

# Introduction

Recently, there has been growing interest in studying the dynamical behaviors of continuous maps of one-dimensional spaces. Especially, one of the central questions in the theory of dynamical systems is how to recognize “chaos”. The theme of this paper is to describe visually the chaoticity of continuous maps of a one-dimensional space into itself. The results in this paper have been obtained in [5] and [7].

In Chapter 1, we give definitions and notations which will be needed in the sequel.

We study the dynamics of continuous maps of a graph into itself in Chapter 2. In [9] and [2, Theorem 4.1], the following result has been shown.

**Theorem.** *Let  $D$  be a dendrite,  $f : D \rightarrow D$  a continuous map and  $P$  a finite subset of  $D$  such that  $f(P) \subset P$ . Then there exist a dendrite  $E$ , a map  $g : E \rightarrow E$  and a semi-conjugacy  $\pi : D \rightarrow E$  (i.e.,  $\pi \circ f = g \circ \pi$ ) such that*

(1)  $g$  is  $\pi(P)$ -expansive, and

(2) if  $x, y, z \in P$  and  $y \in [x, z]$  then  $\pi(y) \in [\pi(x), \pi(z)]$ .

*If, in addition, the Markov graph of  $P$  has no basic intervals of order 0 and no loops of order 1, then  $\pi|_P$  is one-to-one.*

In Chapter 2, we expand a dendrite in the above theorem to a graph, that is,

**Theorem.** *Let  $G$  be a graph,  $f : G \rightarrow G$  a continuous map and  $P$  a finite subset of  $G$  such that  $f(P) \subset P$ . Then there exist a regular continuum  $Z$ , a continuous map  $g : Z \rightarrow Z$  and a semi-conjugacy  $\pi : G \rightarrow Z$  such that*

- (1)  $g$  is  $\pi(P)$ -expansive, and
- (2) if  $p, q \in P$  and  $Q$  is a subset of  $P$  with  $A \cap Q \neq \emptyset$  for any arc  $A$  in  $G$  between  $p$  and  $q$ , then  $A' \cap \pi(Q) \neq \emptyset$  for any arc  $A'$  in  $Z$  between  $\pi(p)$  and  $\pi(q)$ .

*In addition,  $f$  is point-wise  $P$ -expansive if and only if  $\pi|_P$  is one-to-one.*

We show the above by using a more geometrical method than that of Baldwin [9]. We construct the above regular continuum  $Z$  as a subspace of the Euclidean 3-dimensional space and a continuous map  $g$  of  $Z$  by repeating operation of embedding some smaller 3-dimensional balls in numbered 3-dimensional balls corresponding to the components of  $G \setminus P$ . We can visually see the process of the construction of  $Z$ . The new space  $Z$  has a fractal and complicated structure, which implies the chaoticity of  $f$ . The notion of  $P$ -expansiveness plays a very important role in order to investigate complication of the dynamical behavior of  $f$ . Note that we can obtain the same result to this theorem on a continuous map of a graph-separated continuum.

Now our interest is the relationship between the chaoticity of  $f$  and the frac-

tal structure of  $Z$ . In Chapter 3, we investigate the structure of  $Z$  constructed under the restriction that  $G$  is an arc (denote  $G = I$ ). Let  $f : I \longrightarrow I$  be a continuous map and  $P$  a periodic orbit of  $f$  with a period  $n$ . Then note that  $Z$  is a dendrite. We assume the following assumptions in Chapter 3 and 4.

1.  $f$  is pointwise  $P$ -expansive
2.  $f(C)$  is the union of some elements of  $S(I, P)$  for each element  $C$  of  $S(I, P)$ , where  $S(I, P) = P \cup \{C \mid C \text{ is a component of } I \setminus P\}$
3.  $f(C)$  is not one point for each element  $C$  of  $S(I, P) \setminus P$  with  $C \neq \emptyset$

Then we can obtain a necessary and sufficient condition that  $Z$  is the universal dendrite, which is a dendrite such that the set of branch points of  $Z$  is dense in  $Z$  and the order of each branch point of  $Z$  is infinite.

**Theorem.** *There exist two  $n$ -cycles for the orbit of some element of  $P$  if and only if  $Z$  is the universal dendrite.*

Since the fundamental cycle always exists and is unique, we can express the above in other way, that is,

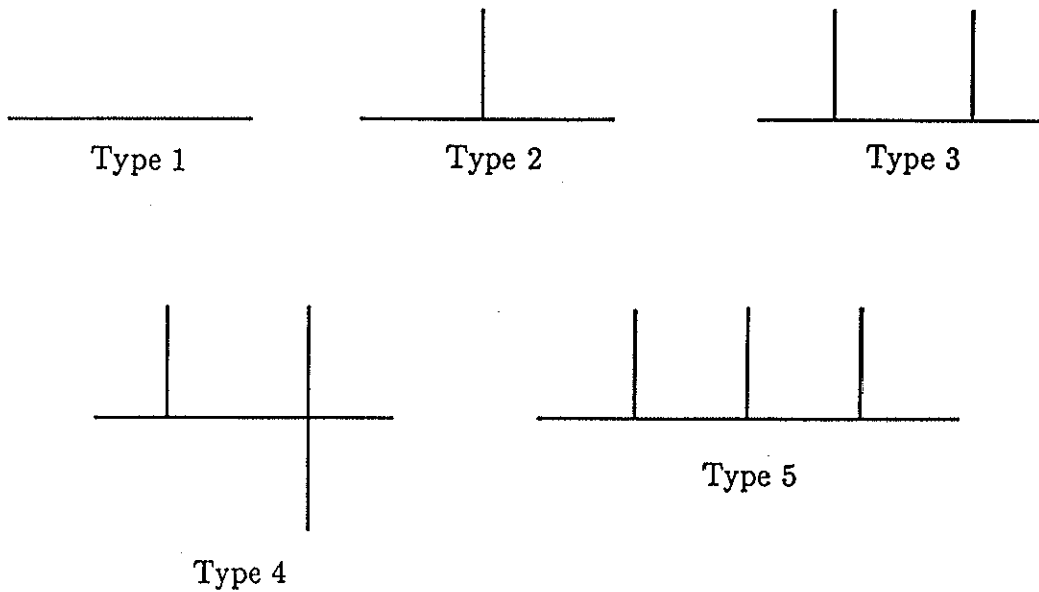
*There exists an  $n$ -cycle for the orbit of some element of  $P$  other than the fundamental cycle if and only if  $Z$  is the universal dendrite.*

Since the structure of  $Z$  is decided according to a continuous map  $f$  and a periodic orbit of  $f$ , it is natural to ask as follows :

*When  $Z$  is not the universal dendrite, how are structures of  $Z$  classified by a continuous map  $f$  and a periodic orbit of  $f$ .*

In Chapter 4, under the same assumption to Chapter 3, we classify types of  $Z$  in case  $Z$  is a tree, i.e.

**Theorem.** *If  $Z$  is a tree, then  $\text{Card}(\text{Br}(Z)) \leq 3$  and  $\text{Ord}(x, Z) \leq 4$  for each element  $x$  of  $Z$ . Furthermore there exist only distinct 5 types of  $Z$*



In fact, we can obtain a continuous map  $f$  and a periodic orbit of  $f$  for each type of  $Z$  in the above (see section 4.2).

In the future, we hope to classify structures of  $Z$  in case  $Z$  is not the universal dendrite and not a tree. At present, we can see as follows :

**Corollary.** *If  $Z$  is not the universal dendrite, then  $\text{Ord}(x, Z) \leq n+1$  for each*

element  $x$  of  $Z$ , where  $n = \text{Card}(P)$ .

This means that there exists no dendrite containing a branch point with infinite order other than the universal dendrite. Moreover we can also obtain the following.

**Corollary.** *Assume that  $Z$  is not the universal dendrite. Then if  $Z$  has an  $m$ -branch point ( $m \geq 3$ ), then  $Z$  has also an  $(m - 1)$ -branch point.*

This implies that  $Z$  is not an  $n$ -od ( $n \geq 4$ ).

The topological entropy, which was introduced by R.L.Adler, A.G.Konheim and M.H.McAndrew[1] in 1965, is one of the best methods of measuring chaoticity. Many authors have made a considerable number of studies on the topological entropy of maps of one-dimensional spaces. We are interested in the relationship between the structure of  $Z$  and the topological entropy of a continuous map  $f$  of the interval into itself. Our objects in future are as follows :

**Conjecture 1.** *Let  $h(f)$  be the topological entropy of a map  $f$  and  $g$  a continuous map of  $Z$ . Then does it hold that  $h(f) = h(g)$*

**Conjecture 2.** *Does it holds the follows :  $h(f) > 0$  if and only if  $Z$  is the universal dendrite?*