Chapter 2. Weakly Hyperbolic Systems with Holder Continuous Coefficients in Time

2.1 Introduction

There are few papers for the weakly hyperbolic systems. K. Kajitani got the Gevrey wellpossedness for the weakly hyperbolic systems with Leray-Volevich's weights(see [Ka1]). As for the analytic wellposedness P. D'Ancona and S. Spagnolo treated the nonlinear weakly hyperbolic systems(see [DS1]). Moreover E. Jannelli treated the weakly hyperbolic systems with the coefficients which belong to L^1(see [J2]).

For the strictly hyperbolic systems E. Jannelli also got the result concerned with the relation between the order of Gevrey classes and the regularity of the coefficients(see [J3]). In this chapter, we shall generalize his result to the weakly hyperbolic systems and investigate the relation among the Gevrey wellposedness and the regularity and the form of the matrices of the coefficients.

We shall consider the following system in $[0,T] \times \mathbf{R}^n_x$

$$(2.1) \qquad \begin{cases} \partial_t u = \sum_{h=1}^{n} A_h(t)\partial_h u + B(t)u \\ u(0,x) = u_0(x), \end{cases}$$

where $A_h(t)(1 \leq h \leq n)$, $B(t)$ are $N \times N$ matrices, while $u(t,x)$, $u_0(x)$ are N-vectors.

We denote by $C^\alpha([0,T])(0 < \alpha \leq 1)$ the space of α-Hölder continuous functions. Now we assume that

$$(2.2) \qquad A_h(t)(1 \leq h \leq n) \in C^\alpha([0,T]), \quad B(t) \in C^0([0,T])$$

and (2.1) is weakly hyperbolic, i.e.,

$$(2.3) \qquad \sum_{h=1}^{n} A_h(t)\xi_h \text{has real eigenvalues (allowing multiplicity) for } {}^\forall t \in [0,T],\ {}^\forall \xi \in \mathbf{R}^n_\xi.$$

We shall treat the following two cases.

CASE 1. No condition is imposed.

CASE 2. There exists a non-singular matrix $P(t,\xi)$ such that

$$P(t,\xi)A(t,\xi)P(t,\xi)^{-1} = \operatorname{diag}\{D_1, D_2, \cdots D_k\} \ (1 \leq^{\exists} k \leq N)$$
$$|P(t,\xi)|+|P(t,\xi)^{-1}| \leq^{\exists} C \quad \text{for} \quad t \in [0,T],\ |\xi| = 1,$$

where $A(t,\xi) = \sum_{h=1}^{n} A_h(t)\xi_h$ and $D_j (1 \leq j \leq k)$ are the triangular matrices whose diagonal components are real and whose sizes are $m_j \times m_j$.

Then we can prove the following theorem.

Theorem 2.1. *Let $0 < \rho_0 < \infty$ and $\nu_0 > 0$. Assume that the coefficients $A_h(t)(1 \leq h \leq n)$ and $B(t)$ satisfy (2.2), (2.3) and CASE 1 (resp. CASE 2). Then there exists $\nu > 0$ such that for any $u_0 \in L^2_{\rho_0,\kappa,\nu_0}(\mathbf{R}^n)$, the Cauchy problem (2.1) has unique solution $u(t,x) \in C^1([0,T],\ L^2_{\rho_1,\kappa,\nu}(\mathbf{R}^n)$, provided*

$$0 < \rho_1 < \rho_0, \quad 1 \leq s < \frac{\mu(1+\alpha^{-1})}{\mu(1+\alpha^{-1})-1}, \tag{2.4}$$

where μ is equal to the dimention of the system, i.e.,

$$\mu = N \tag{2.5}$$

(resp. the maximal sizes of D_j $(1 \leq j \leq k)$, i.e.,

$$\mu = \max_{1\leq i\leq k} m_i \tag{2.6}$$

), and $s = \kappa^{-1}$.

In CASE 1, we find that "No condition is imposed" means that the multiplicity of eigenvalues of $\sum_{h=1}^{n} A_h(t)\xi_h$ is variable. As for CASE 2 the following examples can be also treated.

Example 1. The multiplicity of eigenvalues of $\sum_{h=1}^{n} A_h(t)\xi_h$ is independent of t, ξ, i.e.,

$$\det(\lambda - \sum_{h=1}^{n} A_h(t)\xi_h) = \Pi_{i=1}^{k}(\lambda - \lambda_i(t,\xi))^{m_i} \quad \text{for} \quad {}^{\forall}t \in [0,T],\ {}^{\forall}\xi \in \mathbf{R}^n_\xi$$

with $1 \leq^{\exists} k \leq N$, ${}^{\exists}m_i \in \mathbf{N}^1$ $(1 \leq i \leq k)$, where $\lambda_i(t,\xi)$ $(1 \leq i \leq k$) satisfy that if $i \neq j$, $\lambda_i(t,\xi) \neq \lambda_j(t,\xi)$ for $t \in [0,T]$, $|\xi| = 1$.

We shall show in Appendix that Example 1 is included by CASE 2 and μ is equal to the maximal multiplicity of the eigenvalues of $\sum_{h=1}^{n} A_h(t)\xi_h$, i.e., $\mu = \max_{1\leq i\leq k} m_i$.

Example 2. The multiplicity of factors of all the elementary divisors of $\sum_{h=1}^{n} A_h(t)\xi_h$ is independent of t, ξ, i.e.,

$$e_l(\lambda) = \Pi_{i=1}^{k}(\lambda - \lambda_i(t,\xi))^{m(i,l)} \ (1 \le l \le N) \ \text{ for } \forall t \in [0,T], \ \forall \xi \in \mathbf{R}_\xi^n$$

with $1 \le^{\exists} k \le N$, $^{\exists}m(i,l) \in \mathbf{N}^1(1 \le l \le N, \ 1 \le i \le k)$, where $\lambda_i(t,\xi)(1 \le i \le k)$ satisfy that if $i \ne j$, $\lambda_i(t,\xi) \ne \lambda_j(t,\xi)$ for $t \in [0,T]$, $|\xi| = 1$.

By Jordan normal form, we can see that D_j $(1 \le j \le k)$ are the Jordan blocks whose sizes are $m(i,l) \times m(i,l)$ ($m(i,l)$ denotes the multiplicity of the factor $(\lambda - \lambda_i)$ of the elementary divisors $e_l(\lambda)$ of $\sum_{h=1}^{n} A_h(t)\xi_h$) and μ is equal to the maximal multiplicity of factors of the elementary divisors (or the minimal polynomial) of $\sum_{h=1}^{n} A_h(t)\xi_h$, i.e., $\mu = \max_{1 \le i \le k, 1 \le l \le N} m(i,l)$.

When the maximal multiplicity for factors of the minimal polynomial of $\sum_{h=1}^{n} A_h(t)\xi_h$ is equal to 1 in CASE 2, the system is symetrizable and K.Kajitani proved that the Cauchy problem (2.1) is G^s-wellposed $(1 < s < 1 + \alpha$)(see [Ka3]). Moreover when $\sum_{h=1}^{n} A_h(t)\xi_h$ has real distinct eigenvalues or is Hermitian, the Cauchy problem (2.1) is L^2-wellposed(see [M]). Concerned with the single equations of higher order, the condition corresponding to (2.4) is $1 \le s < 1 + \frac{\sigma}{\mu}$(see [OT]).

2.2 Preliminaries

In this section we shall construct the algebraic lemmas which play an important role to prove the theorem.

Lemma 2.2.A. *Let A be a $N \times N$ constant matrix which has real eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_N$ (allowing multiplicity). Then for $\forall \eta \in (0,1]$, there exists a non-singular matrix P_η such that*

$$P_\eta A P_\eta^{-1} = \tilde{A} + R_\eta \tag{2.7}$$

where $\tilde{A} =$ diag$\{\lambda_1, \lambda_2, \cdots, \lambda_N\}$ is Hermitian, and $P_\eta, P_\eta^{-1}, R_\eta$ satisfy that

$$|P_\eta| \le C_1, \quad |P_\eta^{-1}| \le C_2\eta^{1-N}, \quad |R_\eta| \le C_3\eta. \tag{2.8}$$

The constants $C_1, C_2 > 0$ are independent of A, but $C_3 > 0$ depends on $|A|$.

Proof. From linear algebra we find that there exists a unitary matrix P such that

$$PAP^{-1} = \tilde{A} + R \tag{2.9}$$

where $\tilde{A} = \operatorname{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_N\}$ is Hermitian, and R is a strictly lower triangular matrix with zeroes on the diagonal (see [S]).

Since $|\lambda_i| \leq |A|$ $(1 \leq i \leq N)$, we get

$$|R| \leq |PAP^{-1}| + |\tilde{A}| \leq C_1|A|C_2 + |A| = (C_1C_2 + 1)|A|. \tag{2.10}$$

Defining $Q_\eta = \operatorname{diag}\{1, \eta, \cdots, \eta^{N-1}\}$ and putting $P_\eta = Q_\eta P$, by (2.9) we have

$$P_\eta A P_\eta^{-1} = Q_\eta(\tilde{A} + R)Q_\eta^{-1} = \tilde{A} + R_\eta$$

where $R_\eta = Q_\eta R Q_\eta^{-1}$. Hence we get (2.7).

At last noting that $Q_\eta^{-1} = \operatorname{diag}\{1, \eta^{-1}, \cdots, \eta^{-(N-1)}\}$, we can easily estimate P_η, P_η^{-1} as follows

$$|P_\eta| \leq |Q_\eta||P| \leq 1 \cdot C_1 \equiv C_1.$$

$$|P_\eta^{-1}| \leq |P^{-1}||Q_\eta^{-1}| \leq C_2 \cdot \eta^{1-N} \equiv C_2\eta^{1-N}.$$

Here actually $C_1 = C_2 = 1$ since P is a unitary matrix.

Noting that $(R)_{ij} = 0$ for $j \geq i$ and (2.10), we can estimate R_η as follows.

$$\begin{aligned} |R_\eta| &= \max_{1 \leq j < i \leq N} |(R_\eta)_{ij}| = \max_{1 \leq j < i \leq N} |\eta^{i-j}(R)_{ij}| \\ &\leq \eta \max_{1 \leq j < i \leq N} |(R)_{ij}| = \eta|R| \\ &\leq (C_1C_2 + 1)|A|\eta \equiv C_3\eta. \end{aligned}$$

Hence we get (2.8).

Lemma 2.2.B. *Let $A(\xi)$ be a $N \times N$ matrix which has real eigenvalues $\lambda_1(\xi), \lambda_2(\xi), \cdots, \lambda_n(\xi)$ (allowing multiplicity), and is continuous and homogeneous of degree one in $\xi \in \mathbf{R}^n_\xi$. Then for $^\forall\eta \in (0, 1]$, there exists a non-singular matrix $P_\eta(\xi)$ such that*

$$P_\eta(\xi)A(\xi)P_\eta^{-1}(\xi) = \tilde{A}(\xi) + R_\eta(\xi) \tag{2.11}$$

where $\tilde{A}(\xi)$ is Hermitian, and $P_\eta(\xi), P_\eta^{-1}(\xi), R_\eta(\xi)$ satisfy that

$$|P_\eta(\xi)| \leq C_1, \quad |P_\eta(\xi)^{-1}| \leq C_2\eta^{1-N}, \quad |R_\eta(\xi)| \leq C_5\eta|\xi| \quad \text{for} \quad ^\forall\xi \in \mathbf{R}^n_\xi. \tag{2.12}$$

The constant $C_5 > 0$ is independent of ξ.

Proof. $S^{n-1} = \{\xi \in \mathbf{R}^n_\xi \ ; \ |\xi| = 1\}$ is a compact set, for any fixed $\varepsilon > 0$, there exists a finite partition Γ_i $(1 \le i \le l = l(\varepsilon))$ of S^{n-1} such that

$$\sup_{\xi_1, \xi_2 \in \Gamma_i, 1 \le i \le l} |\xi_1 - \xi_2| \le \varepsilon, \qquad \cup_i \Gamma_i = S^{n-1}.$$

Defining

$$A_\varepsilon(\xi) = \begin{cases} A(\xi^{(i)}) \cdot |\xi| & \text{for} \quad \xi \ne 0, \quad \dfrac{\xi}{|\xi|} \in \Gamma_i \ (1 \le i \le l) \\ 0 & \text{for} \quad \xi = 0, \end{cases}$$

with $^\exists \xi^{(i)} \in \Gamma_i$, we get from the hypotheses

$$|A(\xi) - A_\varepsilon(\xi)| \le C_6 \varepsilon |\xi|. \tag{2.13}$$

Now we apply Lemma 2.2.A to each constant matrix $A(\xi^{(i)})$. We can construct, for $^\exists \eta \in (0, 1]$, non-singular matrix $P_{i,\eta}$ such that

$$P_{i,\eta} A(\xi^{(i)}) P_{i,\eta}^{-1} = \tilde{A}_i + R_{i,\eta} \tag{2.14}$$

where $\tilde{A}_i = diag\{\lambda_1(\xi^{(i)}), \lambda_2(\xi^{(i)}), \cdots, \lambda_N(\xi^{(i)})\}$, $|P_{i,\eta}| \le C_1$, $|P_{i,\eta}^{-1}| \le C_2 \eta^{1-N}$, $|R_{i,\eta}| \le C_3 \eta$. The constant C_3 depends on $|A(\xi^{(i)})|$, however C_3 can be taken independently of ξ since $|A(\xi^{(i)})|$ is bounded for $^\forall \xi^{(i)} \in \Gamma_i$.

Hence, multiplying the both sides of (2.14) by $|\xi|$ and putting

$$P_\eta(\xi) = \begin{cases} P_{i,\eta} & \text{for} \quad \xi \ne 0, \quad \dfrac{\xi}{|\xi|} \in \Gamma_i \ (1 \le i \le l) \\ 0 & \text{for} \quad \xi = 0, \end{cases}$$

$$\tilde{A}(\xi) = \begin{cases} \tilde{A}_i |\xi| & \text{for} \quad \xi \ne 0, \quad \dfrac{\xi}{|\xi|} \in \Gamma_i \ (1 \le i \le l) \\ 0 & \text{for} \quad \xi = 0, \end{cases}$$

$$R'_\eta(\xi) = \begin{cases} R_{i,\eta} |\xi| & \text{for} \quad \xi \ne 0, \quad \dfrac{\xi}{|\xi|} \in \Gamma_i \ (1 \le i \le l) \\ 0 & \text{for} \quad \xi = 0, \end{cases}$$

we obtain

$$\begin{aligned} P_\eta(\xi) A(\xi) P_\eta(\xi)^{-1} &= P_\eta(\xi) A_\varepsilon(\xi) P_\eta(\xi)^{-1} + P_\eta(\xi)(A(\xi) - A_\varepsilon(\xi)) P_\eta(\xi)^{-1} \\ &= \tilde{A}(\xi) + R_\eta(\xi), \end{aligned}$$

where $\tilde{A}(\xi)$ is Hermitian, and $P_\eta(\xi), P_\eta(\xi)^{-1}, R_\eta(\xi)$ satisfy that

$$
\begin{aligned}
&|P_\eta(\xi)| \le C_1, \quad |P_\eta(\xi)^{-1}| \le \eta^{1-N} \\
&|R_\eta(\xi)| = |R'_\eta(\xi) + P_\eta(\xi)(A(\xi) - A_\varepsilon(\xi))P_\eta(\xi)^{-1}| \\
&\qquad \le C_3\eta|\xi| + C_1|A(\xi) - A_\varepsilon(\xi)|C_2\eta^{1-N} \\
&(\text{ using (2.13) and taking } \varepsilon = \eta^N) \\
&\qquad \le (C_3 + C_1C_2C_6)\eta|\xi| \\
&\qquad \equiv C_5\eta|\xi|.
\end{aligned}
$$

Hence we get (2.11), (2.12).

Lemma 2.2.C. *Let $T > 0$, $A(t,\xi)$ be a $N \times N$ matrix which has real eigenvalues $\lambda_1(t,\xi), \lambda_2(t,\xi), \cdots, \lambda_N(t,\xi)$ (allowing multiplicity), and is α-Hölder continuous in $t \in [0,T]$, and continuous and homogeneous of degree one in $\xi \in \mathbf{R}^n_\xi$. Then for $^\forall\eta \in (0,1]$, there exists a non-singular matrix $P_\eta(t,\xi)$ such that*

$$P_\eta(t,\xi)A(t,\xi)P_\eta^{-1}(t,\xi) = \tilde{A}(t,\xi) + R_\eta(t,\xi) \tag{2.15}$$

where $\tilde{A}(t,\xi)$ is Hermitian, and $P_\eta(t,\xi), P_\eta^{-1}(t,\xi), R_\eta(t,\xi)$ satisfy that

$$|P_\eta(t,\xi)| \le C_1, \quad |P_\eta(t,\xi)^{-1}| \le C_2\eta^{1-N}, \quad |R_\eta(t,\xi)| \le C_7\eta|\xi| \tag{2.16}$$

$$\int_0^t \left|\frac{\partial}{\partial s}P_\eta(s,\xi)\right| ds \le 2C_1 t\eta^{-\frac{N}{\alpha}} \tag{2.17}$$

for $^\forall t \in [0,T]$, $\quad ^\forall\xi \in \mathbf{R}^n_\xi$.

Proof. Since $\xi \in \mathbf{R}^n_\xi$ is fixed to the end of the proof, we shall omit the letter ξ.

For any fixed $\tau > 0$, we take a finite collection of disjoint intervals I_i ($1 \le i \le l = [t/\tau] + 1$) of $[0,t]$ such that

$$I_i = \begin{cases} [(i-1)\tau, i\tau) & \text{for } \ 1 \le i \le l-1 \\ [[t/\tau]\tau, t] & \text{for } \ i = l. \end{cases}$$

Defining $A_\tau(t) = A(t^{(i)})$ for $t \in I_i$ $(1 \le i \le l)$ with $^\exists t^{(i)} \in I_i$, we get from the hypothesis,

$$|A(t) - A_\tau(t)| \le C_8\tau^\alpha|\xi|. \tag{2.18}$$

Now applying Lemma 2.2.B to each matrix $A(t^{(i)})$, we can get

$$P_{i,\eta}A(t^{(i)})P_{i,\eta}^{-1} = \tilde{A}_i + R_{i,\eta}$$

where $\tilde{A}_i$ is Hermitian,

$$|P_{i,\eta}| \le C_1, \quad |P_{i,\eta}^{-1}| \le C_2\eta^{1-N}, \quad |R_{i,\eta}| \le C_5\eta|\xi|.$$

Hence putting

$$\begin{aligned} P_\eta(t) &= P_{i,\eta} && \text{for} \quad t \in I_i \quad (1 \le i \le l), \\ \tilde{A}(t) &= \tilde{A}_i && \text{for} \quad t \in I_i \quad (1 \le i \le l), \\ R'_\eta(t) &= R_{i,\eta} && \text{for} \quad t \in I_i \quad (1 \le i \le l), \end{aligned}$$

we obtain

$$P_\eta(t)A(t)P_\eta(t)^{-1} = \tilde{A}(t) + R_\eta(t), \tag{2.19}$$

where $\tilde{A}(t)$ is Hermitian, and

$$\begin{aligned} &|P_\eta(t)| \le C_1, \quad |P_\eta(t)^{-1}| \le C_2\eta^{1-N} \\ &|R_\eta(t)| = |R'_\eta(t) + P_\eta(t)(A(t) - A_\tau(t))P_\eta(t)^{-1} \\ &\qquad \le C_5\eta|\xi| + C_1|A(t) - A_\tau(t)|C_2\eta^{1-N} \\ &(\text{ using (2.18) and taking } \tau = \eta^{\frac{N}{\alpha}}) \\ &\qquad \le C_7 t\eta|\xi|. \end{aligned} \tag{2.20}$$

By (2.19) and (2.20) we get (2.15) and (2.16).

It remains the estimate (2.17). For any fixed $\tau > 0$, defining with Dirac function $\delta(t)$

$$\delta_i(t) = \delta(t - i\tau) \quad \text{for} \quad 1 \le i \le l-1,$$

and noting that $P_\eta(t)$ is the piecewise constant function satisfying

$$|P_{i,\eta} - P_{i-1,\eta}| \le |P_{i,\eta}| + |P_{i-1,\eta}| \le 2C_1 \quad (2 \le i \le l),$$

we obtain

$$\begin{aligned} \int_0^t \left|\frac{\partial}{\partial s}P_\eta(s)\right|ds &\le \int_0^t \sum_{i=1}^{l-1} 2C_1\delta_i(s)ds \\ &= 2C_1(l-1)\int_{-\infty}^{\infty}\delta(s)ds \\ &= 2C_1\left[\frac{t}{\tau}\right] \le 2C_1\frac{t}{\tau} = 2C_1 t\eta^{-\frac{N}{\alpha}}, \end{aligned}$$

here we used $\int_{-\infty}^{\infty}\delta(s)ds = 1$ and $\tau = \eta^{\frac{N}{\alpha}}$. Hence we get (2.17).

Lemma 2.2.D. *Let $T > 0$, $A(t,\xi)$ be a $N \times N$ matrix which has real eigenvalues (allowing multiplicity), and is α-Hölder continuous in $t \in [0,T]$, and continuous and homogeneous of degree one in $\xi \in \mathbf{R}^n_\xi$. Moreover assume that there exists a non-singular matrix $P(t,\xi)$ such that*

$$P(t,\xi)A(t,\xi)P(t,\xi)^{-1} = \operatorname{diag}\{D_1, D_2, \cdots D_k\} \ (1 \leq^{\exists} k \leq N)$$
$$|P(t,\xi)| + |P(t,\xi)^{-1}| \leq^{\exists} C,$$

where $D_j (1 \leq j \leq k)$ are the triangular matrices whose diagonal components are real and whose sizes are $m_j \times m_j$. Then for $^\forall \eta \in (0,1]$, there exists a non-singular matrix $P_\eta(t,\xi)$ such that

$$P_\eta(t,\xi)A(t,\xi)P_\eta^{-1}(t,\xi) = \tilde{A}(t,\xi) + R_\eta(t,\xi), \tag{2.21}$$

where $\tilde{A}(t,\xi)$ is Hermitian, and $P_\eta(t,\xi), P_\eta^{-1}(t,\xi), R_\eta(t,\xi)$ satisfy that

$$|P_\eta(t,\xi)| \leq C_9, \quad |P_\eta(t,\xi)^{-1}| \leq C_{10}\eta^{1-N}, \quad |R_\eta(t,\xi)| \leq C_{11}\eta|\xi| \tag{2.22}$$
$$\int_0^t \left| \frac{\partial}{\partial s} P_\eta(s,\xi) \right| ds \leq 2C_9 t \eta^{-\frac{r}{\alpha}}$$

for $^\forall t \in [0,T]$, $^\forall \xi \in \mathbf{R}^n_\xi$, where $r = \max_{1 \leq j \leq k} m_j$.

Proof. Since $\xi \in \mathbf{R}^n_\xi$ is fixed to the end of the proof, we shall omit the letter ξ.

For $A(t)$ using again the disjoint intervals I_i ($1 \leq i \leq l$) and $A_\tau(t)(= A(t^{(i)})$ for $t \in I_i$ with $^\exists t^{(i)} \in I_i$) of Lemma 2.2.C, we get (2.18).

From the assumption, for each matrix $A(t^{(i)})$, there exists a non-singular matrix P_i such that

$$P_i A(t^{(i)}) P_i^{-1} = \operatorname{diag}\{D_1^{(i)}, D_2^{(i)}, \cdots D_k^{(i)}\} \ (1 \leq^{\exists} k \leq N)$$
$$|P_i| + |P_i^{-1}| \leq^{\exists} C,$$

where $D_j^{(i)} (1 \leq j \leq k)$ are the triangular matrices whose diagonal components are real and whose sizes are $m_j \times m_j$.

Defining

$$Q_\eta = \operatorname{diag}\{1, \cdots, \eta^{m_1-1}, 1, \cdots, \eta^{m_2-1}, \cdots\cdots, 1, \cdots, \eta^{m_k-1}\}.$$

and putting $P_{i,\eta} = Q_\eta P_i$, we obtain

$$\begin{aligned} P_{i,\eta} A(t^{(i)}) P_{i,\eta}^{-1} &= Q_\eta P_i A(t^{(i)}) P_i^{-1} Q_\eta^{-1} \\ &= Q_\eta \operatorname{diag}\{D_1^{(i)}, D_2^{(i)}, \cdots, D_k^{(i)}\} Q_\eta^{-1} \\ &= \tilde{A}_i + R_\eta, \end{aligned}$$

where $\tilde{A}_i$ is Hermitian, and

$$|P_{i,\eta}| \leq C_9, \quad |P_{i,\eta}^{-1}| \leq C_{10}\eta^{1-r}, \quad |R_{i,\eta}| \leq C_{12}\eta|\xi|.$$

Hence we can connect the proof of Lemma 2.2.C and get (2.21), (2.22).

2.3. Proof of Theorem 2.1

For the proof of Theorem for CASE 1 and CASE 2, we use Lemma 2.2.C and Lemma 2.2.D respectively. The defference of two lemmas is only the meaning of the parameter μ. Therefore it is sufficient to prove Theorem 2.1 in CASE 1.

Our task is to derive the energy estimates. By Fourier transform the system (2.1) can be changed to the form

$$\partial_t v = iA(t,\xi)v + B(t)v \tag{2.23}$$

where $A(t,\xi) = \sum_{h=1}^n A_h(t)\xi_h$.

Furtheremore we shall change the system (2.23). With some function $\rho(t) \in C^1([0,\ T])$ and some constant $\kappa \in (0,\ 1]$, putting $w(t,\xi) = P_\eta(t,\xi)e^{\rho(t)\langle\xi\rangle_\nu^\kappa} v(t,\xi)$, and multiplying the both sides of (2.22) by $P_\eta(t,\xi)e^{\rho(t)\langle\xi\rangle_\nu^\kappa}$, we have

$$\begin{aligned} &e^{\rho(t)\langle\xi\rangle_\nu^\kappa} P_\eta(t,\xi)\partial_t\{e^{-\rho(t)\langle\xi\rangle_\nu^\kappa} P_\eta(t,\xi)^{-1} w(t,\xi)\} \\ &\qquad = ie^{\rho(t)\langle\xi\rangle_\nu^\kappa} P_\eta(t,\xi)A(t,\xi)e^{-\rho(t)\langle\xi\rangle_\nu^\kappa} P_\eta(t,\xi)^{-1} w(t,\xi) \\ &\qquad\quad + e^{\rho(t)\langle\xi\rangle_\nu^\kappa} P_\eta(t,\xi)B(t)e^{-\rho(t)\langle\xi\rangle_\nu^\kappa} P_\eta(t,\xi)^{-1} w(t,\xi). \end{aligned}$$

Then we obtain

$$\begin{aligned} \text{left hand side} &= e^{\rho(t)\langle\xi\rangle_\nu^\kappa} P_\eta(t,\xi)(-\rho'(t)\langle\xi\rangle_\nu^\kappa)e^{-\rho(t)\langle\xi\rangle_\nu^\kappa} P_\eta(t,\xi)^{-1} w(t,\xi) \\ &\qquad + e^{\rho(t)\langle\xi\rangle_\nu^\kappa} P_\eta(t,\xi)e^{-\rho(t)\langle\xi\rangle_\nu^\kappa}\partial_t\{P_\eta(t,\xi)^{-1} w(t,\xi)\} \\ &= -\rho'(t)\langle\xi\rangle_\nu^\kappa w(t,\xi) + P_\eta(t,\xi)\partial_t\{P_\eta(t,\xi)^{-1} w(t,\xi)\} \\ &= -\rho'(t)\langle\xi\rangle_\nu^\kappa w(t,\xi) + \partial_t\{P_\eta(t,\xi)P_\eta(t,\xi)^{-1} w(t,\xi)\} \\ &\qquad - \{\partial_t P_\eta(t,\xi)\}\{P_\eta(t,\xi)^{-1} w(t,\xi)\} \\ &= -\rho'(t)\langle\xi\rangle_\nu^\kappa w(t,\xi) + \partial_t w(t,\xi) - \{\partial_t P_\eta(t,\xi)\}\{P_\eta(t,\xi)^{-1} w(t,\xi)\}. \end{aligned}$$

While by Lemma 2.2.C we obtain

$$\text{right hand side} = iP_\eta(t,\xi)A(t,\xi)P_\eta(t,\xi)^{-1}w(t,\xi) + P_\eta(t,\xi)B(t,\xi)P_\eta(t,\xi)^{-1}w(t,\xi)$$
$$= i\tilde{A}(t,\xi)w(t,\xi) + iR_\eta(t,\xi)w(t,\xi) + B(t,\xi)w(t,\xi),$$

where $B(t,\xi) = P_\eta(t,\xi)B(t)P_\eta(t,\xi)^{-1}$.

Thus we get the system

$$\begin{aligned}(2.24)\qquad \partial_t w(t,\xi) =& i\tilde{A}(t,\xi)w(t,\xi) + iR_\eta(t,\xi)w(t,\xi) + \rho'(t)\langle\xi\rangle_\nu^\kappa w(t,\xi)\\ &+ \{\partial_t P_\eta(t,\xi)\}\{P_\eta(t,\xi)^{-1}w(t,\xi)\} + B(t,\xi)w(t,\xi)\end{aligned}$$

Hence we shall derive the energy estimate. Noting that $\tilde{A}(t,\xi)$ is Hermitian, by (2.16) we get the estimate

$$\begin{aligned}(2.25)\qquad \frac{d}{dt}|w(t,\xi)|^2 &= 2Re(\partial_t w(t,\xi), w(t,\xi))\\ &= 2Re(iR_\eta w + \rho'\langle\xi\rangle_\nu^\kappa w + \partial_t P_\eta \cdot P_\eta^{-1}w + Bw, w)\\ &\le 2(C_7\eta|\xi| + \rho'\langle\xi\rangle_\nu^\kappa + C_2|\partial_t P_\eta|\eta^{1-N} + C_1C_2C_{13}\eta^{1-N})|w|^2\end{aligned}$$

where $C_{13} = \max_{0\le t\le T}|B(t)|$.

Writing the left hand side of (2.25) as

$$\frac{d}{dt}|w(t,\xi)|^2 = 2|w(t,\xi)|\frac{d}{dt}|w(t,\xi)|$$

and deviding the both hand sides of (2.25) by $2|w(t,\xi)|$, we get the estimate

$$\frac{d}{dt}|w(t,\xi)| \le (C_7\eta|\xi| + \rho'(t)\langle\xi\rangle_\nu^\kappa + C_2|\partial_t P_\eta(t,\xi)|\eta^{1-N} + C_1C_2C_{13}\eta^{1-N})|w(t,\xi)|.$$

Moreover by Gronwall's inequality and (2.17), we get the estimate

$$\begin{aligned}|w(t,\xi)| &\le |w(0,\xi)|\exp\{\int_0^t (C_7\eta|\xi| + \rho'(s)\langle\xi\rangle_\nu^\kappa\\ &\qquad + C_2|\partial_s P_\eta(s,\xi)|\eta^{1-N} + C_1C_2C_{13}t\eta^{1-N})ds\}\\ &\le |w(0,\xi)|\exp\{C_7t\eta|\xi| + \rho(t)\langle\xi\rangle_\nu^\kappa - \rho(0)\langle\xi\rangle_\nu^\kappa\\ &\qquad + 2C_1C_2t\eta^{1-N(1+\alpha^{-1})} + C_1C_2C_{13}t\eta^{1-N}\}\\ &\le C_{14}|w(0,\xi)|\exp\{\rho'(t)\langle\xi\rangle_\nu^\kappa - \rho(0)\langle\xi\rangle_\nu^\kappa\\ &\qquad + t(C_7\eta|\xi| + 3C_1C_2\eta^{1-N(1+\alpha^{-1})})\},\end{aligned}$$

where $C_{14} = \exp\frac{1}{4}C_1C_2C_{13}^2T$. Here we used

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q \qquad (1 < p,\ q < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1)$$

and supposing $p > 1 + \frac{N-1}{N\alpha^{-1}} \ge 1$,

$$\begin{aligned}\exp\{C_1C_2C_{13}t\eta^{1-N}\} &= \exp\Big\{\{C_{13}^{\frac{1}{q}}q^{-\frac{1}{q}}(C_1C_2C_{13}t\eta^{1-N})^{\frac{1}{p}}\eta^{\frac{1}{q}N\alpha^{-1}}\Big\} \\ &\qquad\times\Big\{C_{13}^{-\frac{1}{q}}q^{\frac{1}{q}}(C_1C_2C_{13}t\eta^{1-N})^{\frac{1}{q}}\eta^{-\frac{1}{q}N\alpha^{-1}}\}\Big\} \\ &\le \exp\Big\{\frac{1}{p}\{\text{ the first factor }\}^p + \frac{1}{q}\{\text{ the second factor }\}^q\Big\} \\ &= \exp\{p^{-1}C_{13}^{\frac{p}{q}}q^{-\frac{p}{q}}C_1C_2C_{13}t\eta^{1-N+\frac{p}{q}N\alpha^{-1}}\}\exp\{C_1C_2t\eta^{1-N(1+\alpha^{-1})}\} \\ &< \exp\{p^{-1}q^{1-p}C_1C_2C_{13}^pT\eta^0\}\exp\{C_1C_2t\eta^{1-N(1+\alpha^{-1})}\}.\end{aligned}$$

Since $1 + \frac{N-1}{N}\alpha^{-1} < 2$ for $0 < \alpha \le 1,\ \ 1 \le N < \infty$, we can take $p = q = 2$, and get $C_{14} = \frac{1}{4}C_1C_2C_{13}^2T$.

Putting

$$0 < \kappa_0 = \frac{N(1+\alpha^{-1}) - 1}{N(1+\alpha^{-1})} < \kappa \le 1, \quad \eta = \langle\xi\rangle_\nu^{-1+\kappa_0} \le 1, \tag{2.26}$$

$$\begin{aligned}|w(t,\xi)| &\le C_{14}|w(0,\xi)|\exp\{\rho(t)\langle\xi\rangle_\nu^\kappa - \rho(0)\langle\xi\rangle_\nu^\kappa \\ &\qquad + t(C_7\langle\xi\rangle_\nu^{-1+\kappa_0}|\xi| + 3C_1C_2\langle\xi\rangle_\nu^{(-1+\kappa_0)(1-N(1+\alpha^{-1}))})\} \\ &\le C_{14}|w(0,\xi)|\exp\{\langle\xi\rangle_\nu^\kappa(\rho(t) - \rho(0)) \\ &\qquad + t(C_7\langle\xi\rangle_\nu^{\kappa_0-\kappa} + 3C_1C_2\langle\xi\rangle_\nu^{\kappa_0-\kappa}\langle\xi\rangle_\nu^{-1+N(1+\alpha^{-1})-\kappa_0N(1+\alpha^{-1})}))\}\end{aligned}$$

using $\langle\xi\rangle_\nu^{\kappa_0-\kappa} \le \nu^{\kappa_0-\kappa}$ and $\langle\xi\rangle_\nu^{-1+N(1+\alpha^{-1})-\kappa_0N(1+\alpha^{-1})} = \langle\xi\rangle_\nu^0 = 1$,

$$\le C_{14}|w(0,\xi)|\exp\{\langle\xi\rangle_\nu^\kappa(\rho(t) - \rho(0) + C_{15}\nu^{\kappa_0-\kappa}t)\},$$

where $C_{15} = C_7 + 3C_1C_2$.

Here if we choose $\rho(t)$ such that in $[0,T]$

$$\begin{cases}\rho(t) - \rho(0) + C_{15}\nu^{\kappa_0-\kappa}t = 0 \\ \rho(0) = \tilde\rho_0, \qquad\qquad \text{i.e.,}\end{cases}$$

$$\rho(t) = \tilde\rho_0 - C_{15}\nu^{\kappa_0-\kappa}t \qquad\qquad (t \in [0,T]), \tag{2.27}$$

where $\tilde\rho_0 = \omega\rho_0 + (1-\omega)\rho_1$ for $0 <^\forall \omega < 1,\ 0 <^\forall \rho_1 < \rho_0$, we have

$$|w(t,\xi)| \le C_{14}|w(0,\xi)|. \tag{2.28}$$

Noting that $|w(0,\xi)| = |P_\eta(0,\xi)e^{\rho(0)\langle\xi\rangle_\nu^\kappa}v(0,\xi)| \le C_1 e^{\tilde\rho_0\langle\xi\rangle_\nu^\kappa}v(0,\xi)|$ and $|v(t,\xi)| = e^{-\rho(t)\langle\xi\rangle_\nu^\kappa}P_\eta(t,\xi)^{-1}w(t,\xi)| \le C_2\eta^{1-N}e^{-\rho(t)\langle\xi\rangle_\nu^\kappa}|w(t,\xi)| = C_2\langle\xi\rangle_\nu^{(-1+\kappa_0)(1-N)}$ $\times e^{-\rho(t)\langle\xi\rangle_\nu^\kappa}|w(t,\xi)|$, (2.28) is changed to the estimate

$$e^{\rho(t)\langle\xi\rangle_\nu^\kappa}|v(t,\xi)| \le C_1C_2C_{14}\langle\xi\rangle^{(1-\kappa_0)(N-1)}e^{\tilde\rho_0\langle\xi\rangle_\nu^\kappa}|v(0,\xi)|. \tag{2.29}$$

It holds generally that

$$e^{-x} \le n!x^{-n} \quad \text{for} \quad x>0, \quad n=\Big[\frac{(1-\kappa_0)(N-1)}{\kappa}\Big], \tag{2.30}$$

and for $\nu_1 \ge \nu_2$

$$\begin{aligned}\langle\xi\rangle_{\nu_1}^\kappa - \langle\xi\rangle_{\nu_2}^\kappa &= (\nu_1-\nu_2)\int_0^1 \partial_\nu\langle\xi\rangle_\nu^\kappa|_{\nu=\nu_2+\theta(\nu_1-\nu_2)}d\theta \\ &= (\nu_1-\nu_2)\int_0^1 \kappa(\nu_2+\theta(\nu_1-\nu_2))\langle\xi\rangle_{\nu_2+\theta(\nu_1-\nu_2)}^{\kappa-2}d\theta \\ &= \kappa(\nu_1-\nu_2)\nu_1\nu_2^{\kappa-2} \le \kappa\nu_1^2\nu_2^{\kappa-2}.\end{aligned} \tag{2.31}$$

If we put $\nu = \left(\frac{C_{15}T}{\omega(\rho_0-\rho_1)}\right)^{\frac{1}{\kappa-\kappa_0}}$ and take $0<\omega<\min\left\{1, \frac{C_{15}T}{(\rho_0-\rho_1)\nu_0^{\kappa-\kappa_0}}\right\}$, we get $\nu>\nu_0$. Hence by (2.30), (2.31) the right hand side of (2.29) is changed to

$$\begin{aligned}\text{right hand side of (2.29)} &\le C_1C_2C_{14}\langle\xi\rangle_\nu^{(1-\kappa_0)(N-1)}e^{-(\rho_0-\tilde\rho_0)\langle\xi\rangle_\nu^\kappa} \\ &\qquad\times e^{\rho_0(\langle\xi\rangle_\nu^\kappa-\langle\xi\rangle_{\nu_0}^\kappa)}e^{\rho_0\langle\xi\rangle_{\nu_0}^\kappa}|v(0,\xi)| \\ &\le C_1C_2C_{14}\langle\xi\rangle_\nu^{(1-\kappa_0)(N-1)}n!\{(\rho_0-\tilde\rho_0)\langle\xi\rangle_\nu^\kappa\}^{-n} \\ &\qquad\times e^{\rho_0\kappa\nu^2\nu_0^{\kappa-2}}e^{\rho_0\langle\xi\rangle_{\nu_0}^\kappa}|v(0,\xi)| \\ &\le C_1C_2C_{14}n!\{(1-\omega)(\rho_0-\rho_1)\}^{-n} \\ &\qquad\times e^{\rho_0\kappa\left(\frac{C_{15}T}{\omega(\rho_0-\rho_1)}\right)^{\frac{1}{\kappa-\kappa_0}}\nu_0^{\kappa-2}}e^{\rho_0\langle\xi\rangle_{\nu_0}^\kappa}|v(0,\xi)|.\end{aligned} \tag{2.32}$$

While, noting that

$$\rho(t) \ge \rho(T) = \tilde\rho_0 - C_{15}\nu^{\kappa_0-\kappa}T = \omega\rho_0-(1-\omega)\rho_1 - C_{15}\Big\{\Big(\frac{C_{15}T}{\omega(\rho_0-\rho_1)}\Big)^{\frac{1}{\kappa-\kappa_0}}\Big\}^{\kappa_0-\kappa}T = \rho_1,$$

the left hand side of (2.29) is changed to

$$\text{left hand side of (2.29)} \ \ge e^{\rho_1\langle\xi\rangle_\nu^\kappa}|v(t,\xi)|. \tag{2.33}$$

Thus by (2.29), (2.32) and (2.33) we get

$$\begin{aligned}e^{\rho_1\langle\xi\rangle_\nu^\kappa}|v(t,\xi)| &\le C_1C_2C_{14}n!\{(1-\omega)(\rho_0-\rho_1)\}^{-n}e^{\rho_0\kappa\left(\frac{C_{15}T}{\omega(\rho_0-\rho_1)}\right)^{\frac{1}{\kappa-\kappa_0}}\nu_0^{\kappa-2}}e^{\rho_0\langle\xi\rangle_{\nu_0}^\kappa}|v(0,\xi)| \\ &\le Ce^{\rho_0\langle\xi\rangle_{\nu_0}^\kappa}|v(0,\xi)| \qquad \text{for} \quad {}^\forall t\in[0,T],\ {}^\forall\xi\in\mathbf{R}_\xi^n,\end{aligned} \tag{2.34}$$

where ρ_1 and κ satisfy

$$0 < \rho_1 < \rho_0, \qquad \frac{N(1+\alpha^{-1})-1}{N(1+\alpha^{-1})} < \kappa \leq 1,$$

respectively from (2.26) and (2.27). This implies (2.4) and (2.5) of CASE 1.Theorem 2.1 under CASE 2 also can be proved quite similarly.

2.4 Appendix

We shall show that the Example 1 is included by CASE 2 and μ is equals to the maximal multiplicity of the eigenvalues of $A_h(t)\xi_h$, i.e., $\mu = \max_{1\leq i\leq k} m_i$. Since the multiplicty of the eigenvalues is constant, it is sufficient to consider the constant matrix A. Moreover for the simplicity we may suppose that the $N \times N$ constant matrix A has two distinct real eigenvalues λ_1 and λ_2 whose multiplicity are m_1 and m_2 respectively. Then similarly as Lemma 2.2.A, we can get a non-singular matrix P such that

$$PAP^{-1} = \tilde{A} + R = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ a_{2,1} & \lambda_1 & \ddots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ a_{m_1,1} & \cdots & a_{m_1,m_1-1} & \lambda_1 & \ddots & & \vdots \\ a_{m_1+1,1} & \cdots & \cdots & a_{m_1+1,m_1} & \lambda_2 & \ddots & \vdots \\ \vdots & & & & & \ddots & 0 \\ a_{N,1} & \cdots & \cdots & \cdots & \cdots & a_{N,N-1} & \lambda_2 \end{pmatrix}$$

$$\equiv \begin{pmatrix} D_1 & 0 \\ E & D_2 \end{pmatrix}.$$

As it is well known, if D_1 and D_2 have no eigenvalues in common, the matrix equation $D_2X - XD_1 = E$ has a unique solution X (see [Ort]). Hence putting $\tilde{P} = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} P$, we find that

$$\tilde{P}A\tilde{P}^{-1} = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ E & D_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ -X & I \end{pmatrix}$$

$$= \begin{pmatrix} D_1 & 0 \\ XD_1 - D_2X + E & D_2 \end{pmatrix} = diag\{D_1, D_2\}.$$

Here we can easily see that D_1 and D_2 are the triangular matrices whose sizes are $m_1 \times m_1$ and $m_2 \times m_2$ respectively. Therefore μ is equals to the maximal multiplicity of the eigenvalues of A.