Chapter 4

The classification of Z in case Z is a tree

4.1 The classification of Z in case Z is a tree

In this chapter, we hope to classify the structures of Z in case Z is a tree under the same assumptions to chapter 3, i.e.

1. $f : I \longrightarrow I$ is pointwise P-expansive.
2. $f(C)$ is the union of some elements of $S(I,P)$ for any $C \in S(I,P)$.
3. $f(C)$ is not one point for each $C \in S(I,P) \setminus P$ with $C \neq \emptyset$.

Let $p_1, p_2, \ldots, p_n$ be the elements of P and $C_0, C_1, \ldots, C_n$ the elements of $S(I,P) \setminus P$ such that $p_1 < p_2 < \cdots < p_n$ and $cl(C_i) \cap cl(C_{i+1}) = \{p_{i+1}\}$ for $i = 0, 1, \ldots, n-1$, but $C_0 = \emptyset$ if p_1 is the endpoint of I and $C_n = \emptyset$ if p_n is the endpoint of I.

Lemma 4.1.1 ([4,p.7]) *Let C_{i_0} be an element of $S(I,P) \setminus P$ and p an element of P such that $C_{i_0} \subset [P]$ and $p \in cl(C_{i_0})$. Then there exists an element $C_{i_1} = (p_{i_1}, p_{i_1+1})$ of $S(I,P) \setminus P$ with $C_{i_1} \subset [P]$ such that $cl(C_{i_0}) \subset \langle f(p_{i_1}), f(p_{i_1+1}) \rangle$, where $\langle f(p_{i_1}), f(p_{i_1+1}) \rangle$ means the closed interval with endpoints $f(p_{i_1})$ and*

$f(p_{i_1+1})$. $\square$

Corollary 4.1.2 *Let C_{i_0} be an element of $S(I,P) \setminus P$ with $C_{i_0} \subset [P]$ and p an element of $P \cap cl(C_{i_0})$. Then*

(1) *there exists an n-cycle $C_{i_0} \longrightarrow C_{i_{n-1}} \longrightarrow \cdots \longrightarrow C_{i_1} \longrightarrow C_{i_0}$ for $O(p,f)$ with $C_{i_\ell} \subset [P]$ for each $\ell = 0, 1, \ldots, n-1$, or*

(2) *there exist k with $k \leq n-1$, elements $C_{i_1}, C_{i_2}, \ldots, C_{i_{k-1}}, C_{i_k}, C_{j_{k-1}}$ of $S(I,P) \setminus P$ with $C_{i_\ell}, C_{j_{k-1}} \subset [P]$ for each $\ell = 1, 2, \ldots, k$, and an element p_∞ of $\mathrm{Int}(B_{i_k}) \cap \pi([P])$ such that*

(i) $C_{i_k} \longrightarrow C_{i_{k-1}} \longrightarrow \cdots \longrightarrow C_{i_1} \longrightarrow C_{i_0}$ *and* $C_{i_k} \longrightarrow C_{j_{k-1}}$

(ii) $f^{n-\ell}(p) \in cl(C_{i_\ell})$ *and* $\{f^{n-k+1}(p)\} = cl(C_{i_{k-1}}) \cap cl(C_{j_{k-1}})$ *for each* $\ell = 1, 2, \ldots, k$

(iii) $g(p_\infty) = \pi \circ f^{n-k+1}(p)$. $\square$

Lemma 4.1.3 *If there exist an element C_{i_0} of $S(I,P) \setminus P$ and an element p_∞ of P_∞ such that $C_{i_0} \subset [P]$, $p_\infty \in \mathrm{Int}(B_{i_0})$ and $\mathrm{Ord}(p_\infty, Z) \geq 3$, then Z is not a tree.*

Proof. Denote $\ell = min\{\ell' \mid p_\infty \in \mathrm{Bd}(B) \cap \mathrm{Bd}(B')$ and B, B' are distinct elements of $\mathbf{A}_{\ell'}\}$. Let $B_{i_0,s_1,s_2,\ldots,s_{\ell-1},\gamma_\ell}, B_{i_0,s_1,s_2,\ldots,s_{\ell-1},\delta_\ell}$ be two distinct elements of $\mathbf{A}_\ell$ such that $\pi(p) \in \mathrm{Bd}(B_{i_0,s_1,s_2,\ldots,s_{\ell-1},\gamma_\ell}) \cap \mathrm{Bd}(B_{i_0,s_1,s_2,\ldots,s_{\ell-1},\delta_\ell})$.

The existence of these elements of $\mathbf{A}_\ell$ implies that $C_{i_0} \longrightarrow C_{s_1} \longrightarrow \cdots \longrightarrow C_{s_{\ell-1}} \longrightarrow C_{\gamma_\ell}, C_{s_{\ell-1}} \longrightarrow C_{\delta_\ell}, \pi \circ f^k(p) \in C_{s_k}$ for each $k = 1, 2, \ldots, \ell - 1$ and $\{\pi \circ f^{\ell-1}(p)\} = cl(C_{\gamma_\ell}) \cap cl(C_{\delta_\ell})$.

We show this lemma by Corollary 4.1.2. Assume that there exists an n-cycle $C_{i_0} \longrightarrow C_{i_1} \longrightarrow \cdots \longrightarrow C_{i_{n-1}} \longrightarrow C_{i_0}$ for $O(p, f)$ with $C_{i_k} \subset [P]$ for each $k = 0, 1, \ldots, n-1$. Then it follows from Theorem 3.1.7 that Z is the universal dendrite. In the other case, we have that $\mathrm{Ord}(p_\infty, Z) = \mathrm{Ord}(g(p_\infty), B_{i_k} \cap Z) + \mathrm{Ord}(g(p_\infty), B_{j_k} \cap Z) = \mathrm{Ord}(\pi(p), B_{i_0, i_1, \ldots, i_{\ell-1}, \gamma_\ell} \cap Z) + \mathrm{Ord}(\pi(p), B_{i_0, i_1, \ldots, i_{\ell-1}, \delta_\ell} \cap Z + \mathrm{Ord}(g(p_\infty), B_{j_k} \cap Z) = \mathrm{Ord}(\pi(p), Z) + \mathrm{Ord}(g(p_\infty), B_{j_k} \cap Z) \geq 3$.

Note that $\ell + m = min\{\ell' \mid h_{i_m, i_{m-1}, \ldots, i_1}(p_\infty) \in \mathrm{Bd}(B) \cap \mathrm{Bd}(B')$ and B, B' are distinct elements of $\mathbf{A}_{\ell'}\}$ for each $m = 1, 2, \ldots$. Thus if $m \neq m'$, then $h_{i_m, i_{m-1}, \ldots, i_1}(p_\infty) \neq h_{i_{m'}, i_{m'-1}, \ldots, i_1}(p_\infty)$. Since $\{h_{i_m, i_{m-1}, \ldots, i_1}(p_\infty) \mid m \geq 1\}$ is infinite and $\mathrm{Ord}(h_{i_m, i_{m-1}, \ldots, i_1}(p_\infty), Z) \geq 3$ for each $m \geq 1$, we see that Z is not a tree. □

Lemma 4.1.4 *If there exist an element p of P and an element C_{i_0} of $S(I, P) \setminus P$ such that $p \in cl(C_{i_0})$, $C_{i_0} \subset [P]$ and* $\mathrm{Ord}(\pi(p), B_{i_0} \cap Z) \geq 2$, *then Z is not a tree.*

Proof. By Collorary 4.2, assume that there exists an n-cycle $C_{i_0} \longrightarrow C_{i_{n-1}} \longrightarrow \cdots \longrightarrow C_{i_1} \longrightarrow C_{i_0}$ for $O(p, f)$ with $C_{i_\ell} \subset [P]$ for each $\ell = 0, 1, \ldots, n-1$. Then Z is the universal dendrite by Theorem 3.1.7.

In the other case, there exist an element C_{i_k} of $S(I, P) \setminus P$ with $C_{i_k} \subset [P]$ and an element p_∞ of P_∞ such that $p_\infty \in \mathrm{Int}(B_{i_k})$ and $\mathrm{Ord}(p_\infty, Z) \geq 3$. Thus by Lemma 4.1.3, Z is not a tree. □

It follows from by Lemma 4.1.3 and 4.1.4 that $\pi([P])$ is an arc if Z is a tree.

Lemma 4.1.5 *Let $\{C_{i_0}, C_{i_1}, C_{i_2}, \ldots\}$ be an infinite sequence of elements of $S(I,P) \setminus P$ with $\cdots \longrightarrow C_{i_2} \longrightarrow C_{i_1} \longrightarrow C_{i_0}$. And let p be an element of P with $p \in cl(C_{i_0})$. If $\mathrm{Ord}(\pi(p), B_{i_0} \cap Z) \geq 2$, then Z is not a tree.*

Proof. By Lemma 4.1.3 and 4.1.4, if $C_{i_k} \subset [P]$ for some $k = 0, 1, \ldots$, then Z is not a tree. Thus we may assume that $C_{i_k} = C_0$ or C_n for each $k = 0, 1, \ldots$. Assume that $C_{i_0} = C_0$.

Case 1. $C_{i_1} = C_0$, i.e. $C_0 \longrightarrow C_0$.
Since $p_1 \not\longrightarrow p_1$, we have that $\pi(p_1) \neq h_0 \circ \pi(p_1) \in \mathrm{Int}(B_0)$. Then it holds that $\mathrm{Ord}(h_0 \circ \pi(p_1), Z) = \mathrm{Ord}(h_0 \circ \pi(p_1), B_{0,0} \cap Z) + \mathrm{Ord}(h_0 \circ \pi(p_1), B_{0,1} \cap Z) = \mathrm{Ord}(\pi(p_1), B_0 \cap Z) + \mathrm{Ord}(\pi(p_1), B_1 \cap Z) = \mathrm{Ord}(\pi(p_1), Z) \geq 3$.

Since $B_{0,0} \cap Z$ is homeomorphic to $B_0 \cap Z$, we have that $h_0 \circ \pi(p_1) \neq h_{0,0} \circ \pi(p_1) \in \mathrm{Int}(B_{0,0})$ and $\mathrm{Ord}(h_{0,0} \circ \pi(p_1), Z) = \mathrm{Ord}(h_{0,0} \circ \pi(p_1), B_{0,0,0} \cap Z) + \mathrm{Ord}(h_{0,0} \circ \pi(p_1), B_{0,0,1} \cap Z) = \mathrm{Ord}(\pi(p_1), B_0 \cap Z) + \mathrm{Ord}(\pi(p_1), B_1 \cap Z) = \mathrm{Ord}(\pi(p_1), Z) \geq 3$.

Repeating this operation, we see that $\{h_{0,0,\ldots,0}(p_1) \mid \{h_{0,0,\ldots,0}(p_1)\} = \mathrm{Int}(B_{0,0,\ldots,0}) \cap \mathrm{Int}(B_{0,0,\ldots,0,1}), B_{0,0,\ldots,0}, B_{0,0,\ldots,0,1} \in \mathbf{A}_m$ and $m \geq 1\}$ is infinite and $\mathrm{Ord}(h_{0,0,\ldots,0}(p_1), Z)$ 3. Thus Z is not a tree (see Figure 4.1.1).

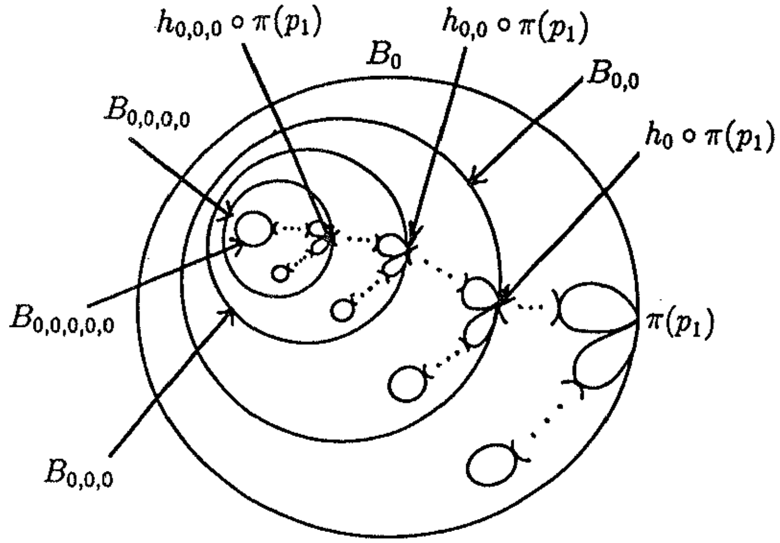


Figure 4.1.1

Case 2. $C_{i_1} = C_n$, i.e. $C_n \longrightarrow C_0$.

(Case 2.1) $C_{i_2} = C_0$, i.e. $C_0 \longrightarrow C_n \longrightarrow C_0$.

Assume that $p_n \not\rightharpoonup p_1$, i.e. $\pi(p_n) \neq h_n \circ \pi(p_1) \in \text{Int}(B_n)$. Then we obtain that $\text{Ord}(h_n \circ \pi(p_1), Z) = \text{Ord}(h_n \circ \pi(p_1), B_{n,0} \cap Z) + \text{Ord}(h_n \circ \pi(p_1), B_{n,1} \cap Z) = \text{Ord}(\pi(p_1), B_0 \cap Z) + \text{Ord}(\pi(p_1), B_1 \cap Z) = \text{Ord}(\pi(p_1), Z) \geq 3$.

Since $C_0 \longrightarrow C_n$, there exists an element $B_{0,n}$ of $\mathbf{A}_1$ such that $h_0 \circ \pi(p_n) \in B_{0,n} \subset B_0$. Since $B_{0,n} \cap Z$ is homeomorphic to $B_n \cap Z$, we have that $h_0 \circ \pi(p_n) \neq h_{0,n} \circ \pi(p_1) \in \text{Int}(B_{0,n})$ and $\text{Ord}(h_{0,n} \circ \pi(p_1), Z) = \text{Ord}(h_{0,n} \circ \pi(p_1), B_{0,n,0} \cap Z) + \text{Ord}(h_{0,n} \circ \pi(p_1), B_{0,n,1} \cap Z) = \text{Ord}(\pi(p_1), B_0 \cap Z) + \text{Ord}(\pi(p_1), B_1 \cap Z) \geq 3$.

Repeating this operation, we obtain that $\{h_{0,n,\ldots,0,n} \circ \pi(p_1) \mid h_{0,n,\ldots,0,n} \circ \pi(p_1) \in \text{Bd}(B_{0,n,\ldots,0,n,0}) \cap \text{Bd}(B_{0,n,\ldots,0,n,1}), B_{0,n,\ldots,0,n,0}, B_{0,n,\ldots,0,n,1} \in \mathbf{A}_{2m} \text{ and } m \geq 1\}$ is infinite and $\text{Ord}(h_{0,n,\ldots,0,n} \circ \pi(p_1), Z) \geq 3$. Thus Z is not a tree (see Figure 4.1.2).

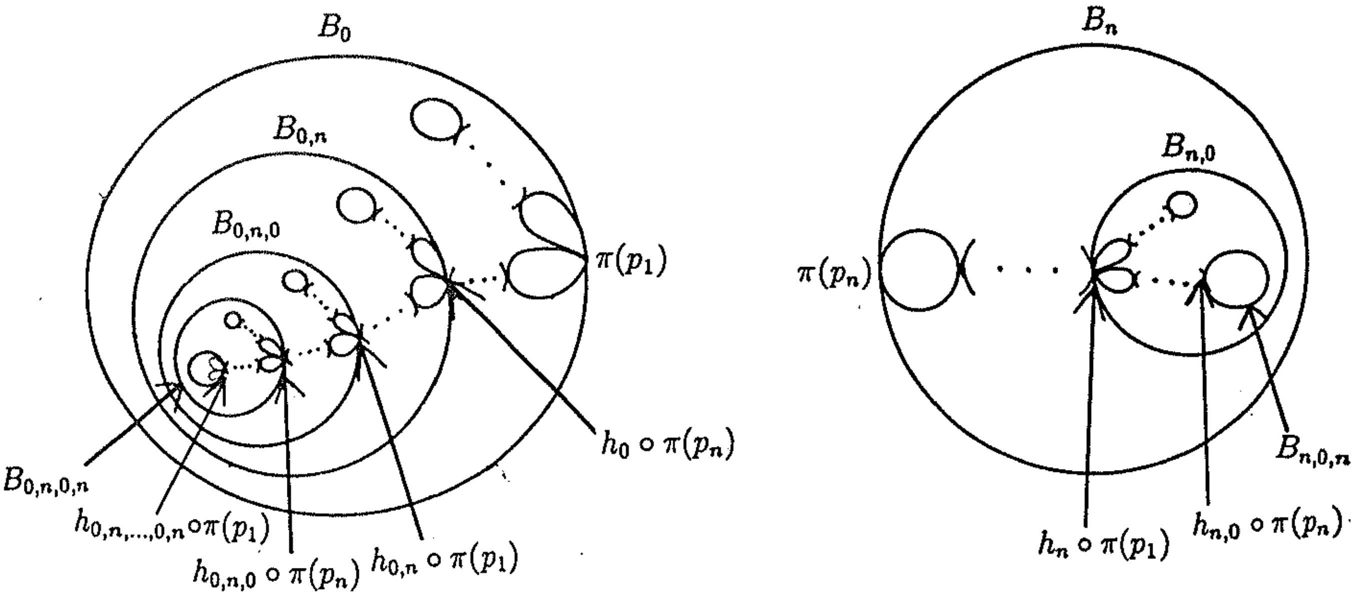


Figure 4.1.2

Assume that $p_n \longrightarrow p_1$. Then we have that $\pi(p_n) = h_n \circ \pi(p_1) \in B_{n,0}$. Thus we see that $\mathrm{Ord}(h_n \circ \pi(p_1), Z) = \mathrm{Ord}(h_n \circ \pi(p_1), B_{n-1} \cap Z) + \mathrm{Ord}(h_n \circ \pi(p_1), B_n \cap Z) \geq 1 + \mathrm{Ord}(h_n \circ \pi(p_1), B_{n,0} \cap Z) = 1 + \mathrm{Ord}(\pi(p_1), B_0 \cap Z) \geq 3$.

Since $p_1 \not\longrightarrow p_n$ and $C_0 \longrightarrow C_n$, we have that $h_0 \circ \pi(p_n) = h_{0,n} \circ \pi(p_1) \in \mathrm{Int}(B_0) \cap \mathrm{Bd}(B_{0,n,0})$. Thus we see that $\mathrm{Ord}(h_{0,n} \circ \pi(p_1), Z) = \mathrm{Ord}(h_0 \circ \pi(p_n), Z) = \mathrm{Ord}(h_0 \circ \pi(p_n), B_{0,n-1} \cap Z) + \mathrm{Ord}(h_0 \circ \pi(p_n), B_{0,n} \cap Z) = \mathrm{Ord}(\pi(p_n), B_{n-1} \cap Z) + \mathrm{Ord}(\pi(p_n), B_n \cap Z) = \mathrm{Ord}(\pi(p_n), Z) \geq 3$. As $B_{0,n,0} \cap Z$ is homeomorphic to $B_0 \cap Z$, we have that $h_{0,n,0} \circ \pi(p_n) = h_{0,n,0,n} \circ \pi(p_1) \in \mathrm{Int}(B_{0,n,0}) \cap \mathrm{Bd}(B_{0,n,0,n,0})$ and $\mathrm{Ord}(h_{0,n,0,n} \circ \pi(p_1), Z) = \mathrm{Ord}(h_{0,n,0} \circ \pi(p_n), Z) \geq 3$.

Repeating this operation, we obtain that $\{h_{0,n,\ldots,0,n} \circ \pi(p_1) \mid h_{0,n,\ldots,0,n} \circ \pi(p_1) \in \mathrm{Bd}(B_{0,n,\ldots,0,n,0}) \cap \mathrm{Bd}(B_{0,n,\ldots,n,0,n-1}), B_{0,n,\ldots,0,n,0}, B_{0,n,\ldots,n,0,n-1} \in \mathbf{A}_{2m} \text{ and } m \geq 0\}$ is infinite and $\mathrm{Ord}(h_{0,n,\ldots,0,n} \circ \pi(p_1), Z) \geq 3$. Thus Z is not a tree (see Figure 4.1.3).

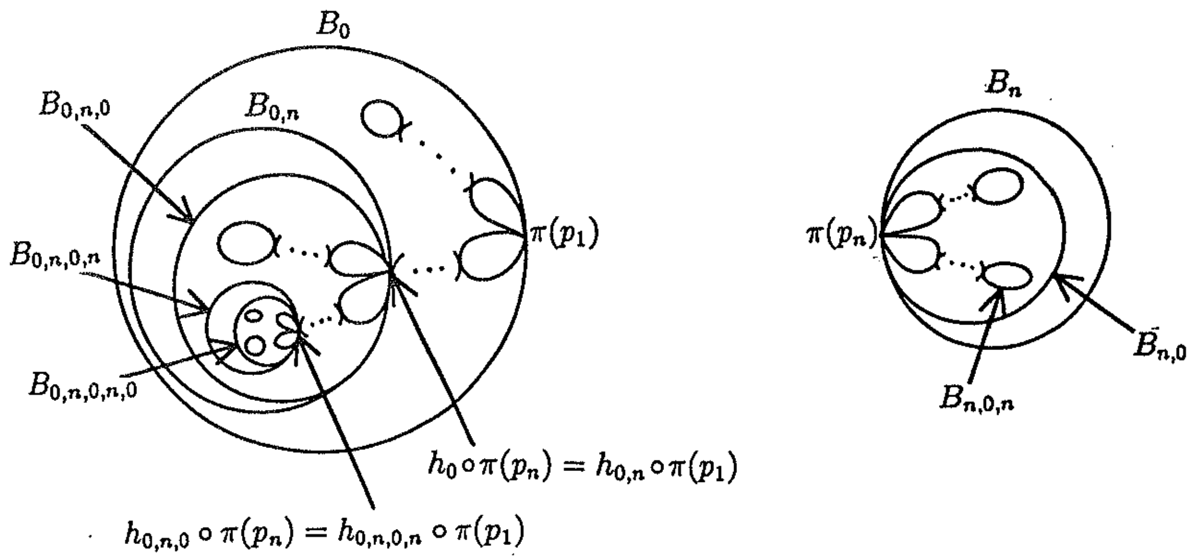


Figure 4.1.3

(Case 2.2) $C_{i_2} = C_n$, i.e. $C_n \longrightarrow C_n \longrightarrow C_0$.

Then since $B_n \cap Z = (B_{n,0} \cup B_{n,1} \cup \cdots \cup B_{n,n}) \cap Z$, we see that $\mathrm{Ord}(\pi(p_n), B_n \cap Z) \geq 2$. Since $p_n \not\longrightarrow p_n$, we have that $\pi(p_n) \neq h_n \circ \pi(p_n) \in \mathrm{Int}(B_n)$. It holds that $\mathrm{Ord}(h_n \circ \pi(p_n), Z) = \mathrm{Ord}(h_n \circ \pi(p_n), B_{n,n-1} \cap Z) + \mathrm{Ord}(h_n \circ \pi(p_n), B_{n,n} \cap Z) = \mathrm{Ord}(\pi(p_n), B_{n-1} \cap Z) + \mathrm{Ord}(\pi(p_n), B_n \cap Z) = \mathrm{Ord}(\pi(p_n), Z) \geq 3$. Since $B_{n,n} \cap Z$ is homeomorphic to $B_n \cap Z$, we see that $h_n \circ \pi(p_n) \neq h_{n,n} \circ \pi(p_n) \in \mathrm{Int}(B_{n,n})$ and $\mathrm{Ord}(h_{n,n} \circ \pi(p_n), Z) = \mathrm{Ord}(h_{n,n} \circ \pi(p_n), B_{n,n,n-1} \cap Z) + \mathrm{Ord}(h_{n,n} \circ \pi(p_n), B_{n,n,n} \cap Z) = \mathrm{Ord}(\pi(p_n), B_{n-1} \cap Z) + \mathrm{Ord}(\pi(p_n), B_n \cap Z) \geq 3$.

Repeating this operation, $\{h_{n,n,\ldots,n} \circ \pi(p_n) \mid h_{n,n,\ldots,n} \circ \pi(p_n) \in B_{n,n,\ldots,n,n-1} \cap B_{n,n,\ldots,n},\ B_{n,n,\ldots,n,n-1}, B_{n,n,\ldots,n} \in \mathrm{A}_m$ and $m \geq 0\}$ is infinite and $\mathrm{Ord}(h_{n,n,\ldots,n} \circ \pi(p_n), Z) \geq 3$. Thus Z is not a tree (see Figure 4.1.4).

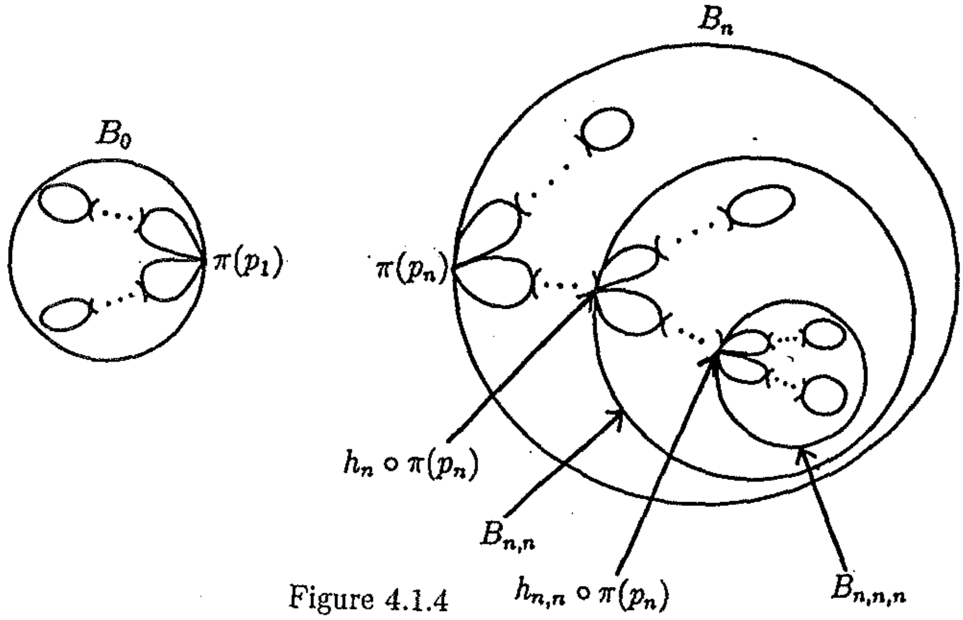


Figure 4.1.4

In a similar way, we can obtain the same result when $C_{i_1} = C_n$. □

Theorem 4.1.6 *If Z is a tree, then* $\mathrm{Card}(\mathrm{Br}(Z)) \leq 3$ *and* $\mathrm{Ord}(x, Z) \leq 4$ *for each element x of Z. Furthermore there exist only distinct 5 types of Z* (see Figure 4.1.5).

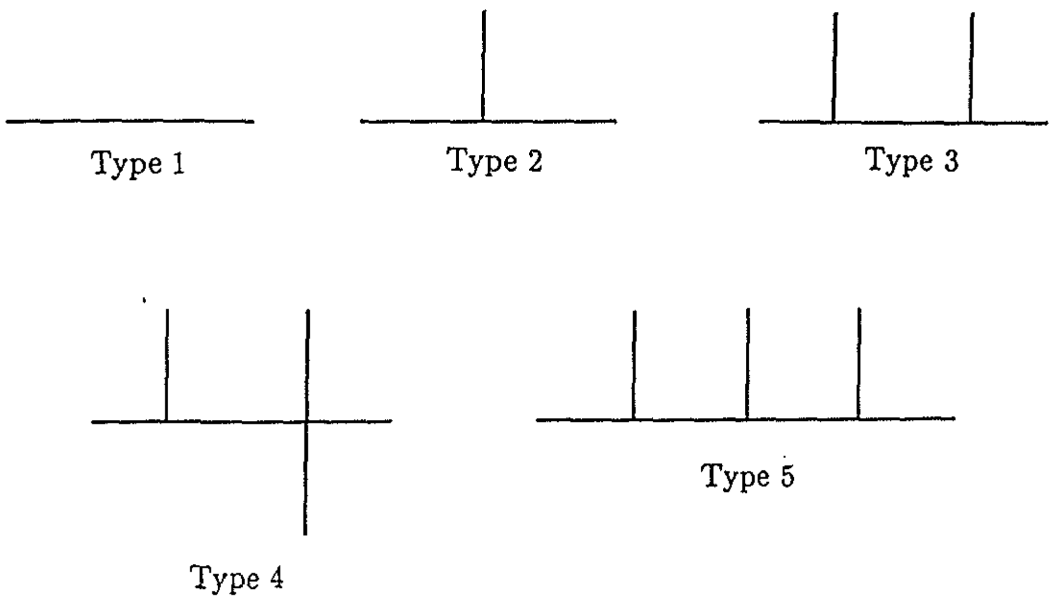


Figure 4.1.5

Proof. By Lemma 4.1.3 and 4.1.4, we see that $\pi([P])$ is an arc. Thus it suffices to investigate $(B_0 \cup B_n) \cap Z$. Note that $C \not\rightharpoonup C_0$ and $C \not\rightharpoonup C_n$ for each element C of $S(I, P) \setminus P$ with $C \subset [P]$ by Lemma 4.1.5. Let $p_k, p_{k'}$ be elements of P such that $f(p_1) = p_k$ and $f(p_n) = p_{k'}$. Note that $\{p_k\} = cl(C_{k-1}) \cap cl(C_k)$ and $\{k'\} = cl(C_{k'-1}) \cap cl(C_{k'})$.

We may assume that $C_0 \neq \emptyset$ or $C_n \neq \emptyset$.

Case 1. $C_0 \neq \emptyset$ and $C_n = \emptyset$.

(Case 1.1) $C_0 \not\rightharpoonup C_{k-1}$ and $C_0 \not\rightharpoonup C_k$.

Assume that $C_0 \longrightarrow C_{k-1}$ or $C_0 \not\rightharpoonup C_k$. Then it holds that $\text{Ord}(\pi(p_1), B_0 \cap Z) = \text{Ord}(\pi(p_1), B_{0,k-1} \cap Z) = \text{Ord}(\pi(p_k), B_{k-1} \cap Z) = 1$. Denote $m_0 = min\{m \mid C_0 \longrightarrow C_m\} \leq k-1$. Then we have $B_0 \cap Z = (B_{0,m_0} \cup B_{0,m_0+1} \cup \cdots \cup B_{0,k-1}) \cap Z$. Assume that $m \neq 0$. Since $(B_{0,m_0} \cup B_{0,m_0+1} \cup \cdots \cup B_{0,k-1}) \cap Z$ is homeomorphic to $(B_{m_0} \cup B_{m_0+1} \cup \cdots \cup B_{k-1}) \cap Z$ which is a connected subspace of $\pi([P])$, we see that $B_0 \cap Z$ is an arc, i.e. Z is homeomorphic to the Type 1.

Assume that $m_0 = 0$. Then we can construct a homeomorphism $\varphi : (B_{0,0} \cup B_{0,1} \cup \cdots \cup B_{0,k-1}) \cap Z \longrightarrow [0,1]$ as follows : $\varphi((B_{0,0,\ldots,0,1} \cup B_{0,0,\ldots,0,2} \cup \cdots \cup B_{0,0,\ldots,0,k-1}) \cap Z) = [\frac{1}{2^t}, \frac{1}{2^{t-1}}]$, if $B_{0,0,\ldots,0,1}, B_{0,0,\ldots,0,2}, \ldots, B_{0,0,\ldots,0,k-1}$ are elements of $\mathbf{A}_t (t \geq 1)$ and $\varphi(\bigcap_{t \geq 1}\{B_{0,0,\ldots,0} \mid B_{0,0,\ldots,0} \in \mathbf{A}_t\} \cap Z) = \{0\}$. Thus $B_0 \cap Z$ is an arc. Since $\text{Ord}(\pi(p_1), B_0 \cap Z) = 1$, we have that Z is an arc, i.e. Z is homeomorphic to the Type 1 (see Figure 4.1.6).

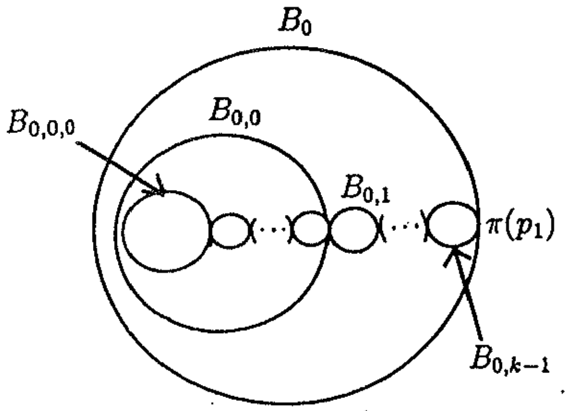


Figure 4.1.6

(Case 1.2) $C_0 \longrightarrow C_{k-1}$ and $C_0 \longrightarrow C_k$.

Then there exist elements $B_{0,k-1}, B_{0,k}$ of $\mathbf{A}_1$ in B_0 such that $\pi(p_1) \in B_{0,k-1} \cap B_{0,k}$.

We have that $\text{Ord}(\pi(p_1), Z) = \text{Ord}(\pi(p_1), B_0 \cap Z) + \text{Ord}(\pi(p_1), B_1 \cap Z) = \text{Ord}(\pi(p_1), B_{0,k-1} \cap Z) + \text{Ord}(\pi(p_1), B_{0,k} \cap Z) + 1 = \text{Ord}(\pi(p_k), B_{k-1} \cap Z) + \text{Ord}(\pi(p_k), B_k \cap Z) + 1 = 3$.

By Lemma 4.1.4, we see that $C \not\longrightarrow C_0$ for each element C of $S(I, P) \setminus P$. Denote $m_0 = min\{m \mid C_0 \longrightarrow C_m\}$ and $m_0' = max\{m \mid C_0 \longrightarrow C_m\}$. Then it holds that $0 < m_0 < m_0' \leq n - 1$. Since $B_0 \cap Z = (B_{0,m_0} \cup B_{0,m_0+1} \cup \cdots \cup B_{0,m_0'}) \cap Z$ is homeomorphic to $(B_{m_0} \cup B_{m_0+1} \cup \cdots \cup B_{m_0'}) \cap Z$ which is a connected subspace of $\pi([P])$, we have that $(B_{0,m} \cup B_{0,m+1} \cup \cdots \cup B_{0,m'}) \cap Z$ is an arc. Thus Z is homeomorphic to the Type 2 (see Figure 4.1.7).

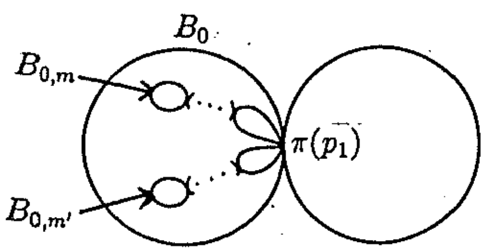


Figure 4.1.7

Case 2. $C_0 \neq \emptyset$ and $C_n \neq \emptyset$.

(Case 2.1) $C_0 \not\to C_{k-1}$ or $C_0 \not\to C_k$. And $C_n \not\to C_{k'-1}$ or $C_n \not\to C_{k'}$.

Then we have that $B_{0,k-1} = \emptyset$ or $B_{0,k} = \emptyset$, and $B_{n,k'-1} = \emptyset$ or $B_{n,k'} = \emptyset$. Denote $m_0 = min\{\ell \mid C_0 \longrightarrow C_\ell\}, m_0' = max\{\ell \mid C_0 \longrightarrow C_\ell\}, m_n = min\{\ell \mid C_n \longrightarrow C_m\}$ and $m_n' = max\{\ell \mid C_n \longrightarrow C_m\}$.

(Case 2.1.1) $B_{0,k-1} \neq \emptyset, B_{0,k} = \emptyset, B_{n,k'-1} \neq \emptyset$ and $B_{n,k'} = \emptyset$.

Note that $0 < k-1 < n$ and $k'-1 < n$. First we show that $\text{Ord}(\pi(p_1), Z) = \text{Ord}(\pi(p_n), Z) = 2$. We have that $\text{Ord}(\pi(p_1), Z) = \text{Ord}(\pi(p_1), B_0 \cap Z) + \text{Ord}(\pi(p_1), B_1 \cap Z) = \text{Ord}(\pi(p_1), B_{0,k-1} \cap Z) + 1 = \text{Ord}(\pi(p_k), B_{k-1} \cap Z) + 1 = 2$. And we see that $\text{Ord}(\pi(p_n), Z) = \text{Ord}(\pi(p_n), B_{n-1} \cap Z) + \text{Ord}(\pi(p_n), B_n \cap Z) = 1 + \text{Ord}(\pi(p_n), B_{n,k'-1} \cap Z) = 1 + \text{Ord}(\pi(p_{k'}), B_{k'-1} \cap Z) = 2$.

Next we show that $B_0 \cap Z$ and $B_n \cap Z$ are arcs. Assume that $m_0 > 0$ and $m_n > 0$. Then it holds that $B_0 \cap Z = (B_{0,m_0} \cup B_{0,m_0+1} \cup \cdots \cup B_{0,k-1}) \cap Z$ and $B_n \cap Z = (B_{n,m_n} \cup B_{n,m_n+1} \cup \cdots \cup B_{n,k'-1}) \cap Z$ are homeomorphic to $(B_{m_0} \cup B_{m_0+1} \cup \cdots \cup B_{k-1}) \cap Z$ and $(B_{m_n} \cup B_{m_n+1} \cup \cdots \cup B_{k'-1}) \cap Z$ respectively. Since $(B_{m_0} \cup B_{m_0+1} \cup \cdots \cup B_{k-1}) \cap Z$ and $(B_{m_n} \cup B_{m_n+1} \cup \cdots \cup B_{k'-1}) \cap Z$ are connected subspaces of $\pi([P])$, we see that $B_0 \cap Z$ and $B_n \cap Z$ are arcs. Thus Z is an arc, i.e. Z is homeomorphic to the Type 1.

Assume that $m_0 = 0$. In a similar way to Case 1.1, we can construct a homeomorphism $\varphi : (B_{0,0} \cup B_{0,1} \cup \cdots \cup B_{0,k-1}) \cap Z \longrightarrow [0,1]$. Thus $B_0 \cap Z$ is an arc. Similarly we see that $B_n \cap Z$ is an arc. Since $\text{Ord}(\pi(p_1), Z) = \text{Ord}(\pi(p_n), Z) = 2$, we have that Z is an arc, i.e. Z is homeomorphic to the Type 1 (see Figure 4.1.8).

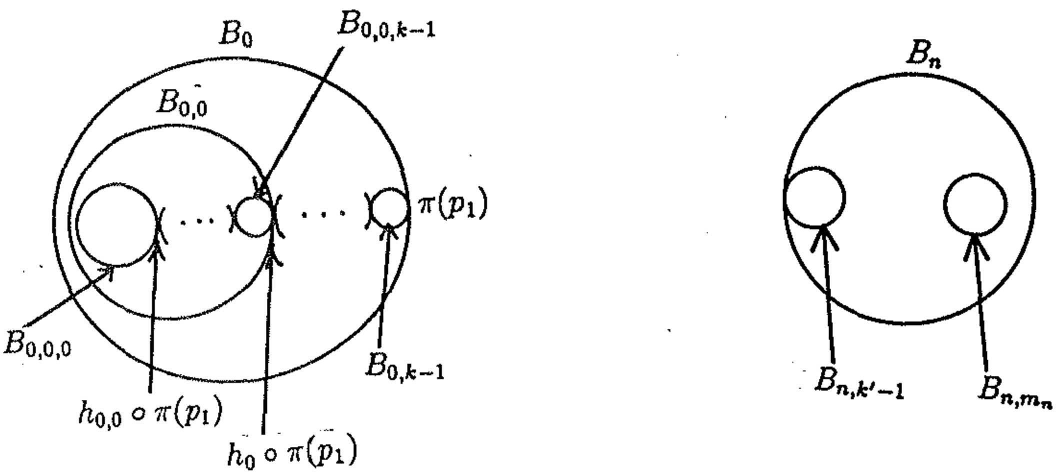


Figure 4.1.8

(Case 2.1.2) $B_{0,k-1}=\emptyset$, $B_{0,k}\neq\emptyset$, $B_{n,k'-1}\neq\emptyset$ and $B_{n,k'}=\emptyset$.

Assume that $m_0'<n$ or $m_n>0$. Then we see that Z is an arc in a similar way to Case 2.1.1.

Assume that $m_0'=n$ and $m_n=0$. Then we can construct a homeomorphism $\varphi: B_0\cap Z\longrightarrow[0,1]$ as follows :

$$\varphi((B_{0,n,0,n,\dots,0,k}\cup B_{0,n,0,n,\dots,0,k+1}\cup\cdots\cup B_{0,n,0,n,\dots,0,n-1})\cap Z)=[\tfrac{1}{2^t},\tfrac{1}{2^{t-1}}],$$

if t is odd and $B_{0,n,0,n,\dots,0,\ell}\in\mathbf{A}_t(\ell=k,k+1,\dots,n-1)$,

$$\varphi((B_{0,n,0,n,\dots,n,1}\cup B_{0,n,0,n,\dots,n,2}\cup\cdots\cup B_{0,n,0,n,\dots,0,k'-1})\cap Z)=[\tfrac{1}{2^t},\tfrac{1}{2^{t-1}}],$$

if t is even and $B_{0,n,0,n,\dots,n,\ell}\in\mathbf{A}_t(\ell=1,2,\dots,k'-1)$ and

$$\varphi(\textstyle\bigcap_{t\geq1}\{B_{0,n,0,n,\dots}\mid B_{0,n,0,n,\dots}\in\mathbf{A}_t\})=\{0\}.$$

Hence we see that $B_0\cap Z$ is an arc. Similarly we have that $B_n\cap Z$ is an arc. Since $\mathrm{Ord}(\pi(p_1),Z)=\mathrm{Ord}(\pi(p_n),Z)=2$, we see that Z is an arc, i.e. Z is homeomorphic to the Type 1 (see Figure 4.1.9).

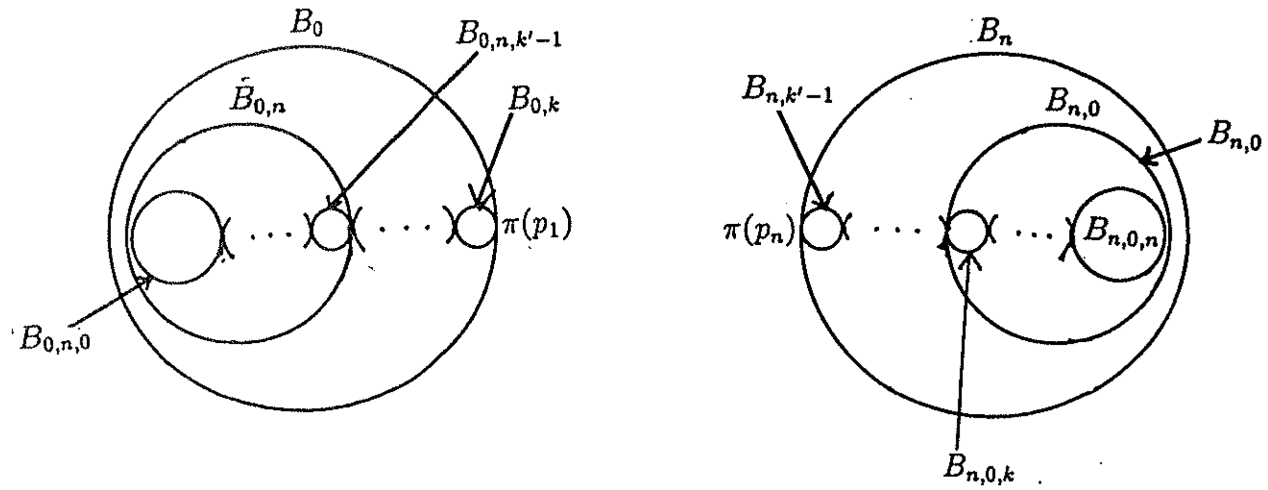


Figure 4.1.9

(Case 2.1.3) $B_{0,k-1} \neq \emptyset, B_{0,k} = \emptyset, B_{n,k'-1} = \emptyset$ and $B_{n,k'} \neq \emptyset$.

(Case 2.1.4) $B_{0,k-1} = \emptyset, B_{0,k} \neq \emptyset, B_{n,k'-1} = \emptyset$ and $B_{n,k'} \neq \emptyset$.

In the same way to Case 2.1.1, we can obtain that Z is an arc, i.e. Z is homeomorphic to the Type 1.

(Case 2.2) $C_0 \longrightarrow C_{k-1}$ and $C_0 \longrightarrow C_k$. And $C_n \not\rightharpoonup C_{k'-1}$ or $C_n \not\rightharpoonup C_{k'}$.

Then there exist elements $B_{0,k-1}, B_{0,k}$ of $\mathbf{A}_1$ in B_0 such that $\pi(p_1) \in B_{0,k-1} \cap B_{0,k}$. By Lemma 4.1.5, we have that $C_0 \not\rightharpoonup C_0$. And we have that $B_{n,k'-1} = \emptyset$ or $B_{n,k'} = \emptyset$.

Assume that $B_{n,k'-1} = \emptyset$ and $B_{n,k'} \neq \emptyset$. Then we have that $B_n \cap Z = (B_{n,k'} \cup B_{n,k'+1} \cup \cdots \cup B_{n,m'_n}) \cap Z$. Since $k' \neq 0$, we can construct a homeomorphism $\varphi : B_n \cap Z \longrightarrow [0,1]$ and it holds that $\mathrm{Ord}(\pi(p_n), Z) = \mathrm{Ord}(\pi(p_n), B_{n-1} \cap Z) + \mathrm{Ord}(\pi(p_n), B_n \cap Z) = 1 + \mathrm{Ord}(\pi(p_n), B_{n,k'} \cap Z) = 1 + \mathrm{Ord}(\pi(p_{k'}), B_{k'} \cap Z) = 2$. Thus we see that $B_n \cap Z$ is an arc.

Assume that $B_{n,k'-1} \neq \emptyset$ and $B_{n,k'} = \emptyset$.

(Case 2.2.1) $C_n \not\rightharpoonup C_0$.

Note that $0 < m_n \leq k' - 1 < n$. Since $B_n \cap Z = (B_{0,m_n} \cup B_{0,m_{n+1}} \cup \cdots \cup B_{0,k'-1}) \cap Z$ is homeomorphic to $(B_{m_n} \cup B_{m_n+1} \cup \cdots \cup B_{k'}) \cap Z$ which is a connected subspace of $\pi([P])$, we see that $B_n \cap Z$ is an arc. And it is clear that $\mathrm{Ord}(\pi(p_n), Z) = 2$. By Lemma 4.1.5, we have that $C_0 \not\rightharpoonup C_0$. Thus we see that $m_0 > 0$. Since $B_0 \cap Z = (B_{0,m_0} \cup B_{0,m_0+1} \cup \cdots \cup B_{0,m'_0}) \cap Z$ is homeomorphic to $(B_{m_0} \cup B_{m_0+1} \cup \cdots \cup B_{m'_0}) \cap Z$ which is a connected subspace of $\pi([P])$, we have that $B_0 \cap Z$ is an arc. And it holds that $\mathrm{Ord}((\pi(p_1), B_0 \cap Z) = \mathrm{Ord}(\pi(p_1), B_{0,k-1} \cap Z) + \mathrm{Ord}(\pi(p_1), B_{0,k} \cap Z) = \mathrm{Ord}(\pi(p_k), B_{k-1} \cap Z) + \mathrm{Ord}(\pi(p_k), B_k \cap Z) = 2$. Thus we obtain that $\mathrm{Ord}(\pi(p_1), Z) = \mathrm{Ord}(\pi(p_1), B_0 \cap Z) + \mathrm{Ord}(\pi(p_1), B_1 \cap Z) = 3$. Hence we see that Z is homeomorphic to the Type 2 (see Figure 4.1.10).

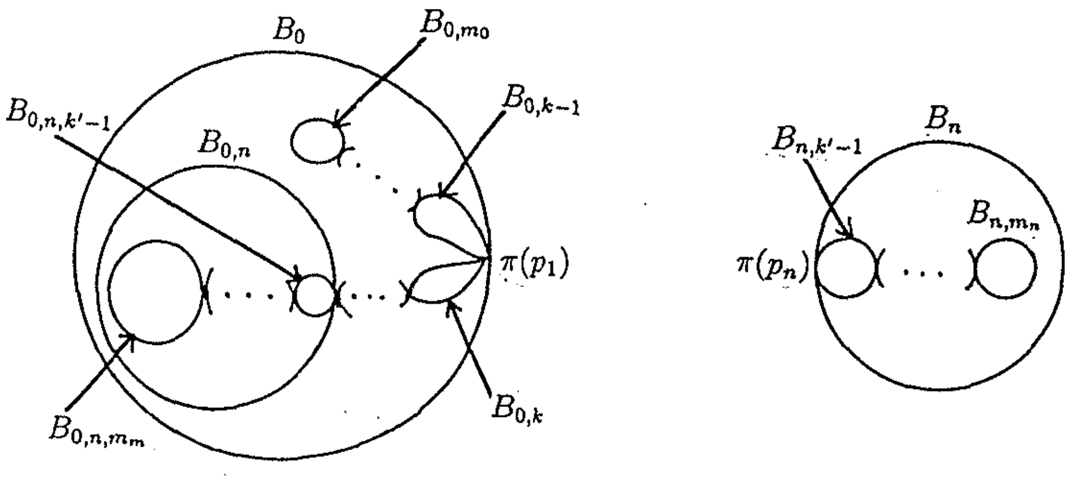


Figure 4.1.10

(Case 2.2.2) $C_n \longrightarrow C_0$.

By Lemma 4.1.5, we see that $C_0 \not\rightharpoonup C_n$ and $C_n \not\rightharpoonup C_n$. Thus we see that $0 < m_0 < m'_0 < n$ and $0 = m_n \leq m'_n = k'-1 < n$. Since $B_0 \cap Z = (B_{0,m_0} \cup B_{0,m_0+1} \cup \cdots \cup B_{0,m'_0}) \cap Z$ is homeomorphic to $(B_{m_0} \cup B_{m_0+1} \cup \cdots \cup B_{m'_0}) \cap Z$ which is a connected subspace of $\pi([P])$, we see that $B_0 \cap Z$ is an arc. And we have that

$\mathrm{Ord}(\pi(p_1), Z) = \mathrm{Ord}(\pi(p_1), B_0 \cap Z) + \mathrm{Ord}(\pi(p_1), B_1 \cap Z) = \mathrm{Ord}(\pi(p_1), B_{0,k-1} \cap Z) + \mathrm{Ord}(\pi(p_1), B_{0,k} \cap Z) + 1 = \mathrm{Ord}(\pi(p_k), B_{k-1} \cap Z) + \mathrm{Ord}(\pi(p_k), B_k \cap Z) + 1 = 3$. Since $C_n \longrightarrow C_0$, there exists an element $B_{n,0}$ of $\mathbf{A}_1$ in B_n such that $h_n \circ \pi(p_1) \in B_{n,0} \subset B_n$. Since $B_n \cap Z = (B_{n,0} \cup B_{n,1} \cup \cdots \cup B_{n,k'-1}) \cap Z$ is homeomorphic to $(B_0 \cup B_1 \cup \cdots \cup B_{k'-1}) \cap Z$, we have that $\mathrm{Ord}(h_n \circ \pi(p_1), Z) = 3$. Hence Z is homeomorphic to the Type 3 (see Figure 4.1.11).

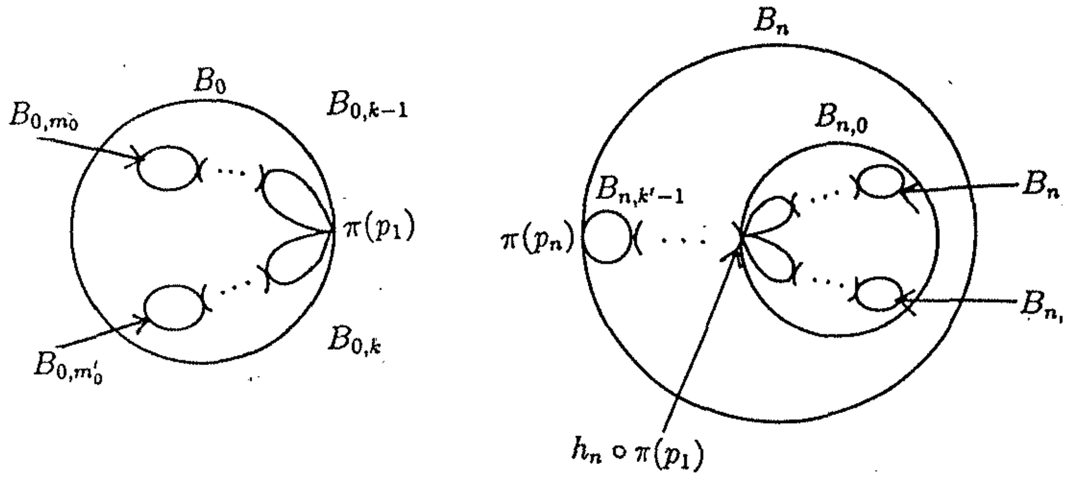


Figure 4.1.11

In reverse case $C_0 \not\longrightarrow C_{k-1}$ or $C_0 \not\longrightarrow C_k$, and $C_n \longrightarrow C_{k'-1}$ and $C_n \longrightarrow C_{k'}$, we can think in the same way to Case 2.2.

(Case 2.3) $C_0 \longrightarrow C_{k-1}, C_0 \longrightarrow C_k, C_n \longrightarrow C_{k'-1}$ and $C_n \longrightarrow C_{k'}$.

There exist elements $B_{0,k-1}, B_{0,k}, B_{n,k'-1}, B_{n,k'}$ of $\mathbf{A}_1$ such that $\pi(p_1) \in B_{0,k-1} \cap B_{0,k}$ and $\pi(p_n) \in B_{n,k'-1} \cap B_{n,k'}$. By Lemma 4.1.5, we see that $C_0 \not\longrightarrow C_0$ and $C_n \not\longrightarrow C_n$. And we also see that $C_0 \not\longrightarrow C_n$ or $C_n \not\longrightarrow C_0$.

(Case 2.3.1) $C_0 \not\longrightarrow C_n$ and $C_n \not\longrightarrow C_0$.

Then we have that $0 < m_0 < m_0' < n$ and $0 < m_n < m_n' < n$. Since $B_0 \cap Z = (B_{0,m_0} \cup B_{0,m_0+1} \cdots \cup B_{0,m_0'}) \cap Z$ is homeomorphic to $(B_{m_0} \cup B_{m_1} \cup \cdots \cup B_{m_0'}) \cap Z$ which is a connected subspace of $\pi([P])$, we have that $B_0 \cap Z$ is an arc. Similarly we see that $B_n \cap Z$ is an arc. And we have that $\mathrm{Ord}(\pi(p_1), Z) = \mathrm{Ord}(\pi(p_1), B_0 \cap Z) + \mathrm{Ord}(\pi(p_1), B_1 \cap Z) = \mathrm{Ord}(\pi(p_1), B_{0,k-1} \cap Z) + \mathrm{Ord}(\pi(p_1), B_{0,k} \cap Z) + 1 = \mathrm{Ord}(\pi(p_k), B_{k-1} \cap Z) + \mathrm{Ord}(\pi(p_k), B_k \cap Z) + 1 = 3$. Similarly we have that $\mathrm{Ord}(\pi(p_n), Z) = 3$. Hence Z is homeomorphic to the Type 3 (see Figure 4.1.12).

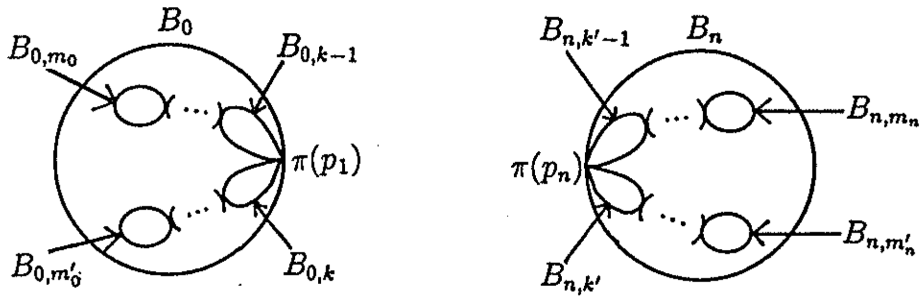


Figure 4.1.12

(Case 2.3.2) $C_0 \not\to C_n$ and $C_n \longrightarrow C_0$.

Since $B_0 \cap Z = (B_{0,m_0} \cup B_{0,m_0+1} \cup \cdots \cup B_{0,m_0'}) \cap Z$ is homeomorphic to $(B_{m_0} \cup B_{m_0+1} \cup \cdots \cup B_{m_0'}) \cap Z$ which is a connected subspace of $\pi([P])$ and it holds that $0 < m_0 < m_0' < n$, we see that $B_0 \cap Z$ is an arc. And we have that $\mathrm{Ord}(\pi(p_1), Z) = \mathrm{Ord}(\pi(p_1), B_0 \cap Z) + \mathrm{Ord}(\pi(p_1), B_1 \cap Z) = \mathrm{Ord}(\pi(p_1), B_{0,k-1} \cap Z) + \mathrm{Ord}(\pi(p_1), B_{0,k} \cap Z) + 1 = \mathrm{Ord}(\pi(p_k), B_{k-1} \cap Z) + \mathrm{Ord}(\pi(p_k), B_k \cap Z) + 1 = 3$. Since $C_n \longrightarrow C_0$ and $C_n \not\to C_n$, we see that $m_n = 0$ and $m_n' < n$.

Assume that $k' - 1 = 0$, i.e. $p_n \longrightarrow p_1$. Since $B_n \cap Z = (B_{n,0} \cup B_{n,1} \cup \cdots \cup B_{n,m_n'}) \cap Z$ is homeomorphic to $(B_0 \cup B_1 \cup \cdots \cup B_{m_n'}) \cap Z$ and $\pi(p_n) = h_n \circ \pi(p_1) \in B_{n,0} \cap B_{n,1}$, it holds that $\mathrm{Ord}(\pi(p_n), Z) = \mathrm{Ord}(\pi(p_n), B_{n-1} \cap$

$Z) + \mathrm{Ord}(\pi(p_n), B_n \cap Z) = 1 + \mathrm{Ord}(\pi(p_n), B_{n,0} \cap Z) + \mathrm{Ord}(\pi(p_n), B_{n,1} \cap Z) = 1 + \mathrm{Ord}(\pi(p_1), B_0 \cap Z) + \mathrm{Ord}(\pi(p_1), B_1 \cap Z) = 1 + \mathrm{Ord}(\pi(p_1), Z) = 4$. Hence we see that Z is homeomorphic to the Type 4 (see Figure 4.1.13).

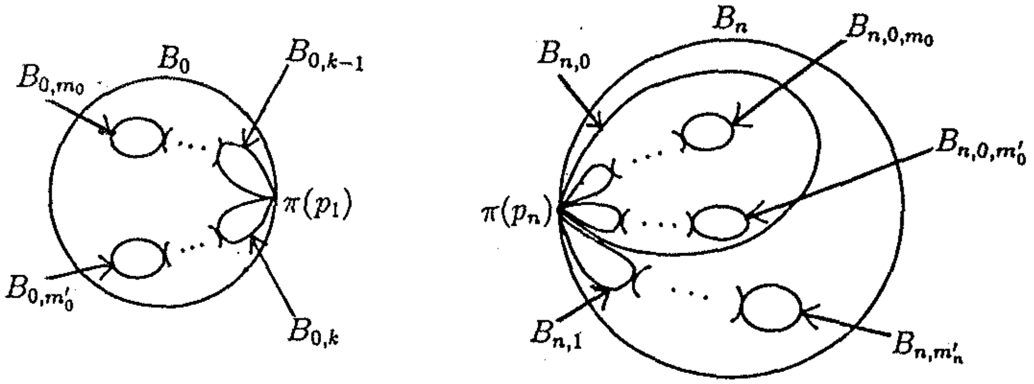


Figure 4.1.13

Assume that $k' - 1 \neq 0$. Then we have that $\pi(p_n) \neq h_n \circ \pi(p_1) \in \mathrm{Int}(B_n)$. We see that $\mathrm{Ord}(\pi(p_n), Z) = \mathrm{Ord}(\pi(p_n), B_{n-1} \cap Z) + \mathrm{Ord}(\pi(p_n), B_n \cap Z) = 1 + \mathrm{Ord}(\pi(p_n), B_{n,k'-1} \cap Z) + \mathrm{Ord}(\pi(p_n), B_{n,k'} \cap Z) = 3$. Since $B_n \cap Z = (B_{n,0} \cup B_{n,1} \cup \cdots \cup B_{n,m'_n}) \cap Z$ is homeomorphic to $(B_0 \cup B_1 \cup \cdots \cup B_{m'_n}) \cap Z$, it holds that $\mathrm{Ord}(h_n \circ \pi(p_1), Z) = \mathrm{Ord}(h_n \circ \pi(p_1), B_{n,0} \cap Z) + \mathrm{Ord}(h_n \circ \pi(p_1), B_{n,1} \cap Z) = \mathrm{Ord}(\pi(p_1), B_0 \cap Z) + \mathrm{Ord}(\pi(p_1), B_1 \cap Z) = \mathrm{Ord}(\pi(p_1), Z) = 3$. Thus we see that Z is homeomorphic to the Type 5 (see Figure 4.1.14).

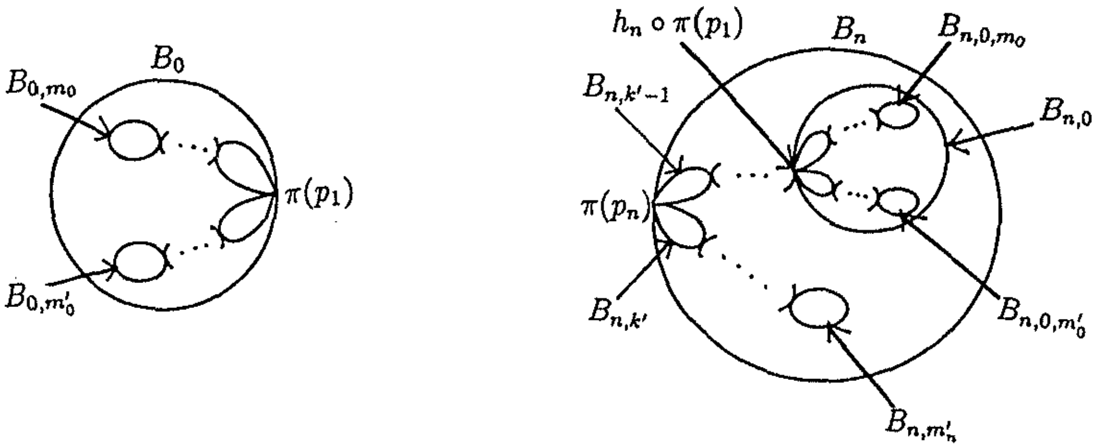


Figure 4.1.14

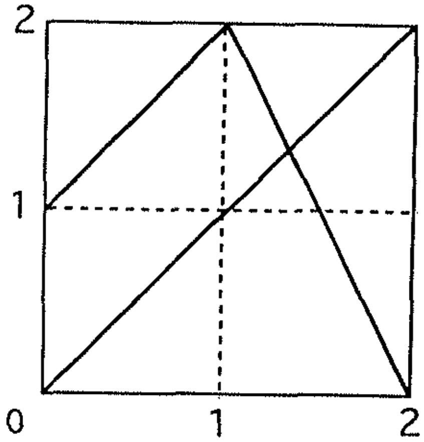
(Case 2.3.3) $C_0 \longrightarrow C_n$ and $C_n \not\to C_0$.

Then we can obtain the same result to Case 2.3.2. □

4.2 Examples.

In fact, we can obtain a continuous map f and a periodic orbit P of f such that Z is homeomorphic to each type in Theorem 4.1.6.

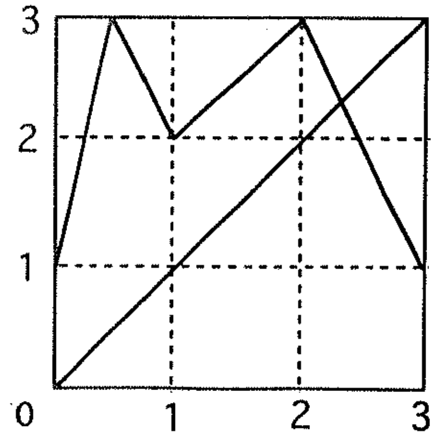
Example 4.2.1 Denote a continious map $f : [0,2] \longrightarrow [0,2]$ as follows : $f(x) = x + 1 (0 \leq x \leq 1)$ and $f(x) = -2x + 4 (1 \leq x \leq 2)$. And let P be a periodic orbit $\{0, 1, 2\}$ (see Graph 4.2.1).



Graph 4.2.1

Then we have that Z is homeomorphic to the type 1 in Theorem 4.1.6.

Example 4.2.2 Denote a continious map $f : [0,3] \longrightarrow [0,3]$ as follows : $f(x) = 4x+1 (0 \leq x \leq \frac{1}{2})$, $f(x) = -2x+4 (\frac{1}{2} \leq x \leq 1)$, $f(x) = x+1 (1 \leq x \leq 2)$ and $f(x) = -2x+7 (2 \leq x \leq 3)$. And let P be a periodic orbit $\{1, 2, 3\}$ Denote $C_0 = [0, 1), C_1 = (1, 2)$ and $C_2 = (2, 3]$ (see Graph 4.2.2).



Graph 4.2.2

The process of the construction of Z is as in Figure 4.2.3.

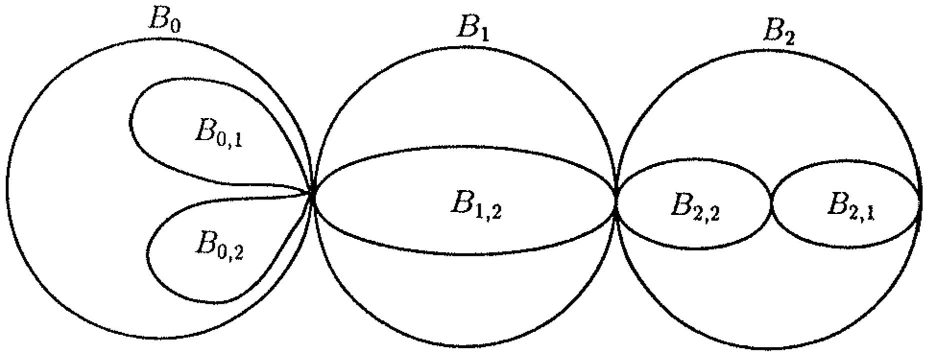
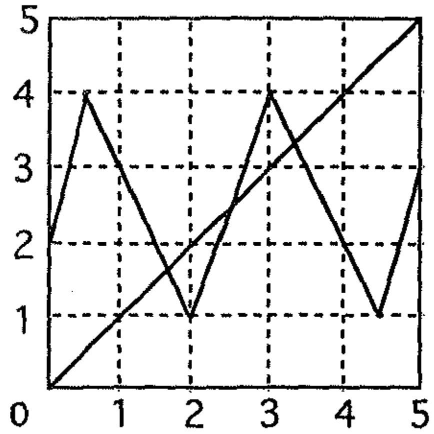


Figure 4.2.3

Then we have that Z is homeomorphic to the type 2 in Theorem 4.1.6.

Example 4.2.3 Denote a continious map $f : [0,5] \longrightarrow [0,5]$ as follows : $f(x) = 4x + 2 (0 \leq x \leq \frac{1}{2})$, $f(x) = -2x + 5 (\frac{1}{2} \leq x \leq 2)$, $f(x) = 3x - 5 (2 \leq x \leq 3)$, $f(x) = -2x + 10 (3 \leq x \leq \frac{9}{2})$ and $f(x) = 4x - 17 (\frac{9}{2} \leq x \leq 5)$. And let P be a periodic orbit $\{1, 2, 3, 4\}$. Denote $C_0 = [0,1), C_1 = (1,2), C_2 = (2,3), C_3 = (3,4)$ and $C_4 = (4,5]$ (see Graph 4.2.4).



Graph 4.2.4

The process of the construction of Z is as Figure 4.2.5.

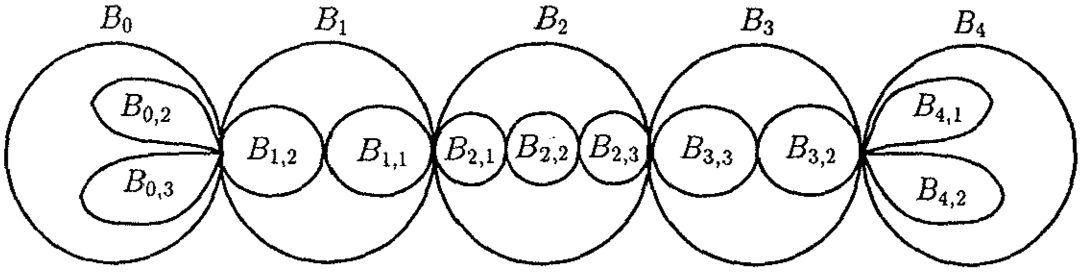
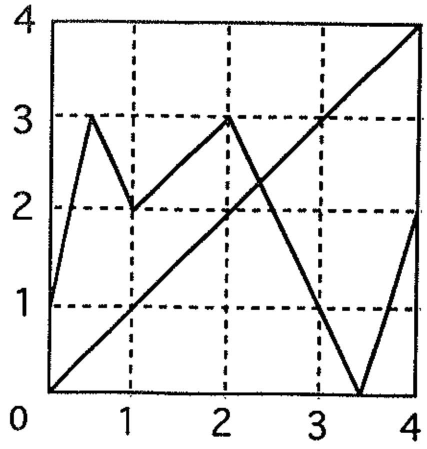


Figure 4.2.5

Then we have that Z is homeomorphic to the type 3 in Theorem 4.1.6.

Example 4.2.4 Denote a continious map $f : [0,4] \longrightarrow [0,4]$ as follows : $f(x) = 4x + 1(0 \leq x \leq \frac{1}{2})$, $f(x) = -2x + 4(\frac{1}{2} \leq x \leq 1)$, $f(x) = x + 1(1 \leq x \leq 2)$, $f(x) = -2x + 7(2 \leq x \leq 3)$, $f(x) = -2x + 7(3 \leq x \leq \frac{7}{2})$ and $f(x) = 4x - 14(\frac{7}{2} \leq x \leq 4)$. And let P be a periodic orbit $\{1,2,3\}$. Denote $C_0 = [0,1), C_1 = (1,2), C_2 = (2,3)$ and $C_3 = (3,4]$ (see Graph 4.2.6).



Graph 4.2.6

The process of the construction of Z is as Figure 4.2.7.

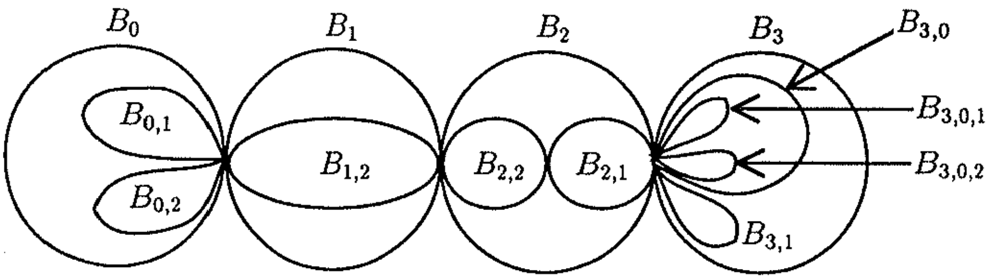
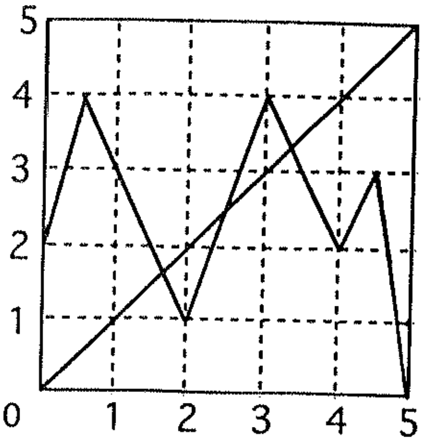


Figure 4.2.7

Then we have that Z is homeomorphic to the type 4 in Theorem 4.1.6.

Example 4.2.5 Denote a continious map $f : [0,5] \longrightarrow [0,5]$ as follows : $f(x) = 4x + 2(0 \leq x \leq \frac{1}{2})$, $f(x) = -2x + 5(\frac{1}{2} \leq x \leq 2)$, $f(x) = 3x - 5(2 \leq x \leq 3)$, $f(x) = -2x + 10(3 \leq x \leq 4)$, $f(x) = 2x - 6(4 \leq x \leq \frac{9}{2})$ and $f(x) = -6x + 30(\frac{9}{2} \leq x \leq 5)$. And let P be a periodic orbit $\{1,2,3,4\}$. Denote $C_0 = [0,1), C_1 = (1,2), C_2 = (2,3), C_3 = (3,4), C_4 = (4,5]$ (see Graph 4.2.8).



Graph 4.2.8

The process of the construction of Z is as Figure 4.2.9.

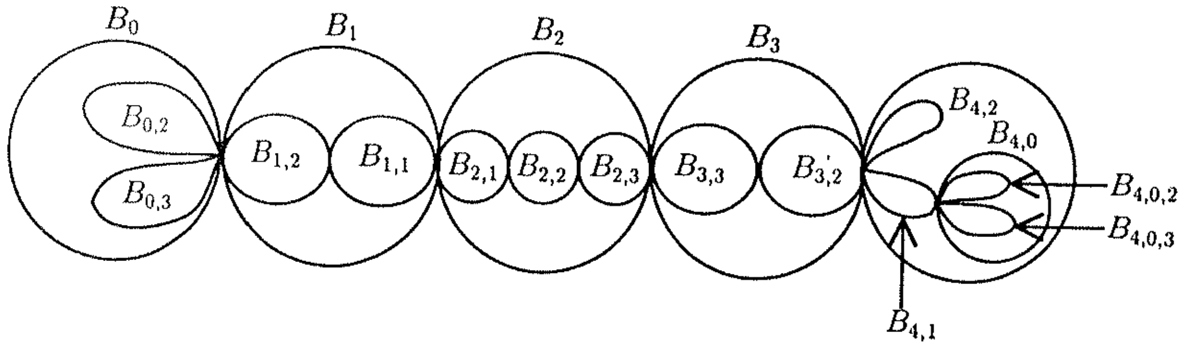


Figure 4.2.9

Then we have that Z is homeomorphic to the type 5 in Theorem 4.1.6.