# Chapter 2

# The construction of $P$-expansive maps of regular continua : A geometric approach

In this chapter, we construct a new space $Z$ from a continuous map $f$ of a graph $G$ and investigate the relationship between the dynamical behavior of $f$ and the structure of $Z$.

Let $G$ be a graph, $f : G \longrightarrow G$ a continuous map and $P$ a finite subset of $G$ such that $f(P) \subset P$. Put $S(G, P) = P \bigcup \{C | C \text{ is a component of } G \setminus P\}$. Given $x \in G$, the *itinerary* of $x$ with respect to $P$ and $f$, written $I_{P,f}(x)$ (or just $I(x)$ if $P$ and $f$ are obvious from context), is defined to be the unique infinite sequence $(C_n)_{n \geq 0}$ from $S(G, P)$ given by the rule $f^n(x) \in C_n$ for all $n \geq 0$. If no two points of $G$ have the same itinerary, then $f$ will be called *$P$-expansive.* And $f$ is *point-wise $P$-expansive* if for each $p, q \in P$, there exists some non-negative integer $m$ such that $A \cap (P \setminus \{f^m(p), f^m(q)\}) \neq \emptyset$ for each arc $A$ in $G$ between $f^m(p)$ and $f^m(q)$.

Let $G$ be a graph, $f : G \longrightarrow G$ a continuous map and $P$ a finite subset of $G$

such that $f(P) \subset P$. We construct new spaces $X_{\rightarrow}$ and $X_{\rightharpoondown}$ from $P$ and $f$.

## 2.1 The constructions of $X_{\rightarrow}$ and $X_{\rightharpoondown}$

First we want to define an equivalence relation $\sim_1$ on $P$. Let $p, q \in P$. If for any non-negative integer $i$, there exists an arc $A_i$ in $G$ between $f^i(p)$ and $f^i(q)$ such that $A_i \cap P = \{f^i(p), f^i(q)\}$, then we put $p \sim_1' q$, where $A_i$ may now consist of a single point. Now, if for $p, q \in P$, there exist some points $p_1, p_2, \ldots, p_k$ of $P$ such that $p \sim_1' p_1 \sim_1' p_2 \sim_1' \cdots \sim_1' p_k \sim_1' q$, then we set $p \sim_1 q$. This relation $\sim_1$ is an equivalence relation on $P$. Let $[p]_1$ be the equivalence class of $p$, $P_1 = \{[p]_1 | p \in P\}$ and $G_1 = G/\sim_1$ the space obtained from $G$ by identifying each equivalence class of $P$. Then we define a continuous map $f_1 : G_1 \longrightarrow G_1$ such that $f_1|_{G_1 \setminus P_1} = f|_{G \setminus P}$ and $f_1([p]_1) = [f(p)]_1$ for $[p]_1 \in P_1$. Similarly, if for any $p, q \in P_1$ and non-negative integer $i$, there exists an arc $A_i$ in $G_1$ between $f_1^i(p)$ and $f_1^i(q)$ such that $A_i \cap P_1 = \{f_1^i(p), f_1^i(q)\}$, then we put $p \sim_2' q$. And if there exist some points $p_1, p_2, \cdots, p_k$ of $P_1$ such that $p \sim_2' p_1 \sim_2' p_2 \sim_2' \cdots \sim_2' p_k \sim_2' q$, then we set $p \sim_2 q$. This relation $\sim_2$ is also an equivalence relation on $P_1$. Let $[p]_2 = \{q | p \sim_2 q \text{ and } p, q \in P_1\}$, $P_2 = \{[p]_2 | p \in P_1\}$ and $G_2 = G_1/\sim_2$ the space obtained from $G_1$ by identifying each equivalence class of $P_1$. Then we define a continuous map $f_2 : G_2 \longrightarrow G_2$ such that $f_2|_{G_2 \setminus P_2} = f_1|_{G_1 \setminus P_1} = f|_{G \setminus P}$ and $f_2([p]_2) = [f_1(p)]_2$ for $[p]_2 \in P_2$. In the same way, we can obtain the space $G_\ell$ and a continuous map $f_\ell : G_\ell \longrightarrow G_\ell$ for $\ell \geq 1$. Since $P$ is finite, there is some natural number $m$ such that $f_m : G_m \longrightarrow G_m$ is point-wise $P$-expansive. There exists a semi-conjugacy $\pi_i$ between $(G_{i-1}, f_{i-1})$ and $(G_i, f_i)$ for $i = 1, 2, \ldots, m$, where $(G_0, f_0) = (G, f)$ (see Graph 2.1.1).

$$
\begin{array}{ccc}
G & \xrightarrow{f} & G \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\
G_1 & \xrightarrow[f_1]{} & G_1 \\
{\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi_2} \\
G_2 & \xrightarrow[f_2]{} & G_2 \\
{\scriptstyle \pi_3}\downarrow & & \downarrow{\scriptstyle \pi_3} \\
\vdots & & \vdots \\
{\scriptstyle \pi_m}\downarrow & & \downarrow{\scriptstyle \pi_m} \\
G_m & \xrightarrow[f_m]{} & G_m \\
{\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi'} \\
Z & \xrightarrow[g]{} & Z
\end{array}
$$

Graph 2.1.1

By the argument above, we may proceed with our construction, under the assumption that $f$ *is point-wise $P$-expansive*, in the rest part of this section.

Let $S(G,P)\backslash P = \{C_1, C_2, \ldots, C_n\}$ and $P = \{p_1, p_2, \ldots, p_k\}$. We will express the relation of elements of $S(G,P)$ as follows : If $p, q \in P$ and $f(p) = q$, then $p \longrightarrow q$. This arrow $\longrightarrow$ defines the Markov graph $P_{\longrightarrow}$ on $P$ (See section 4). If $C_i, C_j \in S(G,P) \setminus P$ and $C_j \subset f(C_i)$, then $C_i \longrightarrow C_j$. If $f(C_i) \cap C_j \neq \emptyset$, then $C_i \rightharpoonup C_j$. These arrows $\longrightarrow$ and $\rightharpoonup$ define the Markov graphs $M_{\longrightarrow}$ and $M_{\rightharpoonup}$ of elements of $S(G,P) \setminus P$ respectively. Note that $\longrightarrow$ implies $\rightharpoonup$.

Now we will construct a new space $X_{\rightharpoonup}$ by using the Markov graphs $M_{\rightharpoonup}$ and $P_{\longrightarrow}$. First we will construct a subspace $X$ which is the union of 3-dimensional balls $B_1, B_2, \ldots, B_n$ in the Euclidean 3-dimensional space $\mathbf{E}^3$ by regarding elements $C_1, C_2, \ldots, C_n$ of $S(G,P) \setminus P$ as 3-dimensional balls $B_1, B_2, \ldots, B_n$ of $\mathbf{E}^3$. That is to say, $X = \bigcup_{i=1}^{n} B_i$, where the relationship of $B_i$ and $B_j$ is decided as follows : If $cl(C_i) \cap cl(C_j) = \emptyset$ for $C_i, C_j \in S(G,P) \setminus P$, then $B_i \cap B_j = \emptyset$. And if $cl(C_i) \cap cl(C_j) = \{q_1, q_2, \ldots, q_\ell\} \subset P$, then $B_i \cap B_j = Bd(B_i) \cap Bd(B_j) = \{q'_1, q'_2, \ldots, q'_\ell\}$, where $Bd(B)$ is the boundary of $B$. Without confusion, we can express elements of $cl(C_i) \cap cl(C_j)$ and $B_i \cap B_j$ in a similar way. And for each $p \in (P \cap cl(C_i)) \setminus \bigcup\{cl(C_j) \cap cl(C_{j'}) | j \neq j' \text{ and } 1 \leq j, j' \leq n\}$, we take a corresponding point $p' \in Bd(B_i) \setminus \bigcup\{B_j \cap B_{j'} | j \neq j' \text{ and } 1 \leq j, j' \leq n\}$. For simplicity, we set $p' = p \in P$ (see Figure 2.1.2).

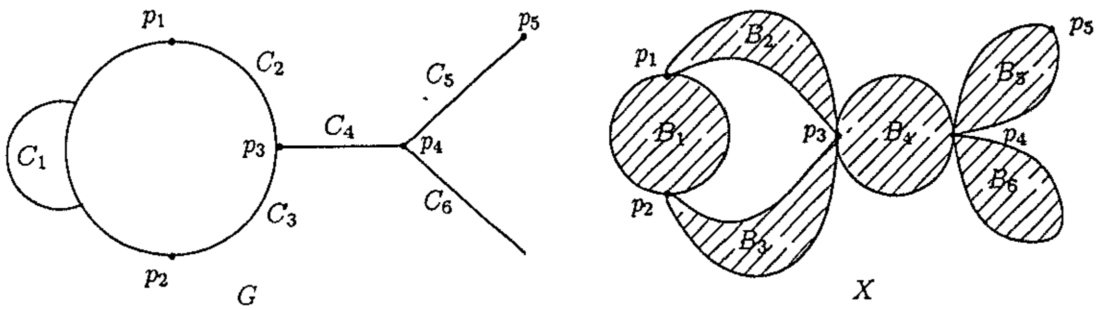



Figure 2.1.2

Put $X_0 = X$. We will construct a subspace $X_1$ contained in $X_0$ by using the Markov graph $M_{\rightharpoonup}$ and $P_{\rightharpoonup}$. For each $i = 1, 2, \ldots, n$, we have an embedding $h_i : X \hookrightarrow B_i$ such that

(1) $h_i(X) \cap Bd(B_i) \subset P$, and

(2) for each $p, q \in P$ with $p \in Bd(B_i)$ and $p \longrightarrow q$, $h_i(q) = p \in Bd(B_i)$.

If $C_i \rightharpoonup C_j$ $(C_i, C_j \in S(G,P) \setminus P)$ in the Markov graph $M_{\rightharpoonup}$, then let $B_{i,j} = h_i(B_j)$ which is a copy of $B_j$. If $C_i \not\rightharpoonup C_j$, then $B_{i,j} = \emptyset$. Let $Y_i = \bigcup_{j=1}^{n} B_{i,j}$, $\mathbf{B}_i = \{B_j | C_i \rightharpoonup C_j\}$ and $(\bigcup \mathbf{B}_i) \cap P = \{p_{t(i:1)}, p_{t(i:2)}, \ldots, p_{t(i:k(i))}\}$, where $t(i : \ell)$

and $k(i)$ are natural numbers with $1 \leq t(i : \ell), k(i) \leq k$ $(1 \leq \ell \leq k(i))$. And put $h_i(p_{t(i:\ell)}) = p_{i,t(i:\ell)}$. Then we obtain a connected subset $X_1 = Y_1 \cup Y_2 \cup \cdots \cup Y_n$ (see Figure 2.1.3).

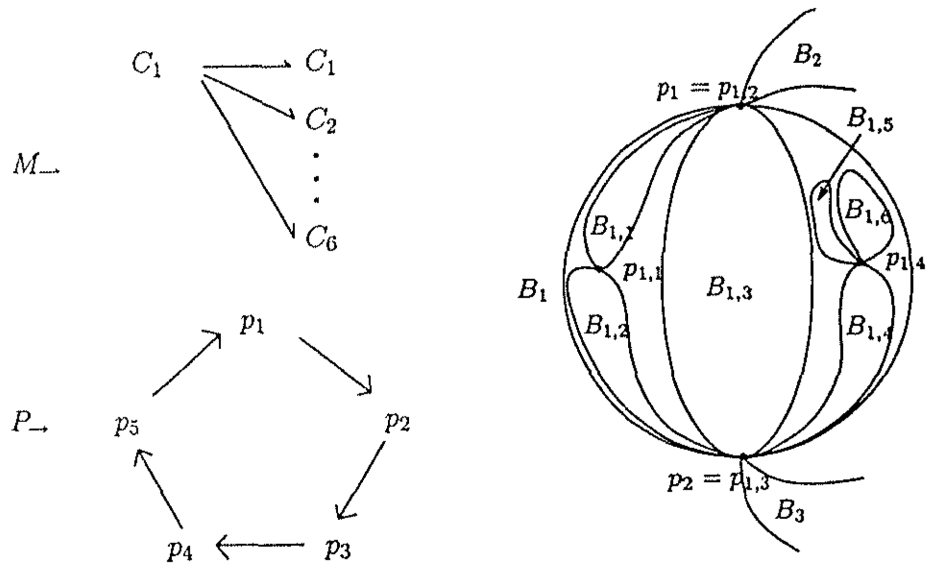



Figure 2.1.3

Similarly, we will construct a subspace $X_2$ in $X_1$. Let $h_{i_0,i_1} : X \hookrightarrow B_{i_0,i_1}$ be an embedding such that

(1) $h_{i_0,i_1}(X) \cap Bd(B_{i_0,i_1}) \subset h_{i_0}(P)$, and

(2) for each $p_{i_0,j} \in Bd(B_{i_0,i_1}) \cap h_{i_0}(P)$ and $q \in P$ with $p_j \longrightarrow q$,

$h_{i_0,i_1}(q) = p_{i_0,j} \in Bd(B_{i_0,i_1})$.

If $C_{i_1} \rightharpoonup C_j$ in the Markov graph $M_{\rightharpoonup}$, then let $B_{i_0,i_1,j} = h_{i_0,i_1}(B_j)$. And if $C_{i_1} \not\rightharpoonup C_j$, then $B_{i_0,i_1,j} = \emptyset$. Let $Y_{i_0,i_1} = \bigcup_{j=1}^{n} B_{i_0,i_1,j}$, $\mathbf{B}_{i_1} = \{B_j | C_{i_1} \rightharpoonup C_j\}$ and $(\bigcup \mathbf{B}_{i_1}) \cap P = \{p_{t(i_0,i_1:1)}, p_{t(i_0,i_1:2)}, \ldots, p_{t(i_0,i_1:k(i_0,i_1))}\}$. Put $h_{i_0,i_1}(p_{t(i_0,i_1:j)}) = p_{i_0,i_1,t(i_0,i_1:j)}$ $(1 \leq j \leq t(i_0, i_1 : k(i_0, i_1))$. Then we obtain $X_2 = \bigcup\{Y_{i_0,i_1} | 1 \leq i_0, i_1 \leq n\}$ (see Figure 2.1.4).

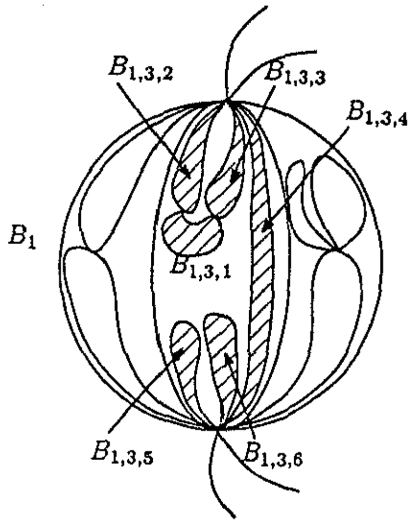



Figure 2.1.4

When this operation is repeated inductively, we obtain $X_0 \supset X_1 \supset X_2 \supset \cdots$ and a subspace $X_{\leftharpoondown} = \bigcap_{i=0}^{\infty} X_i$ of $\mathbf{E}^3$. Note that $X_{\leftharpoondown}$ is connected.

Next let $X_1', X_2', \ldots$ be subspaces constructed in a similar way on basis of the Markov graph $M_{\rightharpoondown}$. Then we obtain a subspace $X_{\rightharpoondown} = \bigcap_{i=1}^{\infty} X_i'$ of $\mathbf{E}^3$. Note that $X_{\rightharpoondown}$ is not always connected.

By using the construction of $X_{\leftharpoondown}$, we show that $lim_{m\to\infty}\ diam(B_{i_0,i_1,\ldots,i_m}) = 0$. Since we have assumed that $f$ is point-wise $P$-expansive, for any distinct points $p, q$ of $P$ there exists a non-negative integer $N_{p,q}$ such that $A \cap (P \setminus \{f^{N_{p,q}}(p), f^{N_{p,q}}(q)\}) \neq \emptyset$ for any arc $A$ in $G$ between $f^{N_{p,q}}(p)$ and $f^{N_{p,q}}(q)$. Let $N = max\{N_{p,q} | p, q \in P \text{ and } p \neq q\}$. Let $m$ be a natural number and $B_{i_0,i_1,\ldots,i_m}$ a 3-dimensional ball from the construction of $X_m$. For any natural numbers $j_{m+1}, j_{m+2}, \ldots, j_{m+N}$ $(1 \leq j_{m+1}, j_{m+2}, \ldots, j_{m+N} \leq n)$, where $n = Card(S(G,P) \setminus P)$, the 3-dimensional ball $B_{i_0,i_1,\ldots,i_m,j_{m+1},j_{m+2},\ldots,j_{m+N}}$ from the constructing of $X_{m+N}$ cannot contain two or more points of $\bigcup\{B_{i_0,i_1,\ldots,i_m} \cap B_{\ell_0,\ell_1,\ldots,\ell_m} | 1 \leq \ell_0, \ell_1, \ldots, \ell_m \leq n \text{ and } (\ell_0, \ell_1, \ldots, \ell_m) \neq (i_0, i_1, \ldots, i_m)\}$. Suppose that there exist distinct points $x, y$ of $\bigcup\{B_{i_0,i_1,\ldots,i_m} \cap B_{\ell_0,\ell_1,\ldots,\ell_m} | 1 \leq \ell_0, \ell_1, \ldots, \ell_m$ $n$ and $(\ell_0, \ell_1, \ldots, \ell_m) \neq (i_0, i_1, \ldots, i_m)\}$ such that $x, y \in B_{i_0,i_1,\ldots,i_m,j_{m+1},\ldots,j_{m+N}}$. Put $x = p_{i_0,i_1,\ldots,i_{m-1},s} = h_{i_0,i_1,\ldots,i_{m-1}}(p_s)$ and $y = p_{i_0,i_1,\ldots,i_{m-1},t} = h_{i_0,i_1,\ldots,i_{m-1}}(p_t)$. Then $p_s, p_t \in P \cap B_{i_m}$. By the construction, for each $i = 0, 1, \ldots, N$, there exists an arc $A_i$ in $G$ between $f^i(p_s)$ and $f^i(p_t)$ such that $A_i \cap P = \{f^i(p_s), f^i(p_t)\}$. This contradicts the definition of $N$. Thus any two points $x, y \in \bigcup\{B_{i_0,i_1,\ldots,i_m} \cap B_{\ell_0,\ell_1,\ldots,\ell_m} | 1 \leq \ell_0, \ell_1, \ldots, \ell_m \leq n$ and $(\ell_0, \ell_1, \ldots, \ell_m) \neq (i_0, i_1, \ldots, i_m)\}$ are connected by the union of two or more 3-dimensional balls $B_{i_0,i_1,\ldots,i_m,j_{m+1},\ldots,j_{m+N}}$ $(1 \leq j_{m+1}, \ldots, j_{m+N} \leq n)$. Hence we may assume that $diam(B_{i_0,i_1,\ldots,i_m,j_{m+N},\ldots,j_{m+N}})$ $\leq \frac{1}{2}\ diam(B_{i_0,i_1,\ldots,i_m})$. Thus we can suppose that $lim_{n\to\infty} diam(B_{i_0,i_1,\ldots,i_n}) = 0$ (see Figure 2.1.5).

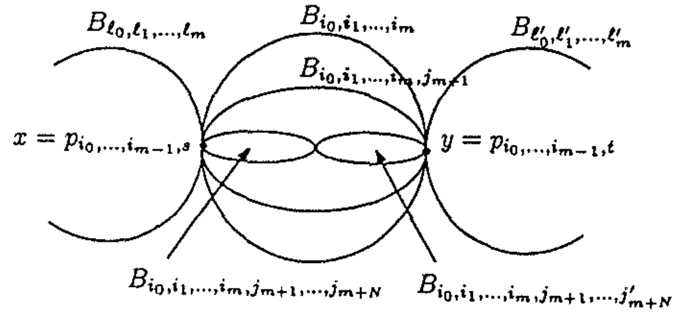



Figure 2.1.5

## 2.2 The construction of $Z$

Let $G$ be a graph, $f : G \longrightarrow G$ a continuous map, $P = \{p_1, p_2, \ldots, p_k\}$ a finite subset of $G$ such that $f(P) \subset P$ and $S(G, P) \setminus P = \{C_1, C_2, \ldots, C_n\}$. We may also assume that $f$ is point-wise $P$-expansive in this section from the argument in section 2. And let $X_{\rightharpoonup}$, $X_{\rightharpoondown}$ be the above spaces constructed by the Markov graphs $(M_{\rightharpoonup}, P_{\rightharpoonup})$, $(M_{\rightharpoondown}, P_{\rightharpoondown})$ on $S(G, P)$ respectively.

Theorem 2.2.1 *The subspace $X_{\perp}$ of $\mathbf{E}^3$ is a regular continuum.*

Proof. Let $\epsilon > 0$ and $x \in X_{\perp}$. As $lim_{m\to\infty} diam(B_{i_0,i_1,\ldots,i_m}) = 0$, for an $\epsilon$-neighbourhood $U_\epsilon(x)$ of $x$ in $X_{\perp}$ there exists a non-negative integer $\ell$ such that $B_{i_0,i_1,\ldots,i_\ell} \cap X_{\perp} \subset U_\epsilon(x)$ for any 3-dimensional ball $B_{i_0,i_1,\ldots,i_\ell}$ containing $x$. Let $B = \bigcup\{B_{i_0,i_1,\ldots,i_\ell} | x \in B_{i_0,i_1,\ldots,i_\ell} \cap X_{\perp} \subset U_\epsilon(x)\}$. Then $B \cap X_{\perp}$ is a neighbourhood of $x$ in $X_{\perp}$ such that $B \cap X_{\perp} \subset U_\epsilon(x)$. By the construction of $X_{\perp}$, the boundary of $B$ has finite cardinality. Thus $X_{\perp}$ is a regular continuum. $\square$

We define a map $\pi : G \longrightarrow X_{\perp}$ as follows : Given $x \in G$, if $f^\ell(x) \in cl(C_{i_\ell})$ for any $\ell = 0, 1, 2, \ldots$, then $\pi(x) = \bigcap_{\ell=0}^{\infty} B_{i_0,i_1,i_2,\ldots,i_\ell}$. We will investigate the uniqueness of $\pi(x)$ for each $x \in G$. Let $\{i_\ell\}_{\ell\geq 0}, \{j_\ell\}_{\ell \geq 0}$ be sequences of natural numbers such that $\{i_\ell\}_{\ell\geq 0} \neq \{j_\ell\}_{\ell\geq 0}$, $f^\ell(x) \in cl(C_{i_\ell}) \cap cl(C_{j_\ell})$ for each $\ell \geq 0$ and $1 \leq i_\ell, j_\ell \leq n$. We will show that $\bigcap_{\ell=0}^{\infty} B_{i_0,i_1,\ldots,i_\ell} = \bigcap_{\ell=0}^{\infty} B_{j_0,j_1,\ldots,j_\ell}$. Let $m = min\{\ell | C_{i_\ell} \neq C_{j_\ell}\}$, then $f^m(x) \in P$. We put $x' = p_{i_0,i_1,\ldots,i_{m-1},t}$, where $p_{i_0,i_1,\ldots,i_{m-1},t} \in B_{i_0,i_1,\ldots,i_{m-1},i_m} \cap B_{j_0,j_1,\ldots,j_{m-1},j_m}$ and $p_t \in P$. Since $f(p_t) \in cl(C_{i_{m+1}}) \cap cl(C_{j_{m+1}})$, $p_{i_0,i_1,\ldots,i_{m-1},t} \in B_{i_0,i_1,\ldots,i_{m-1},i_m,i_{m+1}} \cap B_{j_0,j_1,\ldots,j_{m-1},j_m,j_{m+1}}$ by the construction of $X_{m+1}$. Similarly, since $f^2(p_t) \in cl(C_{i_{m+2}}) \cap cl(C_{j_{m+2}})$, $p_{i_0,i_1,\ldots,i_{m-1},t} \in B_{i_0,i_1,\ldots,i_{m+2}} \cap B_{j_0,j_1,\ldots,j_{m+2}}$. Inductively, for each $\ell \geq 0$, $p_{i_0,i_1,\ldots,i_{m-1},t} \in B_{i_0,i_1,\ldots,i_{m+\ell}} \cap B_{j_0,j_1,\ldots,j_{m+\ell}}$. As $\bigcap_{\ell=0}^{\infty} B_{i_0,i_1,\ldots,i_{m+\ell}}$ and $\bigcap_{\ell=0}^{\infty} B_{j_0,j_1,\ldots,j_{m+\ell}}$ are degenerate, $\{p_{i_0,i_1,\ldots,i_{m-1},t}\} = \bigcap_{\ell=0}^{\infty} B_{i_0,i_1,\ldots,i_{m+\ell}} = \bigcap_{\ell=0}^{\infty} B_{j_0,j_1,\ldots,j_{m+\ell}}$. Thus we can define $\{\pi(x)\} = \bigcap_{\ell=0}^{\infty} B_{i_0,i_1,\ldots,i_{m+\ell}} = \bigcap_{\ell=0}^{\infty} B_{j_0,j_1,\ldots,j_{m+\ell}} = x'$ (see Figure 2.2.1).

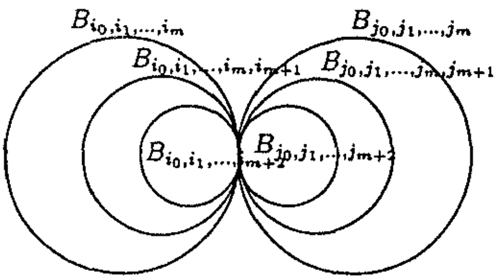


Figure 2.2.1

**Lemma 2.2.2** $\pi : G \longrightarrow X_{\rightharpoondown}$ *is continuous.*

**Proof.** Let $x \in G$ and $V$ be a neighbourhood of $\pi(x)$ in $X_{\rightharpoondown}$.

Case 1. Assume that $f^{\ell}(x) \notin P$ for any $\ell = 0, 1, 2, \ldots$ and $\{\pi(x)\} = \bigcap_{\ell=0}^{\infty} B_{i_0,i_1,\ldots,i_\ell}$. There exists a non-negative integer $\ell$ such that $B_{i_0,i_1,\ldots,i_\ell} \cap X_{\rightharpoondown} \subset V$, where $B_{i_0,i_1,\ldots,i_\ell}$ is a 3-dimensional ball containing $\pi(x)$. Since $C_{i_0,i_1,\ldots,i_\ell} = \{x \in G | x \in C_{i_0}, f(x) \in C_{i_1}, \ldots, f^{\ell}(x) \in C_{i_\ell}\}$ is an open set containing $x$ and $\pi(C_{i_0,i_1,\ldots,i_\ell}) \subset B_{i_0,i_1,\ldots,i_\ell} \cap X_{\rightharpoondown}$, $\pi$ is continuous at $x$.

Case 2. Assume that there exists $m = min\{\ell | f^{\ell}(x) \in P\} < \infty$. There exists $\ell > m$ such that $B_{i_0,i_1,\ldots,i_\ell} \cap X_{\rightharpoondown} \subset V$ for each $B_{i_0,i_1,\ldots,i_\ell}$ containing $\pi(x)$. Let $\mathbf{C}_\ell = \{C_{i_0,i_1,\ldots,i_\ell} | 1 \leq i_0, i_1, \ldots, i_\ell \leq Card(S(G,P))\}$ and $U = \bigcup\{C \in \mathbf{C}_\ell | x \in cl(C)\}$. Since $f$ is continuous, $U$ is a neighbourhood of $x$ such that $\pi(U) \subset V$. Thus $\pi$ is continuous. $\square$

Now we will put $Z = \pi(G)$. Then $X_{\rightarrow} \subset Z \subset X_{\rightharpoondown}$. In general it is difficult to recognize the precise structure of $Z$, but by the above relation $X_{\rightarrow} \subset Z \subset X_{\rightharpoondown}$, we can realize the approximate structure of $Z$. Since $X_{\rightharpoondown}$ is regular, $Z$ is also regular.

Note that by the construction, if for any element $C \in S(G,P) \setminus P$, there exist finitely many elements $C_1, C_2, \ldots, C_m$ of $S(G,P)$ such that $f(C) = \bigcup_{i=1}^{m} C_i$, then $X_{\rightharpoonup} = Z = X_{\rightharpoondown}$.

Define a map $g : X_{\rightharpoondown} \longrightarrow X_{\rightharpoondown}$ as follows : If $\{x\} = \bigcap_{\ell=0}^{\infty} B_{i_0,i_1,\ldots,i_\ell}$, then $\{g(x)\} = g(\bigcap_{\ell=0}^{\infty} B_{i_0,i_1,\ldots,i_\ell}) = \bigcap_{\ell=1}^{\infty} B_{i_1,i_2,\ldots,i_\ell}$. We can investigate the uniqueness of $g$ as we did that of $\pi$. Note that $g(Z) \subset Z$.

**Lemma 2.2.3** *$g : X_{\rightharpoondown} \longrightarrow X_{\rightharpoondown}$ is continuous.*

**Proof.** Let $x \in X_{\rightharpoondown}$ and $V$ be a neighbourhood of $g(x)$ in $X_{\rightharpoondown}$. Then there exists a non-negative integer $\ell$ such that $B_{i_0,i_1,\ldots,i_\ell} \cap X_{\rightharpoondown} \subset V$ for any 3-dimensional ball $B_{i_0,i_1,\ldots,i_\ell}$ containing $g(x)$. Let $B = \bigcup\{B_{j_0,j_1,\ldots,j_{\ell+1}} \mid x \in B_{j_0,j_1,\ldots,j_{\ell+1}}\}$. Then $B$ is a neighbourhood of $x$ and $g(B \cap X_{\rightharpoondown}) \subset V$. Thus $g$ is continuous. □

The following is the main theorem in this paper.

**Theorem 2.2.4** *Let $G$ be a graph, $f : G \longrightarrow G$ a continuous map and $P$ a finite subset of $G$ such that $f(P) \subset P$. Then there exist a regular continuum $Z$, a continuous map $g : Z \longrightarrow Z$ and a semi-conjugacy $\pi : G \longrightarrow Z$ such that*

(1) *$g$ is $\pi(P)$-expansive, and*

(2) *if $p, q \in P$ and $Q$ is a subset of $P$ with $A \cap Q \neq \emptyset$ for any arc $A$ in $G$ between $p$ and $q$, then $A' \cap \pi(Q) \neq \emptyset$ for any arc $A'$ in $Z$ between $\pi(p)$ and $\pi(q)$.*

*In addition, $f$ is point-wise $P$-expansive if and only if $\pi|_P$ is one-to-one.*

**Proof.** Let $\pi$ and $g$ be the above maps. Let $x \in G$ with $f^\ell(x) \in cl(C_{i_\ell})$ for $\ell = 0, 1, 2, \ldots$. Then $\{\pi(x)\} = \bigcap_{\ell=0}^{\infty} B_{i_0,i_1,\ldots,i_\ell}$ and $\{g \circ \pi(x)\} = \bigcap_{\ell=1}^{\infty} B_{i_1,i_2,\ldots,i_\ell} = \{\pi \circ f(x)\}$. Thus $\pi$ is a semi-conjugacy between $(G, f)$ and $(Z, g)$.

We will show that (1) $g$ is $\pi(P)$-expansive. Let $x, y$ be distinct points of $Z$. There exists a 3-dimensional ball $B_{i_0,i_1,\ldots,i_\ell}$ such that $x, y \in B_{i_0,i_1,\ldots,i_{\ell-1}}$, $x \in B_{i_0,i_1,\ldots,i_\ell}$ and $y \notin B_{i_0,i_1,\ldots,i_\ell}$. Then $g^\ell(x) \in B_{i_\ell}$ and $g^\ell(x) \notin B_{i_\ell}$. Thus $I_{\pi(P),g}(x) \neq I_{\pi(P),g}(y)$.

By the construction of $X_{\_}$, we can easily check (2). □

**Proposition 2.2.5** *Let $G$ be a graph, $f : G \longrightarrow G$ a continuous map and $P$ the set of vertices of $G$ with $f(P) \subset P$. If $f$ is point-wise $P$-expansive and $f|_{[p,q]}$ is one-to-one for each edge $[p,q]$ between $p$ and $q$, then $Z$ is homeomorphic to $G$.*

Proof. Let $p, q \in P$ and $[p,q]$ be the edge between $p$ and $q$. Since $f|_{[p,q]}$ is one-to-one, $f([p,q])$ is an arc between $f(p)$ and $f(q)$. Let $\{C_{m_1}, C_{m_2}, \ldots, C_{m_\ell}\}$ be the set of elements of $S(G,P) \setminus P$ which is contained in $f([p,q])$. As $f$ is point-wise $P$-expansive, by the construction of $X$ the 3-dimensional balls $B_{m_1}, B_{m_2}, \ldots, B_{m_\ell}$ corresponding to $C_{m_1}, C_{m_2}, \ldots, C_{m_\ell}$ form a chain between $\pi(p)$ and $\pi(q)$, i.e., $B_{m_i} \cap B_{m_j} \neq \emptyset$ if and only if $|i-j| \leq 1$. Similarly, by the construction of $X_1$, finitely many smaller balls form a chain in each ball $B_{m_i}$ $(i = 1, 2, \ldots, \ell)$, too. When we repeat this operation, $\pi([p,q])$ is an arc between $\pi(p)$ and $\pi(q)$. Thus $Z$ is homeomorphic to $G$. □

## 2.3 Appendix

Let $K$ be a continuum and $P$ a finite subset of $K$. Then we say that $P$ *graph-separates* $K$ if and only if there exists a finite set $S(K,P)$ of subsets of $K$ such that

(1) the element of $S(K,P)$ partition $K$, i.e., every point of $K$ is in exactly one member of $S(K,P)$,

(2) for each $p \in P$, $\{p\} \in S(K,P)$,

(3) for each $A \in S(K,P)$, the closure of $A$ in $K$ is arc-wise connected, and

(4) if $A, B \in S(K,P)$, then the closure of $A$ and $B$ either have empty intersection or intersect in only elements of $P$.

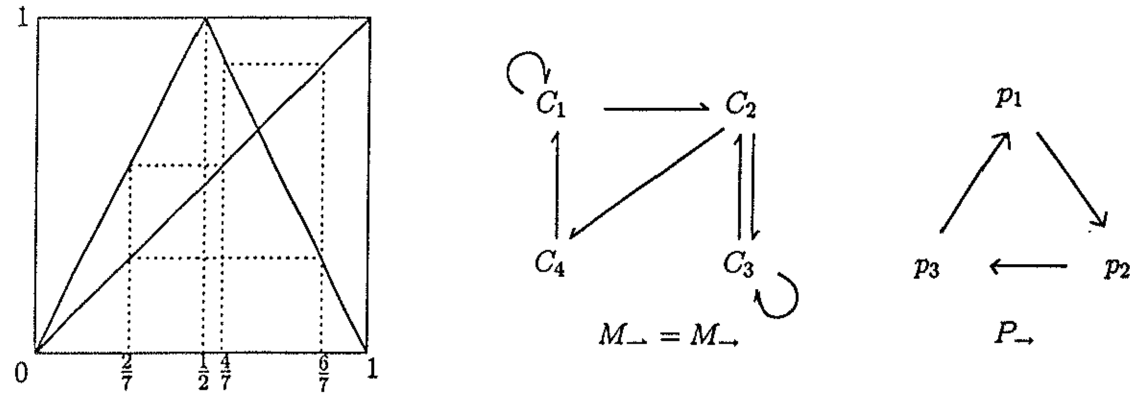
Note that we can also define $P$-expansive for a graph-separated continuum in a similar way.

**Remark.** We can obtain the same result in Theorem 2.2.4 by using a graph-separated continuum instead of a graph. □

## 2.4 Examples

In this section, a few concrete examples will be given to clarify the explanation given so far.

**Example 2.4.1** Let $G$ be the unit interval $[0,1]$. And denote $P = \{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\}$. We define a continuous map $f$ of $G$ into itself such that $f(x) = 2x$ (if $0 \leq x \leq \frac{1}{2}$) and $f(x) = -2x + 2$ (if $\frac{1}{2} \leq x \leq 1$). This map $f$ is point-wise $P$-expansive. Then $S(G,P) = \{[0,\frac{2}{7}), \{\frac{2}{7}\}, (\frac{2}{7},\frac{4}{7}), \{\frac{4}{7}\}, (\frac{4}{7},\frac{6}{7}), \{\frac{6}{7}\}, (\frac{6}{7},1]\}$, where put $C_1 = [0,\frac{2}{7}), C_2 = (\frac{2}{7},\frac{4}{7}), C_3 = (\frac{4}{7},\frac{6}{7}), C_4 = (\frac{6}{7},1], p_1 = \frac{2}{7}, p_2 = \frac{4}{7}$ and $p_3 = \frac{6}{7}$. The Markov graph of $S(G,P) \setminus P$ and $P$ is as in Graph 2.4.1 :




Graph 2.4.1

The above Markov graphs $(M_{\rightharpoondown}, P_{\rightharpoondown})$ will give information useful in constructing the space $Z$. Let $X = B_1 \cup B_2 \cup B_3 \cup B_4$ and $h_i : X \hookrightarrow B_i$ $(i = 1, 2, 3, 4)$ be an embedding such that (1) $h_i(X) \cap Bd(B_i) \subset P$, and (2) for each $p, q \in P$ with $p \in Bd(B_i)$ and $p \longrightarrow q$, $h_i(q) = p \in Bd(B_i)$. Then we describe the union $Y_i$ of finitely many balls in each ball $B_i$. For example when $i = 2$, $Y_2 = B_{2,3} \cup B_{2,4} \subset h_2(X)$, $p_1 = h_2(p_2)$ and $p_2 = h_2(p_3)$, since $C_2 \longrightarrow C_3$, $C_2 \longrightarrow C_4$, $p_1 \longrightarrow p_2$ and $p_2 \longrightarrow p_3$. In this way, we obtain a subspace $X_1 = Y_1 \cup Y_2 \cup Y_3 \cup Y_4$ of $\mathrm{E}^3$ (see Figure 2.4.2).

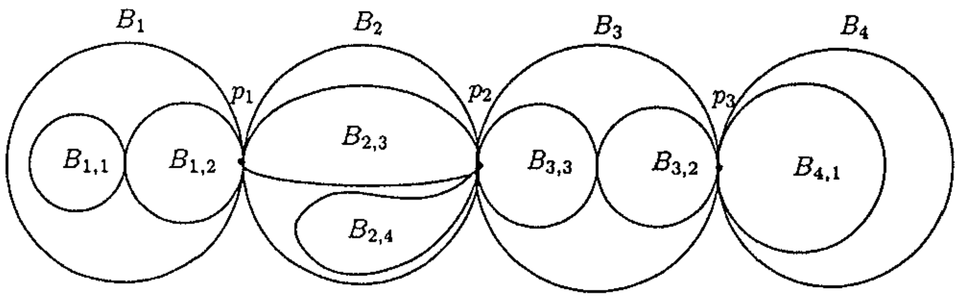



Figure 2.4.2

Next we describe finitely many 3-dimensional balls in $X_1$. Let $h_{i,j} : X \hookrightarrow B_{i,j}$ be an embedding such that (1) $h_{i,j}(X) \cap Bd(B_{i,j}) \subset h_i(P)$, and (2) for each $p, q \in P$ with $h_i(p) \in Bd(B_{i,j})$ and $f^2(p) = q$, $h_{i,j}(q) = h_i(p)$. Then we describe the union $Y_{i,j}$ of finitely many balls in $B_{i,j}$. For example, when $i = 2$ and $j = 3$, $h_{2,3}(X) = B_{2,3,2} \cup B_{2,3,3} = Y_{2,3}$, $h_2(p_2) = h_{2,3}(p_1)$ and $h_2(p_3) = h_{2,3}(p_2)$, since $C_3 \longrightarrow C_2$, $C_3 \longrightarrow C_3$, $f^2(p_2) = p_1$ and $f^2(p_3) = p_2$. Put $X_2 = \bigcup_{i,j=1}^{4} Y_{i,j}$ (see Figure 2.4.3).

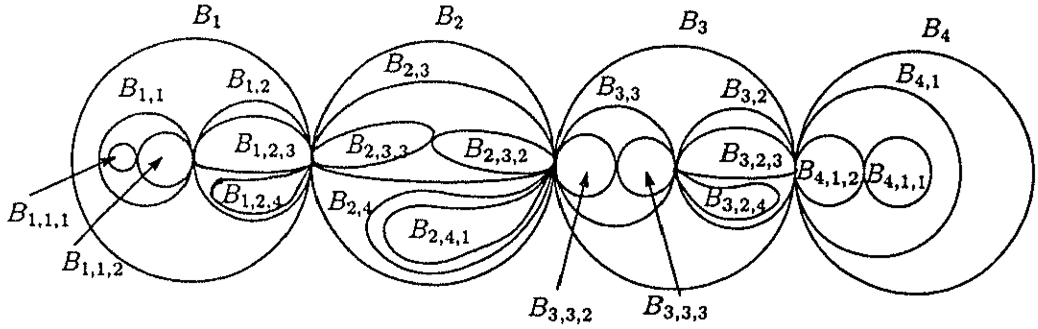



Figure 2.4.3

Similarly, we can describe $X_i (i = 3, 4, \ldots)$. Finally the space $Z = \bigcap_{i=1}^{\infty} X_i$ is the universal dendrite (see Figure 2.4.4).

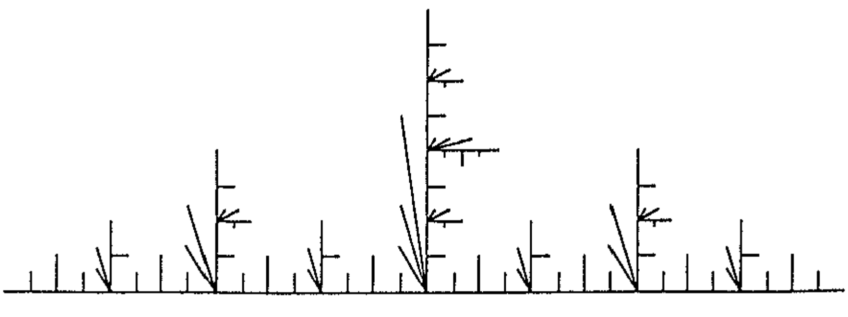


Figure 2.4.4

**Example 2.4.2** Let $G = [0, 1]$ be the unit interval, $P = \{0, \frac{1}{2}, 1\}$ and $f$ the same continuous map of $G$ as in Example 2.4.1. Then $X_{\rightarrow} = Z = X_{\leftarrow}$ and $Z$ is homeomorphic to $G = [0, 1]$. This implies that the structure of $Z$ depends on the way of selecting the points of $P$ (see Proposition 2.2.5).

Example 2.4.3 Let $G = [0,1]$ be the unit interval and $P = \{\frac{1}{2}, \frac{3}{4}, 1\}$. We will define a continuous map $f$ of $G$ as follows : $f(x) = 4x(0 \leq x \leq \frac{1}{4})$, $f(x) = -2x + \frac{3}{2}(\frac{1}{4} \leq x \leq \frac{1}{2})$, $f(x) = 2x - \frac{1}{2}(\frac{1}{2} \leq x \leq \frac{3}{4})$ and $f(x) = -2x + \frac{5}{2}(\frac{3}{4} \leq x \leq 1)$. Then $S(G,P) = \{[0,\frac{1}{2}), \{\frac{1}{2}\}, (\frac{1}{2},\frac{3}{4}), \{\frac{3}{4}\}, (\frac{3}{4},1), \{1\}\}$, where put $C_1 = [0,\frac{1}{2})$, $C_2 = (\frac{1}{2},\frac{3}{4})$, $C_3 = (\frac{3}{4},1)$, $p_1 = \frac{1}{2}$, $p_2 = \frac{3}{4}$ and $p_3 = 1$. The Markov graph of $S(G,P) \setminus P$ and $P$ is as in Figure 2.4.5.

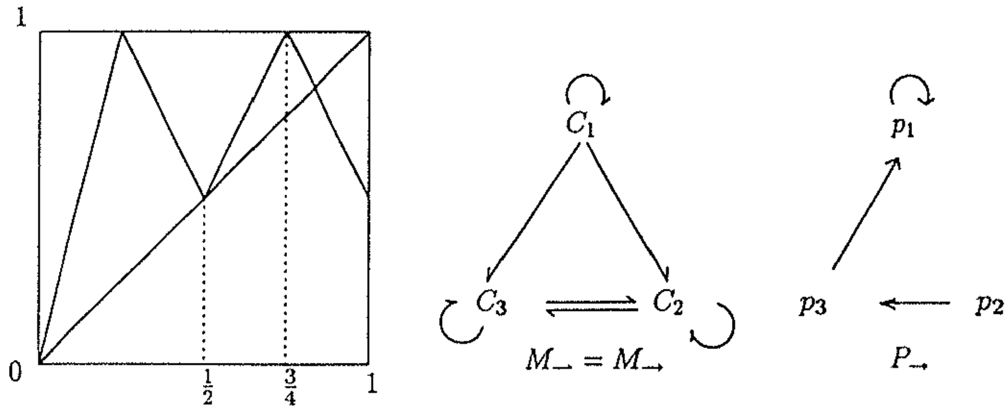



Figure 2.4.5

From the above Markov graph of $S(G,P) \setminus P$, we know that $B_{2,1} = \emptyset$ and $B_{3,1} = \emptyset$. Furthermore, the Markov graph of $P$ suggests the way of connection of each ball $B_{i,j}$, where $i, j = 1, 2, 3$ (see Figure 2.4.6).

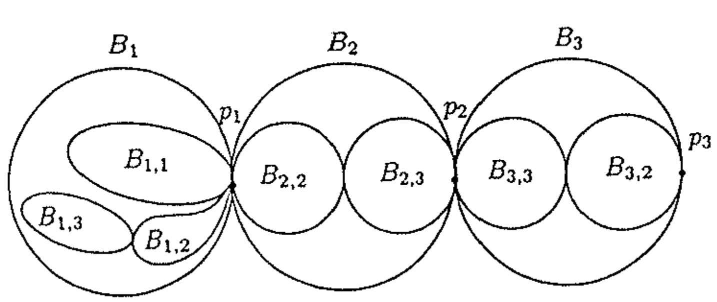



Figure 2.4.6

Finally, $Z$ is the following dendrite (see Figure 2.4.7).

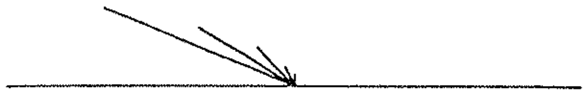


Figure 2.4.7

**Example 2.4.4** Let $G$ be the following graph, $P = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ a finite subset of $G$ and $f : G \longrightarrow G$ a continuous map. And assume that $f(cl(C)) = G$ for any $C \subset S(G, P) \setminus P$, $f(p_1) = p_1 = f(p_4)$, $f(p_3) = p_2 = f(p_6)$ and $f(p_2) = p_3 = f(p_5)$. Note that $f$ is point-wise $P$-expansive (see Figure 2.4.8).

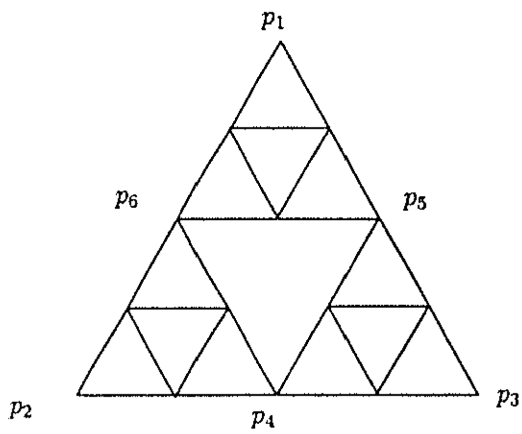



Figure 2.4.8

Then $Z$ is the triangular Sierpinski curve (see Figure 2.4.9).

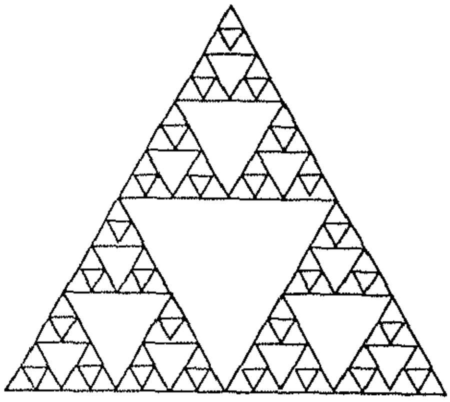
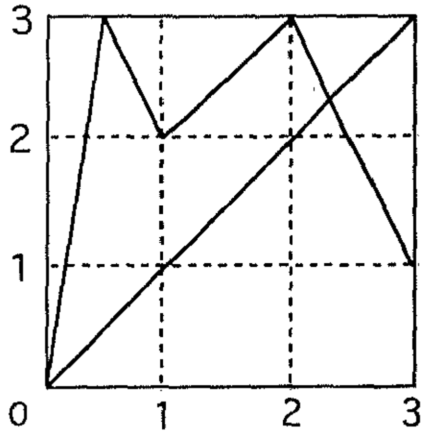


Figure 2.4.9

**Example 2.4.5** Let $f$ be a continuous map of $[0, 3]$ into itself and $P$ a periodic orbit $\{1, 2, 3\}$ of $f$ such that $f(x) = 6x (0 \le x \le \frac{1}{2}), f(x) = -2x + 4(\frac{1}{2} \le x \le 1), f(x) = x + 1(1 \le x \le 2)$ and $f(x) = -2x + 7(2 \le x \le 3)$. Denote $C_0 = [0, 1), C_1 = (1, 2)$ and $C_2 = (2, 3]$ (see Graph 2.4.10).




Graph 2.4.10

The process of the construction of $Z$ is as in Figure 2.4.11.

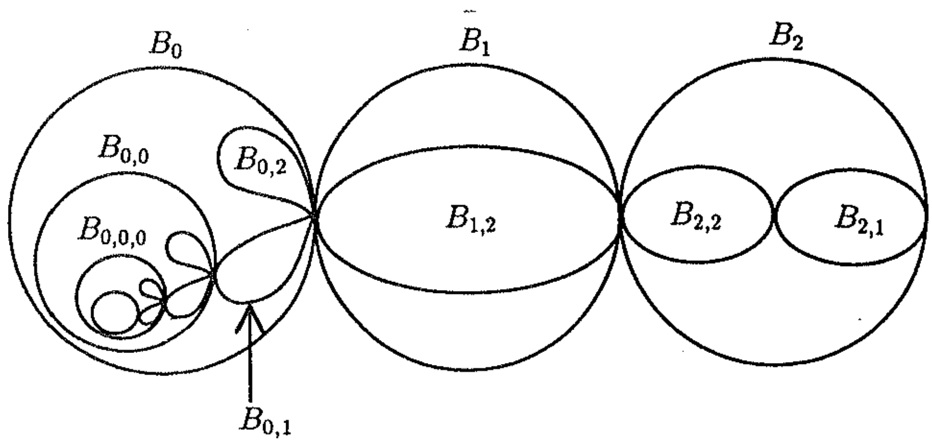



Figure 2.4.11

Then we see that the structure of $Z$ is as in Figure 2.4.12.

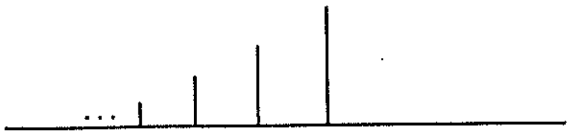
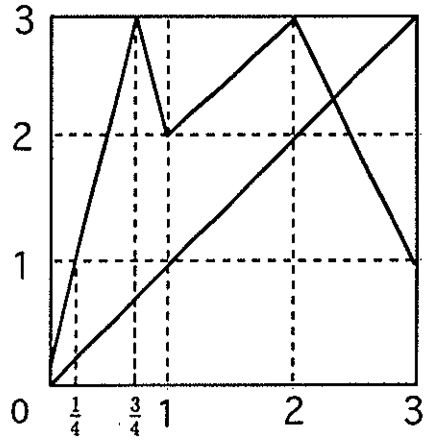



Figure 2.4.12

**Example 2.4.6** Let $f$ be a continuous map of $[0,3]$ into itself and $P$ a periodic orbit $\{1,2,3\}$ of $f$ such that $f(x) = \frac{11}{3}x + \frac{1}{4}(0 \leq x \leq \frac{3}{4}), f(x) = -4x + 6(\frac{3}{4} \leq x \leq 1), f(x) = x + 1(1 \leq x \leq 2)$ and $f(x) = -2x + 7(2 \leq x \leq 3)$. Denote $C_0 = [0,1), C_1 = (1,2)$ and $C_2 = (2,3]$ (see Graph 2.4.13).




Graph 2.4.13

The process of the construction of $Z$ is as in Figure 2.4.14.

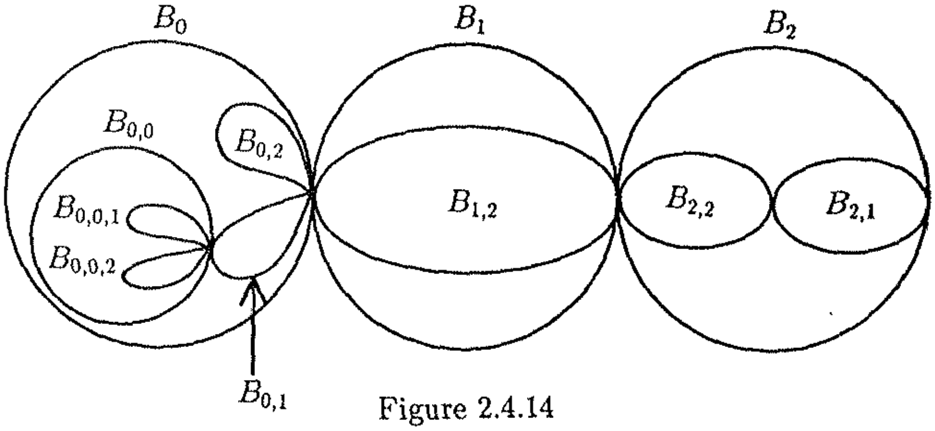



Figure 2.4.14

Then we see that the structure of $Z$ is as in Figure 2.4.15.

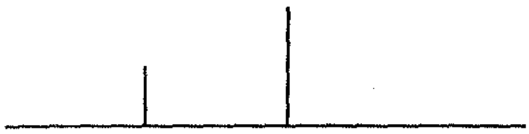


Figure 2.4.15