Chapter 1

Preliminaries

1.1 General definitions and notations

In this section, we introduce some necessary definitions from the theories of continua and dynamical systems.

A *continuum* is a nonempty connected compact metric space. A *subcontinuum* is a continuum which is a subset of a space. A *graph* is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their endpoints. A *tree* is a graph which contains no simple clesed curve. A *dendrite* is a locally connected, uniquely arcwise connected continuum. A continuum X is said to be *regular at* $x \in X$ if for any neighbourhood U of x, there exists a neighbourhood V of x such that $V \subset U$ and the boundary of V has finite cardinality. And X is said to be *regular* provided that X is regular at each of its points. We say that a subcontinuum A of a continuum X is of order less than or equal to β in X, written $\text{Ord}(A, X) \leq \beta$, provided that for each open subset U of X with $A \subset U$ there exists an open subset V of X such that $A \subset V \subset U$ and $\text{Card}(\text{Bd}(V)) \leq \beta$, where $\text{Bd}(V)$ means the boundary of V and $\text{Card}(\text{Bd}(V))$ denotes the cardinarity of $\text{Bd}(V)$. We say that A is of order β in X, written

$\mathrm{Ord}(A, X) = \beta$, provided that $\mathrm{Ord}(A, X) \leq \beta$ and $\mathrm{Ord}(A, X) \not\leq \alpha$ for any cardinal number $\alpha < \beta$. If A is a one-point set $\{p\}$, then we frequently write $\mathrm{Ord}(p, X)$ instead of $\mathrm{Ord}(\{p\}, X)$ and say p is of order ... instead of saying $\{p\}$ is of order .... A point b of X is called a *branch point* of X provided that $\mathrm{Ord}(b, X) \geq 3$. We say that b is a k-branch point in X ($k \geq 3$) provided that $\mathrm{Ord}(b, X) = k$. Let $\mathrm{Br}(X)$ be the set of branch points of X. A dendrite X is called the *universal dendrite* provided that $\mathrm{Br}(X)$ is dense in X and $\mathrm{Ord}(b, X) = \infty$ for any element b of $\mathrm{Br}(X)$.

Let (X, d) be a compact metric space and f a continuous map of X. We define $f^0 = id$ and inductively $f^n = f \circ f^{n-1}$ for a natural number n. The *orbit* $O(x, f)$ of an $x \in X$ is the set $\{f^n(x) \mid n = 0, 1, 2, \cdots\}$. A point $x \in X$ is a *periodic point of f with period n* if $f^n(x) = x$ and $f^i(x) \neq x$ for $1 \leq i \leq n-1$.

Let X_i be a compact metric space and f_i a continuous map of X_i for $i = 1, 2$. We say that (X_1, f_1) is *semi-conjugate* (or *conjugate*, respectively) to (X_2, f_2) if there is a continuous map (or homeomorphism) φ from X_1 onto X_2 such that $\varphi \circ f_1 = f_2 \circ \varphi$. We call φ a *semi-conjugacy* (or *conjugacy*).

1.2 Topological entropy

In this section, we give two definitions of topological entropy, which is the complexity of the map.

First we give the definition using open covers, which was introduced by Adler, Konheim and McAndrew.

Let X be a compact metric space and f a continuous map of X into itself. And let Θ be the collection of all open cover of X. The *topological entropy* of f is given as follows : $h(f) = \sup\{\lim_{n\to\infty} \frac{1}{n} \log N(\vee_{i=0}^{n-1} f^{-i}(\theta)) \mid \theta \in \Theta\}$, where

$\vee_{i=0}^{n-1} f^{-i}(\theta) = \{A_0 \cap f^{-i}(A_1) \cap \cdots \cap f^{-(n-1)}(A_{n-1}) \mid A_0, A_1, \ldots, A_{n-1} \in \theta\}$ and $N(\vee_{i=0}^{n-1} f^{-i}(\theta))$ means the number of sets in a finite subcover of θ with smallest cardinarity.

Next we give the definition of topological entropy using separating and spanning sets, which was introduced by Dinaburg and Bowen.

Similarly let X be a compact metric space with metric d and $f : X \longrightarrow X$ a continuous map. A subset E of X is *(n, ϵ)-separated* if for any $x, y \in E$ with $x \neq y$, there exists an integer j such that $0 \leq j < n$ and $d(f^j(x), f^j(y)) > \epsilon$. A subset F of X is said to *(n, ϵ)-span* another set K if for each $x \in K$ there exists $y \in F$ such that $d(f^j(x), f^j(y)) \leq \epsilon$ for all $0 \leq j < n$. If K is a compact subset of X, we denote by $r_n(\epsilon, K)$ the smallest cardinarity of any subset F of K that (n, ϵ)-spans K and by $s_n(\epsilon, K)$ the largest cardinarity of any subset E of K which is (n, ϵ)-separated. The *topological entropy* of f is $h_d(f) = \sup\{\lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r_n(\epsilon, K) \mid K$ is a compact subset of $X\}$. It is known that $r_n(\epsilon, K) \leq s_n(\epsilon, K) \leq r_n(\frac{1}{2}\epsilon, K) < \infty$ and if $\epsilon_1 < \epsilon_2$, then $r_n(\epsilon_1, K) \geq r_n(\epsilon_2, K)$ and $s_n(\epsilon_1, K) \geq s_n(\epsilon_2, K)$. From this, it follows that the limit, as $\epsilon \longrightarrow 0$, of both $r_n(\epsilon, K)$ and $s_n(\epsilon, K)$ exist and are equal (but this number may be equal to ∞), i.e. $h_d(f) = \sup\{\lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\epsilon, K) \mid K$ is a compact subset of $X\}$. It is known that the two definitions of topological entropy of a continuous map f coincide, that is $h(f) = h_d(f)$ (see [27]).