way we have $\sigma_K(V,W) = \sigma_K(V,g_*W)$. (2) and (3) are obvious from Proposition 2.2.2.

With these notations, the generalized Poincaré formula for homogeneous spaces can be stated as following.

Theorem 2.2.5. ([13]) *Let G/K be a Riemannian homogeneous space, and assume that G is unimodular, that is, $|\det \mathrm{Ad}(g)| = 1$ for all $g \in G$. Then for any submanifolds M and N of G/K with* $\dim M + \dim N \ge \dim(G/K)$

$$\int_G \mathrm{vol}(M \cap gN)dg = \int_{M\times N} \sigma_K(T_x^\perp M, T_y^\perp N)d\mu(x,y).$$

Proof. The proof which here we show was given by Tasaki [32].
We set

$$I(G\times(G/K)^2) = \{(g,x,y) \in G\times(G/K)^2 \mid x = gy\}.$$

At first we show that $I(G\times(G/K)^2)$ is a regular submanifold of $G\times(G/K)^2$. We define the mapping

$$p : G\times(G/K)^2 \to (G/K)^2\ ;\ (g,x,y) \mapsto (x,gy).$$

Then p is a submersion because its differential mapping is surjective. We put

$$D(G/K) = \{(x,x) \in (G/K)^2 \mid x \in G/K\}.$$

Then $D(G/K)$ is a regular submanifold of $(G/K)^2$. Therefore $p^{-1}(D(G/K))$, inverse image of $D(G/K)$ under the submersion p, is a regular submanifold of $G \times (G/K)^2$. By the definitions we have

$$I(G \times (G/K)^2) = p^{-1}(D(G/K)).$$

Thus $I(G \times (G/K)^2)$ is a regular submanifold of $G \times (G/K)^2$. We note that

$$\dim I(G \times (G/K)^2) = \dim G + \dim(G/K).$$

We define the mapping

$$q : I(G \times (G/K)^2) \to (G/K)^2 \ ; \ (g,x,y) \mapsto (x,y).$$

We will show that q gives a fiber bundle with fiber K. For $(x,y) \in (G/K)^2$ we take $g_x, g_y \in G$ with $g_x o = x$ and $g_y o = y$. Then obviously

$$q^{-1}(x,y) \supset (g_x K g_y^{-1}) \times \{(x,y)\}.$$

Conversely, since $x = gy$ for $(g,x,y) \in q^{-1}(x,y)$ we have

$$o = g_x^{-1}x = g_x^{-1}gy = g_x^{-1}gg_y o.$$

This implies $g_x^{-1}gg_y \in K$. Thus

$$q^{-1}(x,y) = (g_x K g_y^{-1}) \times \{(x,y)\}.$$

Therefore $q^{-1}(x,y)$ is diffeomorphic with K.

We denote by $\mathfrak{g}$ Lie algebra of Lie group G. The invariant Riemannian metric on G/K induces $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{g}$. Then we have the direct orthogonal decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$, where $\mathfrak{k}$ is Lie algebra of K. Then we can identify $T_o(G/K)$ with $\mathfrak{m}$. There are open neighborhood U of $0 \in \mathfrak{m}$ and open neighborhood V of $o \in G/K$ such that the mapping

$$U \to V \ ; \ u \mapsto (\exp u)o$$

becomes a diffeomorphism. And then

$$\begin{aligned} U \to g_x V \quad &; \quad u \mapsto g_x(\exp u)o, \\ U \to g_y V \quad &; \quad v \mapsto g_y(\exp v)o \end{aligned}$$

are also diffeomorphisms. So we define the mapping

$$\varphi : K \times g_x V \times g_y V \to q^{-1}(g_x V \times g_y V)$$

by

$$\begin{aligned} & \varphi(k, g_x(\exp u)o, g_y(\exp v)o) \\ = \;& (g_x(\exp u)k \exp(-v)g_y^{-1}, g_x(\exp u)o, g_y(\exp v)o). \end{aligned}$$

This mapping is C^∞-class. And the inverse mapping

$$\varphi^{-1} : q^{-1}(g_x V \times g_y V) \to K \times g_x V \times g_y V$$

is given by

$$\begin{aligned} & \varphi^{-1}(g, g_x(\exp u)o, g_y(\exp v)o) \\ = \;& (\exp(-u)g_x^{-1} g g_y(\exp v), g_x(\exp u)o, g_y(\exp v)o), \end{aligned}$$

this is also C^∞-class. Consequently φ is a diffeomorphism. Therefore φ shows the local triviality of q. Thus q is a fiber bundle with fiber K.

By direct calculation we have

$$\begin{aligned} & \varphi_*((L_k)_* T, (g_x)_* X, (g_y)_* Y) \\ = \;& ((L_{\varphi(k,x,y)})_* \mathrm{Ad}(g_y)(T + \mathrm{Ad}(k^{-1})X - Y), (g_x)_* X, (g_y)_* Y) \end{aligned}$$

for $T \in \mathfrak{k}$ and $X, Y \in \mathfrak{m}$. $q^{-1}(M \times N)$ is a submanifold of $I(G \times (G/K)^2)$ because q is a submersion. We note that

$$\begin{aligned} \dim(q^{-1}(M \times M)) &= \dim M + \dim N + \dim K \\ &\geq \dim(G/K) + \dim K \\ &= \dim G. \end{aligned}$$

Define the mapping

$$f : q^{-1}(M \times N) \to G \; ; \; (g, x, y) \mapsto g,$$

then f is a C^∞-mapping. Applying Theorem 2.1.5 to $q^{-1}(M \times N)$ with $\phi \equiv 1$ we have

$$\int_G \mathrm{vol}(f^{-1}(g))dg = \int_{q^{-1}(M \times N)} Jf d\mu_{q^{-1}(M \times N)}.$$

Here

$$\begin{aligned} f^{-1}(g) &= q^{-1}(M \times N) \cap (\{g\} \times (G/K)^2) \\ &= \{(g, gy, y) \mid gy \in M, y \in N\} \\ &= \{(g, gy, y) \mid gy \in M \cap gN\}, \end{aligned}$$

so the mapping

$$\psi : M \cap gN \to f^{-1}(g) \ ; \ gy \mapsto (g, gy, y)$$

is bijective. If g is a regular value of f, then $f^{-1}(g)$ is a submanifold of $q^{-1}(M \times N)$. Moreover ψ is a diffeomorphism. For $X \in T_y(g^{-1}(M \cap gN))$ we have

$$\psi_*(g_*X) = (0, g_*X, X).$$

Since g_* is a linear isometry, we have

$$\langle \psi_*(g_*X), \psi_*(g_*Y) \rangle = \langle g_*X, g_*Y \rangle + \langle X, Y \rangle = 2\langle X, Y \rangle.$$

Put

$$r = \dim(f^{-1}(g)) = \dim(M \cap gN) = \dim M + \dim N - \dim(G/K).$$

Then we have

$$\mathrm{vol}(f^{-1}(g)) = 2^{r/2}\mathrm{vol}(M \cap gN).$$

Therefore

$$2^{r/2} \int_G \mathrm{vol}(M \cap gN) dg = \int_{q^{-1}(M \times N)} Jf \, d\mu_{q^{-1}(M \times N)}.$$

Now we shall calculate the right hand side of this equation. We take $\{T_a\}$, $\{X_b\}$ and $\{Y_c\}$ orthonormal bases of $\mathfrak{k}$, $(g_x)_*^{-1}(T_xM)$ and $(g_y)_*^{-1}(T_yN)$ respectively. Then for the differential mapping of φ at $(k, x, y) \in K \times M \times N$ we have that

$$\begin{aligned} \varphi_*((L_k)_*T_a, 0, 0) &= \left((L_{\varphi(k,x,y)})_*\mathrm{Ad}(g_y)T_a, 0, 0\right) \\ \varphi_*(0, (g_x)_*X_b, 0) &= \left((L_{\varphi(k,x,y)})_*\mathrm{Ad}(g_y)\mathrm{Ad}(k^{-1})X_b, (g_x)_*X_b, 0\right) \\ \varphi_*(0, 0, (g_y)_*Y_c) &= \left(-(L_{\varphi(k,x,y)})_*\mathrm{Ad}(g_y)Y_c, 0, (g_y)_*Y_c\right) \end{aligned}$$

is a basis of $T_{\varphi(k,x,y)}(q^{-1}(M \times N))$. We note that

$$\begin{aligned} & f_*\varphi_*\left((L_k)_*T_a, (g_x)_*X_b, (g_y)_*Y_c\right) \\ = \; & (L_{\varphi(k,x,y)})_*\mathrm{Ad}(g_y)\left(T_a + \mathrm{Ad}(k^{-1})X_b - Y_c\right). \end{aligned}$$

If we extend $X_d = \mathrm{Ad}(k)Y_d$ $(1 \le d \le r)$ to an orthonormal basis $\{X_b\}$ of $(g_x)_*^{-1}(T_xM)$, then

$$\varphi_*(0, (g_x)_*X_d, (g_y)_*Y_d) = (0, (g_x)_*X_d, (g_y)_*Y_d) \qquad (1 \le d \le r)$$

is a basis of $\mathrm{Ker}(f_*)$. Therefore if we put

$$\begin{aligned} \bar{T}_a &= ((L_k)_*T_a, 0, 0) \\ \bar{X}_b &= (0, (g_x)_*X_b, 0) \qquad (r+1 \le b) \\ \bar{Y}_c &= (0, 0, (g_y)_*Y_c) \qquad (r+1 \le c) \\ \bar{Z}_d &= \frac{1}{\sqrt{2}}(0, (g_x)_*X_d, -(g_y)_*Y_d) \qquad (1 \le d \le r) \end{aligned}$$

then these vectors make a orthonormal system and

$$\varphi_*(\bar{T}_a), \quad \varphi_*(\bar{X}_b), \quad \varphi_*(\bar{Y}_c), \quad \varphi_*(\bar{Z}_d)$$

is a basis of $(\mathrm{Ker}(f_*))^{\perp}$. Moreover we have

$$\begin{aligned} f_*\varphi_*(\bar{T}_a) &= (L_{\varphi(k,x,y)})_*\mathrm{Ad}(g_y)(T_a) \\ f_*\varphi_*(\bar{X}_b) &= (L_{\varphi(k,x,y)})_*\mathrm{Ad}(g_y)(\mathrm{Ad}(k^{-1})X_b) \\ f_*\varphi_*(\bar{Y}_c) &= (L_{\varphi(k,x,y)})_*\mathrm{Ad}(g_y)(-Y_c) \\ f_*\varphi_*(\bar{Z}_d) &= (L_{\varphi(k,x,y)})_*\mathrm{Ad}(g_y)(\sqrt{2}Y_d). \end{aligned}$$

Hence from Proposition 2.1.2 we have

$$\begin{aligned} & Jf \\ = \; & \left\|(L_{\varphi(k,x,y)})_*\mathrm{Ad}(g_y)\left(\wedge_a T_a \wedge \wedge_b \mathrm{Ad}(k^{-1})X_b \wedge \wedge_c(-Y_c) \wedge \wedge_d \sqrt{2}Y_d\right)\right\| \\ = \; & 2^{r/2}\left\|\mathrm{Ad}(g_y)\left(\wedge_a T_a \wedge \wedge_b \mathrm{Ad}(k^{-1})X_b \wedge \wedge_c Y_c \wedge \wedge_d Y_d\right)\right\| \\ = \; & 2^{r/2}|\det \mathrm{Ad}(g_y)|\|\wedge_a T_a \wedge \wedge_b \mathrm{Ad}(k^{-1})X_b \wedge \wedge_c Y_c \wedge \wedge_d Y_d\| \\ & (\text{since } G \text{ is unimodular, } |\det \mathrm{Ad}(g_y)| = 1) \end{aligned}$$

$$
\begin{aligned}
&= 2^{r/2}\| \wedge_a T_a \wedge \wedge_b \mathrm{Ad}(k^{-1})X_b \wedge \wedge_c Y_c \wedge \wedge_d Y_d \| \\
&\quad (\text{since } \mathrm{Ad}(k^{-1})X_b \in \mathfrak{m}) \\
&= 2^{r/2}\| \wedge_b \mathrm{Ad}(k^{-1})X_b \wedge \wedge_c Y_c \wedge \wedge_d Y_d \| \\
&\quad (\text{since } \mathrm{Ad}(k^{-1})X_b \perp \mathrm{Ad}(k^{-1})X_d = Y_d \perp Y_c) \\
&= 2^{r/2}\| \wedge_b \mathrm{Ad}(k^{-1})X_b \wedge \wedge_c Y_c \| \\
&= 2^{r/2}\sigma\left((g_x)_*^{-1}(T_x^\perp M), \mathrm{Ad}(k^{-1})(g_y)_*^{-1}(T_y^\perp N)\right) \\
&= 2^{r/2}\sigma\left((g_x)_*^{-1}(T_x^\perp M), k_*^{-1}(g_y)_*^{-1}(T_y^\perp N)\right).
\end{aligned}
$$

Its integral on K is equal to $2^{r/2}\sigma_K(T_x^\perp M, T_y^\perp N)$. Consequently we have

$$\int_{q^{-1}(M\times N)} Jf\, d\mu_{q^{-1}(M\times N)} = 2^{r/2}\int_{M\times N} \sigma_K(T_x^\perp M, T_y^\perp N)d\mu(x,y),$$

and thus

$$\int_G \mathrm{vol}(M\cap gN)dg = \int_{M\times N} \sigma_K(T_x^\perp M, T_y^\perp N)d\mu(x,y).$$

This completes the proof.

Remark 2.2.6. By Definition 2.2.3, it is clear that if G/K and G'/K' have same isotropy representation then σ_K and $\sigma_{K'}$ coincide with each other. Thus it holds same Poincaré formula in G'/K' as in G/K by Theorem 2.2.5. This is called the "*transfer principle*".

Remark 2.2.7. Since K acts isometrically on $T_o(G/K)$, K acts isometrically on the Grassmannian manifold $G_p(T_o(G/K))$ of all p-dimensional subspaces of $T_o(G/K)$. In the definition of σ_K, we can reduce the integral on K to an integral on an orbit of this action by the fiber integration.

Definition 2.2.8. Let V_o be a p-dimensional subspace of $T_o(G/K)$. Then a p-dimensional submanifold M of G/K is of *type* V_o if and only if for all x in M there exists g in G with $g_*V_o = T_xM$.

If M is a submanifold of G/K of type V_o, then $\sigma_K(T_x^\perp M, T_y^\perp N)$ becomes to an integral on a same orbit of the action in Remark 2.2.7 for any $x \in M$. This implies that $\sigma_K(T_x^\perp M, T_y^\perp N)$ is independent of $x \in M$. Furthermore if N is a submanifold of type W_o, then $\sigma_K(T_x^\perp M, T_y^\perp N)$ is also independent of $y \in N$. So we have the following corollary.

Corollary 2.2.9. *Under the hypothesis of Theorem 2.2.5*

(1) *If M is a submanifold of G/K of type V_o for some subspace V_o of $T_o(G/K)$, then*

$$\int_G \mathrm{vol}(M \cap gN)dg = \mathrm{vol}(M) \int_N \sigma_K(V_o^\perp, T_y^\perp N) d\mu_N(y).$$

(2) *If in addition to the hypothesis of* (1), N is a submanifold of G/K of type W_o for some subspace W_o of $T_o(G/K)$, then

$$\int_G \mathrm{vol}(M \cap gN)dg = \sigma_K(V_o^\perp, W_o^\perp)\mathrm{vol}(M)\mathrm{vol}(N).$$

2.3 Examples of Poicaré formula

However Theorem 2.2.5 is a general shape, it is not easy to get a concrete expression of Poincaré formula in general. In this section we will give some examples which we can obtain the explicit expressions of Poincaré formula applying Theorem 2.2.5.

2.3.1 The case of real space forms

A simply connected, complete Riemannian manifold which has constant sectional curvature is called a *real space form.* Here we consider the Poincaré formula in real space forms. It is well-known fact that multiplying the Riemannian metric by some constants, all n-dimensional real space forms are isometric with one of the following typical cases.

(1) Euclidean space $\mathbb{R}^n$. Put

$$\begin{aligned} G &= \left\{ \begin{bmatrix} A & x \\ O & 1 \end{bmatrix} \;\middle|\; A \in SO(n),\ x \in \mathbb{R}^n \right\}, \\ K &= \left\{ \begin{bmatrix} A & O \\ O & 1 \end{bmatrix} \;\middle|\; A \in SO(n) \right\} \cong SO(n). \end{aligned}$$

Then G can be regarded as the identity component of the group of isometries of $\mathbb{R}^n$, and $\mathbb{R}^n$ is identified with Riemannian homogeneous space G/K.

(2) Unit sphere S^n in $\mathbb{R}^{n+1}$. Put

$$\begin{aligned} G &= SO(n+1), \\ K &= \left\{ \begin{bmatrix} A & O \\ O & 1 \end{bmatrix} \;\middle|\; A \in SO(n) \right\} \cong SO(n). \end{aligned}$$

Then G can be regarded as the identity component of the group of isometries of S^n, and S^n is identified with Riemannian homogeneous space G/K.

(3) Hyperbolic space $\mathbb{R}H^n$ in $\mathbb{R}^{n+1}$, that is

$$\mathbb{R}H^n = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1,\ x_{n+1} > 1\}.$$

Put

$$\begin{aligned} G' &= \{g \in GL(n+1, \mathbb{R}) \mid {}^t g I_{n,1} g = I_{n,1}\}, \\ K &= \left\{ \begin{bmatrix} A & O \\ O & 1 \end{bmatrix} \;\middle|\; A \in SO(n) \right\} \cong SO(n), \end{aligned}$$

where

$$I_{n,1} = \begin{bmatrix} I_n & O \\ O & -1 \end{bmatrix}.$$

Then G' acts on $\mathbb{R}^{n+1}$ preserving the bilinear form defined by $I_{n,1}$. We denote by G the identity component of G'. Then G acts transitively on $\mathbb{R}H^n$ with isotropy K. Therefore $\mathbb{R}H^n$ is identified with Riemannian homogeneous space G/K.

Theorem 2.3.1. ([13]) *Let G/K be an n-dimensional real space form. If M is a p-dimensional submanifold and N is a q-dimensional submanifold of G/K with $p + q \geq n$, then*

$$\int_G \mathrm{vol}(M \cap gN)dg = \frac{\mathrm{vol}(S^{p+q-n})\mathrm{vol}(SO(n+1))}{\mathrm{vol}(S^p)\mathrm{vol}(S^q)} \mathrm{vol}(M)\mathrm{vol}(N).$$

Proof. K acts transitively on the Grassmannian manifold $G_p(T_o(G/K))$ of all p-dimensional subspaces in $T_o(G/K)$ for any p This implies that any p-dimensional submanifold of G/K is type V_o for any p-dimensional subspace

V_o in $T_o(G/K)$, and any q-dimensional submanifold is type W_o for any q-dimensional subspace W_o. Therefore from Corollary 2.2.9 we have

$$\int_G \mathrm{vol}(M \cap gN)dg = \sigma_K(V_o^\perp, W_o^\perp)\mathrm{vol}(M)\mathrm{vol}(N).$$

By Remark 2.2.6, $\sigma_K(V_o^\perp, W_o^\perp)$ coincides for all real space forms. Consider the case where $G/K = S^n$, $M = S^p$ and $N = S^q$. Then $S^p \cap gS^q = S^{p+q-n}$ for almost all $g \in SO(n+1)$. So we have

$$\mathrm{vol}(S^{p+q-n})\mathrm{vol}(SO(n+1)) = \sigma_K(V_o^\perp, W_o^\perp)\mathrm{vol}(S^p)\mathrm{vol}(S^q).$$

Hence

$$\sigma_K(V_o^\perp, W_o^\perp) = \frac{\mathrm{vol}(S^{p+q-n})\mathrm{vol}(SO(n+1))}{\mathrm{vol}(S^p)\mathrm{vol}(S^q)}.$$

This completes the proof.

2.3.2 The case of complex space forms

A simply connected, complete Kähler manifold which has constant holomorphic sectional curvature is called a *complex space form*. Here we consider the Poincaré formula in complex space forms. Multiplying the Riemannian metric by some constants all n-dimensional complex space forms are isometric with one of the following typical cases.

(1) Complex Euclidean space $\mathbb{C}^n$. Put

$$\begin{aligned} G &= \left\{ \begin{bmatrix} A & x \\ O & u \end{bmatrix} \;\middle|\; A \in U(n),\ x \in \mathbb{C}^n,\ u \in U(1) \right\}, \\ K &= \left\{ \begin{bmatrix} A & O \\ O & u \end{bmatrix} \;\middle|\; A \in U(n),\ u \in U(1) \right\} \cong U(n) \times U(1). \end{aligned}$$

Then $\mathbb{C}^n$ is identified with Riemannian homogeneous space G/K.

(2) Complex projective space $\mathbb{C}P^n$. Put

$$\begin{aligned} G &= U(n+1), \\ K &= \left\{ \begin{bmatrix} A & O \\ O & u \end{bmatrix} \;\middle|\; A \in U(n),\ u \in U(1) \right\} \cong U(n) \times U(1). \end{aligned}$$

Then $\mathbb{C}P^n$ is identified with Riemannian homogeneous space G/K.

(3) Complex hyperbolic space $\mathbb{C}H^n$. Put

$$\begin{aligned} G &= \{g \in GL(n+1,\mathbb{C}) \mid {}^t g I_{n,1} \bar{g} = I_{n,1}\}, \\ K &= \left\{ \begin{bmatrix} A & O \\ O & u \end{bmatrix} \;\middle|\; A \in U(n),\ u \in U(1) \right\} \cong U(n) \times U(1). \end{aligned}$$

Then G acts on $\mathbb{C}^{n+1}$ preserving the Hermitian form defined by $I_{n,1}$. Moreover G acts transitively on $\mathbb{R}H^n$. Therefore $\mathbb{C}H^n$ is identified with Riemannian homogeneous space G/K.

Theorem 2.3.2. ([13]) *Let G/K be an n-dimensional complex space form.*

(a) *If M is a complex submanifold in G/K of complex dimension p and N is a complex submanifold of complex dimension q with $p+q \geq n$, then*

$$\int_G \mathrm{vol}(M \cap gN)dg = \frac{\mathrm{vol}(\mathbb{C}P^{p+q-n})\mathrm{vol}(U(n+1))}{\mathrm{vol}(\mathbb{C}P^p)\mathrm{vol}(\mathbb{C}P^q)}\mathrm{vol}(M)\mathrm{vol}(N).$$

(b) *If M is a totally real submanifold in G/K of real dimension p and N is a complex submanifold of complex dimension q with $p+2q \geq 2n$, then*

$$\int_G \mathrm{vol}(M \cap gN)dg = \frac{\mathrm{vol}(\mathbb{R}P^{p+2q-2n})\mathrm{vol}(U(n+1))}{\mathrm{vol}(\mathbb{R}P^p)\mathrm{vol}(\mathbb{C}P^q)}\mathrm{vol}(M)\mathrm{vol}(N).$$

(c) *If M and N are totally real submanifolds in G/K of real dimension n, then*

$$\int_G \sharp(M \cap gN)dg = \frac{(n+1)\mathrm{vol}(U(n+1))}{(\mathrm{vol}(\mathbb{R}P^n))^2}\mathrm{vol}(M)\mathrm{vol}(N).$$

Proof. Since K is isomorphic to $U(n)\times U(1)$, K acts transitively on the complex Grassmannian manifold of all p-dimensional complex subspaces in $T_o(G/K)$ and the set of all q-dimensional totally real subspaces in $T_o(G/K)$ for any p and q. This implies that any p-dimensional complex submanifold of G/K is type V_o for any p-dimensional complex subspace V_o in $T_o(G/K)$, and any q-dimensional totally real submanifold is type W_o for any q-dimensional totally real subspace W_o. Therefore from Corollary 2.2.9 we have

$$\int_G \mathrm{vol}(M \cap gN)dg = \sigma_K(V_o^\perp, W_o^\perp)\mathrm{vol}(M)\mathrm{vol}(N)$$

for each case (a), (b) and (c). By Remark 2.2.6, $\sigma_K(V_o^\perp, W_o^\perp)$ coincides for all complex space forms. Therefore it is enough to consider the case where $G/K = \mathbb{C}P^n$.

(a) Let $M = \mathbb{C}P^p$ and $N = \mathbb{C}P^q$, then $\mathbb{C}P^p \cap g\mathbb{C}P^q = \mathbb{C}P^{p+q-n}$ for almost all $g \in U(n+1)$. So we have

$$\mathrm{vol}(\mathbb{C}P^{p+q-n})\mathrm{vol}(U(n+1)) = \sigma_K(V_o^\perp, W_o^\perp)\mathrm{vol}(\mathbb{C}P^p)\mathrm{vol}(\mathbb{C}P^q).$$

Hence

$$\sigma_K(V_o^\perp, W_o^\perp) = \frac{\mathrm{vol}(\mathbb{C}P^{p+q-n})\mathrm{vol}(U(n+1))}{\mathrm{vol}(\mathbb{C}P^p)\mathrm{vol}(\mathbb{C}P^q)}.$$

(b) Let $M = \mathbb{R}P^p$ and $N = \mathbb{C}P^q$, then $\mathbb{R}P^p \cap g\mathbb{C}P^q = \mathbb{R}P^{p+2q-2n}$ for almost all $g \in U(n+1)$. So we have

$$\mathrm{vol}(\mathbb{R}P^{p+2q-2n})\mathrm{vol}(U(n+1)) = \sigma_K(V_o^\perp, W_o^\perp)\mathrm{vol}(\mathbb{R}P^p)\mathrm{vol}(\mathbb{C}P^q).$$

Hence

$$\sigma_K(V_o^\perp, W_o^\perp) = \frac{\mathrm{vol}(\mathbb{R}P^{p+2q-2n})\mathrm{vol}(U(n+1))}{\mathrm{vol}(\mathbb{R}P^p)\mathrm{vol}(\mathbb{C}P^q)}.$$

(c) Let $M = \mathbb{R}P^n$ and $N = \mathbb{R}P^n$, then $\sharp(\mathbb{R}P^n \cap g\mathbb{R}P^n) = n+1$ for almost all $g \in SO(n+1)$ (see [13] for details). So we have

$$(n+1)\mathrm{vol}(U(n+1)) = \sigma_K(V_o^\perp, W_o^\perp)\mathrm{vol}(\mathbb{R}P^n)\mathrm{vol}(\mathbb{R}P^n).$$

Hence

$$\sigma_K(V_o^\perp, W_o^\perp) = \frac{(n+1)\mathrm{vol}(U(n+1))}{\mathrm{vol}(\mathbb{R}P^p)\mathrm{vol}(\mathbb{R}P^q)}.$$

This completes the proof.

From Remark 2.2.7, the integral in the definition of σ_K is reduced to the integral on K-orbit in $G_p(T_o(G/K))$. Thus parameterizing the orbit space of this action we can formulate the Poincaré formula by this parameter. We shall call this parameter *isotropy invariant.* In the case $G/K = \mathbb{C}P^n$ multiple Kähler angle, which defined by Tasaki [32], becomes the isotropy invariant.

Definition 2.3.3. Let ω denote the Kähler form on $\mathbb{C}^n$. If $1 < k \leq n$, for a real k-dimensional subspace V in $\mathbb{C}$ we can take an orthonormal basis $\alpha^1, \cdots, \alpha^k$ of the dual space V^* which satisfies

(a) $\omega|_V = \displaystyle\sum_{i=1}^{[k/2]} \cos\theta_i \alpha^{2i-1} \wedge \alpha^{2i}$,

(b) $0 \leq \theta_1 \leq \cdots \leq \theta_{[k/2]} \leq \pi/2$.

We put $\theta(V) = (\theta_1, \cdots, \theta_{[k/2]})$ and call it the *multiple Kähler angle* of V. If $n < k < 2n-1$, we define the multiple Kähler angle of real k-dimensional subspace V in $\mathbb{C}^n$ as that of its orthogonal complement $V^\perp$.

Remark 2.3.4. The action of $U(n)$ preserves the multiple Kähler angle. In the case $k = 2$, the multiple Kähler angle becomes exactly the Kähler angle.

The multiple Kähler angle can be regarded as a function defined on the Grassmannian manifold $G_k^{\mathbb{R}}(\mathbb{C}^n)$ of all real k-dimensional subspaces in $\mathbb{C}^n$. If $k \leq n$, for $\theta = (\theta_1, \cdots, \theta_{[k/2]})$ with $0 \leq \theta_1 \leq \cdots \leq \theta_{[k/2]} \leq \pi/2$ we define

$$\begin{aligned} G^n_{k;\theta} &= \{V \in G_k^{\mathbb{R}}(\mathbb{C}^n) \mid \theta(V) = V\}, \\ V^k_\theta &= \sum_{i=1}^{[k/2]} \mathrm{span}_{\mathbb{R}}\{e_{2i-1}, \cos\theta_i \sqrt{-1} e_{2i-1} + \sin\theta_i e_{2i}\} \ (+\mathbb{R}e_k), \end{aligned}$$

where the last term is added only when k is odd. Then $G^n_{k;\theta} = U(n) \cdot V^k_\theta$ holds. We can express

$$G_k^{\mathbb{R}}(\mathbb{C}^n) = O(2n)/(O(k) \times (2n-k))$$

as a compact symmetric space. Here $O(2n)/U(n)$ is also a compact symmetric space. Therefore the action of $U(n)$ on $G_k^{\mathbb{R}}(\mathbb{C}^n)$ is a Hermann action.

Proposition 2.3.5. ([32]) *Let $k \leq n$. The set $\{V^k_\theta \mid \theta \in \mathbb{R}^{[k/2]}\}$ is a flat section of the action of $U(n)$ on $G_k^{\mathbb{R}}(\mathbb{C}^n)$.*

Hence in the viewpoint of isometric transformation group, the multiple Kähler angle can be regarded as a coordinate of the flat section. Namely, it is parameterizing the orbit space of this action. Applying this we can express $\sigma_K(T_x^\perp M, T_y^\perp N)$ by the multiple Kähler angle of $T_x^\perp M$ and $T_y^\perp N$. Furthermore in the case $p = 2$ and $q = 2n-2$ one can obtain the concrete value of σ_K as follows:

Theorem 2.3.6. ([33]) *For any real 2-dimensional submanifold M and real $(2n-2)$-dimensional submanifold N in the complex projective space $\mathbb{C}P^n$ of* *dimension n, we have*

$$\int_{U(n+1)} \sharp(M \cap gN)dg$$
$$= \frac{\mathrm{vol}(U(n+1))}{\mathrm{vol}(\mathbb{C}P^1)\mathrm{vol}(\mathbb{C}P^{n-1})}$$
$$\times \int_{M\times N} \left(\frac{1}{4}(1+\cos^2\theta_x)(1+\cos^2\tau_y) + \frac{n}{4(n-1)}\sin^2\theta_x \sin^2\tau_y\right) d\mu(x,y)$$

where θ_x is the Kähler angle of T_xM and τ_y is the Kähler angle of $T_y^\perp N$.

2.3.3 The case of two point homogeneous spaces

A Riemannian manifold M is called *two-point homogeneous* if for any two pairs $x, y \in M$ and $x', y' \in M$ satisfying $d(x,y) = d(x',y')$ there exists an element $g \in I(M)$, the group of isometries of M, such that $gx = x'$ and $gy = y'$, where d denotes the distance of two points in M. A Riemannian manifold M is called *isotropic* if for each $x \in M$ the isotropy subgroup $I(M)_x$ acts transitively on the unit sphere in the tangent space T_xM. We can see easily that a Riemannian manifold is two-point homogeneous if and only if it is isotropic. These manifolds have been classified by many geometers, and showed that a Riemannian manifold is isotropic if and only if it is either a Euclidean space or a Riemannian symmetric space of rank one (see [9] for reference). We note that in all these cases the isometry group $G = I(M)$ of M is unimodular.

Theorem 2.3.7. ([13]) *Let G/K be a two-point homogeneous space of dimension n. Let M be a p-dimensional submanifold of G/K and N a hypersurface of G/K. Then we have*

$$\int_G \mathrm{vol}(M \cap gN)dg = \frac{\mathrm{vol}(K)\mathrm{vol}(S^{p-1})\mathrm{vol}(S^n)}{\mathrm{vol}(S^p)\mathrm{vol}(S^{n-1})}\mathrm{vol}(M)\mathrm{vol}(N).$$

Remark 2.3.8. We must make special mention that in most cases G does not act transitively on the set of tangent spaces to M.

Proof. Since K acts transitively on the unit sphere in $T_o(G/K)$, all hypersurfaces of G/K are type W_o for any $(n-1)$-dimensional subspace W_o of $T_o(G/K)$. Thus from Corollary 2.2.9 we have

$$\int_G \mathrm{vol}(M \cap gN)dg = \mathrm{vol}(N)\int_M \sigma_K(T_x^\perp M, W_o^\perp)d\mu_M(x).$$

We identify $\mathbb{R}^n$ with the tangent space $T_o(G/K)$. Let $K \rtimes \mathbb{R}^n$ be $K \times \mathbb{R}^n$ with the product Riemannian metric and view it as a group of transformation on $\mathbb{R}^n$ by the rule for $(k, v) \in K \times \mathbb{R}^n$ and $X \in \mathbb{R}^n$

$$(k, v)X = k_*X + v.$$

Then the group $K \rtimes \mathbb{R}^n$ acts isometrically and transitively on $\mathbb{R}^n$. So we can regard $\mathbb{R}^n$ as the homogeneous space $K \rtimes \mathbb{R}^n/K$. Let V be any p-dimensional subspace of $\mathbb{R}^n$ and B^p the unit ball in V. Then the translations of $\mathbb{R}^n$, and thus also $K \rtimes \mathbb{R}^n$, is transitive on the set of tangent spaces to B^p. Because G/K is a two point homogeneous space the group $K \rtimes \mathbb{R}^n$ is transitive on the set of tangent spaces to the unit sphere S^{n-1} of $\mathbb{R}^n$. We note that $SO(n) \rtimes \mathbb{R}^n$ is the group of orientation preserving isometries of $\mathbb{R}^n$, and thus the result of Theorem 2.3.1 apply to this group. From symmetry properties of the sphere S^{n-1} and $\mathrm{vol}(SO(n+1)) = \mathrm{vol}(SO(n))\mathrm{vol}(S^n)$, we have

$$\begin{aligned}
&\sigma_K(V^\perp, W_o^\perp)\mathrm{vol}(B^p)\mathrm{vol}(S^{n-1}) \\
= \;& \int_{\mathbb{R}^n}\int_K \mathrm{vol}(B^p \cap (k_*S^{n-1} + v))\, dk\, dv \\
= \;& \frac{1}{\mathrm{vol}(SO(n))}\int_{\mathbb{R}^n}\int_K\int_{SO(n)} \mathrm{vol}(B^p \cap (k_*a_*S^{n-1} + v))d\mu_{SO(n)}(a)\, dk\, dv \\
= \;& \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))}\int_{\mathbb{R}^n}\int_{SO(n)} \mathrm{vol}(B^p \cap (a_*S^{n-1} + v))d\mu_{SO(n)}(a)\, dv \\
= \;& \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))}\frac{\mathrm{vol}(S^{p-1})\mathrm{vol}(SO(n+1))}{\mathrm{vol}(S^p)\mathrm{vol}(S^{n-1})}\mathrm{vol}(B^p)\mathrm{vol}(S^{n-1}) \\
= \;& \frac{\mathrm{vol}(K)\mathrm{vol}(S^{p-1})\mathrm{vol}(S^n)\mathrm{vol}(B^p)}{\mathrm{vol}(S^p)}.
\end{aligned}$$

Hence

$$\sigma_K(V^\perp, W_o^\perp) = \frac{\mathrm{vol}(K)\mathrm{vol}(S^{p-1})\mathrm{vol}(S^n)}{\mathrm{vol}(S^p)\mathrm{vol}(S^{n-1})}.$$

Since V is an arbitrary p-dimensional subspace of $\mathbb{R}^n = T_o(G/K)$, from the transfer principle it follows that for any $x \in M$ and $y \in N$

$$\sigma_K(T_x^\perp M, T_y^\perp N) = \frac{\mathrm{vol}(K)\mathrm{vol}(S^{p-1})\mathrm{vol}(S^n)}{\mathrm{vol}(S^p)\mathrm{vol}(S^{n-1})}.$$

This completes the proof.

2.3.4 Other examples

Let G/K be a Riemannian symmetric space, let M a curve and N a hypersurface in G/K. Then using the Hodge operator we have

$$\sigma_K(T_x^{\perp}M, T_y^{\perp}N) = \int_K |\langle u, \mathrm{Ad}(k)v\rangle|\, dk,$$

where u and v are unit vectors in $(g_x)_*^{-1}(T_xM)$ and $(g_y)_*^{-1}(T_y^{\perp}N)$ respectively. This implies that we integrate on an R-space. We denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ the canonical decomposition of $\mathfrak{g}$, where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebra of G and K respectively. We take a unit sphere S in $\mathfrak{m}$ and a maximal Abelian subspace $\mathfrak{a}$ in $\mathfrak{m}$. Then $S \cap \mathfrak{a}$ becomes a non-flat section of K-action on S. So linear isotropy actions of symmetric spaces induce non-hyperpolar polar actions on spheres. We can define the isotropy invariant by the coordinate of $S \cap \mathfrak{a}$. In particular if $\mathrm{rank}(G/K) = 2$, then $\dim(S \cap \mathfrak{a}) = 1$, so we can parametrize the orbit space by one parameter.

Example 2.3.9. In the case G/K is the Grassmannian manifold $G_2(\mathbb{R}^4) = SO(4)/SO(2) \times SO(2)$ of all oriented 2-dimensional subspaces in $\mathbb{R}^4$, then

$$\mathfrak{m} = \left\{ \begin{bmatrix} O & X \\ -{}^tX & O \end{bmatrix} \,\middle|\, X \in M_2(\mathbb{R}) \right\}.$$

We can identify $\mathfrak{m}$ with $M_2(\mathbb{R})$ by

$$\mathfrak{m} \to \mathrm{M}_2(\mathbb{R}) \ ; \ \begin{bmatrix} O & X \\ -{}^tX & O \end{bmatrix} \to X.$$

We take a maximal Abelian subspace $\mathfrak{a}$ in $\mathfrak{m}$ as

$$\mathfrak{a} = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}.$$

Then

$$\Sigma = \left\{ \begin{bmatrix} \cos\theta & 0 \\ 0 & \sin\theta \end{bmatrix} \,\middle|\, -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \right\} \subset S \cap \mathfrak{a}$$

is a section of K-action on S, namely $-\pi/4 \leq \theta \leq \pi/4$ is the isotropy invariant and parameterizing the orbit space in this case. We may assume that u and v are vectors in a section. So we put

$$u = \begin{bmatrix} \cos\theta & 0 \\ 0 & \sin\theta \end{bmatrix}, \quad v = \begin{bmatrix} \cos\tau & 0 \\ 0 & \sin\tau \end{bmatrix} \qquad (-\pi/4 \leq \theta, \tau \leq \pi/4).$$

Then we have

$$\begin{aligned}\sigma_K(T_x^\perp M, T_y^\perp N) &= \int_K |\langle u, \mathrm{Ad}(k)v\rangle|\, dk \\ &= \int_{SO(2)\times SO(2)} |\langle uk_2, k_1 v\rangle|\, d\mu(k_1,k_2).\end{aligned}$$

We represent k_1 and k_2 in $SO(2)$ as

$$k_1 = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}, \quad k_2 = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix}.$$

Then

$$k_1 v = \cos\varphi \begin{bmatrix} \cos\tau & 0 \\ 0 & \sin\tau \end{bmatrix} + \sin\varphi \begin{bmatrix} 0 & \sin\tau \\ -\cos\tau & 0 \end{bmatrix}$$

Therefore $k_1 v$ moves on the unit sphere in

$$V_1 = \mathrm{span}\left\{ \begin{bmatrix} \cos\tau & 0 \\ 0 & \sin\tau \end{bmatrix}, \begin{bmatrix} 0 & \sin\tau \\ -\cos\tau & 0 \end{bmatrix} \right\}$$

according to φ. Similarly

$$uk_2 = \cos\psi \begin{bmatrix} \cos\theta & 0 \\ 0 & \sin\theta \end{bmatrix} + \sin\psi \begin{bmatrix} 0 & \cos\theta \\ -\sin\theta & 0 \end{bmatrix},$$

and uk_2 moves on the unit sphere in

$$V_2 = \mathrm{span}\left\{ \begin{bmatrix} \cos\theta & 0 \\ 0 & \sin\theta \end{bmatrix}, \begin{bmatrix} 0 & \cos\theta \\ -\sin\theta & 0 \end{bmatrix} \right\}$$

according to ψ. We define the orthogonal projection $P : V_1 \longrightarrow V_2$. Then

$$P(k_1 v) = \cos\varphi\cos(\theta-\tau) \begin{bmatrix} \cos\theta & 0 \\ 0 & \sin\theta \end{bmatrix} + \sin\varphi\sin(\theta+\tau) \begin{bmatrix} 0 & \cos\theta \\ -\sin\theta & 0 \end{bmatrix}.$$

Thus $P(k_1 v)$ moves on an elliptic curve in V_2 defined by

$$\frac{x^2}{\cos^2(\theta-\tau)} + \frac{y^2}{\sin^2(\theta+\tau)} = 1.$$

We denote this elliptic curve by $\mathrm{Ellip}(\theta,\tau)$. Now we put

$$r_\varphi = \sqrt{\cos^2\varphi\cos^2(\theta-\tau) + \sin^2\varphi\sin^2(\theta+\tau)}.$$

Then

$$
\begin{aligned}
& \sigma_K(T_x^\perp M, T_y^\perp N) \\
= & \int_{SO(2)\times SO(2)} |\langle uk_2, k_1 v\rangle|\, d\mu(k_1,k_2) \\
= & \int_{SO(2)\times SO(2)} |\langle uk_2, P(k_1 v)\rangle|\, d\mu(k_1,k_2) \\
= & \int_0^{2\pi}\int_0^{2\pi} |\langle(\cos\psi,\sin\psi),(\cos(\theta-\tau)\cos\varphi,\sin(\theta+\tau)\sin\varphi)\rangle| d\psi d\varphi \\
= & \int_0^{2\pi}\int_0^{2\pi} |\langle(\cos\psi,\sin\psi),(r_\varphi,0)\rangle| d\psi d\varphi \\
= & \int_0^{2\pi}\int_0^{2\pi} |r_\varphi \cos\psi| d\psi d\varphi \\
= & \int_0^{2\pi} |\cos\psi| d\psi \int_0^{2\pi} r_\varphi d\varphi \\
= & \; 4\mathrm{vol}(\mathrm{Ellip}(\theta,\tau)).
\end{aligned}
$$

Hence we have the Poincaré formula as follows:

Proposition 2.3.10. *Let G/K be the Grassmannian manifold $G_2(\mathbb{R}^4)$ of all oriented 2-dimensional subspaces in $\mathbb{R}^4$. For any curve M and hypersurface N of G/K we have*

$$\int_G \mathrm{vol}(M \cap gN)dg = 4\int_{M\times N} \mathrm{vol}(\mathrm{Ellip}(\theta_x,\tau_y))d\mu(x,y),$$

where θ_x and τ_y are isotropy invariants at $x \in M$ and $y \in N$ respectively.

Remark 2.3.11. This was first obtained by Kang [16]. However, he is considering in $S^2 \times S^2 \cong G_2(\mathbb{R}^4)$ and expressing by elliptic functions.

2.4 Poincaré formula of complex submanifolds

In this section applying the Howard's generalized Poincaré formula, we will show that the Poincaré formula for almost complex submanifolds in an almost Hermitian homogeneous space comes down some representations of the isotropy subgroup

Let (V, J) be a complex vector space with an inner product $\langle \cdot, \cdot \rangle$. We consider the exterior algebra $\wedge^p V^{(1,0)}$ of degree p on holomorphic vector space $V^{(1,0)}$. We extend $\langle \cdot, \cdot \rangle$ to a complex bilinear form on $V^{\mathbb{C}}$, and denote by the same symbol. Note that if X and Y are both in $V^{(1,0)}$ (or $V^{(0,1)}$) then $\langle X, Y \rangle = 0$. Expressing the norm on this exterior algebra by Gramian, we have following lemma immediately.

Lemma 2.4.1. *For each $\xi_1 \wedge \cdots \wedge \xi_p$ in $\wedge^p V^{(1,0)}$*

$$\|\xi_1 \wedge \cdots \wedge \xi_p \wedge \overline{\xi_1 \wedge \cdots \wedge \xi_p}\| = \|\xi_1 \wedge \cdots \wedge \xi_p\|^2.$$

Proposition 2.4.2. *Let G be a unimodular Lie group and G/K an almost Hermitian homogeneous space of complex dimension n. Assume that K acts irreducibly on the exterior algebras $\wedge^p(T_o(G/K))^{(1,0)}$ and $\wedge^q(T_o(G/K))^{(1,0)}$ with $p + q \leq n$. Then there exists a positive constant C such that for any almost complex submanifolds M and N of G/K of complex codimensions p and q respectively*

$$\int_G \mathrm{vol}(M \cap gN)dg = C\mathrm{vol}(M)\mathrm{vol}(N)$$

holds.

Proof. From Theorem 2.2.5 it is sufficient to show that $\sigma_K(T_x^\perp M, T_y^\perp N)$ is a positive constant function on $M \times N$.

Let $\{u_i, Ju_i\}_{1\leq i\leq p}$ and $\{v_i, Jv_i\}_{1\leq i\leq q}$ be orthonormal bases of $(g_x)_*^{-1}(T_x^\perp M)$ and $(g_y)_*^{-1}(T_y^\perp M)$ respectively. We put

$$\xi_i = \frac{1}{\sqrt{2}}\left(u_i - \sqrt{-1}Ju_i\right), \quad \eta_i = \frac{1}{\sqrt{2}}\left(v_i - \sqrt{-1}Jv_i\right).$$

Then $\xi_1, \cdots, \xi_p$ and $\eta_1, \cdots, \xi_q$ are unitary bases of $(g_x)_*^{-1}(T_x^\perp M)^{(1,0)}$ and $(g_y)_*^{-1}(T_y^\perp N)^{(1,0)}$ respectively. We note

$$u_i \wedge Ju_i = -\sqrt{-1}\xi_i \wedge \bar{\xi}_i, \quad v_i \wedge Jv_i = -\sqrt{-1}\eta_i \wedge \bar{\eta}_i.$$

So we have

$$\begin{aligned} & \|u_1 \wedge Ju_1 \wedge \cdots \wedge u_p \wedge Ju_p \wedge \mathrm{Ad}(k)(v_1 \wedge Jv_1 \wedge \cdots \wedge v_q \wedge Jv_q)\| \\ = \ & \|\xi_1 \wedge \bar{\xi}_1 \wedge \cdots \wedge \xi_p \wedge \bar{\xi}_p \wedge \mathrm{Ad}(k)(\eta_1 \wedge \bar{\eta}_1 \wedge \cdots \wedge \eta_q \wedge \bar{\eta}_q)\|. \end{aligned}$$

Now we put

$$\xi = \xi_1 \wedge \cdots \wedge \xi_p, \quad \eta = \eta_1 \wedge \cdots \wedge \eta_q.$$

From Lemma 2.4.1

$$\begin{aligned}\sigma_K(T_x^\perp M, T_y^\perp N) &= \int_K \|\xi \wedge \mathrm{Ad}(k)\eta \wedge \overline{\xi \wedge \mathrm{Ad}(k)\eta}\| dk \\ &= \int_K \|\xi \wedge \mathrm{Ad}(k)\eta\|^2 dk.\end{aligned}$$

Fix η, and define Q_η by

$$Q_\eta(X, Y) = \int_K \langle X \wedge \mathrm{Ad}(k)\eta, \overline{Y \wedge \mathrm{Ad}(k)\eta}\rangle dk$$

for each X, Y in $\wedge^p(T_o(G/K))^{(1,0)}$. Then Q_η is a K-invariant Hermitian form on $\wedge^p(T_o(G/K))^{(1,0)}$. From Schur's lemma, there is a positive constant C_η such that

$$Q_\eta(X, Y) = C_\eta\langle X, \bar{Y}\rangle,$$

since K acts irreducibly on $\wedge^p(T_o(G/K))^{(1,0)}$. So we have

$$\sigma_K(T_x^\perp M, T_y^\perp N) = C_\eta \|\xi\|^2 = C_\eta.$$

This implies that $\sigma_K(T_x^\perp M, T_y^\perp N)$ is independent of $T_x^\perp M$. By the similar way, we can show that $\sigma_K(T_x^\perp M, T_y^\perp N)$ is also independent of $T_y^\perp N$ by the assumption which K acts irreducibly on $\wedge^q(T_o(G/K))^{(1,0)}$. This concludes the proof.

Theorem 2.4.3. ([18]) *Let G be a unimodular Lie group and G/K an almost Hermitian homogeneous space of complex dimension n. Assume that K acts irreducibly on the exterior algebra $\wedge^p(T_o(G/K))^{(1,0)}$. Then for any almost complex submanifolds M and N of G/K with*

$$\dim_{\mathbb{C}} M = p, \quad \dim_{\mathbb{C}} N = n - p,$$

we have

$$\int_G \sharp(M \cap gN) dg = \frac{\mathrm{vol}(K)}{\binom{n}{p}} \mathrm{vol}(M)\mathrm{vol}(N).$$

Proof. From the proof of Proposition 2.4.2 there exists a constant C such that

$$C = \sigma_K(T_x^{\perp}M, T_y^{\perp}N) = \int_K \|\xi \wedge \mathrm{Ad}(k)\eta\| dk.$$

for any $T_x^{\perp}M$ and $T_y^{\perp}N$. Using Hodge star operator we have

$$C = \int_K \left| \left\langle \xi', \overline{\mathrm{Ad}(k)\eta} \right\rangle \right| dk.$$

where ξ' is a vector in $\wedge^p(T_o(G/K))^{(1,0)}$ which corresponds to $(g_x)_*^{-1}(T_xM)$.

Let $X_1, \cdots, X_r$ be a unitary basis of $\wedge^p(T_o(G/K))^{(1,0)}$, where we put

$$r = \dim(\wedge^p(T_o(G/K))^{(1,0)}) = \binom{n}{p}.$$

Since for any i and j

$$\int_K \left| \left\langle X_i, \overline{\mathrm{Ad}(k)X_j} \right\rangle \right|^2 dk = C,$$

we have

$$\begin{aligned} rC &= \sum_{i=1}^{r} \int_K \left| \left\langle X_i, \overline{\mathrm{Ad}(k)X_1} \right\rangle \right|^2 dk \\ &= \int_K \sum_{i=1}^{r} \left| \left\langle X_i, \overline{\mathrm{Ad}(k)X_1} \right\rangle \right|^2 dk \\ &= \int_K \|\mathrm{Ad}(k)X_1\|^2 dk \\ &= \mathrm{vol}(K). \end{aligned}$$

This completes the proof.

We will consider the case where $\wedge^p(T_o(G/K))^{(1,0)}$ is reducible by K-action. The other conditions are same with Theorem 2.4.3.

$$\wedge^p(T_o(G/K))^{(1,0)} = \bigoplus_{i=1}^{s} V_i$$

denotes the irreducible decomposition by K-action. Let X and Y be complex vector subspaces of dimension p in $(T_o(G/K))^{(1,0)}$ and take unitary bases

$\xi_1, \cdots, \xi_p$ and $\eta_1, \cdots, \eta_p$ of X and Y respectively. We denote by $\hat{X}_i$ and $\hat{Y}_i$ be the V_i-components of $\xi = \xi_1 \wedge \cdots \wedge \xi_p$ and $\eta = \eta_1 \wedge \cdots \wedge \eta_p$ respectively. We define $A(X, Y)$ by

$$A(X,Y) = \sum_{i=1}^{s} \frac{\mathrm{vol}(K)}{\dim V_i} \|\hat{X}_i\|^2 \|\hat{Y}_i\|^2.$$

Theorem 2.4.4. ([18]) *If V_i and V_j are not equivalent when $i \neq j$, then we have*

$$\int_G \sharp(M \cap gN) dg = \int_{M \times N} A(T_x M, T_y^{\perp} N) d\mu(x, y).$$

Proof. From the proof of Theorem 2.4.3, we have

$$\begin{aligned} \sigma_K(X,Y) &= \int_K \left| \left\langle \xi, \overline{\mathrm{Ad}(k)\eta} \right\rangle \right|^2 dk \\ &= \int_K \left| \sum_{i=1}^{s} \left\langle \hat{X}_i, \overline{\mathrm{Ad}(k)\hat{Y}_i} \right\rangle \right|^2 dk \\ &= \sum_{i,j=1}^{s} \int_K \left\langle \hat{X}_i, \overline{\mathrm{Ad}(k)\hat{Y}_i} \right\rangle \overline{\left\langle \hat{X}_j, \overline{\mathrm{Ad}(k)\hat{Y}_j} \right\rangle} dk. \end{aligned}$$

The last integrals vanish when $i \neq j$ by the Peter-Weyl theorem. Therefore we get

$$\sigma_K(X,Y) = \sum_{i=1}^{s} \int_K \left| \left\langle \hat{X}_i, \overline{\mathrm{Ad}(k)\hat{Y}_i} \right\rangle \right|^2 dk.$$

By the similar way with Theorem 2.4.3, we can conclude that each integral in just above equation is constant and determine it.

$$\sigma_K(X,Y) = \sum_{i=1}^{s} \frac{\mathrm{vol}(K)}{\dim V_i} \|\hat{X}_i\|^2 \|\hat{Y}_i\|^2 = A(X,Y).$$

In the case where G/K is irreducible Hermitian symmetric spaces, we give p when K acts irreducibly on $\wedge^p(T_o(G/K))^{(1,0)}$ in Table 2.1. Although we show the case of compact type, it is clear that their non-compact duals also give the same result of Table 2.1.

Table 2.1:

	compact type	
A III	$SU(l)/S(U(m) \times U(l-m))$	any p (if $m = 1$) p=1 (if $m \geq 2$)
D III	$SO(2l)/U(l)$	$p = 1, 2$
BD I	$SO(2l)/SO(2) \times SO(2l-2)$	$p \neq l - 1$
	$SO(2l+1)/SO(2) \times SO(2l-1)$	any p
C I	$Sp(l)/U(l)$	$p = 1, 2$
E III	$(\mathfrak{e}_{6(-78)}, \mathfrak{so}(10) + \mathbb{R})$	$p = 1, 2, 3$
E VII	$(\mathfrak{e}_{7(-133)}, \mathfrak{e}_6 + \mathbb{R})$	$p = 1, 2, 3, 4$

Theorem 2.4.5. ([29]) *Let G/K be an irreducible Hermitian symmetric space of complex dimension n. Assume that K acts irreducibly on the exterior algebra $\wedge^p(T_o(G/K))^{(1,0)}$. Then for any complex submanifolds M and N of G/K of complex dimensions $(n-p)$ and $(n-q)$ respectively with $p+q \leq n$, we have*

$$\int_G \mathrm{vol}(M \cap gN)dg = \frac{(n-p)!(n-q)!\mathrm{vol}(K)}{n!(n-p-q)!}\mathrm{vol}(M)\mathrm{vol}(N).$$

Proof. From Table 2.1, if K acts irreducibly on $\wedge^p(T_o(G/K))^{(1,0)}$ for $p \leq n/2$, then K acts irreducibly on $\wedge^r(T_o(G/K))^{(1,0)}$ for any r with $r \leq p$. In addition, if K acts irreducibly on $\wedge^p(T_o(G/K))^{(1,0)}$, then K acts irreducibly on $\wedge^{n-p}(T_o(G/K))^{(1,0)}$, since it is a dual representation of a unitary representation. From these facts, it is sufficient to show the theorem with $p \leq q \leq n-p$.

From the proof of Proposition 2.4.2, we have

$$\sigma_K(T_x^\perp M, T_y^\perp N) = \int_K ||\xi_1 \wedge \cdots \wedge \xi_p \wedge \mathrm{Ad}(k)(\eta_1 \wedge \cdots \wedge \eta_q)||^2 dk.$$

We put a $p \times q$ matrix $A = (a_{ij}) = \left(\langle \xi_i, \overline{\mathrm{Ad}(k)\eta_j}\rangle\right)$, then

$$||\xi_1 \wedge \cdots \wedge \xi_p \wedge \mathrm{Ad}(k)(\eta_1 \wedge \cdots \wedge \eta_q)||^2 = \det \begin{bmatrix} I_p & A \\ A^* & I_q \end{bmatrix},$$

where I_p and I_q are unit matrixes of degree p and q respectively. Expanding with respect to the diagonal element 1, we can expand the right hand side to the sum of minor determinants as follows:

$$\det\begin{bmatrix} I_p & A \\ A^* & I_q \end{bmatrix} = 1 + \sum_{a=1}^{p}\sum_{b=1}^{q}\left(\sum_{\substack{i_1<\cdots<i_a \\ j_1<\cdots<j_b}} \det\begin{bmatrix} O & A^{i_1\cdots i_a}_{j_1\cdots j_b} \\ (A^{i_1\cdots i_a}_{j_1\cdots j_b})^* & O \end{bmatrix}\right),$$

where

$$A^{i_1\cdots i_a}_{j_1\cdots j_b} = \begin{bmatrix} a_{i_1j_1} & \cdots & a_{i_1j_b} \\ \vdots & \ddots & \vdots \\ a_{i_aj_1} & \cdots & a_{i_aj_b} \end{bmatrix}.$$

If $A^{i_1\cdots i_a}_{j_1\cdots j_b}$ is not a square matrix, then

$$\det\begin{bmatrix} O & A^{i_1\cdots i_a}_{j_1\cdots j_b} \\ (A^{i_1\cdots i_a}_{j_1\cdots j_b})^* & O \end{bmatrix} = 0.$$

Therefore we have

$$\begin{aligned}
&\det\begin{bmatrix} I_p & A \\ A^* & I_q \end{bmatrix} \\
&= 1 + \sum_{a=1}^{p}\left(\sum_{\substack{i_1<\cdots<i_a \\ j_1<\cdots<j_a}} \det\begin{bmatrix} O & A^{i_1\cdots i_a}_{j_1\cdots j_a} \\ (A^{i_1\cdots i_a}_{j_1\cdots j_a})^* & O \end{bmatrix}\right) \\
&= 1 + \sum_{a=1}^{p}(-1)^a\left(\sum_{\substack{i_1<\cdots<i_a \\ j_1<\cdots<j_a}} \left|\det A^{i_1\cdots i_a}_{j_1\cdots j_a}\right|^2\right) \\
&= 1 + \sum_{a=1}^{p}(-1)^a\left(\sum_{\substack{i_1<\cdots<i_a \\ j_1<\cdots<j_a}} |\langle \xi_{i_1}\wedge\cdots\wedge\xi_{i_a}, \overline{\mathrm{Ad}(k)(\eta_{j_1}\wedge\cdots\wedge\eta_{j_a})}\rangle|^2\right).
\end{aligned}$$

Since K acts irreducibly on $\wedge^r(T_o(G/K))^{(1,0)}$ for any integer $r \leq p$, by the way of the proof of Theorem 2.4.3 we have

$$\int_K |\langle \xi_{i_1}\wedge\cdots\wedge\xi_{i_a}, \overline{\mathrm{Ad}(k)(\eta_{j_1}\wedge\cdots\wedge\eta_{j_a})}\rangle|^2 dk = \frac{\mathrm{vol(K)}}{\binom{n}{a}}.$$

Thus

$$\sigma_K(T_x^\perp M, T_y^\perp N) = \sum_{a=0}^{p} (-1)^a \frac{\binom{p}{a}\binom{q}{a}}{\binom{n}{a}} \mathrm{vol}(K).$$

Here

$$\sum_{a=0}^{p} (-1)^a \frac{\binom{p}{a}\binom{q}{a}}{\binom{n}{a}}$$

is a constant determined by n, p and q. In the case where G/K is a complex space form, for any n, p and q it satisfies the condition of this theorem. So comparing with the result of the case of $\mathbb{C}P^n$, we have

$$\sigma_K(T_x^\perp M, T_y^\perp N) = \frac{(n-p)!(n-q)!}{n!(n-p-q)!} \mathrm{vol}(K).$$

This completes the proof.

Corollary 2.4.6. *If $p + q \leq n$, then*

$$\sum_{a=0}^{\min\{p,q\}} (-1)^a \frac{\binom{p}{a}\binom{q}{a}}{\binom{n}{a}} = \frac{(n-p)!(n-q)!}{n!(n-p-q)!}.$$