

Chapter 1

Orbits of group actions

1.1 Lie transformation group

In this section we give a brief survey of Lie transformation groups which act on manifolds, and provide some results which we need in this thesis.

Let G be a Lie group and M a smooth manifold. If there is a smooth mapping $\rho : G \times M \rightarrow M$ such that

$$\rho(e, x) = x, \quad \rho(g_1 g_2, x) = \rho(g_1, \rho(g_2, x)) \quad (g_1, g_2 \in G, x \in M)$$

then G is called a *Lie transformation group* of M . And we say that G acts on M or that M is a G -manifold. This defines a group homomorphism from G to the group $\text{Diff}(M)$ of diffeomorphisms of M , namely $\rho(g, x) = \rho(g)x = gx$. Moreover if M is a Riemannian manifold and $\rho(G)$ is included in the group of isometries of M , then we say that G acts isometrically on M or that M is a Riemannian G -manifold.

Definition 1.1.1. If M is a G -manifold and $x \in M$ then we define Gx , the G -orbit through x , and G_x , the *isotropy subgroup* at x by

$$\begin{aligned} Gx &= \{gx \mid g \in G\}, \\ G_x &= \{g \in G \mid gx = x\}. \end{aligned}$$

We denote by \mathfrak{g} the Lie algebra of G . For $X \in \mathfrak{g}$ we define the vector field $X(x)$ on M by

$$X(x) = \left. \frac{d}{dt} \right|_{t=0} \exp(tX)x \quad (x \in M).$$

The following Proposition is easy to prove.

Proposition 1.1.2. *Let M be a G -manifold, then*

- (1) $G_{gx} = gG_xg^{-1}$,
- (2) if $Gx \cap Gy \neq \emptyset$ then $Gx = Gy$,
- (3) $T_x(Gx) = \{X(x) \mid X \in \mathfrak{g}\}$.

The mapping $G/G_x \rightarrow Gx$; $gG_x \mapsto gx$ is clearly bijective. Since G_x is a closed subgroup of G , G/G_x has a smooth quotient manifold structure. This implies that we can regard each orbit as a smooth homogeneous submanifold of M . But in general the topology that Gx inherits from G/G_x does not coincide with the topology induced from M .

Example 1.1.3. For $\alpha \in \mathbb{R}$, the Lie group \mathbb{R} acts on the flat torus T^2 isometrically by

$$\mathbb{R} \times T^2 \rightarrow T^2 ; (t, [x, y]) \mapsto [x + t, y + \alpha t],$$

where $[x, y]$ denotes the image of $(x, y) \in \mathbb{R}^2$ under the projection $\mathbb{R}^2 \rightarrow T^2$. If α is an irrational number, then each orbit of this action is dense in T^2 .

Let M/G denote the set of all orbits of the action of G on M . The set M/G equipped with the quotient topology by the projection $M \rightarrow M/G$; $x \mapsto Gx$ is called the *orbit space* of the G -manifold M . In general M/G is not a Hausdorff space. For instance if α is an irrational number in Example 1.1.3, then T^2/\mathbb{R} is not a Hausdorff space.

Definition 1.1.4. The action of G on M is called *proper*, if for any two distinct points $x, y \in M$ there exist open neighborhoods U_x and U_y of x and y in M respectively such that $\{g \in G \mid U_x \cap gU_y \neq \emptyset\}$ is relative compact in G . This is equivalent to saying that the mapping

$$G \times M \rightarrow M \times M ; (g, x) \mapsto (x, gx)$$

is a proper mapping, that is, the inverse image of each compact set in $M \times M$ is also compact in $G \times M$.

Remark 1.1.5. If G is a compact Lie group then any action of G is proper. Also if G is a closed subgroup of the isometry group of a Riemannian manifold M , then the action of G on M is proper. On the other hand, if a Lie group G acts properly on a manifold M , then all the isotropy subgroup G_x are compact and M/G is a Hausdorff space.

Definition 1.1.6. For $x, y \in M$ if there exists $g \in G$ such that $G_y = gG_xg^{-1}$ then we say that x and y have same *isotropy type*. From Proposition 1.1.2 (1), all points of an orbit have same isotropy type. Thus this defines an equivalence relation among the orbits of G . We denote by $[Gx]$ the corresponding equivalence class, which is called the *orbit type* of Gx .

Now we introduce a partial ordering on the set of all orbit types by saying that $[Gx] \leq [Gy]$ if and only if there exists $g \in G$ such that $G_y \subset gG_xg^{-1}$. An orbit of the largest orbit type is called a *principal orbit*, and the other orbit is called a *singular orbit*. A point $x \in M$ is called a *regular point* if Gx is a principal orbit, and x is called a *singular point* if Gx is a singular orbit.

Theorem 1.1.7. *Let G be a Lie group acting properly on a manifold M . If the orbit space M/G is connected, then there exists a principal orbit. The set of all regular points is open and dense in M and its orbit space is connected.*

Proof. See [21] Section 4.8 or [28] Section 5.4.

Definition 1.1.8. The *cohomogeneity* of the action is the codimension of a principal orbit.

Theorem 1.1.9. ([1], [26]) *Let M be a connected complete Riemannian manifold and G a connected closed subgroup of the isometry group of M acting on M with cohomogeneity one. Then the orbit space M/G is homeomorphic to \mathbb{R} , S^1 , $[0, 1]$ or $[0, \infty)$.*

Definition 1.1.10. Let M be a connected complete Riemannian manifold and G a Lie group acting isometrically on M . Then a connected closed regularly embedded smooth submanifold Σ of M is called a *section* if Σ intersects each orbit of G in M and $T_x(\Sigma) \subset T_x^\perp(Gx)$ for all $x \in \Sigma$. The action of G is called *polar* if it admits a section. A polar action is called *hyperpolar* if it admits a flat section.

Proposition 1.1.11. *Every section of a polar action is totally geodesic.*

Proof. See [28] Section 5.6.

Definition 1.1.12. Let (G, K_1) and (G, K_2) be symmetric pairs of compact type. Then K_2 acts isometrically on a compact symmetric space G/K_1 . This action of K_2 on G/K_1 is called a *Hermann action*.

Definition 1.1.13. Let M_1 be a Riemannian G_1 -manifold and M_2 a Riemannian G_2 -manifold. The action of G_1 on M_1 and the action of G_2 on M_2 are called *isomorphic* if there exists a Lie group isomorphism $\psi : G_1 \rightarrow G_2$ and an isometry $f : M_1 \rightarrow M_2$ such that $f(gx) = \psi(g)f(x)$ for all $g \in G_1$ and $x \in M_1$.

And the action of G_1 on M_1 and the action of G_2 on M_2 are called *orbit equivalent* if there exists an isometry $f : M_1 \rightarrow M_2$ such that $f(G_1x) = G_2f(x)$ for all $x \in M_1$.

Theorem 1.1.14. ([24]) *All hyperpolar actions on the irreducible Riemannian symmetric spaces of compact type are either cohomogeneity one actions or are orbit equivalent to Hermann actions.*

1.2 R -spaces

In this section we shall review some geometric properties of R -spaces as submanifolds in spheres.

Definition 1.2.1. An isotropy representation of a semisimple Riemannian symmetric space is called a *s-representation*. And an orbit of this action is called an *R-space*.

Let (G, K) be a Riemannian symmetric pair and let θ the involutive automorphism of G such that $(K_\theta)_0 \subset K \subset K_\theta$, where K_θ is the set of fixed points of θ and $(K_\theta)_0$ is the identity component of K_θ . We denote by \mathfrak{g} and \mathfrak{k} the Lie algebras of Lie groups G and K respectively. The involutive automorphism θ of G induce involutive automorphism of \mathfrak{g} , and we will denote it by the same symbol. Since θ is involutive, we have the direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m},$$

where we put

$$\mathfrak{m} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}.$$

Then we can identify $T_o(G/K)$ with \mathfrak{m} , and the action of K on $T_o(G/K)$ with the adjoint representation of K on \mathfrak{m} in a natural manner. If we assume G/K

has an invariant Riemannian metric then corresponding $\text{Ad}(K)$ -invariant inner product is induced on \mathfrak{m} . Therefore R -spaces are submanifolds of hypersphere in \mathfrak{m} .

Proposition 1.2.2. *Let \mathfrak{a} be a maximal subspace of \mathfrak{m} then*

$$\mathfrak{m} = \bigcup_{k \in K} \text{Ad}(k)\mathfrak{a},$$

and \mathfrak{a} intersects the orbits $\text{Ad}(K)H$ for all $H \in \mathfrak{a}$ perpendicularly.

This proposition implies that \mathfrak{a} is a section of this action. Moreover since \mathfrak{a} is a subspace of \mathfrak{m} , thus it is flat. Hence the s -representation is a hyperpolar action. Conversely the following theorem had been obtained by Dadok.

Theorem 1.2.3. ([6]) *All polar representations are orbit equivalent to s -representations.*

For $H \in \mathfrak{m}$ we consider the orbit $\text{Ad}(K)H$. We set the isotropy subgroup K_H at H of this action, that is

$$K_H = \{k \in K \mid \text{Ad}(k)H = H\}.$$

Then $\text{Ad}(K)H \rightarrow K/K_H$; $\text{Ad}(k)H \mapsto kK_H$ is a diffeomorphism. We denote the Lie algebra of K_H by \mathfrak{k}_H , then we have

$$\begin{aligned} \mathfrak{k}_H &= \left\{ X \in \mathfrak{k} \mid \left. \frac{d}{dt} \right|_{t=0} \text{Ad}(\exp tX)H = 0 \right\} \\ &= \{X \in \mathfrak{k} \mid [X, H] = 0\}. \end{aligned}$$

Let $(\mathfrak{k}_H)^\perp$ denote the orthogonal complement of \mathfrak{k}_H in \mathfrak{k} . We can identify $T_o(K/K_H)$ with $(\mathfrak{k}_H)^\perp$. And the tangent space of $\text{Ad}(K)H$ at H is given by

$$\begin{aligned} T_H(\text{Ad}(K)H) &= \left\{ \left. \frac{d}{dt} \right|_{t=0} \text{Ad}(\exp tX)H \mid X \in \mathfrak{k} \right\} \\ &= \{\text{ad}(X)H \mid X \in \mathfrak{k}\} \\ &= [\mathfrak{k}, H]. \end{aligned}$$

Since $\text{Ad}(K)H$ is diffeomorphic with K/K_H , there is a linear isomorphism from $[\mathfrak{k}, H]$ to $(\mathfrak{k}_H)^\perp$.

Proposition 1.2.4. ([27]) *Let X be in $(\mathfrak{k}_H)^\perp$ and ξ in $[\mathfrak{k}, H]^\perp$. We put a curve $c(t) = \text{Ad}(\exp tX)H$ in $\text{Ad}(K)H$. Then the normal vector field $\xi(t) = \text{Ad}(\exp tX)\xi$ along $c(t)$ is parallel with respect to the normal connection.*

Proof. The tangent space of $\text{Ad}(K)H$ at $c(t)$ is given by

$$\begin{aligned} T_{c(t)}(\text{Ad}(K)H) &= \left\{ \left. \frac{d}{ds} \right|_{s=0} \text{Ad}(\exp(t+s)X)H \mid X \in \mathfrak{k} \right\} \\ &= \{ \text{Ad}(\exp tX)[X, H] \mid X \in \mathfrak{k} \} \\ &= \text{Ad}(\exp tX)[\mathfrak{k}, H]. \end{aligned}$$

We denote the covariant derivative on \mathfrak{m} by ∇ , then we have

$$\begin{aligned} \nabla_{\dot{c}(t)}\xi(t) &= \left. \frac{d}{ds} \right|_{s=0} \text{Ad}(\exp(t+s)X)\xi \\ &= \text{Ad}(\exp tX)[X, \xi]. \end{aligned}$$

Here we have

$$\begin{aligned} [\mathfrak{k}, H]^\perp &= \{X \in \mathfrak{m} \mid \langle X, [\mathfrak{k}, H] \rangle = 0\} \\ &= \{X \in \mathfrak{m} \mid \langle [X, H], \mathfrak{k} \rangle = 0\} \\ &= \{X \in \mathfrak{m} \mid [X, H] \in \mathfrak{m}\} \\ &= \{X \in \mathfrak{m} \mid [X, H] = 0\}. \end{aligned}$$

Therefore for $Y, Z \in [\mathfrak{k}, H]^\perp$

- (1) $[Y, Z] \in \mathfrak{k}$,
- (2) $[[Y, Z], H] = -[[Z, H], Y] - [[H, Y], Z] = 0$.

This implies that $[Y, Z] \in \mathfrak{k}_H$. Thus

$$\begin{aligned} &[[\mathfrak{k}, H]^\perp, [\mathfrak{k}, H]^\perp] \subset \mathfrak{k}_H \\ \Leftrightarrow &\langle (\mathfrak{k}_H)^\perp, [[\mathfrak{k}, H]^\perp, [\mathfrak{k}, H]^\perp] \rangle = 0 \\ \Leftrightarrow &\langle [(\mathfrak{k}_H)^\perp, [\mathfrak{k}, H]^\perp], [\mathfrak{k}, H]^\perp \rangle = 0 \\ \Leftrightarrow &[(\mathfrak{k}_H)^\perp, [\mathfrak{k}, H]^\perp] \subset [\mathfrak{k}, H]. \end{aligned}$$

Hence $\nabla_{\dot{c}(t)}\xi(t) \in T_{c(t)}(\text{Ad}(K)H)$. So we have $\nabla_{\dot{c}(t)}^\perp \xi(t) = 0$, namely $\xi(t)$ is a parallel vector field along $c(t)$.

From Proposition 1.2.4 we can obtain the following geometric properties of R -spaces.

Theorem 1.2.5. ([22]) *The mean curvature vector field of the R -space is parallel with respect to the normal connection.*

Theorem 1.2.6. ([8]) *The normal holonomy representation of $\text{Ad}(K)H$ is equivalent to the (effectively made) action of a subgroup of K_H on $[\mathfrak{k}, H]^\perp$.*

1.3 Orbits of Hermann actions

In the previous section we review some geometric properties of R -spaces. Here we will express the parallel translations of the normal bundles of the orbits of Hermann actions by the group actions. And we show that the orbits have similar properties to R -spaces.

Let θ_1 and θ_2 be two involutive automorphisms of a compact connected Lie group G furnished with a biinvariant Riemannian metric $\langle \cdot, \cdot \rangle$. We denote by G_{θ_i} ($i = 1, 2$) the closed subgroup consisting of all fixed points of θ_i in G . For a closed subgroup K_i ($i = 1, 2$) of G which lies between G_{θ_i} and the identity component of G_{θ_i} , (G, K_1) and (G, K_2) are Riemannian symmetric pairs. We consider the Hermann action K_2 on a compact symmetric space $M_1 = G/K_1$ with the induced Riemannian metric from the biinvariant Riemannian metric $\langle \cdot, \cdot \rangle$ on G . We denote by \mathfrak{g} , \mathfrak{k}_1 and \mathfrak{k}_2 the Lie algebras of G , K_1 and K_2 , respectively. The involutive automorphisms θ_1 and θ_2 of G induce involutive automorphisms of \mathfrak{g} , also denoted by θ_1 and θ_2 respectively. Since θ_1 and θ_2 are involutive, we have

$$\mathfrak{g} = \mathfrak{k}_1 + \mathfrak{m}_1 = \mathfrak{k}_2 + \mathfrak{m}_2,$$

where we put

$$\mathfrak{m}_i = \{X \in \mathfrak{g} \mid \theta_i(X) = -X\} \quad (i = 1, 2).$$

We can identify \mathfrak{m}_1 with $T_o(M_1)$ in a natural manner. For $H \in \mathfrak{m}_1$, we consider the K_2 -orbit $K_2 \text{Exp} H$ in M_1 , where Exp is the exponential mapping from \mathfrak{m}_1 into M_1 . We define a closed subgroup $N_{K_2}^H[K_1]$ in K_2 by

$$\begin{aligned} N_{K_2}^H[K_1] &= \{k \in K_2 \mid k \text{Exp} H = \text{Exp} H\} \\ &= \{k \in K_2 \mid k \exp H K_1 = \exp H K_1\} \\ &= \{k \in K_2 \mid \exp(-H)k \exp H \in K_1\}. \end{aligned}$$

Then we have the following diffeomorphism from $K_2 \text{Exp} H$ onto $K_2/N_{K_2}^H[K_1]$:

$$K_2 \text{Exp} H \rightarrow K_2/N_{K_2}^H[K_1] ; k \text{Exp} H \mapsto kN_{K_2}^H[K_1].$$

We denote by $\mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1]$ the Lie algebra of $N_{K_2}^H[K_1]$. Then we have

$$\begin{aligned} \mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1] &= \left\{ X \in \mathfrak{k}_2 \mid \left. \frac{d}{dt} \right|_{t=0} \exp(-H) \exp tX \exp H \in \mathfrak{k}_1 \right\} \\ &= \{ X \in \mathfrak{k}_2 \mid \text{Ad}(\exp(-H))X \in \mathfrak{k}_1 \}. \end{aligned}$$

We denote by $(\mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1])^\perp$ the orthogonal complement of $\mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1]$ in \mathfrak{k}_2 . Then we can identify $(\mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1])^\perp$ with $T_o(K_2/N_{K_2}^H(K_1))$. On the other hand, the tangent space of $K_2 \text{Exp} H$ at $\text{Exp} H$ is given by

$$\begin{aligned} T_{\text{Exp} H}(K_2 \text{Exp} H) &= \left\{ \left. \frac{d}{dt} \right|_{t=0} \exp tX \exp H K_1 \mid X \in \mathfrak{k}_2 \right\} \\ &= (\exp H)_* \{ (\text{Ad}(\exp(-H))X)_{\mathfrak{m}_1} \mid X \in \mathfrak{k}_2 \} \\ &= (\exp H)_*(\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}, \end{aligned}$$

where $(\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}$ is the \mathfrak{m}_1 -component of $\text{Ad}(\exp(-H))\mathfrak{k}_2$. Therefore the above diffeomorphism $K_2 \text{Exp} H \cong K_2/N_{K_2}^H[K_1]$ induces a linear isomorphism from $T_{\text{Exp} H}(K_2 \text{Exp} H)$ onto $(\mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1])^\perp$.

Theorem 1.3.1. ([15]) *Let Y be in $(\mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1])^\perp$. We define a curve $c(t)$ in $K_2 \text{Exp} H$ by*

$$c(t) = \exp tY \text{Exp} H.$$

Let ξ be in $(\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}^\perp$. We define a normal vector field $\xi(t)$ of $K_2 \text{Exp} H$ along $c(t)$ by

$$\xi(t) = (\exp tY)_*(\exp H)_*\xi.$$

Then $\xi(t)$ is parallel with respect to the normal connection.

In order to show the theorem we prove the lemmas below.

Lemma 1.3.2. *We denote by ∇^\perp the normal connection of $K_2 \text{Exp} H \subset M_1$. We define a curve $g(t)$ in G by*

$$g(t) = \exp tY \exp H.$$

Then

$$\nabla_{\dot{c}(t)}^\perp \xi(t) = g(t)_*[(\text{Ad}(\exp(-H))Y)_{\mathfrak{k}_1}, \xi]^\perp.$$

Proof. Let π be the natural projection from G onto $M_1 = G/K_1$. We consider the principal fiber bundle $G(M_1, K_1, \pi)$. The canonical decomposition $\mathfrak{g} = \mathfrak{k}_1 + \mathfrak{m}_1$ induces an invariant connection on $G(M_1, K_1, \pi)$. It is known that the Levi-Civita connection of M_1 is reduced to the invariant connection. The tangent bundle $E = TM_1$ of M_1 is the vector bundle associated with $G(M_1, K_1, \pi)$ with standard fiber \mathfrak{m}_1 . We denote by $A^p(E)$ the vector space of E -valued p -forms on M_1 , and by $A_{\text{Ad}}^p(G)$ the vector space of tensorial p -forms $\tilde{\xi}$ of type $\text{Ad}(K_1)$ on G , that is, ξ satisfies the following conditions.

- (1) $R_a^* \tilde{\xi} = \text{Ad}(a^{-1}) \tilde{\xi} \quad (\forall a \in K_1),$
- (2) $\tilde{\xi}(X_1, \dots, X_p) = 0$ when X_1 is vertical $(X_1, \dots, X_p \in T_g G),$

where R denotes the right translation. It is well-known that the linear mapping given by

$$\begin{aligned} A^p(E) &\rightarrow A_{\text{Ad}}^p(G); \xi \mapsto \tilde{\xi} \\ \tilde{\xi}(X_1, \dots, X_p) &= g_*^{-1}(\xi(\pi_* X_1, \dots, \pi_* X_p)) \end{aligned}$$

is an isomorphism. We denote by ∇ the covariant derivative on TM_1 . When X in $A^0(E)$ corresponds to $\tilde{\xi}$ in $A_{\text{Ad}}^0(G)$ by this correspondence, ∇X in $A^1(E)$ corresponds to $d\tilde{\xi} \circ \mathcal{H}$ in $A_{\text{Ad}}^1(G)$ (see [23] Chapter II), where we denote by $\mathcal{H}(Z)$ the horizontal component of $Z \in T_g G$. Therefore for $V \in T_{\pi(g)}(G/K_1)$

$$\nabla_V X = g_*(d\tilde{\xi})(\mathcal{H}(V)) = g_* \mathcal{H}(Z) \tilde{\xi},$$

where $Z \in T_g G$ with $\pi(Z) = V$.

Let $\beta(t)$ be a curve in G such that $\beta(0) = g$ and $\dot{\beta}(0) = Z$, and A in \mathfrak{k} such that $g_* A = \mathcal{V}(Z)$, where we denote by $\mathcal{V}(Z)$ the vertical component of $Z \in T_g G$. Then we have

$$\begin{aligned} \nabla_V X &= g_*(Z\tilde{\xi} - \mathcal{V}(Z)\tilde{\xi}) \\ &= g_* \left(\left. \frac{d}{dt} \right|_{t=0} \tilde{\xi}(\beta(t)) - \left. \frac{d}{dt} \right|_{t=0} \tilde{\xi}(g \exp tA) \right) \\ &= g_* \left(\left. \frac{d}{dt} \right|_{t=0} \tilde{\xi}(\beta(t)) - \left. \frac{d}{dt} \right|_{t=0} \text{Ad}(\exp(-tA)) \tilde{\xi}(g) \right) \\ &= g_* \left(\left. \frac{d}{dt} \right|_{t=0} \tilde{\xi}(\beta(t)) + [A, \tilde{\xi}(g)] \right) \\ &= g_* \left(\left. \frac{d}{dt} \right|_{t=0} \beta(t)_*^{-1} X_{\pi(\beta(t))} + [A, g_*^{-1} X_{\pi(g)}] \right). \end{aligned}$$

In particular we put $V = g_*v$ for $v \in \mathfrak{m}_1$, and a curve $\alpha(t)$ in G such that $\alpha(0) = e$ and $\dot{\alpha}(0) = v + A$. If we set $\beta(t) = g\alpha(t)$ then $\beta(t)$ satisfies above conditions. Hence we have

$$\nabla_{g_*v}X = g_* \left(\left. \frac{d}{dt} \right|_{t=0} \alpha(t)^{-1} g_*^{-1} X_{\pi(g\alpha(t))} + [A, g_*^{-1} X_{\pi(g)}] \right).$$

For fixed t , we define $\alpha(s)$ by $\alpha(s) = g(t)^{-1}g(t+s)$. Then $\alpha(0) = e$ and $\dot{\alpha}(0) = \text{Ad}(\exp(-H))Y$. Here we have

$$\begin{aligned} \dot{c}(t) &= \left. \frac{d}{ds} \right|_{s=0} \exp(t+s)Y \text{Exp}H \\ &= g(t)_* \left(\left. \frac{d}{ds} \right|_{s=0} \exp(-H) \exp sY \exp HK_1 \right) \\ &= g(t)_* (\text{Ad}(\exp(-H))Y)_{\mathfrak{m}_1} \end{aligned}$$

Thus we have

$$\begin{aligned} &\nabla_{\dot{c}(t)}\xi(t) \\ &= g(t)_* \left\{ \left. \frac{d}{ds} \right|_{s=0} \alpha(s)^{-1} g(t)^{-1} \xi(t+s) + [(\text{Ad}(\exp(-H))Y)_{\mathfrak{k}_1}, g(t)^{-1} \xi(t)] \right\} \\ &= g(t)_* \left\{ \left. \frac{d}{ds} \right|_{s=0} g(t+s)^{-1} \xi(t+s) + [(\text{Ad}(\exp(-H))Y)_{\mathfrak{k}_1}, \xi] \right\} \\ &= g(t)_* [(\text{Ad}(\exp(-H))Y)_{\mathfrak{k}_1}, \xi]. \end{aligned}$$

Consequently we obtain the lemma.

Hence in order to show the theorem it is sufficient to prove

$$[(\text{Ad}(\exp(-H))(\mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1])^\perp)_{\mathfrak{k}_1}, (\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}^\perp] \subset (\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}.$$

The following lemma is trivial.

Lemma 1.3.3.

$$\text{Ad}(\exp H)(\mathfrak{n}_{\mathfrak{k}_1}^{-H}[\mathfrak{k}_2]) = \mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1],$$

where $\mathfrak{n}_{\mathfrak{k}_1}^{-H}[\mathfrak{k}_2] = \{X \in \mathfrak{k}_1 \mid \text{Ad}(\exp H)X \in \mathfrak{k}_2\}$.

Lemma 1.3.4.

$$(\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}^\perp = \{X \in \mathfrak{m}_1 \mid \text{Ad}(\exp H)X \in \mathfrak{m}_2\}.$$

In particular $(\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}^\perp$ is a Lie triple system in \mathfrak{m}_1 (cf. Proposition 1.1.11).

Proof.

$$\begin{aligned} (\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}^\perp &= \{X \in \mathfrak{m}_1 \mid \langle X, (\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1} \rangle = 0\} \\ &= \{X \in \mathfrak{m}_1 \mid \langle X, \text{Ad}(\exp(-H))\mathfrak{k}_2 \rangle = 0\} \\ &= \{X \in \mathfrak{m}_1 \mid \langle \text{Ad}(\exp H)X, \mathfrak{k}_2 \rangle = 0\} \\ &= \{X \in \mathfrak{m}_1 \mid \text{Ad}(\exp H)X \in \mathfrak{m}_2\}. \end{aligned}$$

Hence the lemma is proved.

The following lemma immediately follows from the lemma above.

Lemma 1.3.5.

$$[(\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}^\perp, (\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}^\perp] \subset \mathfrak{n}_{\mathfrak{k}_1}^{-H}[\mathfrak{k}_2].$$

Lemma 1.3.6.

$$(\text{Ad}(\exp(-H))(\mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1])_{\mathfrak{k}_1}^\perp)_{\mathfrak{k}_1} \subset (\mathfrak{n}_{\mathfrak{k}_1}^{-H}[\mathfrak{k}_2])^\perp.$$

Proof.

$$\begin{aligned} &\langle (\text{Ad}(\exp(-H))(\mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1])_{\mathfrak{k}_1}^\perp)_{\mathfrak{k}_1}, \mathfrak{n}_{\mathfrak{k}_1}^{-H}[\mathfrak{k}_2] \rangle \\ &= \langle \text{Ad}(\exp(-H))(\mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1])_{\mathfrak{k}_1}^\perp, \mathfrak{n}_{\mathfrak{k}_1}^{-H}[\mathfrak{k}_2] \rangle \quad (\text{by } \mathfrak{n}_{\mathfrak{k}_1}^{-H}[\mathfrak{k}_2] \subset \mathfrak{k}_1) \\ &= \langle (\mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1])_{\mathfrak{k}_1}^\perp, \text{Ad}(\exp H)\mathfrak{n}_{\mathfrak{k}_1}^{-H}[\mathfrak{k}_2] \rangle \\ &= \langle (\mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1])_{\mathfrak{k}_1}^\perp, \mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1] \rangle \quad (\text{by Lemma 1.3.3}) \\ &= \{0\}. \end{aligned}$$

Hence the lemma is proved.

Lemma 1.3.7.

$$[(\mathfrak{n}_{\mathfrak{k}_1}^{-H}[\mathfrak{k}_2])^\perp, (\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}^\perp] \subset (\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}.$$

Proof.

$$\begin{aligned}
& \langle [(\mathfrak{n}_{\mathfrak{k}_1}^{-H}[\mathfrak{k}_2])^\perp, (\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}^\perp], (\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}^\perp \rangle \\
&= \langle (\mathfrak{n}_{\mathfrak{k}_1}^{-H}[\mathfrak{k}_2])^\perp, [(\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}^\perp, (\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}^\perp] \rangle \\
&\subset \langle (\mathfrak{n}_{\mathfrak{k}_1}^{-H}[\mathfrak{k}_2])^\perp, \mathfrak{n}_{\mathfrak{k}_1}^{-H}[\mathfrak{k}_2] \rangle \quad (\text{by Lemma 1.3.5}) \\
&= \{0\}.
\end{aligned}$$

Hence the lemma is proved.

By Lemmas 1.3.6 and 1.3.7 we have

$$[(\text{Ad}(\exp(-H))(\mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1])^\perp)_{\mathfrak{k}_1}, (\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}^\perp] \subset (\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}.$$

This completes the proof of Theorem 1.3.1.

Corollary 1.3.8. *The mean curvature vector of $K_2\text{Exp}H \subset M_1$ is parallel with respect to the normal connection.*

Proof. We denote by \vec{H} the mean curvature vector of $K_2\text{Exp}H \subset M_1$. Since $\vec{H}_{\text{Exp}tX\text{Exp}H} = (\exp tX)_* \vec{H}_{\text{Exp}H}$ ($X \in (\mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1])^\perp$), we have $\nabla_X^\perp \vec{H} = 0$ by Theorem 1.3.1. Hence $(\nabla^\perp \vec{H})_{\text{Exp}H} = 0$. Therefore $\nabla^\perp \vec{H}$ vanishes everywhere by the homogeneity of $K_2\text{Exp}H$.

The decomposition

$$\mathfrak{k}_2 = \mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1] + (\mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1])^\perp$$

defines an invariant connection ∇^c of $K_2\text{Exp}H$. We denote by h the second fundamental form of $K_2\text{Exp}H \subset M_1$. We define $\nabla^c h$ by

$$(\nabla_X^c h)(Y, Z) = \nabla_X^\perp(h(Y, Z)) - h(\nabla_X^c Y, Z) - h(Y, \nabla_X^c Z).$$

Corollary 1.3.9.

$$\nabla^c h = 0.$$

Proof. Let X, Y and Z be elements of $(\mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1])^\perp$. Then the vector fields $(\exp tX)_*(\exp H)_*Y$ and $(\exp tX)_*(\exp H)_*Z$ of $K_2\text{Exp}H$ along a curve $\exp tX\text{Exp}H$ are ∇^c -parallel. Thus we get

$$\begin{aligned}
& (\nabla_{(\exp H)_*X}^c h)(Y, Z) \\
&= \nabla_{(\exp H)_*X}^\perp(h((\exp tX)_*(\exp H)_*Y, (\exp tX)_*(\exp H)_*Z)) \\
&= \nabla_{(\exp H)_*X}^\perp((\exp tX)_*(h((\exp H)_*Y, (\exp H)_*Z))) \\
&= 0 \quad (\text{Theorem 1.3.1}).
\end{aligned}$$

Hence we have $(\nabla^c h)_{\text{Exp}H} = 0$. By homogeneity we have $\nabla^c h = 0$.

By Corollary 1.3.9, for any vector fields X, Y and Z of $K_2 \text{Exp}H$ we have

$$(\nabla_X^\perp(h(Y, Z)))_p \in \text{span}\{h(Y, Z) \mid Y, Z \in T_p(K_2 \text{Exp}H)\}.$$

In other words we get:

Corollary 1.3.10. *The degree of $K_2 \text{Exp}H \subset M_1$ is at most 2.*

We consider the normal holonomy representation of $K_2 \text{Exp}H \subset M_1$. By Lemma 1.3.4, we can define an action ρ of $N_{K_2}^H[K_1]$ on $(\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}^\perp$ by

$$\rho(k)X = \text{Ad}(\exp(-H)k \exp H)X$$

for $k \in N_{K_2}^H[K_1]$, $X \in (\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}^\perp$. This action is equivalent to the differential representation of $N_{K_2}^H[K_1]$ on $T_{\text{Exp}H}^\perp(K_2 \text{Exp}H)$.

Corollary 1.3.11. *The normal holonomy representation of $K_2 \text{Exp}H \subset M_1$ is equivalent to the (effectively made) action of a subgroup of $N_{K_2}^H[K_1]$ on $(\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}^\perp$.*

Proof. Every geodesic $c(t)$ of $K_2/N_{K_2}^H[K_1]$ through the origin $\text{Exp}H$ with respect to the normal homogeneous Riemannian metric is given by

$$c(t) = \exp tY \text{Exp}H \quad \text{for some } Y \in (\mathfrak{n}_{\mathfrak{k}_2}^H[\mathfrak{k}_1])^\perp.$$

By Theorem 1.3.1, the parallel translation along $c(t)$ with respect to the normal connection is given by $(\exp tY)_*$. Now any curve in $K_2 \text{Exp}H$ can be approximated by broken geodesics with respect to the normal homogeneous Riemannian metric. It follows that the normal holonomy representation is equivalent to the action of K/L on $(\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}^\perp$ where

$$L = \{k \in N_{K_2}^H[K_1] \mid \rho(k) = 1 \text{ on } (\text{Ad}(\exp(-H))\mathfrak{k}_2)_{\mathfrak{m}_1}^\perp\}$$

and where K is a subgroup of $N_{K_1}^H[K_2]$ with $L \subset K$.

We shall prove that the mean curvature vector of any orbit of any hyperpolar action is parallel with respect to the normal connection. In order to do this we review a result of Hsiang.

Let G be a compact, connected Lie group of isometries of a Riemannian manifold M . Each orbit of G has a well defined volume by the induced metric as a submanifold. The volume of an orbit $N_0 = Gx_0$ is said to be *extremal among nearby orbits of the same type* if

$$\left. \frac{d}{dt} \right|_{t=0} \text{vol}(N_t) = 0$$

for all smooth families N_t , $|t| < \varepsilon$, of G -orbits of the same type on M .

Theorem 1.3.12. ([14]) *Let G be a compact connected Lie group of isometries of a Riemannian manifold M . Then any orbit of G whose volume is extremal among nearby orbits of the same type is a minimal submanifold of M . In particular if there exists a neighborhood of N_0 in which there are no other orbits of the same type, then N_0 is a minimal submanifold of M .*

Corollary 1.3.13. *The mean curvature vector of any orbit of any hyperpolar action on a compact symmetric space is parallel with respect to the normal connection.*

Proof. From Theorem 1.1.14 all hyperpolar actions on compact symmetric spaces are either cohomogeneity one actions or orbit equivalent to Hermann actions. We have already proved that the mean curvature vector of any orbit of any Hermann action is parallel in Corollary 1.3.8, so it is sufficient to consider the case of cohomogeneity one actions.

We assume that the cohomogeneity of the action is equal to one. Since the codimensions of principal orbits are equal to one, they have parallel mean curvature vectors. From Theorem 1.1.9 the cohomogeneity one action has at most two singular orbits, corresponding to the boundary points of the orbit space. Thus singular orbits of the cohomogeneity one actions are minimal by Theorem 1.3.12. Hence we have the conclusion.