Chapter 3

Dendrites constructed by P-expansive maps of the interval into itself

In this chapter, we investigate the structure of Z under the restriction that f is a continuous map of the interval into itself, where Z is a regular continuum constructed in Chapter 2.

Denote $I = [0, 1]$. And let $f : I \longrightarrow I$ be a continuous map and P a periodic orbit of f. Let A be a subset of I containing more than one point, then we will use $[A]$ to denote the smallest subinterval containing A. If $A = \{a, b\}$, then denote $[A]$ by $[a, b]$. And we define $(a, b) = [a, b] \setminus \{a, b\}, (a, b] = [a, b] \setminus \{a\}$ and $[a, b) = [a, b] \setminus \{b\}$. Put $S(I, P) = P \cup \{C \mid C \text{ is a component of } I \setminus P\}$. Given $x \in I$, the *itinerary* of x with respect to P and f is defined to be the unique infinite sequence $(C_n)_{n \geq 0}$ from $S(I, P)$ given by the rule $f^n(x) \in C_n$ for all $n \geq 0$. If no two points of I have the same itinerary, then f will be called *P-expansive*. And we say that f is *pointwise P-expansive* if for each $p, q \in P$, there exists some non-negative integer m such that $(f^m(p), f^m(q)) \cap P \neq \emptyset$ (see Chapter 2).

Notices. In this chapter, we assume the followings :

1. $f : I \longrightarrow I$ is pointwise P-expansive.
2. $f(C)$ is the union of some elements of $S(I,P)$ for any $C \in S(I,P)$.
3. $f(C)$ is not one point for each $C \in S(I,P) \setminus P$ with $C \neq \emptyset$.

Recall the following facts shown in Chapter 2.

Fact 1. Z is a dendrite.

Fact 2. $g|_{B_{s_0,s_1,\dots,s_\ell}\cap Z} : B_{s_0,s_1,\dots,s_\ell} \cap Z \longrightarrow B_{s_1,s_2,\dots,s_\ell} \cap Z$ is a homeomorphism $(0 \leq s_0, s_1, \dots, s_\ell \leq n)$.

Fact 3. Denote $P_\infty = \{h_{s_0,s_1,\dots,s_\ell}(P) \mid \ell \geq 0 \text{ and } 0 \leq s_0, s_1, \dots, s_\ell \leq n\}$. Then P_∞ is dense in Z and $\mathrm{Br}(Z) \subset P_\infty$.

The first part can be proved by the last part of section 2 in [2]. We show that $\mathrm{Br}(Z) \subset P_\infty$. Let x be an element of $Z \setminus P_\infty$. Then it suffices to show that $\mathrm{Ord}(x,Z) \leq 2$. By the way of the construction of Z, there exists an infinite sequence $s_0, s_1, \dots$ such that $x \in \mathrm{Int}(B_{s_0,s_1,\dots,s_k} \cap Z)$ for each $k \geq 0$, where $\mathrm{Int}(B_{s_0,s_1,\dots,s_k} \cap Z)$ means the interior of the set $B_{s_0,s_1,\dots,s_k} \cap Z$. For any neighborhood U of x, there exists $\ell \geq 0$ such that $x \in B_{s_0,s_1,\dots,s_\ell} \cap Z \subset U$. We see that $\mathrm{Bd}(B_{s_0,s_1,\dots,s_\ell} \cap Z) \leq 2$. Thus we have $\mathrm{Ord}(x,Z) \leq 2$.

Fact 4. $\mathrm{Ord}(p_{s_0,s_1,\dots,s_\ell}, B_{s_0,s_1,\dots,s_{\ell-1},t_\ell} \cap Z) = \mathrm{Ord}(p_{s_1,s_2,\dots,s_\ell}, B_{s_1,s_2,\dots,s_{\ell-1},t_\ell} \cap Z)$ for each element $p_{s_0,s_1,\dots,s_\ell}$ of P_∞ with $p_{s_0,s_1,\dots,s_\ell} \in \mathrm{Bd}(B_{s_0,s_1,\dots,s_{\ell-1},t_\ell})$. Hence we have $\mathrm{Ord}(p_{s_0,s_1,\dots,s_\ell}, B_{s_0,s_1,\dots,s_{\ell-1},t_\ell} \cap Z) = \mathrm{Ord}(p_{s_\ell}, B_{t_\ell} \cap Z)$ inductively.

In fact since $g|_{B_{s_0,s_1,\dots,s_{\ell-1},t_\ell}\cap Z} : B_{s_0,s_1,\dots,s_{\ell-1},t_\ell} \cap Z \longrightarrow B_{s_1,s_2,\dots,s_{\ell-1},t_\ell} \cap Z$ is a homeomorphism by Fact 2, we see that $\mathrm{Ord}(p_{s_0,s_1,\dots,s_\ell}, B_{s_0,s_1,\dots,s_{\ell-1},t_\ell} \cap Z)$ $= \mathrm{Ord}(p_{s_1,s_2,\dots,s_\ell}, B_{s_1,s_2,\dots,s_{\ell-1},t_\ell} \cap Z)$ for each element $p_{s_0,s_1,\dots,s_\ell}$ of P_∞ with $p_{s_0,s_1,\dots,s_\ell} \in \mathrm{Bd}(B_{s_0,s_1,\dots,s_{\ell-1},t_\ell})$.

3.1 A necessary and sufficient condition that Z is the universal dendrite

Let P be a periodic orbit of f with period n. We say that $C_{i_0} \longrightarrow C_{i_1} \longrightarrow \cdots \longrightarrow C_{i_{n-1}} \longrightarrow C_{i_0}$ is a *cycle* of length n for the orbit $O(p, f)$ of an element p of P in $S(I,P)\backslash P$, written an n-cycle for $O(p,f)$, provided that $f^k(p) \in cl(C_{i_k})$ for each $k = 0, 1, \ldots, n-1$. In this chapter, C_{i_k} means $C_{i_{k(mod\ n)}}$ for each natural number k.

Let p, q be elements of P and C_{α_0} the element of $S(I,P) \setminus P$ such that p is an endpoint of $[P]$ and $C_{\alpha_0} = (p, q)$. Let C_{α_1} be the element of $S(I,P) \setminus P$ such that $f(p) \in cl(C_{\alpha_1})$ and $C_{\alpha_1} \subset (f(p), f(q))$. Then we have the element q_1 of P such that $C_{\alpha_1} = (f(p), q_1)$. Let C_{α_2} be the element of $S(I,P) \setminus P$ such that $f^2(p) \in cl(C_{\alpha_2})$ and $C_{\alpha_2} \subset (f^2(p), f(q_1))$. Then there exists the element q_2 of P such that $C_{\alpha_2} = (f^2(p), q_2)$. Similarly let C_{α_3} be the element of $S(I,P) \setminus P$ such that $f^3(p) \in cl(C_{\alpha_3})$ and $C_{\alpha_3} \subset (f^3(p), f(q_2))$. And let q_3 be the element of P with $C_{\alpha_3} = (f^2(p), q_3)$. Repeating this operation, we can have a cycle $C_{\alpha_0} \longrightarrow C_{\alpha_1} \longrightarrow \cdots \longrightarrow C_{\alpha_{n_1}} \longrightarrow C_{\alpha_0}$ of length n such that $f^k(p) \in cl(C_{\alpha_k})$ for each $k = 0, 1, \ldots, n-1$. We say that this cycle $C_{\alpha_0} \longrightarrow C_{\alpha_1} \longrightarrow \cdots \longrightarrow C_{\alpha_{n_1}} \longrightarrow C_{\alpha_0}$ is a *fundamental cycle* for $O(p, f)$. A fundamental cycle always exists and is unique (see [4, p.8]). Note that $C_{\alpha_k} \subset [P]$ for any $k = 0, 1, \ldots, n-1$. Let C_{β_k} be the element of $S(I,P)\backslash P$ with $\{f^k(p)\} = cl(C_{\alpha_k}) \cap cl(C_{\beta_k})$ for each $k = 0, 1, \ldots, n-1$, but $C_{\beta_k} = \emptyset$ if $f^k(p)$ is an endpoint of I.

Lemma 3.1.1 *If there exists an element p of P such that* $\mathrm{Ord}(\pi(p), Z) < \infty$, *then we have that* $\mathrm{Ord}(\pi \circ f^k(p), Z) < \infty$ *for each* $k = 0, 1, \ldots, n-1$.

Proof. Let C_{i_k}, C_{j_k} be elements of $S(I, P) \setminus P$ such that $\{f^k(p)\} = cl(C_{i_k}) \cap cl(C_{j_k})$, but $C_{j_k} = \emptyset$ if $f^k(p)$ is an endpoint of I. Assume that $\mathrm{Ord}(\pi(p), Z) < \infty$. Then it is clear that $\mathrm{Ord}(\pi(p), B_{i_0} \cap Z) < \infty$ and $\mathrm{Ord}(\pi(p), B_{j_0} \cap Z) < \infty$.

Since $C_{i_0} \longrightarrow C_{i_1}$ or $C_{i_0} \longrightarrow C_{j_1}$, we see that $\mathrm{Ord}(\pi(p), B_{i_0} \cap Z) = \mathrm{Ord}(\pi(p), B_{i_0,i_1} \cap Z) + \mathrm{Ord}(\pi(p), B_{i_0,j_1} \cap Z)$ by the way of the construction of Z, where $\mathrm{Ord}(\pi(p), B_{i_0,i_1} \cap Z) = 0$ if $B_{i_0,i_1} = \emptyset$, and $\mathrm{Ord}(\pi(p), B_{i_0,j_1} \cap Z) = 0$ if $B_{i_0,j_1} = \emptyset$ (see Figure 3.1.1). Thus we have that $\mathrm{Ord}(\pi(p), B_{i_0,i_1} \cap Z) < \infty$ and $\mathrm{Ord}(\pi(p), B_{i_0,j_1} \cap Z) < \infty$.

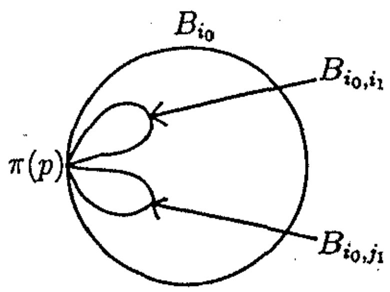


Figure 3.1.1

By Fact 4, it holds that $\mathrm{Ord}(\pi(p), B_{i_0,i_1} \cap Z) = \mathrm{Ord}(\pi \circ f(p), B_{i_1} \cap Z) < \infty$ and $\mathrm{Ord}(\pi(p), B_{i_0,j_1} \cap Z) = \mathrm{Ord}(\pi \circ f(p), B_{j_1} \cap Z) < \infty$. Hence we have that $\mathrm{Ord}(\pi \circ f(p), Z) = \mathrm{Ord}(\pi \circ f(p), B_{i_1} \cap Z) + \mathrm{Ord}(\pi \circ f(p), B_{j_1} \cap Z) < \infty$. Similarly since $\mathrm{Ord}(\pi \circ f(p), B_{i_1} \cap Z) = \mathrm{Ord}(\pi \circ f(p), B_{i_1,i_2} \cap Z) + \mathrm{Ord}(\pi \circ f(p), B_{i_1,j_2} \cap Z)$, we have that $\mathrm{Ord}(\pi \circ f(p), B_{i_1,i_2} \cap Z) < \infty$ and $\mathrm{Ord}(\pi \circ f(p), B_{i_1,j_2} \cap Z) < \infty$. By Fact 4, it holds that $\mathrm{Ord}(\pi \circ f(p), B_{i_1,i_2} \cap Z) = \mathrm{Ord}(\pi \circ f^2(p), B_{i_2} \cap Z) < \infty$ and $\mathrm{Ord}(\pi \circ f(p), B_{i_1,j_2} \cap Z) = \mathrm{Ord}(\pi \circ f^2(p), B_{j_2} \cap Z) < \infty$. Thus we have that $\mathrm{Ord}(\pi \circ f^2(p), Z) = \mathrm{Ord}(\pi \circ f^2(p), B_{i_2} \cap Z) + \mathrm{Ord}(\pi \circ f^2(p), B_{j_2} \cap Z) < \infty$. Repeating this operation, we obtain that $\mathrm{Ord}(\pi \circ f^k(p), Z) < \infty$ for each $k = 0, 1, \ldots, n-1$. □

Lemma 3.1.2 *If we have that* $\mathrm{Ord}(\pi\circ f^k(p), Z) < \infty$ *for each* $k = 0, 1, \ldots, n-1$, *then it holds that* $\mathrm{Ord}(x, Z) < \infty$ *for each element* x *of* P_∞.

Proof. Let $p_{s_0,s_1,\ldots,s_\ell}$ be an element of P_∞ and $B_{s_0,,s_1,\ldots,s_{\ell-1},t_\ell}$ an element of $\mathbf{A}_\ell$ such that $p_{s_0,s_1,\ldots,s_\ell} \in \mathrm{Bd}(B_{s_0,s_1,\ldots,s_{\ell-1},t_\ell})$. By Fact 4, we have that $\mathrm{Ord}(p_{s_0,s_1,\ldots,s_\ell}, B_{s_0,s_1,\ldots,s_{\ell-1},t_\ell} \cap Z) = \mathrm{Ord}(p_{s_\ell}, B_{t_\ell} \cap Z) \le \mathrm{Ord}(p_{s_\ell}, Z) < \infty$. This means that $\mathrm{Ord}(p_{s_0,s_1,\ldots,s_\ell}, Z) < \infty$. □

Lemma 3.1.3 *If there exists an element* x *of* P_∞ *such that* $\mathrm{Ord}(x, Z) = \infty$, *then we have that* $\mathrm{Ord}(y, Z) = \infty$ *or* $\mathrm{Ord}(y, Z) \le 2$ *for each element* y *of* P_∞.

Proof. It is sufficient to show that if $\mathrm{Ord}(y, Z) \ge 3$ for some element y of P_∞, then $\mathrm{Ord}(y, Z) = \infty$. Let y be an element of P_∞ such that $\mathrm{Ord}(y, Z) \ge 3$. Then there exist a natural number ℓ and an element $B_{s_0,s_1,\ldots,s_\ell}$ of $\mathbf{A}_\ell$ such that $y \in \mathrm{Bd}(B_{s_0,s_1,\ldots,s_\ell})$ and $\mathrm{Ord}(y, B_{s_0,s_1,\ldots,s_\ell} \cap Z) \ge 2$. Denote $m = min\{\ell' \mid$ there exist two distinct elements B, B' of $\mathbf{A}_{\ell'}$ in $B_{s_0,s_1,\ldots,s_\ell}$ such that $y \in B \cap B'\} \ge \ell$. Let $B_{s_0,s_1,\ldots,s_{m-1},\gamma_m}$, $B_{s_0,s_1,\ldots,s_{m-1},\delta_m}$ be distinct elements of $\mathbf{A}_m$ such that $y \in B_{s_0,s_1,\ldots,s_{m-1},\gamma_m} \cap B_{s_0,s_1,\ldots,s_{m-1},\delta_m}$ (see Figure 3.1.2).

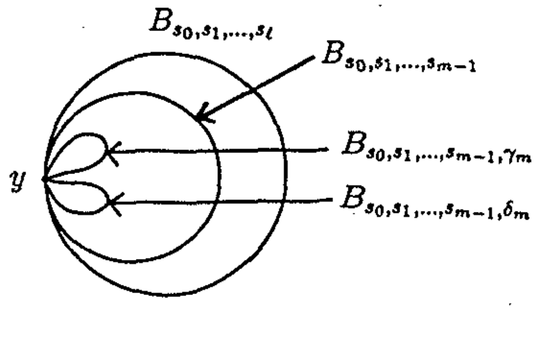


Figure 3.1.2

Then we have that $\mathrm{Ord}(y, B_{s_0,s_1,\ldots,s_\ell} \cap Z) = \mathrm{Ord}(y, B_{s_0,s_1,\ldots,s_{m-1},\gamma_m} \cap Z) + \mathrm{Ord}(y, B_{s_0,s_1,\ldots,s_{m-1},\delta_m} \cap Z) = \mathrm{Ord}(g^m(y), B_{\gamma_m} \cap Z) + \mathrm{Ord}(g^m(y), B_{\delta_m} \cap Z) = \mathrm{Ord}(g^m(y), Z) = \infty$ by Lemma 3.1.1 and 3.1.2, since $g^m(y)$ is an element of $\pi(P)$. Thus there exists no element y of P_∞ such that $3 \leq \mathrm{Ord}(y, Z) < \infty$. □

Lemma 3.1.4 *If we have that* $\mathrm{Ord}(\pi \circ f^k(p), Z) = \infty$ *for each* $k = 0, 1, \ldots, n-1$, *then* Z *is the universal dendrite.*

Proof. By Lemma 3.1.3, there exists no element y of Z such that $3 \leq \mathrm{Ord}(y, Z) < \infty$. Thus it is sufficient to show that the set $\{x \in Z \mid \mathrm{Ord}(x, Z) = \infty\}$ is dense in Z. By the way of the construction of Z, for any open subset U of Z, there exist an element $p_{s_0,s_1,\ldots,s_\ell}$ of P_∞ and elements $B_{s_0,s_1,\ldots,s_{\ell-1},\gamma_\ell}, B_{s_0,s_1,\ldots,s_{\ell-1},\delta_\ell}$ of $\mathbf{A}_\ell$ such that $p_{s_0,s_1,\ldots,s_\ell} \in U$ and $p_{s_0,s_1,\ldots,s_\ell} \in \mathrm{Bd}(B_{s_0,s_1,\ldots,s_{\ell-1},\gamma_\ell}) \cap \mathrm{Bd}(B_{s_0,s_1,\ldots,s_{\ell-1},\delta_\ell})$. Then we have that $\mathrm{Ord}(p_{s_0,s_1,\ldots,s_\ell}, Z) \geq \mathrm{Ord}(p_{s_0,s_1,\ldots,s_\ell}, B_{s_0,s_1,\ldots,s_{\ell-1},\gamma_\ell} \cap Z) + \mathrm{Ord}(p_{s_0,s_1,\ldots,s_\ell}, B_{s_0,s_1,\ldots,s_{\ell-1},\delta_\ell} \cap Z) = \mathrm{Ord}(p_{s_\ell}, B_{\gamma_\ell} \cap Z) + \mathrm{Ord}(p_{s_\ell}, B_{\delta_\ell} \cap Z) = \mathrm{Ord}(p_{s_\ell}, Z) = \infty$. □

Lemma 3.1.5 *Let* p *be an element of* P *and* $C_{i_0} \longrightarrow C_{i_1} \longrightarrow \cdots \longrightarrow C_{i_{n-1}} \longrightarrow C_{i_0}$ *an* n*-cycle for* $O(p, f)$. *And let* C_{j_k} *be the element of* $S(I, P) \setminus P$ *such that* $\{f^k(p)\} = cl(C_{i_k}) \cap cl(C_{j_k})$, *but* $C_{j_k} = \emptyset$ *if* $f^k(p)$ *is an endpoint of* I. *If* $C_{i_k} \not\rightarrow C_{j_{k+1}}$ *for each* $k = 0, 1, \ldots, n-1$, *then we have that* $\mathrm{Ord}(\pi \circ f^k(p), B_{i_k} \cap Z) = 1$ *for each* $k = 0, 1, \ldots, n-1$. *Moreover* $\mathrm{Ord}(\pi \circ f^k(p), Z) = \mathrm{Ord}(\pi \circ f^k(p), B_{j_k} \cap Z) + 1$.

Proof. Assume that $\mathrm{Ord}(\pi \circ f^m(p), B_{i_m} \cap Z) \geq 2$ for some $m = 0, 1, \ldots, n-1$. Then there exists a natural number ℓ such that $\mathrm{Card}\{B \in \mathbf{A}_\ell \mid \pi \circ$

$f^m(p) \in B \subset B_{i_m}\} = 2$ and $\mathrm{Card}\{B \in \mathbf{A}_{\ell-1} \mid \pi \circ f^m(p) \in B \subset B_{i_m}\} = 1$. Let $B_{s_0,s_1,\dots,s_{\ell-1},\gamma_\ell}, B_{s_0,s_1,\dots,s_{\ell-1},\delta_\ell}$ be distinct elements of $\mathbf{A}_\ell$ in B_{i_m} such that $\{\pi \circ f^m(p)\} = B_{s_0,s_1,\dots,s_{\ell-1},\gamma_\ell} \cap B_{s_0,s_1,\dots,s_{\ell-1},\delta_\ell}$. This implies that $C_{s_0} = C_{i_m}, C_{s_0} \longrightarrow C_{s_1} \longrightarrow \cdots \longrightarrow C_{s_{\ell-1}} \longrightarrow C_{\gamma_\ell}, C_{s_{\ell-1}} \longrightarrow C_{\delta_\ell}, f^{m+k}(p) \in C_{s_k}$ for $k = 0, 1, \dots, \ell-1$ and $\{f^{m+\ell}(p)\} = cl(C_{\gamma_\ell}) \cap cl(C_{\delta_\ell})$. Since $C_{i_k} \not\longrightarrow C_{j_{k+1}}$ for each $k = 0, 1, \dots, n-1$, we see that $C_{s_k} = C_{i_{m+k}}$ for each $k = 0, 1, \dots, \ell-1$ and $\{C_{\gamma_\ell}, C_{\delta_\ell}\} = \{C_{i_{m+\ell}}, C_{j_{m+\ell}}\}$. Since $B_{\gamma_\ell}, B_{\delta_\ell}$ are distinct elements of $\mathbf{A}_0$ such that $\{\pi \circ f^{m+\ell}(p)\} = \mathrm{Bd}(B_{\gamma_\ell}) \cap \mathrm{Bd}(B_{\delta_\ell})$, we see that $\{B_{\gamma_\ell}, B_{\delta_\ell}\} = \{B_{i_{m+\ell}}, B_{j_{m+\ell}}\}$. Thus we have that $C_{i_{m+\ell-1}} \longrightarrow C_{j_{m+\ell}}$. This is a contradiction. Hence it holds that $\mathrm{Ord}(\pi \circ f^k(p), B_{i_k} \cap Z) = 1$ for each $k = 0, 1, \dots, n-1$.

Note that $\mathrm{Ord}(\pi \circ f^k(p), Z) = \mathrm{Ord}(\pi \circ f^k(p), B_{i_k} \cap Z) + \mathrm{Ord}(\pi \circ f^k(p), B_{j_k} \cap Z) = \mathrm{Ord}(\pi \circ f^k(p), B_{j_k} \cap Z) + 1$. □

Lemma 3.1.6 *Let p be an element of P and $C_{i_0} \longrightarrow C_{i_1} \longrightarrow \cdots \longrightarrow C_{i_{n-1}} \longrightarrow C_{i_0}$ an n-cycle for $O(p, f)$. And let C_{j_k} be the element of $S(I, P) \setminus P$ with $\{f^k(p)\} = cl(C_{i_k}) \cap cl(C_{j_k})$ for each $k = 0, 1, \dots, n-1$, but $C_{j_k} = \emptyset$ if $f^k(p)$ is an endpoint of I. If there exists $k = 0, 1, \dots, n-1$ such that $C_{i_k} \longrightarrow C_{j_{k+1}}$, then Z is the universal dendrite.*

Proof. Without loss of generality, we may assume that $C_{i_{n-1}} \longrightarrow C_{j_0}$. Then $B_{i_{n-1}}$ contains two distinct elements $B_{i_{n-1},i_0}, B_{i_{n-1},j_0}$ of $\mathbf{A}_1$ such that $\pi \circ f^{n-1}(p) \in B_{i_{n-1},i_0} \cap B_{i_{n-1},j_0}$ (see Figure 3.1.3).

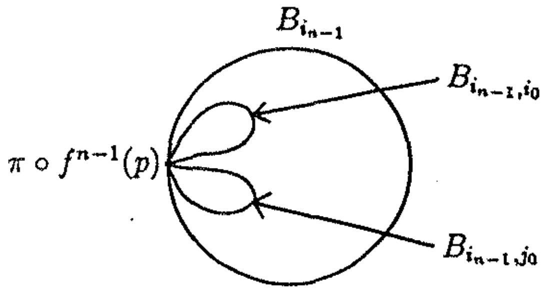


Figure 3.1.3

Since $f^{n-2}(p) \longrightarrow f^{n-1}(p)$ and $C_{i_{n-2}} \longrightarrow C_{i_{n-1}}$, $B_{i_{n-2}}$ contains an element $B_{i_{n-2},i_{n-1}}$ of $\mathbf{A}_1$ such that $\pi \circ f^{n-2}(p) \in B_{i_{n-2},i_{n-1}}$. As $B_{i_{n-2},i_{n-1}} \cap Z$ is homeomorphic to $B_{i_{n-1}} \cap Z$, $B_{i_{n-2},i_{n-1}}$ contains two distinct elements B_{i_{n-2},i_{n-1},i_0}, B_{i_{n-2},i_{n-1},j_0} of $\mathbf{A}_2$ such that $\pi \circ f^{n-2}(p) \in B_{i_{n-2},i_{n-1},i_0} \cap B_{i_{n-2},i_{n-1},j_0}$ (see Figure 3.1.4).

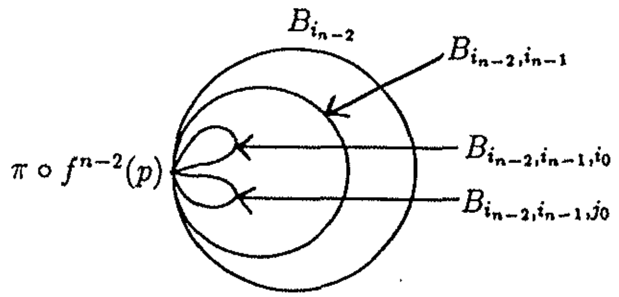


Figure 3.1.4

Similarly B_{i_k} contains two distinct elements $B_{i_k,i_{k+1},\ldots,i_{n-1},i_0}$, $B_{i_k,i_{k+1},\ldots,i_{n-1},j_0}$ of $\mathbf{A}_{n-k}$ such that $\pi \circ f^k(p) \in B_{i_k,i_{k+1},\ldots,i_{n-1},i_0} \cap B_{i_k,i_{k+1},\ldots,i_{n-1},j_0}$ for each $k = 0, 1, \ldots, n-1$ (see Figure 3.1.5).

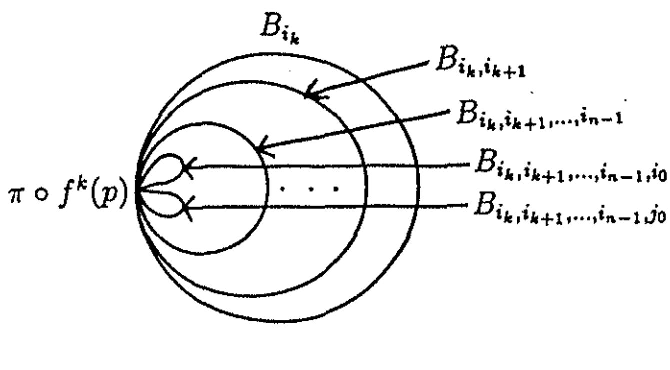


Figure 3.1.5

Since $B_{i_{n-1},i_0} \cap Z$ is homeomorphic to $B_{i_0} \cap Z$, B_{i_{n-1},i_0} contains two distinct elements $B_{i_{n-1},i_0,i_1,\ldots,i_{n-1},i_0}, B_{i_{n-1},i_0,i_1,\ldots,i_{n-1},j_0}$ of $\mathbf{A}_{n+1}$ such that $\pi \circ f^{n-1}(p) \in B_{i_{n-1},i_0,i_1,\ldots,i_{n-1},i_0} \cap B_{i_{n-1},i_0,i_1,\ldots,i_{n-1},j_0}$. Similarly $B_{i_{n-1},i_0,i_1,\ldots,i_{n-1},i_0}$ contains two distinct elements $B_{i_{n-1},i_0,\ldots,i_{n-1},i_0,\ldots,i_{n-1},i_0}, B_{i_{n-1},i_0,\ldots,i_{n-1},i_0,\ldots,i_{n-1},j_0}$ of $\mathbf{A}_{2n+1}$ such that $\pi \circ f^{n-1}(p) \in B_{i_{n-1},i_0,\ldots,i_{n-1},i_0,\ldots,i_{n-1},i_0} \cap B_{i_{n-1},i_0,\ldots,i_{n-1},i_0,\ldots,i_{n-1},j_0}$ (see Figure 3.1.6).

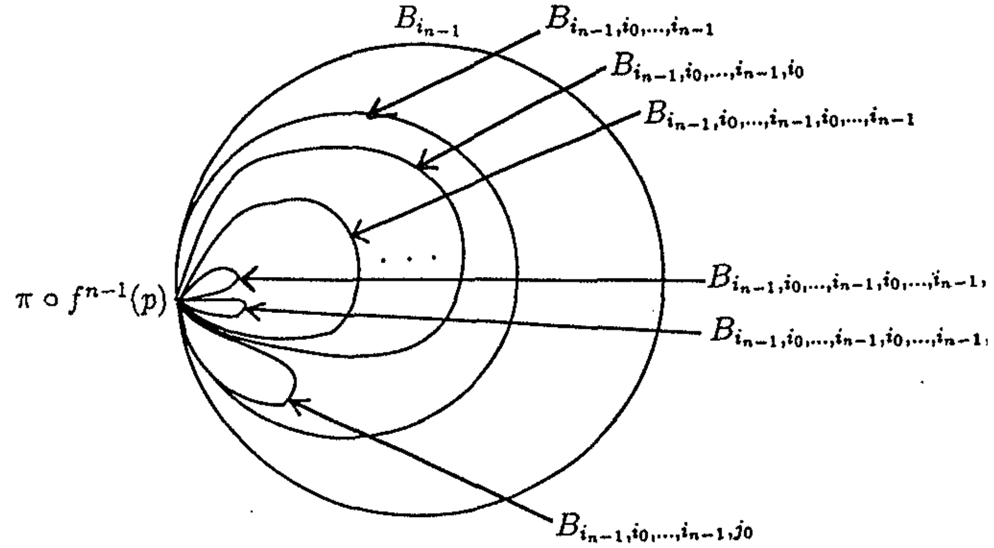


Figure 3.1.6

Repeating this operation, we have that $\text{Ord}(\pi \circ f^{n-1}(p), B_{i_{n-1}} \cap Z) = \infty$. By Lemma 3.1.1 and 3.1.4, we see that Z is the universal dendrite. □

The following result is the main theorem in this chapter.

Theorem 3.1.7 *There exist two n-cycles for the orbit of some element of P if and only if Z is the universal dendrite.*

Proof. Let p be an element of P such that p is an endpoint of $[P]$ and $f^{n-1}(p)$ is not an endpoint of $[P]$. And let $C_{\alpha_0} \longrightarrow C_{\alpha_1} \longrightarrow \cdots \longrightarrow C_{\alpha_{n-1}} \longrightarrow C_{\alpha_0}$ be the fundamental cycle for $O(p,f)$ and C_{β_k} an element of $S(I,P) \setminus P$ with $\{f^k(p)\} = cl(C_{\alpha_k}) \cap cl(C_{\beta_k})$ for each $k = 0,1,\ldots,n-1$, but $C_{\beta_k} = \emptyset$ if $f^k(p)$ is the endpoint of I.

(Sufficiency) : Suppose that there exists an n-cycle for $O(p,f)$ other than the fundamental cycle. Note that there exists an n-cycle for $O(p,f)$ other than the fundamental cycle if and only if $C_{\alpha_k} \longrightarrow C_{\beta_{k+1}}$ for some $k = 0,1,\ldots,n-1$, or there exists the n-cycle $C_{\beta_0} \longrightarrow C_{\beta_1} \longrightarrow \cdots \longrightarrow C_{\beta_{n-1}} \longrightarrow C_{\beta_0}$.

Let $C_{\gamma_0} \longrightarrow C_{\gamma_1} \longrightarrow \cdots \longrightarrow C_{\gamma_{n-1}} \longrightarrow C_{\gamma_0}$ be an n-cycle for $O(p,f)$ other than the fundamental cycle. Then we have that $C_{\gamma_\ell} = C_{\beta_\ell}$ for some $\ell = 0,1,\ldots,n-1$. Assume that $C_{\gamma_k} = C_{\alpha_k}$ for each $k = 0,1,\ldots,n-1$, i.e. $C_{\beta_0} \longrightarrow C_{\beta_1} \longrightarrow \cdots \longrightarrow C_{\beta_{n-1}} \longrightarrow C_{\beta_0}$. Since $C_{\beta_{n-1}} \longrightarrow C_{\beta_0}$ and $C_{\beta_0} \not\subset [P]$, we have that $C_{\beta_{n-1}} \longrightarrow C_{\alpha_0}$. Thus Z is the universal dendrite by Lemma 3.1.6. Assume that $C_{\gamma_k} = C_{\alpha_k}$ for some $k = 0,1,\ldots,n-1$. Since $C_{\gamma_\ell} = C_{\beta_\ell}$, there exists a natural number k' such that $C_{\alpha_{k'}} \longrightarrow C_{\beta_{k'+1}}$. Thus we see that Z is the universal dendrite by Lemma 3.1.6.

(Necessity) : We prove by the reductive absurdity. Assume that there exists no n-cycle for $O(p,f)$ other than the fundamental cycle. This implies that $C_{\alpha_k} \not\longrightarrow C_{\beta_{k+1}}$ for each $k = 0,1,\ldots,n-1$ and $C_{\beta_{k'}} \not\longrightarrow C_{\beta_{k'+1}}$ for some $k' = 0,1,\ldots,n-1$.

Since $C_{\alpha_k} \not\longrightarrow C_{\beta_{k+1}}$ for each $k = 0,1,\ldots,n-1$, we see that $\mathrm{Ord}(\pi \circ f^k(p), B_{\alpha_k} \cap Z) = 1$ for each $k = 0,1,\ldots,n-1$ by Lemma 3.1.5.

Assume that $C_{\beta_{k'}} \neq \emptyset$. Since $C_{\beta_{k'}} \not\longrightarrow C_{\beta_{k'+1}}$, we have that $\mathrm{Ord}(\pi \circ f^{k'}(p), Z) = \mathrm{Ord}(\pi \circ f^{k'}(p), B_{\alpha_{k'}} \cap Z) + \mathrm{Ord}(\pi \circ f^{k'}(p), B_{\beta_{k'}} \cap Z) = 1 + \mathrm{Ord}(\pi \circ f^{k'}(p), B_{\beta_{k'},\alpha_{k'+1}} \cap$

$Z) = 1 + \mathrm{Ord}(\pi \circ f^{k'+1}(p), B_{\alpha_{k'+1}} \cap Z) = 2.$

When $C_{\beta_{k'}} = \emptyset$, we have that $\mathrm{Ord}(\pi \circ f^{k'}(p), Z) = \mathrm{Ord}(\pi \circ f^{k'}(p), B_{\alpha_{k'}} \cap Z) = 1$. By Lemma 3.1.1 and 3.1.2, we obtain that $\mathrm{Ord}(x, Z) < \infty$ for each element x of P_∞.

Since $\mathrm{Br}(Z) \subset P_\infty$, we see that Z is not the universal dendrite. □

Let P be a periodic orbit of f with period n, p an element of P and $C_{\alpha_0} \longrightarrow C_{\alpha_1} \longrightarrow \cdots \longrightarrow C_{\alpha_{n-1}} \longrightarrow C_{\alpha_0}$ the fundamental cycle for $O(p, f)$ in $S(I, P) \backslash P$. And let C_{β_k} be the element of $S(I, P) \setminus P$ with $\{f^k(p)\} = cl(C_{\alpha_k}) \cap cl(C_{\beta_k})$ for any $k = 0, 1, \ldots, n-1$, but $C_{\beta_k} = \emptyset$ if $f^k(p)$ is an endpoint of I.

Proposition 3.1.8 *Assume that Z is not the universal dendrite. Then the following statements are equivalent :*

(1) *there exist $\ell + 1$ elements $C_{\beta_k}, C_{\beta_{k+1}}, \ldots, C_{\beta_{k+\ell}}$ of $S(I, P) \setminus P$ ($\ell < n$) and a natural number m such that $C_{\beta_k} \longrightarrow C_{\beta_{k+1}} \longrightarrow \cdots \longrightarrow C_{\beta_{k+\ell}}$ and $\mathrm{Card}\{C_{\beta_s} \mid C_{\beta_s} \longrightarrow C_{\alpha_{s+1}} \text{ and } k \leq s < k + \ell\} \geq m$.*

(2) *Z has an $(m + 2)$-branch point.*

Proof. (1) $\Longrightarrow$ (2) : We may assume that $k = 0$. Denote $\ell_0 = max\{\ell' \mid C_{\beta_0} \longrightarrow C_{\beta_1} \longrightarrow \cdots \longrightarrow C_{\beta_{\ell'}}\}$ and $m_0 = \mathrm{Card}\{C_{\beta_s} \mid C_{\beta_s} \longrightarrow C_{\alpha_{s+1}}$ and $0 \leq s \leq \ell_0 - 1\} \geq m$. Let $C_{\beta_{s_{m-m_0+1}}}, C_{\beta_{s_{m-m_0+2}}}, \ldots, C_{\beta_{s_{-1}}}, C_{\beta_{s_0}}, C_{\beta_{s_1}}, \ldots, C_{\beta_{s_m}}$ be elements of $\{C_{\beta_0}, C_{\beta_1}, \ldots, C_{\beta_{\ell_0-1}}\}$ such that $C_{\beta_{s_k}} \longrightarrow C_{\alpha_{s_k+1}}$ for each $k = s_{m-m_0+1}, s_{m-m_0+2}, \ldots, s_m$ ($s_{m-m_0+1} < s_{m-m_0+2} < \ldots < s_m$) (see Graph 3.1.7).

$$
\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & C_{\alpha_{s_{k-1}}} & \longrightarrow & C_{\alpha_{s_{k-1}+1}} & \longrightarrow & \cdots & \longrightarrow & C_{\alpha_{s_k}} & \longrightarrow & C_{\alpha_{s_k+1}} & \longrightarrow & \cdots \\
 & & & \nearrow & & & & & & \nearrow & & & \\
\cdots & \longrightarrow & C_{\beta_{s_{k-1}}} & \longrightarrow & C_{\beta_{s_{k-1}+1}} & \longrightarrow & \cdots & \longrightarrow & C_{\beta_{s_k}} & \longrightarrow & C_{\beta_{s_k+1}} & \longrightarrow & \cdots
\end{array}
$$

Graph 3.1.7

We show that $\mathrm{Ord}(\pi \circ f^{s_1}(p), Z) = m + 2$. Since Z is not the universal dendrite, we have that $C_{\alpha_k} \not\to C_{\beta_{k+1}}$ for each $k = 0, 1, \ldots, n-1$ by the proof of Theorem 3.1.7. Thus it holds that $\mathrm{Ord}(\pi \circ f^k(p), B_{\alpha_k} \cap Z) = 1$ for each $k = 0, 1, \ldots, n-1$ by Lemma 3.1.5. Since $\mathrm{Ord}(\pi \circ f^{s_1}(p), Z) = \mathrm{Ord}(\pi \circ f^{s_1}(p), B_{\alpha_{s_1}} \cap Z) + \mathrm{Ord}(\pi \circ f^{s_1}(p), B_{\beta_{s_1}} \cap Z)$, it suffices to show that $\mathrm{Ord}(\pi \circ f^{s_1}(p), B_{\beta_{s_1}} \cap Z) = m + 1$.

Since $C_{\beta_{\ell_0}} \longrightarrow C_{\alpha_{\ell_0+1}}$ and $C_{\beta_{\ell_0}} \not\to C_{\beta_{\ell_0+1}}$, we see that $\{B \in \mathbf{A}_1 \mid \pi \circ f^{\ell_0}(p) \in B \subset B_{\beta_{\ell_0}}\} = \{B_{\beta_{\ell_0}, \alpha_{\ell_0+1}}\}$. Thus it holds that $\mathrm{Ord}(\pi \circ f^{\ell_0}(p), B_{\beta_{\ell_0}} \cap Z) = \mathrm{Ord}(\pi \circ f^{\ell_0}(p), B_{\beta_{\ell_0}, \alpha_{\ell_0+1}} \cap Z) = \mathrm{Ord}(\pi \circ f^{\ell_0+1}(p), B_{\alpha_{\ell_0+1}} \cap Z) = 1$. Since $C_{\beta_{s_m+1}} \longrightarrow C_{\beta_{s_m+2}} \longrightarrow \cdots \longrightarrow C_{\beta_{\ell_0}}$ and $C_{\beta_k} \not\to C_{\alpha_{k+1}}$ for $k = s_m + 1, s_m + 2, \ldots, \ell_0 - 1$, we have that $\mathrm{Ord}(\pi \circ f^k(p), B_{\beta_k} \cap Z) = \mathrm{Ord}(\pi \circ f^k(p), B_{\beta_k, \beta_{k+1}, \ldots, \beta_{\ell_0}} \cap Z) = \mathrm{Ord}(\pi \circ f^{\ell_0}(p), B_{\beta_{\ell_0}} \cap Z) = 1$ for each $k = s_m + 1, s_m + 2, \ldots, \ell_0 - 1$. Since $C_{\beta_{s_m}} \longrightarrow C_{\alpha_{s_m+1}}$ and $C_{\beta_{s_m}} \longrightarrow C_{\beta_{s_m+1}}$, we see that $\{B \in \mathbf{A}_1 \mid \pi \circ f^{s_m}(p) \in B \subset B_{\beta_{s_m}}\} = \{B_{\beta_{s_m}, \alpha_{s_m+1}}, B_{\beta_{s_m}, \beta_{s_m+1}}\}$. Thus it holds that $\mathrm{Ord}(\pi \circ f^{s_m}(p), B_{\beta_{s_m}} \cap Z) = \mathrm{Ord}(\pi \circ f^{s_m}(p), B_{\beta_{s_m}, \alpha_{s_m+1}} \cap Z) + \mathrm{Ord}(\pi \circ f^{s_m}(p), B_{\beta_{s_m}, \beta_{s_m+1}} \cap Z) = \mathrm{Ord}(\pi \circ f^{s_m+1}(p), B_{\alpha_{s_m+1}} \cap Z) + \mathrm{Ord}(\pi \circ f^{s_m+1}(p), B_{\beta_{s_m+1}} \cap Z) = 2$ (see Figure 3.1.8).

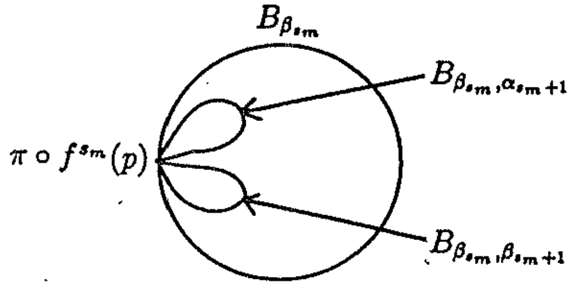


Figure 3.1.8

Since $C_{\beta_{s_m-1}} \not\to C_{\alpha_{s_m}}$ and $C_{\beta_{s_m-1}} \longrightarrow C_{\beta_{s_m}}$, we have that $\{B \in \mathbf{A}_1 \mid \pi \circ f^{s_m-1}(p) \in B_{\beta_{s_m-1}}\} = \{B_{\beta_{s_m-1},\beta_{s_m}}\}$. Thus we see that $\mathrm{Ord}(\pi \circ f^{s_m-1}(p), B_{\beta_{s_m-1}} \cap Z) = \mathrm{Ord}(\pi \circ f^{s_m-1}(p), B_{\beta_{s_m-1},\beta_{s_m}} \cap Z) = \mathrm{Ord}(\pi \circ f^{s_m}(p), B_{\beta_{s_m}} \cap Z) = 2$.

Similarly we have $\mathrm{Ord}(\pi \circ f^k(p), B_{\beta_k} \cap Z) = 2$ for $k = s_{(m-1)} + 1, s_{(m-1)} + 2, \ldots, s_m - 1$. Since $C_{\beta_{s_{(m-1)}}} \longrightarrow C_{\alpha_{s_{(m-1)}+1}}$ and $C_{\beta_{s_{(m-1)}}} \longrightarrow C_{\beta_{s_{(m-1)}+1}}$, $\{B \in \mathbf{A}_1 \mid \pi \circ f^{s_{(m-1)}}(p) \in B \subset B_{\beta_{s_{(m-1)}}}\} = \{B_{\beta_{s_{(m-1)}},\alpha_{s_{(m-1)}+1}}, B_{\beta_{s_{(m-1)}},\beta_{s_{(m-1)}+1}}\}$. Thus we have that $\mathrm{Ord}(\pi \circ f^{s_{(m-1)}}(p), B_{\beta_{s_{(m-1)}}} \cap Z) = \mathrm{Ord}(\pi \circ f^{s_{(m-1)}}(p), B_{\beta_{s_{(m-1)}},\alpha_{s_{(m-}}}$ $Z) + \mathrm{Ord}(\pi \circ f^{s_{(m-1)}}(p), B_{\beta_{s_{(m-1)}},\beta_{s_{(m-1)}+1}} \cap Z) = \mathrm{Ord}(\pi \circ f^{s_{(m-1)}+1}(p), B_{\alpha_{s_{(m-1)}+1}} \cap Z) + \mathrm{Ord}(\pi \circ f^{s_{(m-1)}+1}(p), B_{\beta_{s_{(m-1)}+1}} \cap Z) = 3$ (see Figure 3.1.9).

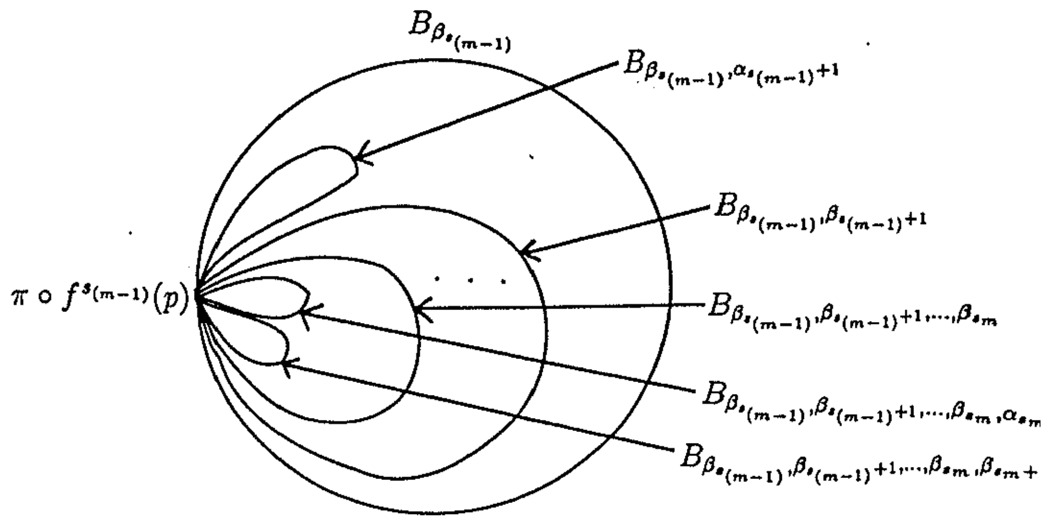


Figure 3.1.9

Repeating this operation, we have $\mathrm{Ord}(\pi \circ f^{s_1}(p), B_{\beta_{s_1}} \cap Z) = m + 1$

$(2) \Longrightarrow (1)$: By the reductive absurdity, we show that $(2) \Longrightarrow (1)$. Assume that for each $\ell + 1$ elements $C_{\beta_k}, C_{\beta_{k+1}}, \ldots, C_{\beta_{k+\ell}}$ of $S(I,P) \setminus P$ with $C_{\beta_k} \longrightarrow C_{\beta_{k+1}} \longrightarrow \cdots \longrightarrow C_{\beta_{k+\ell}}$, it holds that $\mathrm{Card}\{C_{\beta_s} \mid C_{\beta_s} \longrightarrow C_{\alpha_{s+1}}$ and $k \leq s < k + \ell\} < m$ $(m \leq \ell)$. Then by the proof of $(1) \Longrightarrow (2)$, we see that $\mathrm{Ord}(\pi \circ f^{k'}(p), Z) < m + 2$ for each $k' = k, k+1, \ldots, k+\ell$. This implies that $\mathrm{Ord}(\pi \circ f^{k}(p), Z) < m + 2$ for each $k = 0, 1, \ldots, n-1$.

Now we prove that $\mathrm{Ord}(p_\infty, Z) < m+2$ for each element p_∞ of P_∞. We may assume that $p_\infty \notin \pi(P)$. Then there exist a natural number ℓ' and an element B of $\mathbf{A}_{\ell'}$ such that $p_\infty \in \mathrm{Bd}(B)$ and $p_\infty \in \mathrm{Int}(B')$ for each element B' of $\mathbf{A}_{\ell'-1}$. Denote $p_\infty = p_{s_0,s_1,\ldots,s_{\ell'}}$. And let $B_{s_0,s_1,\ldots,s_{\ell'-1},\gamma_{\ell'}}, B_{s_0,s_1,\ldots,s_{\ell'-1},\delta_{\ell'}}$ be elements of $\mathbf{A}_{\ell'}$ such that $\{p_{s_0,s_1,\ldots,s_{\ell'}}\} = \mathrm{Bd}(B_{s_0,s_1,\ldots,s_{\ell'-1},\gamma_{\ell'}}) \cap \mathrm{Bd}(B_{s_0,s_1,\ldots,s_{\ell'-1},\delta_{\ell'}})$. Then we have that $\mathrm{Ord}(p_{s_0,s_1,\ldots,s_{\ell'}}, Z) = \mathrm{Ord}(p_{s_0,s_1,\ldots,s_{\ell'}}, B_{s_0,s_1,\ldots,s_{\ell'-1},\gamma_{\ell'}} \cap Z) + \mathrm{Ord}(p_{s_0,s_1,\ldots,s_{\ell'}}, B_{s_0,s_1,\ldots,s_{\ell'-1},\delta_{\ell'}} \cap Z) = \mathrm{Ord}(p_{s_{\ell'}}, B_{\gamma_{\ell'}} \cap Z) + \mathrm{Ord}(p_{s_{\ell'}}, B_{\delta_{\ell'}} \cap Z) = \mathrm{Ord}(p_{s_{\ell'}}, Z) < m + 2$, since $p_{s_{\ell'}}$ is an element of $\pi(p)$.

Since $\mathrm{Br}(Z) \subset P_\infty$ by Fact 3, we have that $\mathrm{Ord}(b, Z) < m + 2$ for each element b of $\mathrm{Br}(Z)$. □

Corollary 3.1.9 *If Z is not the universal dendrite, then* $\mathrm{Ord}(x, Z) \leq n + 1$ *for each element x of Z, where $n = \mathrm{Card}(P)$.*

Proof. By Theorem 3.1.7, it holds that $C_{\alpha_k} \not\longrightarrow C_{\beta_{k+1}}$ for each $k = 0, 1, \ldots, n-1$ and $C_{\beta_{k'}} \not\longrightarrow C_{\beta_{k'+1}}$ for some $k' = 0, 1, \ldots, n-1$. Thus $max\{\ell \mid C_{\beta_k} \longrightarrow C_{\beta_{k+1}} \longrightarrow \cdots \longrightarrow C_{\beta_{k+\ell}}\} \leq n - 1$ for each $k = 0, 1, \ldots, n-1$. By Proposition

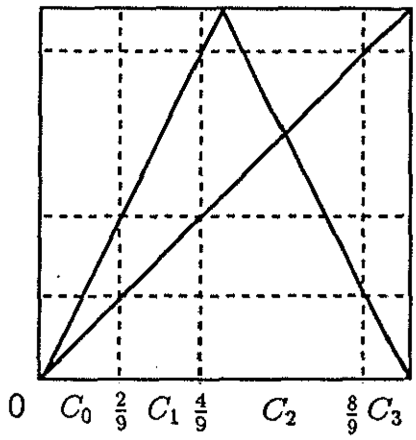
3.1.8, we see that $\mathrm{Ord}(x, Z) \leq n+1$ for each $x \in Z$. □

Corollary 3.1.10 *Assume that Z is not the universal dendrite. Then if Z has an m-branch point ($m \geq 3$), then Z has also an ($m-1$)-branch point.* □

3.2 Examples

In this section, we give some concrete examples .

Example 3.2.1 Denote that $I = [0,1]$ and $f : I \longrightarrow I$ a continuous map such that $f(x) = 2x(0 \leq x \leq \frac{1}{2})$ and $f(x) = -2x + 2(\frac{1}{2} \leq x \leq 1)$. Let P be a periodic orbit $\{\frac{2}{9}, \frac{4}{9}, \frac{8}{9}\}$, $C_0 = [0, \frac{2}{9}), C_1 = (\frac{2}{9}, \frac{4}{9}), C_2 = (\frac{4}{9}, \frac{8}{9})$ and $C_3 = (\frac{8}{9}, 1]$ (see Graph 3.2.1).



Graph 3.2.1

By the definition, $C_1 \longrightarrow C_2 \longrightarrow C_2 \longrightarrow C_1$ is the fundamental cycle. The Markov graph of $S(I, P) \setminus P$ is as in Graph 3.2.2.

$$\begin{array}{ccccccc} C_1 & \longrightarrow & C_2 & \longrightarrow & C_2 & \longrightarrow & C_1 \\ & & & \searrow & & & \\ C_0 & \longrightarrow & C_1 & & C_3 & \longrightarrow & C_0 \end{array}$$

Graph 3.2.2

Since we have that $C_2 \longrightarrow C_2$ and $C_2 \longrightarrow C_3$, we see that Z is the universal dendrite by Lemma 3.1.6 (see Figure 3.2.3).

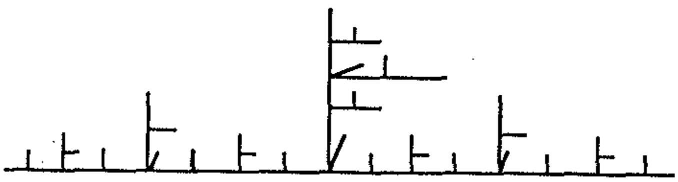
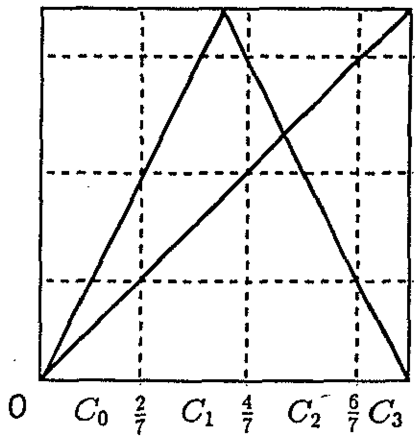


Figure 3.2.3

Example 3.2.2 Let f be the same map as Example 5.1. Denote $P = \{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\}$, $[0, \frac{2}{7}), C_1 = (\frac{2}{7}, \frac{4}{7}), C_2 = (\frac{4}{7}, \frac{6}{7})$ and $C_3 = (\frac{6}{7}, 1]$ (see Graph 3.2.4).



Graph 3.2.4

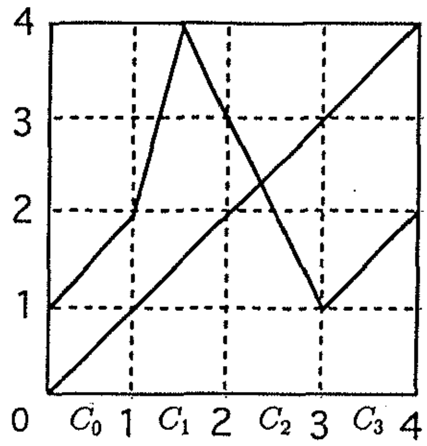
The fundamental cycle is $C_1 \longrightarrow C_2 \longrightarrow C_2 \longrightarrow C_1$. The Markov graph of $S(I,P) \setminus P$ is as in Graph 3.2.5.

$$\begin{array}{c} C_1 \longrightarrow C_2 \longrightarrow C_2 \longrightarrow C_1 \\ \nearrow \\ C_0 \longrightarrow C_1 \longrightarrow C_3 \longrightarrow C_0 \end{array}$$

Graph 3.2..5

Since there exists an 3-cycle $C_0 \longrightarrow C_1 \longrightarrow C_3 \longrightarrow C_0$ other than the fundamental cycle, we see that Z is the universal dendrite by Theorem 3.1.7 (see Figure 3.2.3).

Example 3.2.3 Denote $I = [0,4]$, $f : I \longrightarrow I$ a continuous map and $P = \{1,2,3\}$ a 3-periodic orbit of f with $f(i) = i + 1 \pmod 3$ for $i = 1,2,3$. And denote $C_0 = [0,1), C_1 = (1,2), C_2 = (2,3)$ and $C_3 = (3,4]$ (see Graph 3.2.6).



Graph 3.2.6

By the definition, $C_1 \longrightarrow C_2 \longrightarrow C_2 \longrightarrow C_1$ is the fundamental cycle for $O(1,f)$. Assume that there exist no cycles for $O(1,f)$ other than the fundamental cycle. The Markov graph of $S(I,P) \setminus P$ is as in Graph 3.2.7.

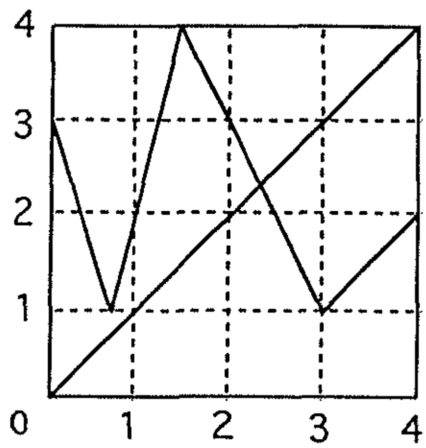
$$\begin{array}{ccccccc} C_1 & \longrightarrow & C_2 & \longrightarrow & C_2 & \longrightarrow & C_1 \\ & & & \nearrow & & \nearrow & \\ C_0 & \longrightarrow & C_1 & \longrightarrow & C_3 & & C_0 \end{array}$$

Graph 3.2.7

Then it holds that $\mathrm{Ord}(x,Z) \leq 3$ for any $x \in Z$. And $\mathrm{Br}(Z)$ is dense in Z (see Figure 3.2.8).

Figure 3.2.8

Example 3.2.4 Let P be the same periodic orbit as Example 3.2.3. And let $f : I \longrightarrow I$ be a continuous map as in Graph 3.2.9.



Graph 3.2.9

The Markov graph of $S(I, P) \setminus P$ is as in Graph 3.2.10.

$$\begin{array}{ccccccc} C_1 & \longrightarrow & C_2 & \longrightarrow & C_2 & \longrightarrow & C_1 \\ & \nearrow & & \nearrow & & \nearrow & \\ C_0 & \longrightarrow & C_1 & \longrightarrow & C_3 & & C_0 \end{array}$$

Graph 3.2.10

Then it holds that $\mathrm{Ord}(x, Z) \leq 4$ for any $x \in Z$. There exists only one 4-branch point $\pi(1)$ in Z and the set of 3-branch points of Z is dense in Z (Figure 3.2.11).

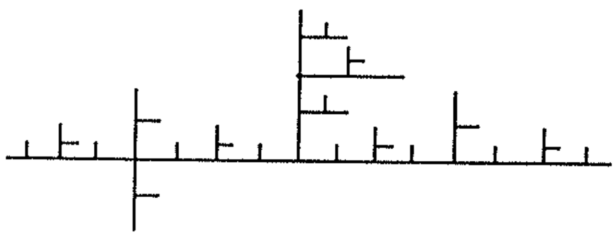
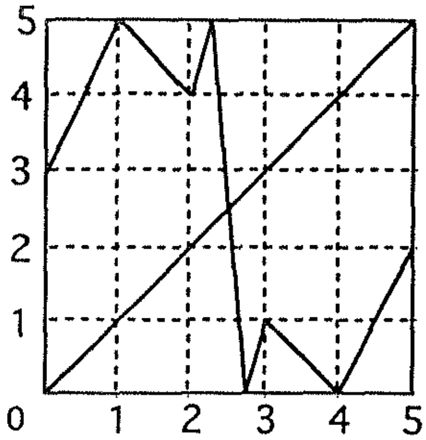


Figure 3.2.11

Example 3.2.5 Denote $I = [0, 5]$. Define a continuous map $f : I \longrightarrow I$ as follows : $f(x) = 2x + 3$ $(0 \leq x \leq 1)$, $f(x) = -x + 6$ $(1 \leq x \leq 2)$, $f(x) = 3x - 2$ $(2 \leq x \leq \frac{7}{3})$, $f(x) = -15x + 40$ $(\frac{7}{3} \leq x \leq \frac{8}{3})$, $f(x) = 3x - 8$ $(\frac{8}{3} \leq x \leq 3)$, $f(x) = -x + 4$ $(3 \leq x \leq 4)$ and $f(x) = 2x - 8$ $(4 \leq x \leq 5)$. Denote $P = \{0, 1, 2, 3, 4, 5\}$, $C_0 = [0, 1), C_1 = (1, 2), C_2 = (2, 3), C_3 = (3, 4), C_4 = (4, 5]$ (see Graph 3.2.12).



Graph 3.2.12

The fundamental cycle for $O(0,f)$ is as in Graph 3.2.13.

$$
\begin{array}{ccccccccccccc}
C_0 & \longrightarrow & C_3 & \longrightarrow & C_0 & \longrightarrow & C_4 & \longrightarrow & C_1 & \longrightarrow & C_4 & \longrightarrow & C_0 \\
 & & \nearrow & & \nearrow & & & & \nearrow & & \nearrow & & \\
 & C_2 & \longrightarrow & C_1 & & & & C_2 & \longrightarrow & C_3 & & &
\end{array}
$$

Graph 3.2.13

Then there exist infinite 3-branch points in $B_3 \cap Z$, but there exist no branch points in $Z \setminus (B_3 \cap Z)$. The structure of Z is as in Figure 3.2.14.

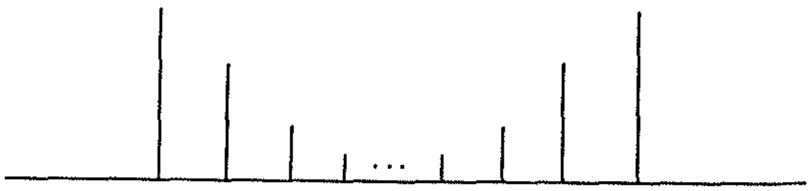


Figure 3.2.14