Chapter 3. Weakly Hyperbolic Equations of Third Order with Analytic Coefficients in Time

3.1 Introduction

F. Colombini and N. Orrú studied the third order hyperbolic equation

$$(P) \qquad u_{ttt} + a(t)u_{xxx} + b(t)u_{txx} + c(t)u_{ttx} + M = 0,$$

where M is an operator of order ≤ 2 and the characteristic roots $\tau_1(t)$, $\tau_2(t)$ and $\tau_3(t)(\tau_1(t) \leq \tau_2(t) \leq \tau_3(t))$ satisfy $\tau_2(t) - \tau_1(t) \sim t^{k_1}$ and $\tau_3(t) - \tau_2(t) \sim t^{k_2}$ with $k_1, k_2 \geq 1$. Then they showed that the Cauchy problem (P) is wellposed in G^s if $1 \leq s < \frac{3k}{2k-1}$, where $k = \min\{k_1, k_2\}$(see [CO1], [CO2]).

From the results of F. Colombini and N. Orrú we find that the differences of the characteristic roots play an important role in the treatment of the weakly hyperbolic equations. As the relations between the characteristic roots and the coefficients are very complicated in general, it is difficult to give conditions in terms of the coefficients. In this chapter we shall consider the third order equation without the term u_{xxx}, since the relations between the characteristic roots and the coefficients become simple(see (3.5)). This kind of equation is also considered in Chapter 1 and [DS2] and [Yn]. Then we shall investigate the influences of the coefficients on the Gevrey wellposedness.

We shall consider the third order hyperbolic equation in $[0,T] \times \mathbf{R}_x$

$$(3.1) \qquad \begin{cases} u_{ttt} - t^{\alpha}u_{txx} + t^{\beta}u_{ttx} + t^{\eta}u_{xx} + t^{\lambda}u_{tx} + t^{\sigma}u_{tt} + t^{\mu}u_x + t^{\omega}u_t + t^{\theta}u = 0 \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad u_{tt}(0,x) = u_2(x), \end{cases}$$

where α, β, η, λ, σ, μ, ω and θ are integers satisfying

$$(3.2) \qquad \gamma \equiv \min\{\alpha,\ 2\beta\} \geq 2, \quad \eta \geq 0, \quad \lambda \geq 0, \quad \sigma \geq 0, \quad \mu \geq 0, \quad \omega \geq 0, \quad \theta \geq 0,$$

$$(3.3) \qquad \alpha - \eta - 1 > 0, \quad \gamma - 2\lambda - 2 > 0, \quad \alpha - 2\mu - 4 > \frac{\alpha - \gamma}{\gamma + 1},$$

$$(3.4) \qquad \left(\text{ resp. } \alpha - \eta - 1 \leq 0, \quad \gamma - 2\lambda - 2 \leq 0, \quad \alpha - 2\mu - 4 \leq \frac{\alpha - \gamma}{\gamma + 1}. \right)$$

Since (3.1) doesn't include the term $t^{\zeta}u_{xxx}$ in the principal part, the characteristic equation is $\tau^3 + t^{\beta}\tau^2 - t^{\alpha}\tau = 0$ and we can easily obtain the characteristic roots$(\tau_1(t) \leq \tau_2(t) \leq \tau_3(t))$

$$(3.5)\qquad \tau_1(t)=\frac{1}{2}\{-t^\beta-\sqrt{t^{2\beta}+4t^\alpha}\,\},\quad \tau_2(t)\equiv 0\quad \tau_3(t)=\frac{1}{2}\{-t^\beta+\sqrt{t^{2\beta}+4t^\alpha}\,\}.$$

Noting that $t^{2\beta}+4t^\alpha\geq 0$ for ${}^\forall t\in[0,T]$, we see that (3.1) is weakly hyperbolic.

Then we can prove the following theorem.

Theorem 3.1. *Let $T>0$. Assume that α, β, η, λ, μ, σ, ω and θ satisfy (3.2) and (3.3)(resp. (3.4)). Then the Cauchy problem (3.1) is wellposed in G^s (resp. C^∞), provided*

$$(3.6)\ 1\leq s<\min\Big\{\frac{3\alpha-2\eta}{2(\alpha-\eta-1)}+\frac{\alpha-\gamma}{2(\gamma+1)(\alpha-\eta-1)},\frac{2(\gamma-\lambda)}{\gamma-2\lambda-2},$$
$$\frac{3\alpha-2\mu}{\alpha-2\mu-4}+\frac{4(\alpha-\mu-1)(\alpha-\gamma)}{(\alpha-2\mu-4)(\gamma+1)\{(\alpha-2\mu-4)-\frac{\alpha-\gamma}{\gamma+1}\}}\Big\}.$$

Remark 1. The estimate (3.6) is independent of σ, ω and θ.

Remark 2. In case of $\alpha\leq 2\beta$, i.e., $\gamma=\alpha$, the third condition of (3.3)(resp. (3.4)) becomes $\alpha-2\mu-4>0$ (resp. $\alpha-2\mu-4\leq 0$) and (3.6) becomes simply

$$(3.7)\qquad 1\leq s<\min\Big\{\frac{3\alpha-2\eta}{2(\alpha-\eta-1)},\frac{2(\alpha-\lambda)}{\alpha-2\lambda-2},\frac{3\alpha-2\mu}{\alpha-2\mu-4}\Big\}.$$

Remark 3. In case of $\alpha=2\beta$ and $\eta=0$, (3.6) coincides with

$$(3.8)\qquad 1\leq s<\frac{3\beta}{2\beta-1}.$$

Since the equation (3.1) is (P) with $a(t)\equiv 0$, $b(t)=-t^{2\beta}$ and $c(t)=t^\beta$(the characteristic roots $\tau_1(t)$, $\tau_2(t)$ and $\tau_3(t)$ satisfy $\tau_2(t)-\tau_1(t)$, $\tau_3(t)-\tau_2(t)\sim t^\beta$), applying the result of F. Colombini and N. Orrú with $k(=k_1=k_2)=\beta$, we can also find that the Cauchy problem (3.1) is wellposed in G^s, provided (3.8).

Remark 4. If we replace the condition in (3.3) by the contrary condtion in (3.4), we can remove the corresponding component(in the minimum) in (3.6) and (3.7).

For the proof of the theorem, we shall approximate the weakly hyperbolic equation to the strictly one most suitably, considering that the coefficients of (3.1) are analytic and degenerate in only one point $t=0$. Therefore we use the energy of the third order hyperbolic equations, which is a little different from the energy used in Chapter 1.

3.2. Preliminaries

In this section we shall invistigate some functions which are entered into the energy defined in the next section.

We first define

$$\chi_\delta(t) = \begin{cases} \delta^\alpha - t^\alpha & \text{for } t \in [0,\delta] \\ 0 & \text{for } t \in [-\infty, 0] \cup [\delta, \infty], \end{cases}$$

$$\tilde{\chi}_\delta(t) = \begin{cases} \delta^\gamma - t^\gamma (= \delta^\alpha - t^\alpha) & \text{for } t \in [0,\delta] \quad \text{if } \alpha \leq 2\beta \\ \frac{1}{4}\delta^\gamma - \frac{1}{4}t^\gamma \Big(= \frac{1}{4}\delta^{2\beta} - \frac{1}{4}t^{2\beta}\Big) & \text{for } t \in [0,\delta] \quad \text{if } \alpha > 2\beta \\ 0 & \text{for } t \in [-\infty, 0] \cup [\delta, \infty]. \end{cases}$$

Moreover using $\chi_\delta(t)$ and $\tilde{\chi}_\delta(t)$, we also define

$$\omega_\delta(t) = \frac{1}{\delta^\alpha} \int_{-\infty}^{\infty} \chi_\delta(t+\tau)\varphi\Big(\frac{\tau}{\delta^\alpha}\Big) d\tau, \tag{3.9}$$

$$\pi_\delta(t) = \frac{1}{\delta^\gamma} \int_{-\infty}^{\infty} \tilde{\chi}_\delta(t+\tau)\varphi\Big(\frac{\tau}{\delta^\gamma}\Big) d\tau, \tag{3.10}$$

where

$$\varphi(t) = \begin{cases} 2t^3 - 3t^2 + 1 & \text{for } t \in [0,1] \\ -2t^3 - 3t^2 + 1 & \text{for } t \in [-1,0] \\ 0 & \text{for } t \in [-\infty, -1] \cup [1, \infty]. \end{cases} \tag{3.11}$$

We can easily see that $\varphi(t)$ is a even function and $\varphi(t) \in C_0^1(\mathbf{R}_t^1)$ and $\int_{-1}^{1} \varphi(\tau)d\tau = 1$. In particular we remark that for any $\delta > 0$

$$\frac{1}{\delta^\alpha} \int_{-\infty}^{\infty} \varphi\Big(\frac{\tau}{\delta^\alpha}\Big) d\tau = 1. \tag{3.12}$$

Furtheremore we also remark that $\omega_\delta(t), \pi_\delta(t)$ also belong $C_0^1(\mathbf{R}_t^1)$.

Then we shall prove the following lemma.

Lemma 3.2.A. *Assume that*

$$\gamma = \min\{\alpha, 2\beta\} \geq 2 \tag{3.13}$$

and

$$0 < \delta \leq \gamma^{\frac{1}{1-\gamma}} (< 1). \tag{3.14}$$

Then there exists $0 < C_0 < 1$ *such that*

$$
\tag{3.15} |\omega_\delta(t) - \chi_\delta(t)| \begin{cases} \leq C_0\delta^\alpha & \text{for } t \in [0, \delta + \delta^\alpha] \\ = 0 & \text{for } t \in [\delta + \delta^\alpha, T], \end{cases}
$$

$$
\tag{3.16} |\pi_\delta(t) - \tilde{\chi}_\delta(t)| \begin{cases} \leq C_0\delta^\gamma & \text{for } t \in [0, \delta + \delta^\gamma] \\ = 0 & \text{for } t \in [\delta + \delta^\gamma, T]. \end{cases}
$$

Proof. It is sufficient to prove only (3.15). We shall first treat the case $t \in [0, \delta]$. Noting that $0 \leq \frac{t}{\delta} \leq 1$, we obtain from (3.13) and (3.14)

$$
\tag{3.17} \begin{aligned} \int_{-\frac{t}{\delta^\alpha}}^{\frac{\delta - t}{\delta^\alpha}} \varphi(\tau)d\tau &= \int_{-\delta^{1-\alpha}(\frac{t}{\delta})}^{\delta^{1-\alpha}(1-\frac{t}{\delta})} \varphi(\tau)d\tau \geq \int_{-\alpha(\frac{t}{\delta})}^{\alpha(1-\frac{t}{\delta})} \varphi(\tau)d\tau \geq \int_{-2(\frac{t}{\delta})}^{2(1-\frac{t}{\delta})} \varphi(\tau)d\tau \\ &= \int_{-1}^{1} \varphi\Big(\tau + 1 - 2\frac{t}{\delta}\Big)d\tau \geq \int_{-1}^{1} \varphi(\tau+1)d\tau \quad \text{or} \int_{-1}^{1} \varphi(\tau - 1)d\tau \\ &= \frac{1}{2}\int_{-1}^{1} \varphi(\tau)d\tau\Big(= \frac{1}{2}\Big). \end{aligned}
$$

From (3.13) and (3.14) we also obtain for $t \in [0, \delta]$ and $\tau \in [-t, \delta - t]$

$$
\tag{3.18} \big|t^\alpha - (t+\tau)^\alpha\big| = |\tau| \sum_{m=0}^{\alpha-1} t^m (t+\tau)^{\alpha-1-m} \leq |\tau| \sum_{m=0}^{\alpha-1} \delta^m \delta^{\alpha-1-m} = \alpha|\tau|\delta^{\alpha-1} \leq |\tau|.
$$

Taking the support of $\chi_\delta(t)$ into consideration, we have by (3.12), (3.17), (3.18)

$$
\begin{aligned} |\omega_\delta(t) - \chi_\delta(t)| &= \Big|\frac{1}{\delta^\alpha}\int_{-\infty}^{\infty} \{\chi_\delta(t+\tau) - \chi_\delta(t)\}\varphi\Big(\frac{\tau}{\delta^\alpha}\Big)d\tau\Big| \\ &\leq \frac{1}{\delta^\alpha}\int_{-t}^{\delta - t} |\chi_\delta(t+\tau) - \chi_\delta(t)|\varphi\Big(\frac{\tau}{\delta^\alpha}\Big)d\tau \\ &\qquad + \frac{1}{\delta^\alpha}\int_{\mathbf{R}^1\setminus[-t,\delta-t]} |\chi_\delta(t+\tau) - \chi_\delta(t)|\varphi\Big(\frac{\tau}{\delta^\alpha}\Big)d\tau \\ &= \frac{1}{\delta^\alpha}\int_{-t}^{\delta-t} |t^\alpha - (t+\tau)^\alpha|\varphi\Big(\frac{\tau}{\delta^\alpha}\Big)d\tau \\ &\qquad + \frac{1}{\delta^\alpha}\int_{\mathbf{R}^1\setminus[-t,\delta-t]} (\delta^\alpha - t^\alpha)\varphi\Big(\frac{\tau}{\delta^\alpha}\Big)d\tau \\ &\leq \frac{1}{\delta^\alpha}\int_{-t}^{\delta-t} |\tau|\varphi\Big(\frac{\tau}{\delta^\alpha}\Big)d\tau + \int_{\mathbf{R}^1\setminus[-t,\delta-t]} \varphi\Big(\frac{\tau}{\delta^\alpha}\Big)d\tau \\ &= \delta^\alpha \int_{-\frac{t}{\delta^\alpha}}^{\frac{\delta-t}{\delta^\alpha}} |\tau|\varphi(\tau)d\tau + \delta^\alpha\int_{\mathbf{R}^1}\varphi(\tau)d\tau - \delta^\alpha\int_{-\frac{t}{\delta^\alpha}}^{\frac{\delta-t}{\delta^\alpha}} \varphi(\tau)d\tau \\ &\leq \delta^\alpha\int_{-1}^{1} |\tau|\varphi(\tau)d\tau + \delta^\alpha\int_{-1}^{1}\varphi(\tau)d\tau - \frac{\delta^\alpha}{2}\int_{-1}^{1}\varphi(\tau)d\tau \\ &= \Big(\frac{3}{10} + \frac{1}{2}\Big)\delta^\alpha = \frac{4}{5}\delta^\alpha. \end{aligned}
$$

We shall next treat the case $t \in [\delta, T]$. Noting that $\chi_\delta(t) = 0$ for $t \in [\delta, T]$, we have

$$|\omega_\delta(t) - \chi_\delta(t)| = |\omega_\delta(t)| = \frac{1}{\delta^\alpha}\int_{-t}^{\delta - t}\{\delta^\alpha - (t+\tau)^\alpha\}\varphi\Big(\frac{\tau}{\delta^\alpha}\Big)d\tau$$

$$\leq \int_{-t}^{\delta-t}\varphi\Big(\frac{\tau}{\delta^\alpha}\Big)d\tau = \delta^\alpha \int_{-\frac{t}{\delta^\alpha}}^{\frac{\delta-t}{\delta^\alpha}}\varphi(\tau)d\tau$$

$$\leq \begin{cases} \delta^\alpha \displaystyle\int_{-\infty}^{0}\varphi(\tau)d\tau \leq \frac{1}{2}\delta^\alpha\Big(\leq \frac{4}{5}\delta^\alpha\Big) & \text{for } t \in [\delta, \delta + \delta^\alpha] \\ \delta^\alpha \displaystyle\int_{-\infty}^{-1}\varphi(\tau)d\tau = 0 & \text{for } t \in [\delta + \delta^\alpha, T]. \end{cases}$$

Thus putting $C_0 = \frac{4}{5}$, we can get (3.15). Similarly we can also get (3.16) with $C_0 = \frac{1}{5}$.

Secondly we shall prove the following lemma.

Lemma 3.2.B. *Assume that (3.13) and (3.14). Then there exists $C_1 > 0$ such that*

$$t^\alpha + \omega_\delta(t) \geq C_1 \delta^{\alpha^2} \tag{3.19}$$

$$\frac{1}{4}t^{2\beta} + t^\alpha + \pi_\delta(t) \geq C_1 \delta^{\gamma^2}. \tag{3.20}$$

The estimates (3.19) and (3.20) are not be optimal. But we the power α^2 and γ^2 on the right hands of (3.19) and (3.20) do not play an important role in the proof of Theorem 3.1.

Proof. In order to show (3.19), we shall first treat the case $t \in [0, \frac{1}{2}\delta^\alpha]$. Since $(t \in)[0, \frac{1}{2}\delta^\alpha] \subset [0, \delta]$, we can use (3.18) again. By (3.12)-(3.14) and (3.18) we have

$$t^\alpha + \omega_\delta(t) = \frac{1}{\delta^\alpha}\int_{-\infty}^{\infty}\{t^\alpha + \chi_\delta(t+\tau)\}\varphi\Big(\frac{\tau}{\delta^\alpha}\Big)d\tau \geq \frac{1}{\delta^\alpha}\int_{-t}^{\delta-t}\{t^\alpha + \delta^\alpha - (t+\tau)^\alpha\}\varphi\Big(\frac{\tau}{\delta^\alpha}\Big)d\tau$$

$$\geq \frac{1}{\delta^\alpha}\int_{-t}^{\delta-t}\{\delta^\alpha - |t^\alpha - (t+\tau)^\alpha|\}\varphi\Big(\frac{\tau}{\delta^\alpha}\Big)d\tau$$

$$\geq \frac{1}{\delta^\alpha}\int_{-t}^{\delta-t}(\delta^\alpha - |\tau|)\varphi\Big(\frac{\tau}{\delta^\alpha}\Big)d\tau \geq \frac{1}{\delta^\alpha}\int_{-t}^{\delta-t}(\delta^\alpha - |-t|)\varphi\Big(\frac{\tau}{\delta^\alpha}\Big)d\tau$$

$$\geq \frac{1}{\delta^\alpha}\int_{-t}^{\delta-t}\frac{1}{2}\delta^\alpha\varphi\Big(\frac{\tau}{\delta^\alpha}\Big)d\tau = \frac{1}{2}\delta^\alpha\int_{-\frac{t}{\delta^\alpha}}^{\frac{\delta-t}{\delta^\alpha}}\varphi(\tau)d\tau \geq \frac{1}{2}\delta^\alpha\int_0^{\delta^{1-\alpha}-\frac{1}{2}}\varphi(\tau)d\tau$$

$$\geq \frac{1}{2}\delta^\alpha\int_0^{\alpha-\frac{1}{2}}\varphi(\tau)d\tau \geq \frac{1}{2}\delta^\alpha\int_0^{\frac{3}{2}}\varphi(\tau)d\tau = \frac{1}{4}\delta^\alpha\Big(\geq \frac{1}{2^\alpha}\delta^{\alpha^2}\Big).$$

We shall next treat the case $t \in [\frac{1}{2}\delta^\alpha, T]$. We easily obtain

$$t^\alpha + \omega_\delta(t) \geq \left(\frac{1}{2}\delta^\alpha\right)^\alpha + 0 = \frac{1}{2^\alpha}\delta^{\alpha^2}.$$

Thus putting $C_1 = \frac{1}{2^\alpha}$, we get (3.19). As to (3.20), we see that

$$\frac{1}{4}t^{2\beta} + t^\alpha + \pi_\delta(t) \geq \begin{cases} t^\alpha + \pi_\delta(t) & \text{if } \alpha \leq 2\beta \\ \frac{1}{4}t^{2\beta} + \pi_\delta(t) & \text{if } \alpha > 2\beta. \end{cases}$$

Therefore similarly we can also get (3.20) with $C_1 = \frac{1}{2^{\gamma+2}}$.

Furthermore we shall intoroduce the algebraic property of $\omega_\delta(t)$ and $\pi_\delta(t)$.

Lemma 3.2.C. *Assume that (3.13) and (3.14). Then $\omega'_\delta(t)$ and $\pi'_\delta(t)$ are continuous in the interval $[0,T]$ and have at most a finite number(independent of δ) of zeros in the interval where $\omega'_\delta(t)$, $\pi'_\delta(t)$ are not identically equal to zero.*

We remark that the assumptions (3.13) and (3.14) are not essential for this lemma, but they simplify the proof.

Proof. It is sufficient to invistige only $\omega'_\delta(t)$. Since $\omega_\delta(t)$ can be wirtten as

$$\omega_\delta(t) = \frac{1}{\delta^\alpha}\int_{-\infty}^{\infty} \chi_\delta(\tau)\varphi\Big(\frac{\tau - t}{\delta^\alpha}\Big)d\tau,$$

we have

$$\begin{aligned}\omega'_\delta(t) &= -\frac{1}{\delta^{2\alpha}}\int_{-\infty}^{\infty} \chi_\delta(\tau)\varphi'\Big(\frac{\tau - t}{\delta^\alpha}\Big)d\tau \\ &= -\frac{1}{\delta^{2\alpha}}\int_0^\delta (\delta^\alpha - \tau^\alpha)\varphi'\Big(\frac{\tau - t}{\delta^\alpha}\Big)d\tau \\ &= -\frac{1}{\delta^\alpha}\int_{-\frac{t}{\delta^\alpha}}^{\frac{\delta - t}{\delta^\alpha}} \{\delta^\alpha - (t + \tau\delta^\alpha)^\alpha\}\varphi'(\tau)d\tau.\end{aligned}$$

By (3.11) it holds that

$$\varphi'(t) = \begin{cases} 6t^2 - 6t & \text{for } t \in [0,1] \\ -6t^2 - 6t & \text{for } t \in [-1,0] \\ 0 & \text{for } t \in [-\infty,-1] \cup [1,\infty]. \end{cases} \tag{3.21}$$

From the assumptions (3.13) and (3.14) we know that $\delta^\alpha \leq \delta - \delta^\alpha$. Hence we can consider the following intervals of t

$$[0,\delta^\alpha],\quad [\delta^\alpha, \delta - \delta^\alpha],\quad [\delta - \delta^\alpha, \delta],\quad [\delta, \delta + \delta^\alpha],\quad [\delta + \delta^\alpha, T].$$

When $t \in [0, \delta^\alpha]$, we find that $\frac{\delta - t}{\delta^\alpha} \geq 1$ and $-1 \leq -\frac{t}{\delta^\alpha} \leq 0$. Therefore by (3.21) we get

$$
\begin{aligned}
(3.22)\quad \omega_\delta'(t) = &-\frac{1}{\delta^\alpha}\int_0^1 \{\delta^\alpha - (t + \tau\delta^\alpha)^\alpha\}(6\tau^2 - 6\tau)d\tau \\
&\qquad - \frac{1}{\delta^\alpha}\int_{-\frac{t}{\delta^\alpha}}^0 \{\delta^\alpha - (t + \tau\delta^\alpha)^\alpha\}(-6\tau^2 - 6\tau)d\tau \\
&= \frac{12\{t(t+\delta^\alpha)^{\alpha+2} - 2t^{\alpha+3}\}}{(\alpha+1)(\alpha+2)(\alpha+3)\delta^{4\alpha}} - \frac{6(t+\delta^\alpha)^{\alpha+2}}{(\alpha+2)(\alpha+3)\delta^{3\alpha}} + \frac{2t^3}{\delta^{3\alpha}} - \frac{3t^2}{\delta^{2\alpha}} + 1.
\end{aligned}
$$

Here we remark that the term of highest degree in $\omega_\delta'(t)$ is ${}^3C_{\alpha\delta}t^{\alpha+3}$.

When $t \in [\delta^\alpha, \delta - \delta^\alpha]$, we find that $\frac{\delta - t}{\delta^\alpha} \geq 1$ and $-\frac{t}{\delta^\alpha} \leq -1$. Therefore by (3.21) we get

$$
\begin{aligned}
(3.23)\qquad \omega_\delta'(t) = &-\frac{1}{\delta^\alpha}\int_0^1 \{\delta^\alpha - (t + \tau\delta^\alpha)^\alpha\}(6\tau^2 - 6\tau)d\tau \\
&\qquad - \frac{1}{\delta^\alpha}\int_{-1}^0 \{\delta^\alpha - (t + \tau\delta^\alpha)^\alpha\}(-6\tau^2 - 6\tau)d\tau \\
&= \frac{-24t^{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)\delta^{4\alpha}} + \frac{6t\{(t+\delta^\alpha)^{\alpha+2} + (t-\delta^\alpha)^{\alpha+2}\}}{(\alpha+1)(\alpha+2)\delta^{4\alpha}} \\
&\qquad - \frac{6\{(t+\delta^\alpha)^{\alpha+3} + (t-\delta^\alpha)^{\alpha+3}\}}{(\alpha+2)(\alpha+3)\delta^{4\alpha}}.
\end{aligned}
$$

When $t \in [\delta - \delta^\alpha, \delta]$, we find that $0 \leq \frac{\delta - t}{\delta^\alpha} \leq 1$ and $-\frac{t}{\delta^\alpha} \leq -1$. Therefore by (3.21) we get

$$
\begin{aligned}
(3.24)\quad \omega_\delta'(t) = &-\frac{1}{\delta^\alpha}\int_0^{\frac{\delta-t}{\delta^\alpha}} \{\delta^\alpha - (t + \tau\delta^\alpha)^\alpha\}(6\tau^2 - 6\tau)d\tau \\
&\qquad - \frac{1}{\delta^\alpha}\int_{-1}^0 \{\delta^\alpha - (t + \tau\delta^\alpha)^\alpha\}(-6\tau^2 - 6\tau)d\tau \\
&= \frac{-24t^{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)\delta^{4\alpha}} + \frac{12t(t-\delta^\alpha)^{\alpha+2}}{(\alpha+1)(\alpha+2)(\alpha+3)\delta^{4\alpha}} \\
&\quad + \frac{6(t-\delta^\alpha)^{\alpha+2}}{(\alpha+2)(\alpha+3)\delta^{3\alpha}} + \frac{6t^2}{(\alpha+1)\delta^{3\alpha-1}} + 6\Big(\frac{1}{(\alpha+1)\delta^{2\alpha-1}} - \frac{2}{(\alpha+2)\delta^{3\alpha-2}}\Big)t \\
&\quad + 6\Big(\frac{1}{(\alpha+3)\delta^{3\alpha-3}} - \frac{1}{(\alpha+2)\delta^{2\alpha-2}}\Big) - 2\Big(\frac{\delta-t}{\delta^\alpha}\Big)^3 + 3\Big(\frac{\delta-t}{\delta^\alpha}\Big)^2 - 1.
\end{aligned}
$$

When $t \in [\delta, \delta + \delta^\alpha]$, we find that $-1 \leq \frac{\delta - t}{\delta^\alpha} \leq 0$ and $-\frac{t}{\delta^\alpha} \leq -1$. Therefore by (3.21) we get

$$
(3.25)\quad \omega_\delta'(t) = -\frac{1}{\delta^\alpha}\int_{-1}^{\frac{\delta-t}{\delta^\alpha}} \{\delta^\alpha - (t + \tau\delta^\alpha)^\alpha\}(-6\tau^2 - 6\tau)d\tau
$$

$$= -\frac{6(t-\delta^\alpha)^{\alpha+3}}{(\alpha+2)(\alpha+3)\delta^{4\alpha}} + \frac{6t(t-\delta^\alpha)^{\alpha+2}}{(\alpha+1)(\alpha+2)\delta^{4\alpha}}$$
$$- \frac{6t^2}{(\alpha+1)\delta^{3\alpha-1}} + 6\Big(\frac{1}{(\alpha+1)\delta^{2\alpha-1}} + \frac{2}{(\alpha+2)\delta^{3\alpha-2}}\Big)t$$
$$- 6\Big(\frac{1}{(\alpha+3)\delta^{3\alpha-3}} + \frac{1}{(\alpha+2)\delta^{2\alpha-2}}\Big) + 2\Big(\frac{\delta-t}{\delta^\alpha}\Big)^3 + 3\Big(\frac{\delta-t}{\delta^\alpha}\Big)^2 - 1.$$

Finally when $t \in [\delta+\delta^\alpha, T]$, we find that $\frac{\delta-t}{\delta^\alpha} \le -1$ and $-\frac{t}{\delta^\alpha} \le -1$. Therefore by (3.21) we get

$$\omega_\delta'(t) = -\frac{1}{\delta^\alpha}\int_{-1}^{-1}\{\delta^\alpha - (t+\tau\delta^\alpha)^\alpha\}(-6\tau^2 - 6\tau)d\tau \equiv 0. \tag{3.26}$$

Thus thanks to (3.22)-(3.26), we see that the function $\omega'(t)$ is continuous in $[0,T]$ and the equation $\omega'(t) = 0$ has at most $4(\alpha+3)$ algebraic roots in $[0, \delta+\delta^\alpha]$. Similarly we also see that the function $\pi'(t)$ is continuous in $[0,T]$ and the equation $\pi'(t) = 0$ has at most $4(\gamma+3)$ roots in $[0, \delta+\delta^\gamma]$. This concludes the proof of the lemma.

3.3 Estimate of the energy

The coefficients of (3.1) belong to the analytic class. Especially when $s = 1$, the initial data also belong to the analytic class. Therefore from the Cauchy-Kowalevski Theorem we can see that the Cauchy problem (3.1) is wellposed in the analytic class. Thus we may suppose $s > 1$ for the proof.

In virtue of Holmgren's Theorem we get the uniqueness of solutions to (3.1) and can suppose that $u_0(x)$, $u_1(x)$ and $u_2(x)$ belong to G_0^s. Hence by Paley-Wiener Theorem we shall more precisely assume that

$$\sup_{\xi\in\mathbf{R}} e^{\rho_0\langle\xi\rangle_\nu^{\frac{1}{s}}}\big(\langle\xi\rangle_\nu^4|\hat{u}_0|^2 + \langle\xi\rangle_\nu^2|\hat{u}_1|^2 + |\hat{u}_2|^2\big) \le C_2$$
$$\Big(\text{or } \sup_{\xi\in\mathbf{R}} e^{\frac{\rho_0}{2}\langle\xi\rangle_\nu^{\frac{1}{s}}}\big(\langle\xi\rangle_\nu^2|\hat{u}_0| + \langle\xi\rangle_\nu|\hat{u}_1| + |\hat{u}_2|\big) \le C_2\Big).$$

Moreover Ovciannikov Theorem gives the existence of solutions(see [CJS], [J3], [Ov]). Therefore our task is to derive the energy inequality and investigate the regularity of the solution.

Similarly as the proof in [Ki5], we know that the terms $t^\sigma u_{tt}$, $t^\omega u_t$ and $t^\theta u$ in the equation (3.1) do not influence the Gevrey wellposedness. For simplicity we shall exclude these terms from (3.1) in advance.

By Fourier transform the Cauchy problem (3.1) is changed to

$$
\begin{cases} v_{ttt} + t^{\alpha}\xi^2 v_t + it^{\beta}\xi v_{tt} - t^{\eta}\xi^2 v + it^{\lambda}\xi v_t + it^{\mu}\xi v = 0 \\ v(0,\xi) = v_0(\xi), \quad v_t(0,\xi) = v_1(\xi), \quad v_{tt}(0,\xi) = v_2(\xi), \end{cases} \tag{3.27}
$$

where $v = \hat{u}$, and $v_l = \hat{u}_l$ $(l = 0, 1, 2)$.

For this equation we shall define the following energy with $\omega_{\delta_1}(t)$ and $\pi_{\delta_2}(t)$ defined in (3.9) and (3.10).

$$
\begin{aligned} E_{\delta_1\delta_2}(t,\xi)^2 =& e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Big\{ \left| v_{tt} + it^{\beta}\xi v_t + t^{\alpha}\xi^2 v + \omega_{\delta_1}(t)\xi^2 v \right|^2 \\ & + \left| v_{tt} + \frac{i}{2}t^{\beta}\xi v_t \right|^2 + \Big(\frac{1}{4}t^{2\beta} + t^{\alpha} + \pi_{\delta_2}(t)\Big)\xi^2|v_t|^2 \Big\}. \end{aligned} \tag{3.28}
$$

Here the function $\rho(t)$ and the parameter ν are positive and determined later on. Noting that $\omega_{\delta_1}(t)$, $\pi_{\delta_2}(t) \geq 0$, we show that this energy is bounded from below by the absolute values of v, v_t and v_{tt}. We also see easily that this energy is bounded (from above) in terms of their the absolute values. Therefore the energy inequality based on (3.28) can be changed into one based on the absolute values of v, v_t and v_{tt}.

The author used this type of energy of third order equations in Chapter 1. However the terms in Chapter 1 corresponding to $\omega_{\delta_1}(t)$ and $\pi_{\delta_2}(t)$ in (3.28), are independent of t, since the coefficients of the equation in Chapter 1 may degenerate in an infinite number of points.

Defferentiating (3.28) in t, by (3.27) we get

$$
\begin{aligned} & \frac{d}{dt}(E^2_{\delta_1\delta_2}) \\ =& \rho'(t)\langle\xi\rangle^{\frac{\kappa}{\nu}} E^2_{\delta_1\delta_2} \\ & + 2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(i\beta t^{\beta-1}\xi v_t + \alpha t^{\alpha-1}\xi^2 v + \omega'_{\delta_1}\xi^2 v + \omega_{\delta_1}\xi^2 v_t + t^{\eta}\xi^2 v - it^{\lambda}\xi v_t - it^{\mu}\xi v, k(t,\xi) \Big) \\ & + 2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(-t^{\alpha}\xi^2 v_t - it^{\beta}\xi v_{tt} + t^{\eta}\xi^2 v - it^{\lambda}\xi v_t - it^{\mu}\xi v + \frac{i}{2}\beta t^{\beta-1}\xi v_t + \frac{i}{2}t^{\beta}\xi v_{tt}, l(t,\xi) \Big) \\ & + 2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Big(\frac{1}{4}t^{2\beta} + t^{\alpha} + \pi_{\delta_2}\Big)\xi^2\Re(v_t,\ v_{tt}) + e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Big(\frac{1}{2}\beta t^{2\beta-1} + \alpha t^{\alpha-1} + \pi'_{\delta_2}\Big)\xi^2|v_t|^2 \\ & (\equiv \rho'(t)\langle\xi\rangle^{\frac{\kappa}{\nu}} E^2_{\delta_1\delta_2} + I + II + III + IV), \end{aligned} \tag{3.29}
$$

where $k(t,\xi) = v_{tt} + it^{\beta}\xi v_t + t^{\alpha}\xi^2 v + \omega_{\delta_1}\xi^2 v$ and $l(t,\xi) = v_{tt} + \frac{i}{2}t^{\beta}\xi v_t$.

In order to further estimate the derivative of $E^2_{\delta_1\delta_2}(t,\xi)$ in t, we shall first study the term I and separately examine some terms which I is composed of.

Estimate of $2e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(i\beta t^{\beta-1}\xi v_t, k(t,\xi)\Big)$

We obtain

$$
\begin{aligned}
(3.30)\qquad & 2e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(i\beta t^{\beta-1}\xi v_t, k(t,\xi)\Big)\\
=&2\beta t^{\beta-1}\xi e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Im\Big(\Big(\frac{1}{2}t^{\beta}|\xi|+\delta_2^{\frac{\gamma}{2}}|\xi|\Big)^{\frac{1}{2}}v_t, \Big(\frac{1}{2}t^{\beta}|\xi|+\delta_2^{\frac{\gamma}{2}}|\xi|\Big)^{-\frac{1}{2}}k\Big)\\
\leq&\beta t^{\beta-1}|\xi|e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{\Big(\frac{1}{2}t^{\beta}|\xi|+\delta_2^{\frac{\gamma}{2}}|\xi|\Big)|v_t|^2+\Big(\frac{1}{2}t^{\beta}|\xi|+\delta_2^{\frac{\gamma}{2}}|\xi|\Big)^{-1}|k|^2\Big\}\\
=&\frac{\big\{(\frac{1}{2}t^{\beta}+\delta_2^{\frac{\gamma}{2}})^2\big\}'}{(\frac{1}{2}t^{\beta}+\delta_2^{\frac{\gamma}{2}})^2}\Big(\frac{1}{2}t^{\beta}+\delta_2^{\frac{\gamma}{2}}\Big)^2\xi^2e^{\rho(t)\langle\xi\rangle_\nu^\kappa}|v_t|^2+2\frac{(\frac{1}{2}t^{\beta}+\delta_2^{\frac{\gamma}{2}})'}{(\frac{1}{2}t^{\beta}+\delta_2^{\frac{\gamma}{2}})}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}|k|^2.
\end{aligned}
$$

If $t\in[0,\delta_2]$ and $\alpha\leq 2\beta$, by (3.15) we obtain

$$
\begin{aligned}
\frac{1}{4}t^{2\beta}+t^{\alpha}+\pi_{\delta_2}(t)&=\frac{1}{4}t^{2\beta}+\delta_2^{\alpha}-\chi_{\delta_2}(t)+\omega_{\delta_2}(t)\geq\delta_2^{\alpha}-|\chi_{\delta_2}(t)-\omega_{\delta_2}(t)|\\
&\geq\delta_2^{\alpha}-C_0\delta_2^{\alpha}=C_3\delta_2^{\alpha}\quad(0<C_3<1).
\end{aligned}
$$

Similarly ,if $t\in[0,\delta_2]$ and $\alpha>2\beta$, by (3.16) we obtain

$$\frac{1}{4}t^{2\beta}+t^{\alpha}+\pi_{\delta_2}(t)\geq C_3\delta_2^{2\beta}.$$

Therefore in the case of $t\in[0,\delta_2]$, one has

$$\frac{1}{4}t^{2\beta}+t^{\alpha}+\pi_{\delta_2}(t)\geq C_3\delta_2^{\gamma}. \tag{3.31}$$

In the case of $t\in[\delta_2,T]$, since $\pi_{\delta_2}(t)\geq 0$, it holds that

$$\frac{1}{4}t^{2\beta}+t^{\alpha}+\pi_{\delta_2}(t)\geq\frac{1}{4}\delta_2^{2\beta}+\delta_2^{\alpha}+0\geq C_4\delta_2^{\gamma}\quad(0<C_4<1). \tag{3.32}$$

By (3.31), (3.32) we get for $t\in[0,T]$

$$\frac{1}{4}t^{2\beta}+t^{\alpha}+\pi_{\delta_2}(t)\geq C_5\delta_2^{\gamma}\quad(0<C_5<1). \tag{3.33}$$

Hence we have

$$
\begin{aligned}
(3.34)\qquad \Big(\frac{1}{2}t^{\beta}+\delta_2^{\frac{\gamma}{2}}\Big)^2&\leq 2\Big(\frac{1}{4}t^{2\beta}+\delta_2^{\gamma}\Big)\leq 2\Big\{\frac{1}{4}t^{2\beta}+C_5^{-1}\Big(\frac{1}{4}t^{2\beta}+t^{\alpha}+\pi_{\delta_2}(t)\Big)\Big\}\\
&\leq C_6\Big(\frac{1}{4}t^{2\beta}+t^{\alpha}+\pi_{\delta_2}(t)\Big).
\end{aligned}
$$

Noting the difinition of the energy, we have by (3.30), (3.34)

$$2e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(i\beta t^{\beta-1}\xi v_t, k(t,\xi)\Big)\leq C_6\frac{\big\{(\frac{1}{2}t^{\beta}+\delta_2^{\frac{\gamma}{2}})^2\big\}'}{(\frac{1}{2}t^{\beta}+\delta_2^{\frac{\gamma}{2}})^2}E_{\delta_1\delta_2}(t,\xi)^2+2\frac{(\frac{1}{2}t^{\beta}+\delta_2^{\frac{\gamma}{2}})'}{(\frac{1}{2}t^{\beta}+\delta_2^{\frac{\gamma}{2}})}E_{\delta_1\delta_2}(t,\xi)^2 \tag{3.35}$$

Estimate of $2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big((\alpha t^{\alpha-1}+\omega'_{\delta_1})\xi^2 v, k(t,\xi)\Big)$

We first rewrite as follows

$$
\begin{aligned}
&2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big((\alpha t^{\alpha-1}+\omega'_{\delta_1})\xi^2 v, k(t,\xi)\Big)\\
=&2\frac{\alpha t^{\alpha-1}+\omega'_{\delta_1}}{t^\alpha+\omega_{\delta_1}}e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(-v_{tt}-\frac{i}{2}t^\beta\xi v_t, k\Big)+2\frac{\alpha t^{\alpha-1}+\omega'_{\delta_1}}{t^\alpha+\omega_{\delta_1}}e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(-\frac{i}{2}t^\beta\xi v_t, k\Big)\\
&+2\frac{\alpha t^{\alpha-1}+\omega'_{\delta_1}}{t^\alpha+\omega_{\delta_1}}e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}|k|^2.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
(3.36)\ &2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big((\alpha t^{\alpha-1}+\omega'_{\delta_1})\xi^2 v, k(t,\xi)\Big)\\
&\le\frac{|(t^\alpha+\omega_{\delta_1})'|}{(t^\alpha+\omega_{\delta_1})}e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\left\{\left|v_{tt}+\frac{i}{2}t^\beta\xi v_t\right|^2+\frac{1}{4}t^{2\beta}\xi^2|v_t|^2+4|k|^2\right\}\le 4\frac{|(t^\alpha+\omega_{\delta_1})'|}{(t^\alpha+\omega_{\delta_1})}E_{\delta_1\delta_2}(t,\xi)^2.
\end{aligned}
$$

Esitimate of $2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(\omega_{\delta_1}\xi^2 v_t, k(t,\xi)\Big)$

Using (3.33) again, we have

$$
\begin{aligned}
(3.37)\quad 2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(\omega_{\delta_1}\xi^2 v_t, k(t,\xi)\Big)&=2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\omega_{\delta_1}\xi^2\Re\Big(\delta_2^{\frac{\gamma}{4}}|\xi|^{\frac12}v_t, \delta_2^{-\frac{\gamma}{4}}|\xi|^{-\frac12}k\Big)\\
&\le e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\omega_{\delta_1}\xi^2\delta_2^{-\frac{\gamma}{2}}|\xi|^{-1}\Big\{\delta_2^{\gamma}\xi^2|v_t|^2+|k|^2\Big\}\\
&\le e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\omega_{\delta_1}\delta_2^{-\frac{\gamma}{2}}|\xi|\Big\{C_5^{-1}\Big(\frac14 t^{2\beta}+t^\alpha+\pi_{\delta_2}\Big)\xi^2|v_t|^2+|k|^2\Big\}\\
&\le C_5^{-1}\omega_{\delta_1}\delta_2^{-\frac{\gamma}{2}}|\xi|E_{\delta_1\delta_2}(t,\xi)^2.
\end{aligned}
$$

Estimate of $2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(t^\eta\xi^2 v, k(t,\xi)\Big)$

We obtain

$$
\begin{aligned}
(3.38)\quad 2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(t^\eta\xi^2 v, k(t,\xi)\Big)&=2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}t^\eta\xi^2\Re\Big((t^\alpha\xi^2+\chi_{\delta_1}\xi^2)^{\frac12}v, (t^\alpha\xi^2+\chi_{\delta_1}\xi^2)^{-\frac12}k\Big)\\
&\le e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}t^\eta(t^\alpha+\chi_{\delta_1})^{-1}\Big\{(t^\alpha\xi^2+\chi_{\delta_1}\xi^2)^2|v|^2+|k|^2\Big\}\\
&\le e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}t^\eta(t^\alpha+\chi_{\delta_1})^{-1}\Big\{2(t^\alpha\xi^2+\omega_{\delta_1}\xi^2)^2|v|^2+|k|^2\Big\}\\
&\quad+2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}t^\eta(t^\alpha+\chi_{\delta_1})^{-1}\Big(\frac{\chi_{\delta_1}-\omega_{\delta_1}}{t^\alpha+\omega_{\delta_1}}\Big)^2(t^\alpha\xi^2+\omega_{\delta_1}\xi^2)^2|v|^2,
\end{aligned}
$$

where we have used

$$
\begin{aligned}
(t^\alpha\xi^2+\chi_{\delta_1}\xi^2)^2&=\Big\{t^\alpha\xi^2+\omega_{\delta_1}\xi^2+(\chi_{\delta_1}-\omega_{\delta_1})\xi^2\Big\}^2\\
&\le 2(t^\alpha\xi^2+\omega_{\delta_1}\xi^2)^2+2(\chi_{\delta_1}-\omega_{\delta_1})^2\xi^4\\
&=2(t^\alpha\xi^2+\omega_{\delta_1}\xi^2)^2+2\Big(\frac{\chi_{\delta_1}-\omega_{\delta_1}}{t^\alpha+\omega_{\delta_1}}\Big)^2(t^\alpha\xi^2+\omega_{\delta_1}\xi^2)^2.
\end{aligned}
$$

In the case of $t \in [0, \delta_1]$, by (3.15) one has

$$\begin{aligned}\frac{|\chi_{\delta_1} - \omega_{\delta_1}|}{t^\alpha + \omega_{\delta_1}} &= \frac{|\chi_{\delta_1} - \omega_{\delta_1}|}{\delta_1^\alpha - \chi_{\delta_1} + \omega_{\delta_1}} \le \frac{|\chi_{\delta_1} - \omega_{\delta_1}|}{\delta_1^\alpha - |\chi_{\delta_1} - \omega_{\delta_1}|} \\ &\le \frac{C_0 \delta_1^\alpha}{\delta_1^\alpha - C_0 \delta_1^\alpha} = \frac{C_0}{1 - C_0}.\end{aligned}$$

In the case of $t \in [\delta_1, T]$, since $\omega_{\delta_1}(t) \ge 0$, by (3.15) we have

$$\frac{|\chi_{\delta_1} - \omega_{\delta_1}|}{t^\alpha + \omega_{\delta_1}} \le \frac{C_0 \delta_1^\alpha}{\delta_1^\alpha + 0} = C_0.$$

Therefore we get for $t \in [0, T]$

$$\frac{|\chi_{\delta_1} - \omega_{\delta_1}|}{t^\alpha + \omega_{\delta_1}} \le C_7. \tag{3.39}$$

From the definition of the energy, we obtain

$$\begin{aligned}E_{\delta_1\delta_2}(t,\xi)^2 &\ge e^{\rho(t)\langle\xi\rangle^\kappa_\nu} \Big\{ \Big| \Big(v_{tt} + \frac{i}{2} t^\beta \xi v_t\Big) + \Big(\frac{i}{2} t^\beta \xi v_t + t^\alpha \xi^2 v + \omega_{\delta_1}(t) \xi^2 v\Big)\Big|^2 \\ &\qquad + \Big| v_{tt} + \frac{i}{2} t^\beta \xi v_t \Big|^2 + \frac{1}{4} t^{2\beta} \xi^2 |v_t|^2 \Big\} \\ &= e^{\rho(t)\langle\xi\rangle^\kappa_\nu} \Big\{ \Big| \sqrt{2}\Big(v_{tt} + \frac{i}{2} t^\beta \xi v_t\Big) + \frac{1}{\sqrt{2}}\Big(\frac{i}{2} t^\beta \xi v_t + t^\alpha \xi^2 v + \omega_{\delta_1} \xi^2 v\Big)\Big|^2 \\ &\qquad + \frac{1}{2}\Big| \frac{i}{2} t^\beta \xi v_t + t^\alpha \xi^2 v + \omega_{\delta_1} \xi^2 v \Big|^2 + \frac{1}{4} t^{2\beta} \xi^2 |v_t|^2 \Big\} \\ &\ge e^{\rho(t)\langle\xi\rangle^\kappa_\nu} \Big\{ \frac{1}{2}\Big| \frac{i}{2} t^\beta \xi v_t + t^\alpha \xi^2 v + \omega_{\delta_1} \xi^2 v \Big|^2 + \frac{1}{4} t^{2\beta} \xi^2 |v_t|^2 \Big\} \\ &= e^{\rho(t)\langle\xi\rangle^\kappa_\nu} \Big\{ \frac{1}{2}\Big| \frac{\sqrt{3} i}{2} t^\beta \xi v_t + \frac{1}{\sqrt{3}}(t^\alpha \xi^2 v + \omega_{\delta_1} \xi^2 v) \Big|^2 + \frac{1}{3} |t^\alpha \xi^2 v + \omega_{\delta_1} \xi^2 v|^2 \Big\} \\ &\ge e^{\rho(t)\langle\xi\rangle^\kappa_\nu} \Big\{ \frac{1}{3} |t^\alpha \xi^2 v + \omega_{\delta_1} \xi^2 v|^2 \Big\} \\ &= \frac{1}{3} e^{\rho(t)\langle\xi\rangle^\kappa_\nu} (t^\alpha \xi^2 + \omega_{\delta_1} \xi^2)^2 |v|^2.\end{aligned} \tag{3.40}$$

Thus by (3.38), (3.39) and (3.40) we have

$$2 e^{\rho(t)\langle\xi\rangle^\kappa_\nu} \Re\Big(t^\eta \xi^2 v, k(t,\xi)\Big) \le C_8 \psi_1(t) E_{\delta_1\delta_2}(t,\xi)^2, \tag{3.41}$$

where

$$\psi_1(t) = t^\eta (t^\alpha + \chi_{\delta_1})^{-1} = \begin{cases} \delta_1^{-\alpha} t^\eta & \text{for } t \in [0, \delta_1] \\ t^{\eta - \alpha} & \text{for } t \in [\delta_1, T]. \end{cases} \tag{3.42}$$

Estimate of $2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(-it^\lambda\xi v_t, k(t,\xi)\Big)$

We obtain

$$
\begin{aligned}
&2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(-it^\lambda\xi v_t, k(t,\xi)\Big)\\
=&2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}(-t^\lambda\xi)\Im\Big(\Big\{\frac{1}{4}t^{2\beta}\xi^2+t^\alpha\xi^2+\tilde{\chi}_{\delta_2}\xi^2\Big\}^{\frac{1}{4}}v_t, \Big\{\frac{1}{4}t^{2\beta}\xi^2+t^\alpha\xi^2+\tilde{\chi}_{\delta_2}\xi^2\Big\}^{-\frac{1}{4}}k\Big)\\
\le& e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}t^\lambda\Big(\frac{1}{4}t^{2\beta}+t^\alpha+\tilde{\chi}_{\delta_2}\Big)^{-\frac{1}{2}}\Big\{\Big(\frac{1}{4}t^{2\beta}\xi^2+t^\alpha\xi^2+\tilde{\chi}_{\delta_2}\xi^2\Big)|v_t|^2+|k|^2\Big\}\\
\le& e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}t^\lambda\Big(\frac{1}{4}t^{2\beta}+t^\alpha+\tilde{\chi}_{\delta_2}\Big)^{-\frac{1}{2}}\Big\{\Big(\frac{1}{4}t^{2\beta}\xi^2+t^\alpha\xi^2+\pi_{\delta_2}\xi^2\Big)|v_t|^2+|k|^2\Big\}\\
&+e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}t^\lambda\Big(\frac{1}{4}t^{2\beta}+t^\alpha+\tilde{\chi}_{\delta_2}\Big)^{-\frac{1}{2}}\Big(\frac{|\tilde{\chi}_{\delta_2}-\pi_{\delta_2}|}{\frac{1}{4}t^{2\beta}+t^\alpha+\pi_{\delta_2}}\Big)\Big(\frac{1}{4}t^{2\beta}\xi^2+t^\alpha\xi^2+\pi_{\delta_2}\xi^2\Big)|v_t|^2.
\end{aligned}
$$

Noting that

$$
\frac{|\tilde{\chi}_{\delta_2}-\pi_{\delta_2}|}{\frac{1}{4}t^{2\beta}+t^\alpha+\pi_{\delta_2}}\le\begin{cases}\dfrac{|\tilde{\chi}_{\delta_2}-\pi_{\delta_2}|}{t^\alpha+\pi_{\delta_2}} & \text{if } \alpha\le 2\beta\\[2mm] \dfrac{|\tilde{\chi}_{\delta_2}-\pi_{\delta_2}|}{\frac{1}{4}t^{2\beta}+\pi_{\delta_2}} & \text{if } \alpha> 2\beta,\end{cases}
$$

we get in similar fashion to (3.39)

$$
\frac{|\tilde{\chi}_{\delta_2}-\pi_{\delta_2}|}{\frac{1}{4}t^{2\beta}+t^\alpha+\pi_{\delta_2}}\le C_7. \tag{3.43}
$$

Moreover if $\alpha\le 2\beta$, one has

$$
t^\lambda\Big(\frac{1}{4}t^{2\beta}+t^\alpha+\tilde{\chi}_{\delta_2}\Big)^{-\frac{1}{2}}\le t^\lambda(t^\alpha+\tilde{\chi}_{\delta_2})^{-\frac{1}{2}}=\begin{cases}\delta_2^{-\frac{\alpha}{2}}t^\lambda & \text{for } t\in[0,\delta_2]\\ t^{\lambda-\frac{\alpha}{2}} & \text{for } t\in[\delta_2,T].\end{cases}
$$

If $\alpha>2\beta$, we have

$$
t^\lambda\Big(\frac{1}{4}t^{2\beta}+t^\alpha+\tilde{\chi}_{\delta_2}\Big)^{-\frac{1}{2}}\le t^\lambda\Big(\frac{1}{4}t^{2\beta}+\tilde{\chi}_{\delta_2}\Big)^{-\frac{1}{2}}=\begin{cases}2\delta_2^{-\beta}t^\lambda & \text{for } t\in[0,\delta_2]\\ 2t^{\lambda-\beta} & \text{for } t\in[\delta_2,T].\end{cases}
$$

Therefore we get

$$
t^\lambda\Big(\frac{1}{4}t^{2\beta}+t^\alpha+\tilde{\chi}_{\delta_2}\Big)^{-\frac{1}{2}}\le\begin{cases}2\delta_2^{-\frac{\gamma}{2}}t^\lambda & \text{for } t\in[0,\delta_2]\\ 2t^{\lambda-\frac{\gamma}{2}} & \text{for } t\in[\delta_2,T].\end{cases} \tag{3.44}
$$

Thus by (3.43), (3.44) we have

$$2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(-it^{\lambda}\xi v_t, k(t,\xi)\Big) \le C_9\psi_2(t)E_{\delta_1\delta_2}(t,\xi)^2, \tag{3.45}$$

where

$$\psi_2(t) = \begin{cases} \delta_2^{-\frac{\gamma}{2}}t^{\lambda} & \text{for } t \in [0,\delta_2] \\ t^{\lambda-\frac{\gamma}{2}} & \text{for } t \in [\delta_2, T]. \end{cases} \tag{3.46}$$

Estimate of $2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(-it^{\mu}\xi v, k(t,\xi)\Big)$

In similare fashion to the estimete of $2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(t^{\eta}\xi^2 v, k(t,\xi)\Big)$, we have

$$\begin{aligned}2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(-it^{\mu}\xi v, k(t,\xi)\Big) &= (-|\xi|^{-1})2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Im\Big(t^{\mu}\xi^2 v, k\Big) \\ &\le C_8\psi_3(t)|\xi|^{-1}E_{\delta_1\delta_2}(t,\xi)^2,\end{aligned} \tag{3.47}$$

where

$$\psi_3(t) = \begin{cases} \delta_1^{-\alpha}t^{\mu} & \text{for } t \in [0,\delta_1] \\ t^{\mu-\alpha} & \text{for } t \in [\delta_1, T]. \end{cases} \tag{3.48}$$

Summing up (3.35)-(3.37), (3.41), (3.45) and (3.47), we have the following estimate of the term I

$$\begin{aligned} I \le & C_6\frac{\{(\frac{1}{2}t^{\beta}+\delta_2^{\frac{\gamma}{2}})^2\}'}{(\frac{1}{2}t^{\beta}+\delta_2^{\frac{\gamma}{2}})^2}E_{\delta_1\delta_2}(t,\xi)^2 + 2\frac{(\frac{1}{2}t^{\beta}+\delta_2^{\frac{\gamma}{2}})'}{(\frac{1}{2}t^{\beta}+\delta_2^{\frac{\gamma}{2}})}E_{\delta_1\delta_2}(t,\xi)^2 \\ &+ 4\frac{|(t^{\alpha}+\omega_{\delta_1})'|}{(t^{\alpha}+\omega_{\delta_1})}E_{\delta_1\delta_2}(t,\xi)^2 + C_5^{-1}\omega_{\delta_1}\delta_2^{-\frac{\gamma}{2}}|\xi|E_{\delta_1\delta_2}(t,\xi)^2 \\ &+ C_8\psi_1(t)E_{\delta_1\delta_2}(t,\xi)^2 + C_9\psi_2(t)E_{\delta_1\delta_2}(t,\xi)^2 + C_8\psi_3(t)|\xi|^{-1}E_{\delta_1\delta_2}(t,\xi)^2. \end{aligned} \tag{3.49}$$

Estimate of II, III and IV

We shall next estimate the terms II, III and IV. We first rewrite the term II as follows

$$\begin{aligned} II =& 2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(t^{\eta}\xi^2 v - it^{\lambda}\xi v_t - it^{\mu}\xi v, l(t,\xi)\Big) \\ &+ 2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(-t^{\alpha}\xi^2 v_t - it^{\beta}\xi v_{tt} + \frac{i}{2}\beta t^{\beta-1}\xi v_t + \frac{i}{2}t^{\beta}\xi v_{tt}, l(t,\xi)\Big) \\ \equiv& II_1 + II_2. \end{aligned}$$

We can find that $l(t,\xi)$ in the estimate of the term II_1 plays the same role as $k(t,\xi)$ in the previous estimates. Therefore similarly we have the estimate of the term II_1

$$II_1 \le C_8\psi_1(t)E_{\delta_1\delta_2}(t,\xi)^2 + C_9\psi_2(t)E_{\delta_1\delta_2}(t,\xi)^2 + C_8\psi_3(t)|\xi|^{-1}E_{\delta_1\delta_2}(t,\xi)^2. \tag{3.50}$$

Combining II_2 with III, we can rewrite $II_2 + III$ as follows

$$\begin{aligned} II_2 + III &= 2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Big\{\Re\Big(-t^\alpha\xi^2 v_t, v_{tt}\Big) + \Re\Big(-\frac{1}{2}t^{2\beta}\xi^2 v_t, v_{tt}\Big) + \Re\Big(\frac{i}{2}\beta t^{\beta-1}\xi v_t, v_{tt}\Big) \\ &\quad + \frac{1}{4}\beta t^{2\beta-1}\xi^2|v_t|^2 + \Re\Big(\frac{1}{4}t^{2\beta}\xi^2 v_t, v_{tt}\Big)\Big\} + III \\ &= 2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(\Big\{-t^\alpha\xi^2 - \frac{1}{4}t^{2\beta}\xi^2\Big\}v_t, v_{tt}\Big) + e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(i\beta t^{\beta-1}\xi v_t, v_{tt}\Big) \\ &\quad + e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\frac{1}{2}\beta t^{2\beta-1}\xi^2|v_t|^2 + III \\ &= 2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(\pi_{\delta_2}\xi^2 v_t, l(t,\xi)\Big) + \frac{1}{2}\Big\{2e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\Re\Big(i\beta t^{\beta-1}\xi v_t, l(t,\xi)\Big)\Big\}. \end{aligned} \tag{3.51}$$

Similarly we have the estimate of the term $II_2 + III$

$$II_2 + III \le C_5^{-1}\pi_{\delta_2}\delta_2^{-\frac{\gamma}{2}}|\xi|E_{\delta_1\delta_2}^2 + \frac{C_6\{(\frac{1}{2}t^\beta + \delta_2^{\frac{\gamma}{2}})^2\}'}{2(\frac{1}{2}t^\beta + \delta_2^{\frac{\gamma}{2}})^2}E_{\delta_1\delta_2}^2 + \frac{(\frac{1}{2}t^\beta + \delta_2^{\frac{\gamma}{2}})'}{(\frac{1}{2}t^\beta + \delta_2^{\frac{\gamma}{2}})}E_{\delta_1\delta_2}^2. \tag{3.52}$$

Finally we have the estimate of the term IV

$$IV = e^{\rho(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}}\frac{(\frac{1}{4}t^{2\beta} + t^\alpha + \pi_{\delta_2})'}{(\frac{1}{4}t^{2\beta} + t^\alpha + \pi_{\delta_2})}\Big(\frac{1}{4}t^{2\beta} + t^\alpha + \pi_{\delta_2}\Big)\xi^2|v_t|^2 \le \frac{|(\frac{1}{4}t^{2\beta} + t^\alpha + \pi_{\delta_2})'|}{(\frac{1}{4}t^{2\beta} + t^\alpha + \pi_{\delta_2})}E_{\delta_1\delta_2}^2. \tag{3.53}$$

Estimate of $\frac{d}{dt}E_{\delta_1\delta_2}(t,\xi)^2$

Thus by (3.49), (3.50), (3.52) and (3.53) we have the estimate

$$\begin{aligned} \frac{d}{dt}(E_{\delta_1\delta_2}^2) \le\, &\rho'(t)\langle\xi\rangle^{\frac{\kappa}{\nu}}E_{\delta_1\delta_2}^2 + \frac{3}{2}C_6\frac{\{(\frac{1}{2}t^\beta + \delta_2^{\frac{\gamma}{2}})^2\}'}{(\frac{1}{2}t^\beta + \delta_2^{\frac{\gamma}{2}})^2}E_{\delta_1\delta_2}^2 + 3\frac{(\frac{1}{2}t^\beta + \delta_2^{\frac{\gamma}{2}})'}{(\frac{1}{2}t^\beta + \delta_2^{\frac{\gamma}{2}})}E_{\delta_1\delta_2}^2 \\ &+ 4\frac{|(t^\alpha + \omega_{\delta_1})'|}{(t^\alpha + \omega_{\delta_1})}E_{\delta_1\delta_2}^2 + \frac{|(\frac{1}{4}t^{2\beta} + t^\alpha + \pi_{\delta_2})'|}{(\frac{1}{4}t^{2\beta} + t^\alpha + \pi_{\delta_2})}E_{\delta_1\delta_2}^2 + C_5^{-1}\omega_{\delta_1}\delta_2^{-\frac{\gamma}{2}}|\xi|E_{\delta_1\delta_2}^2 \\ &+ C_5^{-1}\pi_{\delta_2}\delta_2^{-\frac{\gamma}{2}}|\xi|E_{\delta_1\delta_2}^2 + 2C_8\psi_1(t)E_{\delta_1\delta_2}^2 + 2C_9\psi_2(t)E_{\delta_1\delta_2}^2 + 2C_8\psi_3(t)|\xi|^{-1}E_{\delta_1\delta_2}^2. \end{aligned}$$

Hence Gronwall's inequality yields

$$
\begin{aligned}
(3.54)\quad E_{\delta_1\delta_2}(t,\xi)^2 \le & E_{\delta_1\delta_2}(0,\xi)^2 \exp\Big[\int_0^t \rho'(\tau)\langle\xi\rangle_\nu^\kappa d\tau \\
& + C_{10}\Big\{\int_0^t \frac{\{(\frac{1}{2}\tau^\beta+\delta_2^{\frac{7}{2}})^2\}'}{(\frac{1}{2}\tau^\beta+\delta_2^{\frac{7}{2}})^2}d\tau + \int_0^t \frac{(\frac{1}{2}\tau^\beta+\delta_2^{\frac{7}{2}})'}{(\frac{1}{2}\tau^\beta+\delta_2^{\frac{7}{2}})}d\tau + \int_0^t \frac{|(\tau^\alpha+\omega_{\delta_1}(\tau))'|}{(\tau^\alpha+\omega_{\delta_1}(\tau))}d\tau \\
& + \int_0^t \frac{|(\frac{1}{4}\tau^{2\beta}+\tau^\alpha+\pi_{\delta_2}(\tau))'|}{(\frac{1}{4}\tau^{2\beta}+\tau^\alpha+\pi_{\delta_2}(\tau))}d\tau + \int_0^t \omega_{\delta_1}(\tau)\delta_2^{-\frac{7}{2}}|\xi|d\tau + \int_0^t \pi_{\delta_2}(\tau)\delta_2^{-\frac{7}{2}}|\xi|d\tau \\
& + \int_0^t \psi_1(\tau)d\tau + \int_0^t \psi_2(\tau)d\tau + \int_0^t \psi_3(\tau)|\xi|^{-1}d\tau\Big\}\Big].
\end{aligned}
$$

Furthermore we shall examine the terms in the parenthesis { } in the right hand of (3.54) separately.

Obviously we have

$$
(3.55)\qquad \int_0^t \rho'(\tau)\langle\xi\rangle_\nu^\kappa d\tau = (\rho(t)-\rho_0)\langle\xi\rangle_\nu^\kappa.
$$

Estimate of $\int_0^t \frac{\{(\frac{1}{2}\tau^\beta+\delta_2^{\frac{7}{2}})^2\}'}{(\frac{1}{2}\tau^\beta+\delta_2^{\frac{7}{2}})^2}d\tau$

We can easily get

$$
(3.56)\qquad \int_0^t \frac{\{(\frac{1}{2}\tau^\beta+\delta_2^{\frac{7}{2}})^2\}'}{(\frac{1}{2}\tau^\beta+\delta_2^{\frac{7}{2}})^2}d\tau = 2\Big[\log\Big(\frac{1}{2}\tau^\beta+\delta_2^{\frac{7}{2}}\Big)\Big]_{\tau=0}^{\tau=t} = 2\log\Big(\frac{1}{2}t^\beta\delta_2^{-\frac{7}{2}}+1\Big).
$$

Estimate of $\int_0^t \frac{(\frac{1}{2}\tau^\beta+\delta_2^{\frac{7}{2}})'}{(\frac{1}{2}\tau^\beta+\delta_2^{\frac{7}{2}})}d\tau$

As in the previous estimate, we get

$$
(3.57)\qquad \int_0^t \frac{(\frac{1}{2}\tau^\beta+\delta_2^{\frac{7}{2}})'}{(\frac{1}{2}\tau^\beta+\delta_2^{\frac{7}{2}})}d\tau \le \log\Big(\frac{1}{2}t^\beta\delta_2^{-\frac{7}{2}}+1\Big).
$$

Estimate of $\int_0^t \frac{|(\tau^\alpha+\omega_{\delta_1}(\tau))'|}{(\tau^\alpha+\omega_{\delta_1}(\tau))}d\tau$

From Lemma 3.2.C we know that the signature of the function $\alpha\tau^{\alpha-1}+\omega'_{\delta_1}(\tau)$ also changes at most a finite number of times for $\tau\in[0,t]$. Now we may suppose without loss of generality that the signature of $\alpha\tau^{\alpha-1}+\omega'_{\delta_1}(\tau)$ changes $2N$ times for $\tau\in[0,t]$. Let t_n $(0\le n\le 2N+1)$ be real numbers such that

i) $\quad 0=t_0<t_1<\cdots<t_{2N+1}=t$

ii) $\quad \alpha t_n^{\alpha-1}+\omega'_{\delta_1}(t_n)=0 \quad (1\le n\le 2N)$.

From (3.13), (3.14) and (3.22) it holds that

$$\tag{3.58} \begin{aligned} \omega_{\delta_1}'(0) &= 1 - \frac{6\delta_1^{\alpha(\alpha+2)}}{(\alpha+2)(\alpha+3)\delta_1^{3\alpha}} = 1 - \frac{6(\delta_1^{\alpha-1})^\alpha}{(\alpha+2)(\alpha+3)} \\ &\geq 1 - \frac{6(\alpha^{-1})^\alpha}{(\alpha+2)(\alpha+3)} \geq 1 - \frac{6(2^{-1})^2}{(2+2)(2+3)} = \frac{37}{40} \geq 0.) \end{aligned}$$

Since (3.58) implies $t_0^\alpha + \omega_{\delta_1}(t_0) \leq t_1^\alpha + \omega_{\delta_1}(t_1)$, we find that $t_{2n}^\alpha + \omega_{\delta_1}(t_{2n}) \leq t_{2n+1}^\alpha + \omega_{\delta_1}(t_{2n+1})$ $(0 \leq n \leq N)$ and $t_{2n}^\alpha + \omega_{\delta_1}(t_{2n}) \leq t_{2n-1}^\alpha + \omega_{\delta_1}(t_{2n-1})$ $(1 \leq n \leq N)$. Therefore we obtain

$$\tag{3.59} \begin{aligned} &\int_0^t \frac{|(\tau^\alpha + \omega_{\delta_1})'|}{(\tau^\alpha + \omega_{\delta_1})} d\tau = \int_0^t \frac{|(\alpha\tau^{\alpha-1} + \omega_{\delta_1}'|}{(\tau^\alpha + \omega_{\delta_1})} d\tau \\ &= \sum_{n=0}^N \log \frac{t_{2n+1}^\alpha + \omega_{\delta_1}(t_{2n+1})}{t_{2n}^\alpha + \omega_{\delta_1}(t_{2n})} + \sum_{n=1}^N \log \frac{t_{2n-1}^\alpha + \omega_{\delta_1}(t_{2n-1})}{t_{2n}^\alpha + \omega_{\delta_1}(t_{2n})} \\ &= \sum_{n=0}^N \log\Big\{ \frac{t_{2n+1}^\alpha - t_{2n}^\alpha + \omega_{\delta_1}(t_{2n+1}) - \omega_{\delta_1}(t_{2n})}{t_{2n}^\alpha + \omega_{\delta_1}(t_{2n})} + 1 \Big\} \\ &\qquad + \sum_{n=1}^N \log\Big\{ \frac{t_{2n-1}^\alpha - t_{2n}^\alpha + \omega_{\delta_1}(t_{2n-1}) - \omega_{\delta_1}(t_{2n})}{t_{2n}^\alpha + \omega_{\delta_1}(t_{2n})} + 1 \Big\} \\ &= \sum_{n=0}^N \log\Big\{ (t_{2n+1} - t_{2n}) \frac{\sum_{m=0}^{\alpha-1} t_{2n+1}^m t_{2n}^{\alpha-1-m} + \omega_{\delta_1}'\big(t_{2n} + \theta_1(t_{2n+1} - t_{2n})\big)}{t_{2n}^\alpha + \omega_{\delta_1}(t_{2n})} + 1 \Big\} \\ &\quad + \sum_{n=1}^N \log\Big\{ (t_{2n} - t_{2n-1}) \frac{-\sum_{m=0}^{\alpha-1} t_{2n-1}^m t_{2n}^{\alpha-1-m} - \omega_{\delta_1}'\big(t_{2n-1} + \theta_2(t_{2n} - t_{2n-1})\big)}{t_{2n}^\alpha + \omega_{\delta_1}(t_{2n})} + 1 \Big\}, \end{aligned}$$

where $0 < \theta_1,\ \theta_2 < 1$.

Noting that $\chi_{\delta_1} \leq \delta_1^\alpha$, we see that

$$\tag{3.60} \begin{aligned} |\omega_{\delta_1}'(t)| &= \Big| -\frac{1}{\delta_1^{2\alpha}} \int_{-\infty}^\infty \chi_{\delta_1}(\tau) \varphi'\Big(\frac{t-\tau}{\delta_1^\alpha}\Big) d\tau \Big| \leq \frac{1}{\delta_1^\alpha} \int_{-\infty}^\infty \Big|\varphi'\Big(\frac{t-\tau}{\delta_1^\alpha}\Big)\Big| d\tau \\ &= \frac{1}{\delta_1^\alpha} \int_\infty^{-\infty} |\varphi'(\tau)|(-\delta_1^\alpha) d\tau = \int_{-\infty}^\infty |\varphi'(\tau)| d\tau = 2. \end{aligned}$$

By (3.59), (3.60) and Lemma 3.2.B we get

$$\tag{3.61} \begin{aligned} \int_0^t \frac{|(\tau^\alpha + \omega_{\delta_1})'|}{(\tau^\alpha + \omega_{\delta_1})} d\tau &\leq \sum_{n=0}^N \log\Big\{ t \frac{\alpha T^{\alpha-1} + 2}{C_1 \delta_1^{\alpha^2}} + 1 \Big\} + \sum_{n=1}^N \log\Big\{ t \frac{2}{C_1 \delta_1^{\alpha^2}} + 1 \Big\} \\ &= C_{11} \log(C_{12} t \delta_1^{-\alpha^2} + 1). \end{aligned}$$

Estimate of $\int_0^t \frac{|(\frac{1}{4}\tau^{2\beta}+\tau^\alpha+\pi_{\delta_2}(\tau))'|}{(\frac{1}{4}\tau^{2\beta}+\tau^\alpha+\pi_{\delta_2}(\tau))}d\tau$

In similar fashion to the previous estimate we get

$$\int_0^t \frac{|(\frac{1}{4}\tau^{2\beta}+\tau^\alpha+\pi_{\delta_2})'|}{(\frac{1}{4}\tau^{2\beta}+\tau^\alpha+\pi_{\delta_2})}d\tau \le C_{13}\log(C_{14}t\delta_2^{-\gamma^2}+1). \tag{3.62}$$

Estimate of $\int_0^t \omega_{\delta_1}(\tau)\delta_2^{-\frac{\gamma}{2}}|\xi|d\tau$

When $t\in[0,\delta_1+\delta_1^\alpha]$, we see that $t^{\frac{1}{1-\varepsilon}} \le 2t^{\frac{\varepsilon}{1-\varepsilon}}\delta_1$ since $t\le \delta_1+\delta_1^\alpha \le 2\delta_1$. Then we have for all $0<\varepsilon<1$

$$t \le 2^{1-\varepsilon}t^\varepsilon\delta_1^{1-\varepsilon} \le 2t^\varepsilon\delta_1^{1-\varepsilon}. \tag{3.63}$$

Noting that $\omega_{\delta_1}(t)\le\delta_1^\alpha$, we obtain

$$\int_0^t \omega_{\delta_1}\delta_2^{-\frac{\gamma}{2}}|\xi|d\tau \le \int_0^t \delta_1^\alpha\delta_2^{-\frac{\gamma}{2}}|\xi|d\tau \le t\delta_1^\alpha\delta_2^{-\frac{\gamma}{2}}|\xi| \le 2t^\varepsilon\delta_1^{\alpha+1-\varepsilon}\delta_2^{-\frac{\gamma}{2}}|\xi|. \tag{3.64}$$

When $t\in[\delta_1+\delta_1^\alpha,T]$, one has $\int_{\delta_1+\delta_1^\alpha}^t \omega_{\delta_1}\delta_2^{-\frac{\gamma}{2}}|\xi|d\tau=0$ and $\delta_1\le t$. Hence using also (3.64), we obtain

$$\begin{aligned}\int_0^t \omega_{\delta_1}\delta_2^{-\frac{\gamma}{2}}|\xi|d\tau &\le \int_0^{\delta_1+\delta_1^\alpha} \omega_{\delta_1}\delta_2^{-\frac{\gamma}{2}}|\xi|d\tau + \int_{\delta_1+\delta_1^\alpha}^t \omega_{\delta_1}\delta_2^{-\frac{\gamma}{2}}|\xi|d\tau\\ &\le (\delta_1+\delta_1^\alpha)\delta_1^\alpha\delta_2^{-\frac{\gamma}{2}}|\xi| \le 2\delta_1^\varepsilon\delta_1^{\alpha+1-\varepsilon}\delta_2^{-\frac{\gamma}{2}}|\xi| \le 2t^\varepsilon\delta_1^{\alpha+1-\varepsilon}\delta_2^{-\frac{\gamma}{2}}|\xi|.\end{aligned}$$

Thus we get

$$\int_0^t \omega_{\delta_1}\delta_2^{-\frac{\gamma}{2}}|\xi|d\tau \le C_{15}t^\varepsilon\delta_1^{\alpha+1-\varepsilon}\delta_2^{-\frac{\gamma}{2}}|\xi|, \tag{3.65}$$

where $C_{15}=2$.

Estimate of $\int_0^t \pi_{\delta_2}(\tau)\delta_2^{-\frac{\gamma}{2}}|\xi|d\tau$

In fashion similar to the previous estimate, we get

$$\int_0^t \pi_{\delta_2}\delta_2^{-\frac{\gamma}{2}}|\xi|d\tau \le C_{16}t^\varepsilon\delta_2^{\gamma+1-\varepsilon}\delta_2^{-\frac{\gamma}{2}}|\xi| = C_{16}t^\varepsilon\delta_2^{\frac{\gamma}{2}+1-\varepsilon}|\xi|. \tag{3.66}$$

Estimate of $\int_0^t \psi_1(\tau)d\tau$

When $t\in[0,\delta_1]$, we can obtain $t\le t^\varepsilon\delta_1^{1-\varepsilon}$ as in (3.63). Hence by (3.42) we have

$$\int_0^t \psi_1(\tau)d\tau = \delta_1^{-\alpha}\int_0^t \tau^\eta d\tau = \delta_1^{-\alpha}\frac{t^{\eta+1}}{\eta+1} \le t\,\frac{\delta_1^{-\alpha+\eta}}{\eta+1} \le t^\varepsilon\frac{\delta_1^{-\alpha+\eta+1-\varepsilon}}{\eta+1}. \tag{3.67}$$

When $t \in [\delta_1, T]$, taking the first condition of (3.3) into consideration, we obtain by (3.42)

$$\text{(3.68)} \quad \int_0^t \psi_1(\tau)d\tau = \delta_1^{-\alpha}\int_0^{\delta_1}\tau^\eta d\tau + \int_{\delta_1}^t \tau^{\eta-\alpha}d\tau = \delta_1^{\eta-\alpha+1}\frac{\alpha}{(\eta+1)(\alpha-\eta-1)} - \frac{t^{\eta-\alpha+1}}{\alpha-\eta-1}$$
$$\leq \delta_1^{\eta-\alpha+1}\left\{\frac{\alpha}{(\eta+1)(\alpha-\eta-1)} - \frac{1}{\alpha-\eta-1}\right\} = \frac{\delta_1^{-\alpha+\eta+1}}{\eta+1} = \delta_1^\varepsilon\frac{\delta_1^{-\alpha+\eta+1-\varepsilon}}{\eta+1}$$
$$\leq t^\varepsilon\frac{\delta_1^{-\alpha+\eta+1-\varepsilon}}{\eta+1}.$$

Thus by (3.67), (3.68) we get

$$\text{(3.69)} \quad \int_0^t \psi_1(\tau)d\tau \leq t^\varepsilon\frac{\delta_1^{-\alpha+\eta+1-\varepsilon}}{\eta+1}.$$

Estimate $\int_0^t \psi_2(\tau)d\tau$

When $t \in [0, \delta_2]$, in fashion similar to (3.63) we can obtain $t \leq t^\varepsilon\delta_2^{1-\varepsilon}$. Hence by (3.46) we have

$$\text{(3.70)} \quad \int_0^t \psi_2(\tau)d\tau = \delta_2^{-\frac{\gamma}{2}}\int_0^t \tau^\lambda d\tau = \delta_2^{-\frac{\gamma}{2}}\frac{t^{\lambda+1}}{\lambda+1} \leq t\,\frac{\delta_2^{-\frac{\gamma}{2}+\lambda}}{\lambda+1} \leq t^\varepsilon\frac{\delta_2^{-\frac{\gamma}{2}+\lambda+1-\varepsilon}}{\lambda+1}.$$

When $t \in [\delta_2, T]$, taking the second condition of (3.3) into consideration, we obtain by (3.46)

$$\text{(3.71)} \quad \int_0^t \psi_2(\tau)d\tau = \delta_2^{-\frac{\gamma}{2}}\int_0^{\delta_2}\tau^\lambda d\tau + \int_{\delta_2}^t \tau^{\lambda-\frac{\gamma}{2}}d\tau = \delta_2^{-\frac{\gamma}{2}+\lambda+1}\frac{\gamma}{(\lambda+1)(\gamma-2\lambda-2)} - \frac{2t^{\lambda-\frac{\gamma}{2}+1}}{\gamma-2\lambda-2}$$
$$\leq \delta_2^{\lambda-\frac{\gamma}{2}+1}\left\{\frac{\gamma}{(\lambda+1)(\gamma-2\lambda-2)} - \frac{2}{\gamma-2\lambda-2}\right\} = \frac{\delta_2^{-\frac{\gamma}{2}+\lambda+1}}{\lambda+1} = \delta_2^\varepsilon\frac{\delta_2^{-\frac{\gamma}{2}+\lambda+1-\varepsilon}}{\lambda+1}$$
$$\leq t^\varepsilon\frac{\delta_2^{-\frac{\gamma}{2}+\lambda+1-\varepsilon}}{\lambda+1}.$$

Thus by (3.70), (3.71) we get

$$\text{(3.72)} \quad \int_0^t \psi_2(\tau)d\tau \leq t^\varepsilon\frac{\delta_2^{-\frac{\gamma}{2}+\lambda+1-\varepsilon}}{\lambda+1}.$$

estimate of $\int_0^t \psi_3(\tau)|\xi|^{-1}d\tau$

Finally as in the estimate of $\int_0^t \psi_1(\tau)d\tau$, taking the third conditon of (3.3) into consideration, we see that $\alpha-\mu-1(>\mu+3+\frac{\alpha-\gamma}{\gamma+1}) > 0$ and by (3.48) get

$$\text{(3.73)} \quad \int_0^t \psi_3(\tau)|\xi|^{-1}d\tau \leq t^\varepsilon\frac{\delta_1^{-\alpha+\mu+1-\varepsilon}|\xi|^{-1}}{\mu+1}.$$

Estimate of $E_{\delta_1\delta_2}(t,\xi)^2$

Thus by (3.54)-(3.57), (3.61), (3.62), (3.63), (3.66), (3.69), (3.72) and (3.73) we have

$$
\begin{aligned}
(3.74)\quad E_{\delta_1\delta_2}(t,\xi)^2 \le& E_{\delta_1\delta_2}(0,\xi)^2 \exp\Big[(\rho(t)-\rho_0)\langle\xi\rangle_\nu^\kappa \\
&+ C_{10}\Big\{3\log\Big(\frac{1}{2}t^\beta\delta_2^{-\frac{\gamma}{2}}+1\Big) + C_{11}\log(C_{12}t\delta_1^{-\alpha^2}+1) \\
&+ C_{13}\log(C_{14}t\delta_2^{-\gamma^2}+1) + C_{15}t^\varepsilon\delta_1^{\alpha+1-\varepsilon}\delta_2^{-\frac{\gamma}{2}}|\xi| + C_{16}t^\varepsilon\delta_2^{\frac{\gamma}{2}+1-\varepsilon}|\xi| \\
&+ t^\varepsilon\frac{\delta_1^{-\alpha+\eta+1-\varepsilon}}{\eta+1} + t^\varepsilon\frac{\delta_2^{-\frac{\gamma}{2}+\lambda+1-\varepsilon}}{\lambda+1} + t^\varepsilon\frac{\delta_1^{-\alpha+\mu+1-\varepsilon}|\xi|^{-1}}{\mu+1}\Big\}\Big] \\
\le& C_{17}E_{\delta_1\delta_2}(0,\xi)^2\exp\Big[(\rho(t)-\rho_0)\langle\xi\rangle_\nu^\kappa + C_{18}\log\big(g_1(t)\delta_1^{-1}+1\big) \\
&+ C_{19}\log\big(g_2(t)\delta_2^{-1}+1\big) + h(t)\big\{\delta_1^{\alpha+1-\varepsilon}\delta_2^{-\frac{\gamma}{2}}|\xi| + \delta_2^{\frac{\gamma}{2}+1-\varepsilon}|\xi| \\
&+ \delta_1^{-\alpha+\eta+1-\varepsilon} + \delta_2^{-\frac{\gamma}{2}+\lambda+1-\varepsilon} + \delta_1^{-\alpha+\mu+1-\varepsilon}|\xi|^{-1}\big\}\Big],
\end{aligned}
$$

where $g_1(t)$, $g_2(t)$ and $h(t)$ are increasing functions satisfying $g_1(0)=0$, $g_2(0)=0$ and $h(0)=0$.

3.4 Energy inequality in case of $\alpha \le \beta$

If $\alpha \le 2\beta$, we can see $\gamma=\alpha$. Supossing that $\delta_1=\delta_2$, (3.74) is changed into

$$
\begin{aligned}
E_{\delta_1\delta_2}(t,\xi)^2 \le& C_{17}E_{\delta_1\delta_2}(0,\xi)^2\exp\Big[(\rho(t)-\rho_0)\langle\xi\rangle_\nu^\kappa + (C_{18}+C_{19})\log\big(g(t)\delta_1^{-1}+1\big) \\
&+ h(t)\delta_1^{-\varepsilon}\big\{2\delta_1^{\frac{\alpha}{2}+1}|\xi| + \delta_1^{-\alpha+\eta+1} + \delta_1^{-\frac{\alpha}{2}+\lambda+1} + \delta_1^{-\alpha+\mu+1}|\xi|^{-1}\big\}\Big],
\end{aligned}
$$

where $g(t)=g_1(t)+g_2(t)$.

Put

$$
(3.75)\qquad \delta_1(=\delta_2) = \langle\xi\rangle_{\nu_1}^{-\min\{\frac{1}{3\alpha-2\eta},\frac{1}{\alpha-\lambda},\frac{4}{3\alpha-2\mu}\}},
$$

where $\nu_1>0$ is determined sufficiently large such that δ_1 and δ_2 satisfy (3.14).

Then we obtain

$$
\begin{aligned}
&E_{\delta_1\delta_2}(t,\xi)^2 \\
\le& C_{17}E_{\delta_1\delta_2}(0,\xi)^2\exp\Big[(\rho(t)-\rho_0)\langle\xi\rangle_\nu^\kappa \\
&+ (C_{18}+C_{19})\min\Big\{\frac{1}{3\alpha-2\eta},\frac{1}{\alpha-\lambda},\frac{4}{3\alpha-2\mu}\Big\}\log\big(g(t)\langle\xi\rangle_{\nu_1}+1\big) \\
&+ h(t)\delta_1^{-\varepsilon}\Big\{2\langle\xi\rangle_{\nu_1}^{-\min\{-\frac{2(\alpha-\eta-1)}{3\alpha-2\eta},-\frac{\frac{\alpha}{2}-\lambda-1}{\alpha-\lambda},-\frac{\alpha-2\mu-4}{3\alpha-2\mu}\}} + \langle\xi\rangle_{\nu_1}^{\min\{\frac{2(\alpha-\eta-1)}{3\alpha-2\eta},\frac{\alpha-\eta-1}{\alpha-\lambda},\frac{4(\alpha-\eta-1)}{3\alpha-2\mu}\}} \\
&+ \langle\xi\rangle_{\nu_1}^{\min\{\frac{2(\frac{\alpha}{2}-\lambda-1)}{3\alpha-2\eta},\frac{\frac{\alpha}{2}-\lambda-1}{\alpha-\lambda},\frac{4(\frac{\alpha}{2}-\lambda-1)}{3\alpha-2\mu}\}} + \langle\xi\rangle_{\nu_1}^{\min\{\frac{2\eta-\alpha-2\mu-2}{3\alpha-2\eta},\frac{\lambda-\mu-1}{\alpha-\lambda},\frac{\alpha-2\mu-4}{3\alpha-2\mu}\}}\Big\}\Big]
\end{aligned}
$$

$$\le C_{17}E_{\delta_1\delta_2}(0,\xi)^2\exp\Big[(\rho(t)-\rho_0)\langle\xi\rangle_\nu^\kappa + C_{20}\log\big(g(t)\langle\xi\rangle_{\nu_1}+1\big)$$
$$+ h(t)\delta_1^{-\varepsilon}\Big\{2\langle\xi\rangle_{\nu_1}^{\max\{\frac{2(\alpha-\eta-1)}{3\alpha-2\eta},\frac{\frac{\alpha}{2}-\lambda-1}{\alpha-\lambda},\frac{\alpha-2\mu-4}{3\alpha-2\mu}\}} + \langle\xi\rangle_{\nu_1}^{\frac{2(\alpha-\eta-1)}{3\alpha-2\eta}} + \langle\xi\rangle_{\nu_1}^{\frac{\frac{\alpha}{2}-\lambda-1}{\alpha-\lambda}} + \langle\xi\rangle_{\nu_1}^{\frac{\alpha-2\mu-4}{3\alpha-2\mu}}\Big\}\Big]$$
$$\le C_{17}E_{\delta_1\delta_2}(0,\xi)^2\exp\Big[(\rho(t)-\rho_0)\langle\xi\rangle_\nu^\kappa + C_{20}\log\big(g(t)\langle\xi\rangle_{\nu_1}+1\big)$$
$$+ C_{21}h(t)\langle\xi\rangle_{\nu_1}^{\max\{\frac{2(\alpha-\eta-1)}{3\alpha-2\eta},\frac{\frac{\alpha}{2}-\lambda-1}{\alpha-\lambda},\frac{\alpha-2\mu-4}{3\alpha-2\mu}\}+M\varepsilon}\Big].$$

Lemma 3.4. $ab+1\le\big(a^p+1\big)^{\frac{b^p}{p}}$ for $^\forall a\ge 0,\ ^\forall b\ge 1,\ 0<^\forall p<1.$

Proof. We first invistigate the function $f(x)=\big(x+1\big)^{b^p}-\big(xb^p+1\big)$. Defferentiating $f(x)$ in x, we get $f'(x)=b^p\big(x+1\big)^{b^p-1}-b^p=b^p\big\{\big(x+1\big)^{b^p-1}-1\big\}$. Since $b^p-1\ge 0$, we find that $f'(x)\ge 0$ for $^\forall x\ge 0$. Noting that $f(0)=0$ and putting $x=a^p$, we get $f\big(a^p\big)\ge 0$ i.e., $\big(a^p+1\big)^{b^p}\ge\big(a^pb^p+1\big)$. Hence we have

$$ab+1\le\big(a^pb^p+1\big)^{\frac{1}{p}}\le\big(a^p+1\big)^{\frac{b^p}{p}}.$$

From Lemma 3.4 we put $a=g(t)$, $b=\langle\xi\rangle_{\nu_1}$, $p=\max\{\frac{2(\alpha-\eta-1)}{3\alpha-2\eta},\frac{\frac{\alpha}{2}-\lambda-1}{\alpha-\lambda},\frac{\alpha-2\mu-4}{3\alpha-2\mu}\}$ and get $\log\big(g(t)\langle\xi\rangle_{\nu_1}+1\big)\le\frac{1}{p}\langle\xi\rangle_{\nu_1}^{\max\{\frac{2(\alpha-\eta-1)}{3\alpha-2\eta},\frac{\frac{\alpha}{2}-\lambda-1}{\alpha-\lambda},\frac{\alpha-2\mu-4}{3\alpha-2\mu}\}}\log\big(g(t)^p+1\big)$. Thus we have for $\nu\ge\nu_1$

$$E_{\delta_1\delta_2}(t,\xi)^2\le C_{17}E_{\delta_1\delta_2}(0,\xi)^2\exp\Big[\langle\xi\rangle_\nu^\kappa\Big\{(\rho(t)-\rho_0)+\Big(\frac{C_{20}}{p}\log\big(g(t)^p+1\big)+C_{21}h(t)\Big)$$
$$\times\langle\xi\rangle_\nu^{\max\{\frac{2(\alpha-\eta-1)}{3\alpha-2\eta},\frac{\frac{\alpha}{2}-\lambda-1}{\alpha-\lambda},\frac{\alpha-2\mu-4}{3\alpha-2\mu}\}+M\varepsilon-\kappa}\Big\}\Big].$$

Now define the function $\rho(t)$ by

$$\rho(t)=\rho_0-C_T\Big(\frac{C_{20}}{p}\log\big(g(t)^p+1\big)+C_{21}h(t)\Big),$$

where C_T is the constant such that $\rho(T)>0$. Then

$$E_{\delta_1\delta_2}(t,\xi)^2\le C_{17}E_{\delta_1\delta_2}(0,\xi)^2\exp\Big[\langle\xi\rangle_\nu^\kappa\Big(\frac{C_{20}}{p}\log\big(g(t)^p+1\big)+C_{21}h(t)\Big)$$
$$\times\Big(\langle\xi\rangle_\nu^{\max\{\frac{2(\alpha-\eta-1)}{3\alpha-2\eta},\frac{\frac{\alpha}{2}-\lambda-1}{\alpha-\lambda},\frac{\alpha-2\mu-4}{3\alpha-2\mu}\}+M\varepsilon-\kappa}-C_T\Big)\Big].$$

Therefore if

$$\kappa>\max\Big\{\frac{2(\alpha-\eta-1)}{3\alpha-2\eta},\frac{\frac{\alpha}{2}-\lambda-1}{\alpha-\lambda},\frac{\alpha-2\mu-4}{3\alpha-2\mu}\Big\}, \tag{3.76}$$

by taking $\varepsilon > 0$ small enough and $\nu(\geq \nu_1)$ large enough, we have the energy inequality

$$E_{\delta_1\delta_2}(t,\xi)^2 \leq C_{17} E_{\delta_1\delta_2}(0,\xi)^2 \quad \text{for } {}^\forall t \in [0,T]. \tag{3.77}$$

Since $s = \kappa^{-1}$ and $\alpha = \gamma$, the inequality (3.76) can be also written by

$$s < \min\Big\{\frac{3\alpha - 2\eta}{2(\alpha-\eta-1)}, \frac{2(\gamma-\lambda)}{\gamma-2\lambda-2}, \frac{3\alpha-2\mu}{\alpha-2\mu-4}\Big\}. \tag{3.78}$$

3.5 Energy inequality in case of $\alpha > \beta$

If $\alpha > 2\beta$, we see $\gamma = 2\beta$. If we suposse that $\delta_1 = \delta_2^{\frac{2\beta+1}{\alpha+1}}$, (3.74) is changed into

$$\begin{aligned} E_{\delta_1\delta_2}(t,\xi)^2 \leq & C_{17} E_{\delta_1\delta_2}(0,\xi)^2 \exp\Big[(\rho(t)-\rho_0)\langle\xi\rangle_\nu^\kappa + (C_{18}+C_{19})\log\big(g(t)\delta_2^{-1}+1\big) \\ & + h(t)\delta_2^{-\varepsilon}\big\{2\delta_2^{\beta+1}|\xi| + \delta_2^{\frac{(2\beta+1)(\eta-\alpha+1)}{\alpha+1}} + \delta_2^{\lambda-\beta+1} + \delta_2^{\frac{(2\beta+1)(\mu-\alpha+1)}{\alpha+1}}|\xi|^{-1}\big\}\Big], \end{aligned}$$

here we have used $\delta_1^{-1} < \delta_2^{-1}$ and $\delta_1^{-\varepsilon} < \delta_2^{-\varepsilon}$.

We put

$$\begin{aligned} &\delta_2 = \langle\xi\rangle_{\nu_1}^{-\min\{\frac{\alpha+1}{3\alpha\beta-2\eta\beta-\beta-\eta+2\alpha}, \frac{1}{2\beta-\lambda}, \frac{2(\alpha+1)}{3\alpha\beta-2\mu\beta-\beta-\mu+2\alpha}\}} \\ &\Big(\delta_1 = \langle\xi\rangle_{\nu_1}^{-\min\{\frac{2\beta+1}{3\alpha\beta-2\eta\beta-\beta-\eta+2\alpha} \frac{2\beta+1}{(2\beta-\lambda)(\alpha+1)}, \frac{2(2\beta+1)}{3\alpha\beta-2\mu\beta-\beta-\mu+2\alpha}\}}\Big), \end{aligned} \tag{3.79}$$

where $\nu_1 > 0$ is determined sufficiently large such that δ_1 and δ_2 satisfy (3.14).

Then by Lemma 3.4 we obtain for $\nu \geq \nu_1$

$$\begin{aligned} &E_{\delta_1\delta_2}(t,\xi)^2 \\ &\leq C_{17} E_{\delta_1\delta_2}(0,\xi)^2 \exp\Big[(\rho(t)-\rho_0)\langle\xi\rangle_\nu^\kappa + (C_{18}+C_{19})\log\big(g(t)\langle\xi\rangle_{\nu_1}+1\big) \\ &\quad \times \min\Big\{\frac{\alpha+1}{3\alpha\beta-2\eta\beta-\beta-\eta+2\alpha}, \frac{1}{2\beta-\lambda}, \frac{2(\alpha+1)}{3\alpha\beta-2\mu\beta-\beta-\mu+2\alpha}\Big\} \\ &\quad + h(t)\delta_2^{-\varepsilon}\Big\{2\langle\xi\rangle_{\nu_1}^{-\min\{-\frac{(2\beta+1)(\alpha-\eta-1)}{3\alpha\beta-2\eta\beta-\beta-\eta+2\alpha}, -\frac{\beta-\lambda-1}{2\beta-\lambda}, -\frac{\alpha\beta-2\mu\beta-3\beta-\mu-2}{3\alpha\beta-2\mu\beta-\beta-\mu+2\alpha}\}} \\ &\qquad + \langle\xi\rangle_{\nu_1}^{\min\{\frac{(2\beta+1)(\alpha-\eta-1)}{3\alpha\beta-2\eta\beta-\beta-\eta+2\alpha}, \frac{(2\beta+1)(\alpha-\eta-1)}{(2\beta-\lambda)(\alpha+1)}, \frac{2(2\beta+1)(\alpha-\eta-1)}{3\alpha\beta-2\mu\beta-\beta-\mu+2\alpha}\}} \\ &\qquad + \langle\xi\rangle_{\nu_1}^{\min\{\frac{(\alpha+1)(\beta-\lambda-1)}{3\alpha\beta-2\eta\beta-\beta-\eta+2\alpha}, \frac{\beta-\lambda-1}{2\beta-\lambda}, \frac{2(\alpha+1)(\beta-\lambda-1)}{3\alpha\beta-2\mu\beta-\beta-\mu+2\alpha}\}} \\ &\qquad + \langle\xi\rangle_{\nu_1}^{\min\{\frac{-\alpha\beta-2\mu\beta+2\eta\beta-\beta-\alpha+\eta-\mu-1}{3\alpha\beta-2\eta\beta-\beta-\eta+2\alpha}, \frac{\alpha\lambda-2\beta\mu-4\beta+\alpha+\lambda-\mu-1}{(2\beta-\lambda)(\alpha+1)}, \frac{\alpha\beta-2\mu\beta-3\beta-\mu-2}{3\alpha\beta-2\mu\beta-\beta-\mu+2\alpha}\}}\Big\}\Big] \end{aligned} \tag{3.80}$$

$$
\begin{aligned}
\leq & C_{17} E_{\delta_1\delta_2}(0,\xi)^2 \exp\Big[(\rho(t)-\rho_0)\langle\xi\rangle_\nu^\kappa + C_{22}\log\big(g(t)\langle\xi\rangle_{\nu_1}+1\big) \\
&\quad + h(t)\delta_2^{-\varepsilon}\Big\{2\langle\xi\rangle_{\nu_1}^{\max\{\frac{(2\beta+1)(\alpha-\eta-1)}{3\alpha\beta-2\eta\beta-\beta-\eta+2\alpha},\frac{\beta-\lambda-1}{2\beta-\lambda},\frac{\alpha\beta-2\mu\beta-3\beta-\mu-2}{3\alpha\beta-2\mu\beta-\beta-\mu+2\alpha}\}} \\
&\quad + \langle\xi\rangle_{\nu_1}^{\frac{(2\beta+1)(\alpha-\eta-1)}{3\alpha\beta-2\eta\beta-\beta-\eta+2\alpha}} + \langle\xi\rangle_{\nu_1}^{\frac{\beta-\lambda-1}{2\beta-\lambda}} + \langle\xi\rangle_{\nu_1}^{\frac{\alpha\beta-2\mu\beta-3\beta-\mu-2}{3\alpha\beta-2\mu\beta-\beta-\mu+2\alpha}}\Big\}\Big] \\
\leq & C_{17} E_{\delta_1\delta_2}(0,\xi)^2 \exp\Big[(\rho(t)-\rho_0)\langle\xi\rangle_\nu^\kappa + C_{22}\log\big(g(t)\langle\xi\rangle_{\nu_1}+1\big) \\
&\quad + C_{23}h(t)\langle\xi\rangle_{\nu_1}^{\max\{\frac{(2\beta+1)(\alpha-\eta-1)}{3\alpha\beta-2\eta\beta-\beta-\eta+2\alpha},\frac{\beta-\lambda-1}{2\beta-\lambda},\frac{\alpha\beta-2\mu\beta-3\beta-\mu-2}{3\alpha\beta-2\mu\beta-\beta-\mu+2\alpha}\}+M'\varepsilon}\Big] \\
\leq & C_{17} E_{\delta_1\delta_2}(0,\xi)^2 \exp\Big[\langle\xi\rangle_\nu^\kappa\Big\{(\rho(t)-\rho_0) + \Big(\frac{C_{22}}{q}\log\big(g(t)^q+1\big) + C_{23}h(t)\Big) \\
&\quad \times \langle\xi\rangle_\nu^{\max\{\frac{(2\beta+1)(\alpha-\eta-1)}{3\alpha\beta-2\eta\beta-\beta-\eta+2\alpha},\frac{\beta-\lambda-1}{2\beta-\lambda},\frac{\alpha\beta-2\mu\beta-3\beta-\mu-2}{3\alpha\beta-2\mu\beta-\beta-\mu+2\alpha}\}+M'\varepsilon-\kappa}\Big\}\Big],
\end{aligned}
$$

where $q=\max\{\frac{(2\beta+1)(\alpha-\eta-1)}{3\alpha\beta-2\eta\beta-\beta-\eta+2\alpha},\frac{\beta-\lambda-1}{2\beta-\lambda},\frac{\alpha\beta-2\mu\beta-3\beta-\mu-2}{3\alpha\beta-2\mu\beta-\beta-\mu+2\alpha}\}$.

Now define the function $\rho(t)$ by

$$
\rho(t)=\rho_0-C'_T\Big(\frac{C_{22}}{q}\log\big(g(t)^q+1\big)+C_{23}h(t)\Big),
$$

where C'_T is the constant such that $\rho(T)>0$. Hence by (3.80) we get

$$
\begin{aligned}
E_{\delta_1\delta_2}(t,\xi)^2 \leq & C_{17}E_{\delta_1\delta_2}(0,\xi)^2\exp\Big[\langle\xi\rangle_\nu^\kappa\Big(\frac{C_{22}}{q}\log\big(g(t)^q+1\big)+C_{23}h(t)\Big) \\
&\quad \times\Big(\langle\xi\rangle_\nu^{\max\{\frac{(2\beta+1)(\alpha-\eta-1)}{3\alpha\beta-2\eta\beta-\beta-\eta+2\alpha},\frac{\beta-\lambda-1}{2\beta-\lambda},\frac{\alpha\beta-2\mu\beta-3\beta-\mu-2}{3\alpha\beta-2\mu\beta-\beta-\mu+2\alpha}\}+M'\varepsilon-\kappa}-C'_T\Big)\Big].
\end{aligned}
$$

Therefore if

$$
\kappa>\max\Big\{\frac{(2\beta+1)(\alpha-\eta-1)}{3\alpha\beta-2\eta\beta-\beta-\eta+2\alpha},\frac{\beta-\lambda-1}{2\beta-\lambda},\frac{\alpha\beta-2\mu\beta-3\beta-\mu-2}{3\alpha\beta-2\mu\beta-\beta-\mu+2\alpha}\Big\}, \tag{3.81}
$$

by taking $\varepsilon>0$ small enough and $\nu(\geq\nu_1)$ large enough, we have the energy inequality

$$
E_{\delta_1\delta_2}(t,\xi)^2\leq C_{17}E_{\delta_1\delta_2}(0,\xi)^2 \quad \text{for } {}^\forall t\in[0,T]. \tag{3.82}
$$

Since $s = \kappa^{-1}$ and $2\beta = \gamma$, (3.81) can be also written as

$$(3.83)\quad s < \min\Big\{\frac{3\alpha\beta - 2\eta\beta - \beta - \eta + 2\alpha}{(2\beta+1)(\alpha-\eta-1)}, \frac{2\beta-\lambda}{\beta-\lambda-1}, \frac{3\alpha\beta - 2\mu\beta - \beta - \mu + 2\alpha}{\alpha\beta - 2\mu\beta - 3\beta - \mu - 2}\Big\}$$
$$= \min\Big\{\frac{3\alpha-2\eta}{2(\alpha-\eta-1)} + \frac{\alpha-2\beta}{2(2\beta+1)(\alpha-\eta-1)}, \frac{2(2\beta-\lambda)}{2\beta-2\lambda-2},$$
$$\frac{3\alpha-2\mu}{\alpha-2\mu-4} + \frac{4(\alpha-\mu-1)(\alpha-2\beta)}{(\alpha-2\mu-4)(2\beta+1)\{(\alpha-2\mu-4)-\frac{\alpha-2\beta}{2\beta+1}\}}\Big\}$$
$$= \min\Big\{\frac{3\alpha-2\eta}{2(\alpha-\eta-1)} + \frac{\alpha-\gamma}{2(\gamma+1)(\alpha-\eta-1)}, \frac{2(\gamma-\lambda)}{\gamma-2\lambda-2},$$
$$\frac{3\alpha-2\mu}{\alpha-2\mu-4} + \frac{4(\alpha-\mu-1)(\alpha-\gamma)}{(\alpha-2\mu-4)(\gamma+1)\{(\alpha-2\mu-4)-\frac{\alpha-\gamma}{\gamma+1}\}}\Big\}.$$

3.6 Proof of Theorem 3.1

At last by (3.77), (3.78), (3.82) and (3.83) we have the energy inequality

$$(3.84)\qquad E_{\delta_1\delta_2}(t,\xi)^2 \le C_{17} E_{\delta_1\delta_2}(0,\xi)^2 \quad \text{for } {}^\forall t \in [0,T],$$

if s satisfies

$$(3.85)\quad s < \min\Big\{\frac{3\alpha-2\eta}{2(\alpha-\eta-1)} + \frac{\alpha-\gamma}{2(\gamma+1)(\alpha-\eta-1)}, \frac{2(\gamma-\lambda)}{\gamma-2\lambda-2},$$
$$\frac{3\alpha-2\mu}{\alpha-2\mu-4} + \frac{4(\alpha-\mu-1)(\alpha-\gamma)}{(\alpha-2\mu-4)(\gamma+1)\{(\alpha-2\mu-4)-\frac{\alpha-\gamma}{\gamma+1}\}}\Big\}.$$

We remark that in case of $\alpha \le 2\beta$, i.e., $\gamma = \alpha$, (3.85) becomes (3.78).

Furtheremore we shall change (3.84) into an energy inequality based on v, v_t and v_{tt}. We first show that $E_{\delta_1\delta_2}(t,\xi)^2$ is bounded from below by the absolute values of these terms. By (3.15) and (3.40), we find that

$$(3.86)\quad E_{\delta_1\delta_2}(t,\xi)^2 \ge \frac{1}{3} e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\big(t^\alpha\xi^2 + \omega_{\delta_1}\xi^2\big)^2|v|^2 = \frac{1}{3} e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\big\{t^\alpha + \chi_{\delta_1} + (\omega_{\delta_1} - \chi_{\delta_1})\big\}^2\xi^4|v|^2$$
$$\ge \frac{1}{3} e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\big\{t^\alpha + \chi_{\delta_1} - |\omega_{\delta_1} - \chi_{\delta_1}|\big\}^2\xi^4|v|^2 \ge \frac{1}{3} e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\big(\delta_1^\alpha - C_0\delta_1^\alpha\big)^2\xi^4|v|^2$$
$$= \frac{1}{3}(1-C_0)^2\delta_1^{2\alpha} e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\xi^4|v|^2 \quad (0 < C_0 < 1).$$

Similarly by (3.16) we also find that

$$(3.87)\qquad \begin{aligned} E_{\delta_1\delta_2}(t,\xi)^2 &\geq e^{\rho(t)\langle\xi\rangle_\nu^{\kappa}}\Big(\frac{1}{4}t^{2\beta}+t^\alpha+\pi_{\delta_2}(t)\Big)\xi^2|v_t|^2 \\ &\geq (1-C_0)\delta_2^\gamma e^{\rho(t)\langle\xi\rangle_\nu^{\kappa}}\xi^2|v_t|^2 \quad (0<C_0<1). \end{aligned}$$

As for v_{tt}, we obtain

$$(3.88)\qquad \begin{aligned} E_{\delta_1\delta_2}(t,\xi)^2 &\geq e^{\rho(t)\langle\xi\rangle_\nu^{\kappa}}\Big(\Big|v_{tt}+\frac{i}{2}t^\beta\xi v_t\Big|^2+\frac{1}{4}t^{2\beta}\xi^2|v_t|^2\Big) \\ &= e^{\rho(t)\langle\xi\rangle_\nu^{\kappa}}\Big(\frac{1}{2}|v_{tt}|^2+\frac{1}{2}|v_{tt}|^2+\Re\big(v_{tt},it^\beta\xi v_t\big)+\frac{1}{2}t^{2\beta}\xi^2|v_t|^2\Big) \\ &= e^{\rho(t)\langle\xi\rangle_\nu^{\kappa}}\Big(\frac{1}{2}|v_{tt}|^2+\frac{1}{2}|v_{tt}+it^\beta\xi v_t|^2\Big) \geq \frac{1}{2}e^{\rho(t)\langle\xi\rangle_\nu^{\kappa}}|v_{tt}|^2. \end{aligned}$$

While $E_{\delta_1\delta_2}(0,\xi)^2$ is dominated by the absolute values of the initial data. From the definition of $\omega_{\delta_1}(t)$, we see that $\omega_{\delta_1}(0)=\frac{1}{\delta_1^\alpha}\int_0^{\delta_1}(\delta_1^\alpha-\tau^\alpha)\varphi\Big(\frac{\tau}{\delta_1^\alpha}\Big)d\tau \leq \int_0^{\delta_1}\varphi\Big(\frac{\tau}{\delta_1^\alpha}\Big)d\tau \leq \frac{1}{2}\delta_1^\alpha$. Similarly, from the definition of $\pi_{\delta_2}(t)$, we also see that $\pi_{\delta_2}(0)\leq\frac{1}{2}\delta_2^\gamma$. Therefore we have

$$(3.89)\qquad \begin{aligned} E_{\delta_1\delta_2}(0,\xi)^2 =&e^{\rho_0\langle\xi\rangle_\nu^{\kappa}}\big\{\big|v_2+\omega_{\delta_1}(0)\xi^2v_0\big|^2+|v_2|^2+\pi_{\delta_2}(0)\xi^2|v_1|^2\big\} \\ \leq&e^{\rho_0\langle\xi\rangle_\nu^{\kappa}}\big\{3|v_2|^2+2\omega_{\delta_1}(0)^2\xi^4|v_0|^2+\pi_{\delta_2}(0)\xi^2|v_1|^2\big\} \\ \leq&e^{\rho_0\langle\xi\rangle_\nu^{\kappa}}\big\{3|v_2|^2+\frac{1}{2}\delta_1^{2\alpha}\xi^4|v_0|^2+\frac{1}{2}\delta_2^\gamma\xi^2|v_1|^2\big\}. \end{aligned}$$

Consequently if s satisfies (3.85), by (3.84) and (3.86)-(3.89) we also have the energy inequality based on v, v_t and v_{tt}

$$\begin{aligned} e^{\rho(t)\langle\xi\rangle_\nu^{\frac{1}{s}}}\big(\delta_1^{2\alpha}\xi^4|v|^2+\delta_2^\gamma\xi^2|v_t|^2+|v_{tt}|^2\big) &\leq C_{24}e^{\rho_0\langle\xi\rangle_\nu^{\frac{1}{s}}}\big(\delta_1^{2\alpha}\xi^4|v_0|^2+\delta_2^\gamma\xi^2|v_1|^2+|v_2|^2\big) \\ &\leq C_{25}e^{\rho_0\langle\xi\rangle_\nu^{\frac{1}{s}}}\big(\langle\xi\rangle_\nu^4|v_0|^2+\langle\xi\rangle_\nu^2|v_1|^2+|v_2|^2\big) \\ &\leq C_{25}C_2, \end{aligned}$$

where δ_1 and δ_2 are defined in (3.75) or (3.79).

Hence in vitue of Paley-Wiener theorem, $\{u(\cdot,t);t\in[0,T]\}$ is bounded in G_0^s. Thus taking into account that u is a solution of the equation (3.1), we find that $u\in C^3([0,T],G_0^s)$. This concludes the proof of Theorem.