$K^+ \to \pi^+ \pi^0$ decay amplitude with the Wilson quark action in quenched lattice QCD

S. Aoki,[1,*] M. Fukugita,[2] S. Hashimoto,[3,†] N. Ishizuka,[1,4] Y. Iwasaki,[1,4] K. Kanaya,[1,4] Y. Kuramashi,[5] M. Okawa,[5] A. Ukawa,[1] and T. Yoshié[1,4]

(JLQCD Collaboration)

[1]*Institute of Physics, University of Tsukuba, Tsukuba, Ibaraki 305, Japan*

[2]*Institute for Cosmic Ray Research, University of Tokyo, Tanashi, Tokyo 188, Japan*

[3]*Computing Research Center, High Energy Accelerator Research Organization (KEK), Tsukuba, Ibaraki 305, Japan*

[4]*Center for Computational Physics, University of Tsukuba, Tsukuba, Ibaraki 305, Japan*

[5]*Institute of Particle and Nuclear Studies, High Energy Accelerator Research Organization (KEK), Tsukuba, Ibaraki 305, Japan*

(Received 25 November 1997; published 29 July 1998)

We present a calculation for the $K^+ \to \pi^+ \pi^0$ decay amplitude using a quenched simulation of lattice QCD with the Wilson quark action at $\beta = 6/g^2 = 6.1$. The decay amplitude is extracted from the ratio, the $K \to \pi\pi$ three-point function divided by either K and π meson two-point functions or K meson two-point function and $I=2$ $\pi\pi$ four-point function; the two different methods yield consistent results. Finite size effects are examined with calculations made on $24^3 \times 64$ and $32^3 \times 64$ lattices, and are shown that they are explained by one-loop effects of chiral perturbation theory. The lattice amplitude is converted to the continuum value by employing a one-loop calculation of chiral perturbation theory, yielding a value in agreement with experiment if extrapolated to the chiral limit. We also report on the K meson B parameter B_K obtained from the $K^+ \to \pi^+ \pi^0$ amplitude using chiral perturbation theory. [S0556-2821(98)03417-1]

PACS number(s): 12.38.Gc, 11.15.Ha, 12.38.Aw

I. INTRODUCTION

Despite the full understanding of the fundamental theory of weak interactions, the non-leptonic decay of hadrons still remains to be the least understood weak process, the most notable problems being the $\Delta I = 1/2$ rule and the calculation of ϵ'/ϵ. The predicament originates from the difficulty of evaluating the hadronic matrix element of the product of currents. Much work has already been done to attack this problem using lattice QCD simulations [1–4], but they have not yielded satisfactory results.

Difficulties have proven to be especially severe for the $\Delta I = 1/2$ amplitudes [1–3]. From the computational point of view the problem lies in a calculation of the so-called eye diagram, which suffers from extremely large statistical fluctuations [1–3,5]. Theoretically, this may be related to mixing of the dimension six weak operator responsible for the decay with operators of lower dimensions whose coefficients diverge linearly in the continuum limit. At a more fundamental level, there is the difficulty [6] that the $K \to \pi\pi$ 3-point function evaluated in Euclidean space-time does not yield information on the phase of the decay amplitude.

Calculation of the $\Delta I = 3/2$ process is known to be easier than that of the $\Delta I = 1/2$ process. For this case mixing of lower dimension operators is absent, and the so-called figure-eight diagrams which represent the $\Delta I = 3/2$ amplitude, have clear signals in numerical simulations. Indeed lattice calculations have been reported by several groups [2,4] quite a long time ago. The problem, however, was that the results turned out to be inconsistent with experiment: lattice calculations have given the amplitude roughly a factor of two larger than experiment.

Two potential origins are suspected to give this discrepancy. One is an issue in the matching of the lattice and continuum operators. Early studies employed the factor $\sqrt{2\kappa}$ for the quark wave function normalization and the bare lattice coupling constant for estimating the renormalization factor of the four-quark weak operator. It is by now well known that the Kronfeld-Lepage-Mackenzie (KLM) factor $\sqrt{1-3\kappa/4\kappa_c}$ [7] and tadpole-improved perturbation theory [8] are more adequate for the operator matching.

Another problem concerns the use of chiral perturbation theory (CHPT) to convert lattice results into the physical amplitude. Only the tree-level formula was known and used in the previous work. The meson mass dependence of lattice calculations appeared consistent with the prediction of the tree-level formula, allowing, however, for large statistical errors. It was probably necessary to use the formula including higher order CHPT effects, but its necessity was not manifest. An interesting development in this connection is a recent calculation of one-loop corrections to the $K^+ \to \pi^+ \pi^0$ amplitude in CHPT by Golterman and Leung [9]. Applying their results to the old data obtained by Bernard and Soni [2], they found that one-loop effects decrease the physical amplitude by about 30%.

With the hope to improve the problems posed here, we have carried out a high statistics simulation of the $K^+ \to \pi^+ \pi^0$ amplitude in quenched lattice QCD, incorporating various theoretical and technical developments made in recent years. In particular, we discuss in detail how one-loop corrections of CHPT affect physical predictions for the decay amplitude from lattice QCD simulations. We also report on

*Present address: Max-Planck-Institut für Physik, Föhringer Ring 6, D-80805 München, Germany.

†Present address: Theoretical Physics Department, Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510.

the K meson B parameter B_K obtained from the $K^+\to\pi^+\pi^0$ amplitude using CHPT.

This paper is organized as follows. After a brief description of simulation parameters in Sec. II, we explain our method for extracting the decay amplitude in Sec. III. Our results for the $K^+\to\pi^+\pi^0$ amplitude are presented in Sec. IV with discussion made on one-loop effects of CHPT. Results for B_K are given in Sec. V. Section VI summarizes our conclusions.

II. SIMULATION PARAMETERS

Our simulation is carried out in quenched lattice QCD employing the standard plaquette action for gluons at $\beta=6.1$ and the Wilson action for quarks. We take up, down and strange quarks to be degenerate, and make measurements at four values of the common hopping parameter, $\kappa=0.1520$, 0.1530, 0.1540, and 0.1543, which correspond to $M_\pi/M_\rho=0.797$, 0.734, 0.586 and 0.515. In order to examine finite-size effects, simulations are carried out for two lattice sizes, 120 configurations on $24^3\times64$ and 65 configurations on $32^3\times64$. Gluon configurations are separated by 2000 pseudo heat bath sweeps. Quark propagators are solved with the Dirichlet boundary condition imposed in the time direction and the periodic boundary condition in the space directions.

We adopt $1/a=2.67(10)$ GeV for the physical scale of lattice spacing estimated from the ρ meson mass, and $\kappa_c=0.15499(2)$ for the critical hopping parameter, which were obtained in our previous study [10]. Our calculations are carried out on the Fujitsu VPP500/80 supercomputer at KEK.

III. METHODS

A. Extraction of decay amplitude

Let us consider the four-quark operator defined by

$$Q_+=\frac{1}{2}\cdot[\ \bar{s}\gamma_\mu(1-\gamma_5)d\ \ \bar{u}\gamma_\mu(1-\gamma_5)u+\bar{s}\gamma_\mu(1-\gamma_5)u\ \ \bar{u}\gamma_\mu(1-\gamma_5)d\ \ ], \tag{1}$$

which is relevant to $\Delta I=3/2$ two-pion decay of the K meson. We first discuss our method for extracting the lattice matrix element of the operator Q_+, deferring the question of matching the lattice and continuum operators to Sec. III B.

We extract the decay amplitude from the 4-point correlation function

$$G_Q(t_+,t_0;t;t_K)=\langle 0|W_+(t_+)W_0(t_0)Q_+(t)W_K(t_K)|0\rangle. \tag{2}$$

In order to enhance signals we construct wall sources (denoted by W) for all external mesons, and fix gauge configurations to the Coulomb gauge. The wall sources W_K, W_+, and W_0 for K^+, π^+, and π^0 are placed at the time slices t_K, t_+ and t_0 such that $t_K\ll t\ll t_+,t_0$. All mesons are at rest, and the 4-quark operator Q_+ is projected to zero spatial momentum.

In our calculation for temporal lattice size $T=64$, we place the K meson at $t_K=4$. The two π mesons are placed at different time slices, $t_+=59$ and $t_0=60$ to avoid contaminations from Fierz-rearranged terms in the two-pion state that would occur for the choice $t_+=t_0$.

The correlation function G_Q behaves for $t_K\ll t\ll t_+\sim t_0$ as

$$\begin{aligned}G_Q(t_+,t_0;t;t_K)&=\langle 0|W_+(0)W_0(t_0-t_+)|\pi^+\pi^0\rangle\\&\times\frac{1}{N_{\pi\pi}}\langle\pi^+\pi^0|Q_+(0)|K^+\rangle\\&\times\frac{1}{N_K}\langle K^+|W_K(0)|0\rangle e^{M_K(t_K-t)}e^{(t-t_+)M_{\pi\pi}},\end{aligned} \tag{3}$$

where N_K denotes the normalization factor of the K meson state, $|\pi^+\pi^0\rangle$ represents the $I=2$ two-pion state with a mass $M_{\pi\pi}$ and a state normalization factor $N_{\pi\pi}$.

In order to remove the normalization factors in G_Q we calculate the product of the meson 2-point functions given by

$$\begin{aligned}G_W(t_+,t_0;t;t_K)&=\langle 0|W_0(t_0)\pi^0(t)|0\rangle\langle 0|W_+(t_+)\pi^+(t)|0\rangle\\&\times\langle 0|K^+(t)W_K(t_K)|0\rangle.\end{aligned} \tag{4}$$

Defining a ratio $R_W=G_Q/G_W$, we find

$$R_W(t_+,t_0;t;t_K)=S_W\frac{\langle\pi^+\pi^0|Q_+|K^+\rangle}{\langle\pi|\pi|0\rangle^3}e^{(t-t_+)\Delta}, \tag{5}$$

where

$$\Delta=M_{\pi\pi}-2M_\pi \tag{6}$$

is a mass shift due to a finite spatial lattice size, and S_W is defined by

$$S_W=\frac{N_\pi^2}{N_{\pi\pi}}\frac{\langle 0|W_+(0)W_0(t_0-t_+)|\pi^+\pi^0\rangle}{\langle 0|W_+(0)|\pi^+\rangle\langle 0|W_0(t_0-t_+)|\pi^0\rangle}, \tag{7}$$

where $t_0-t_+=1$ in our calculation. The value of S_W should converge to unity for infinite volume.

In Fig. 1 we plot $\langle\pi|\pi|0\rangle^3R_W$ at $\kappa=0.1530$ as a function of time t of the weak operator, where we calculate $\langle\pi|\pi|0\rangle$ from the pion 2-point function for point source and point sink. We observe a clear nonvanishing slope, which means the mass shift Δ being positive. Numerical values of Δ and the decay amplitude $\langle\pi^+\pi^0|Q_+|K^+\rangle$ obtained by a single exponential fit for the time range $t=18-46$ are tabulated in Table I. Here we assume $S_W=1$, whose justification will be discussed below.

According to Lüscher's formula [11], the finite-size mass shift of the two-pion state is written

$$\Delta=M_{\pi\pi}-2M_\pi=-\frac{4\pi a_0}{M_\pi(aL)^3}+O(L^{-4}), \tag{8}$$

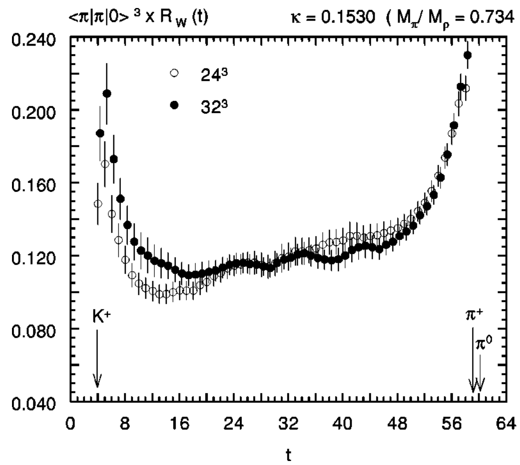

FIG. 1. $\langle \pi | \pi | 0 \rangle^3 R_W(t_+, t_0; t; t_K)$ at $\kappa = 0.153$. Open and filled circles refer to data for 24^3 and 32^3 lattices.

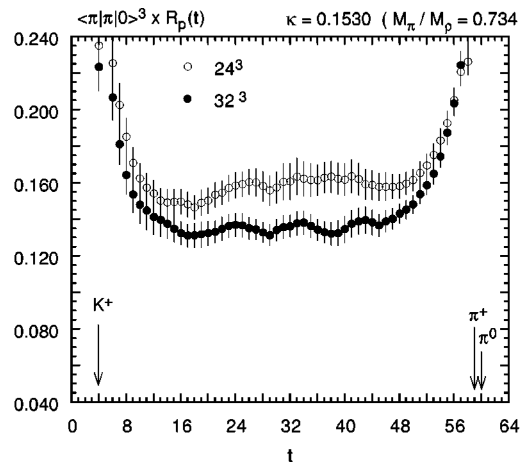

FIG. 2. $\langle \pi | \pi | 0 \rangle^3 R_P(t_+, t_0; t; t_K)$ at $\kappa = 0.153$. Open and filled circles refer to data for 24^3 and 32^3 lattices.

where a_0 is the s-wave scattering length and L is the spatial size. This formula was previously employed to calculate the s-wave $\pi\pi$ scattering length in quenched lattice QCD [12–14]. It was found that lattice calculations give a_0 in good agreement with the prediction of current algebra. Using the current algebra formula $a_0^{I=2} = M_\pi / (8\pi F_\pi^2)$ with $F_\pi = 132$ MeV and $1/a = 2.67(10)$ GeV, we obtain $a\Delta = 0.015$ for $L = 24$ and 0.006 for $L = 32$. Considering uncertainties arising from terms of $O(L^{-4})$ and the difference between the physical and measured values of F_π, we regard this estimate being consistent with the measured $a\Delta$ (see Table I).

As an alternative method we may remove the normalization factors of the 4-point function G_Q with

$$G_P(t_+, t_0; t; t_K) = \langle 0 | W_+(t_+) W_0(t_0) \pi^+(t) \pi^0(t) | 0 \rangle \times \langle 0 | K^+(t) W_K(t_K) | 0 \rangle. \tag{9}$$

The ratio $R_P = G_Q / G_P$ is independent of t and it does not depend on the wall sources for $t_K \ll t \ll t_+ \sim t_0$;

TABLE I. The mass shift $\Delta = M_{\pi\pi} - 2M_\pi$ and $\langle \pi^+ \pi^0 | Q_+ | K^+ \rangle$ from R_W and R_P. Here we assume $S_W = S_P = 1$. These values are obtained by a single exponential fit over $t = 18-46$ for R_W and by a constant fit over $t = 22-42$ for R_P.

κ	aM_π	$a\Delta$ (10^{-3})	$\langle \pi^+ \pi^0 \vert Q_+ \vert K^+ \rangle$ from R_W	from R_P
$L = 24$				
0.1520	0.3440(14)	7.9(1.7)	0.261(20)	0.271(19)
0.1530	0.2776(17)	8.6(2.0)	0.151(14)	0.160(13)
0.1540	0.1967(19)	9.3(2.9)	0.0617(81)	0.0680(69)
0.1543	0.1653(21)	8.6(3.6)	0.0382(58)	0.0434(49)
$L = 32$				
0.1520	0.3459(10)	3.6(1.4)	0.229(17)	0.234(16)
0.1530	0.2784(11)	4.2(1.5)	0.132(13)	0.135(11)
0.1540	0.1914(13)	5.7(2.1)	0.0565(71)	0.0573(55)
0.1543	0.1651(15)	7.1(2.8)	0.0383(55)	0.0380(37)

$$R_P(t_+, t_0; t; t_K) = S_P^{-1} \frac{\langle \pi^+ \pi^0 | Q_+ | K^+ \rangle}{\langle \pi | \pi | 0 \rangle^3}, \tag{10}$$

where

$$S_P = \frac{\langle \pi^+ \pi^0 | \pi^+ \pi^0 | 0 \rangle}{\langle \pi | \pi | 0 \rangle^2}, \tag{11}$$

which should become unity for infinite spatial lattice.

The dependence of R_P on the time t of the weak operator is shown in Fig. 2 for $\kappa = 0.153$, the same hopping parameter as in Fig. 1 for R_W. As expected, a clear plateau is seen for $t \approx 20-40$, where effects of excited states near the lattice boundaries already disappear.

In Table I we list $\langle \pi^+ \pi^0 | Q_+ | K^+ \rangle$ obtained by fitting R_P to a constant over $t = 22-42$ assuming $S_P = 1$. The results from the two methods show good mutual agreement, well within the statistical error of $10-15\,\%$. We note that statistical errors for R_P are smaller, and therefore adopt the matrix elements from R_P to obtain the physical decay amplitude below.

We still have to justify the assumption $S_W = S_P = 1$ used above. This is not *a priori* obvious, especially for S_W, since wall sources are uniformly extended across the spatial lattice, although a good agreement of $\langle \pi^+ \pi^0 | Q_+ | K^+ \rangle$ from R_W and R_P implies $S_W S_P$ close to unity. For S_P chiral perturbation theory predicts a finite-size correction of the form [9]

$$S_P = 1 + \frac{M_\pi^2}{24 F_\pi^2 (M_\pi a L)^3}. \tag{12}$$

This formula indicates that the deviation of S_P from unity would be less than 1% in our simulation. Hence we conclude $S_W \approx 1$.

B. Operator matching

For the quark field normalization we employ the KLM factor [7]

$$\psi^{continuum} = \sqrt{1-\frac{3\kappa}{4\kappa_c}}\ \psi^{lattice}\ . \tag{13}$$

Because of CPS symmetry the weak operator Q_+ defined in Eq. (1) does not mix with other operators [2]. With the tadpole improvement (with the factor $u_0=1/8\kappa_c$) the multiplicative renormalization factor for Q_+, which relates the lattice operator to the continuum one at a scale μ, is given by [15,16]

$$Z(\mu)=1+\frac{g^2_{\overline{\rm MS}}(\mu)}{16\pi^2}[-4\log(\mu a/\pi)-21.140], \tag{14}$$

where the naive dimensional regularization (NDR) is taken with the modified minimum subtraction scheme ($\overline{\rm MS}$) in the continuum.

We employ the $\overline{\rm MS}$ coupling constant estimated as follows. First we obtain g_V^2 by [8]

$$-\log P=\frac{1}{3}g_V^2(3.41/a)\left\{1-(1.19+0.017N_f)\frac{g_V^2(3.41/a)}{4\pi}+O(g_V^4)\right\} \tag{15}$$

with P the average plaquette. Next we calculate Λ_V from g_V^2 using

$$\log\left(\frac{3.41/a}{\Lambda_V}\right)^2=\frac{1}{\beta_0 x}+\frac{\beta_1}{\beta_0^2}\log\frac{\beta_1 x/\beta_0}{1+\beta_1 x/\beta_0}, \tag{16}$$

where $\beta_0=11-2N_f/3$, $\beta_1=102-38N_f/3$, and $x=g_V^2(3.41/a)/(4\pi)^2$. A perturbative relation $\Lambda_{\overline{\rm MS}}=0.6252\Lambda_V$ then yields $\Lambda_{\overline{\rm MS}}$, with which we can calculate $g^2_{\overline{\rm MS}}(\mu)$ at any scale μ. In the present calculation we find $\Lambda_{\overline{\rm MS}}=293(11)$ MeV with $P=0.605$ and $1/a=2.67(10)$ GeV at $\beta=6.1$.

Let A_2 be the physical amplitude for $\Delta I=3/2$ $K\to\pi\pi$ decay. Experimentally,

$$\sqrt{\frac{3}{2}}{\rm Re}A_2\left[\frac{G_F}{\sqrt{2}}V_{us}^*V_{ud}\right]^{-1}=10.4\times10^{-3}\ \ {\rm GeV}^3. \tag{17}$$

The relation of the decay amplitude to the matrix element of Q_+ is

$$\sqrt{\frac{3}{2}}{\rm Re}A_2\left[\frac{G_F}{\sqrt{2}}V_{us}^*V_{ud}\right]^{-1}=C_+^{(N_f)}(\mu)\langle\pi^+\pi^0|Q_+^{(N_f)}(\mu)|K^+\rangle. \tag{18}$$

On the right-hand side $C_+^{(N_f)}(\mu)$ and $Q_+^{(N_f)}(\mu)$ are the Wilson coefficient function and the renormalized weak operator at a scale μ with superscript N_f the number of quark flavors appropriate for the scale μ. We choose $\mu=2$ GeV to estimate the physical amplitude, and hence $N_f=4$.

In our calculation, matching of the lattice operator $Q_+^{lattice}$ to the continuum operator $Q_+^{(4)}(2\ \ {\rm GeV})$ is not straightforward since the simulation is carried out in quenched QCD ($N_f=0$). To treat this problem we proceed in the following way. We first match the lattice operator to the continuum operator $Q_+^{(0)}$ for $N_f=0$ at a scale q^* using the renormalization factor $Z(q^*)$ in Eq. (14): $Q_+^{(0)}(q^*)=Z(q^*)Q_+^{lattice}$. The operator $Q_+^{(0)}(\mu)$ at any scale μ can then be obtained by renormalization group evolution in the continuum:

$$Q_+^{(0)}(\mu)=U^{(0)}(\mu,q^*)Q_+^{(0)}(q^*)=U^{(0)}(\mu,q^*)Z(q^*)Q_+^{lattice}, \tag{19}$$

where $U^{(N_f)}(\mu,\mu')$ is the two-loop renormalization group running factor from scale μ' to μ and it is given by

$$U^{(N_f)}(\mu,\mu')=\left(\frac{g^2(\mu)}{g^2(\mu')}\right)^{\gamma_0/2\beta_0}\left[1+\frac{g^2(\mu)-g^2(\mu')}{16\pi^2}\times\left(\frac{\gamma_1\beta_0-\gamma_0\beta_1}{2\beta_0^2}\right)\right]. \tag{20}$$

Here $\gamma_0=4$ and $\gamma_1=-7+4N_f/9$ are the one- and two-loop anomalous dimensions for Q_+ [17].

In the spirit of tadpole improvement, the matching point q^* from the lattice to the continuum operator should be chosen to minimize higher order contributions in the renormalization factor $Z(q^*)$. Since an estimate of this value is not available, however, we take $q^*=1/a$ or π/a and investigate the q^* dependence of the decay amplitude.

We still need to relate the operator $Q_+^{(0)}$ of the $N_f=0$ theory to the operator $Q_+^{(4)}$ of the $N_f=4$ theory. Whether such a matching is possible is a problem generally encountered in quenched QCD calculations of weak matrix elements. As a working hypothesis, we assume that there is a scale k^*, typical of the $K^+\to\pi^+\pi^0$ process, at which the $N_f=0$ operator matches with the $N_f=4$ operator,

$$U^{(0)}(k^*,q^*)Q_+^{(0)}(q^*)=Q_+^{(4)}(k^*). \tag{21}$$

We then estimate the decay amplitude for the $N_f=4$ theory by

$$\begin{aligned}C_+^{(4)}(\mu)\langle\pi^+\pi^0|Q_+^{(4)}(\mu)|K^+\rangle &=C_+^{(4)}(\mu)U^{(4)}(\mu,k^*)\langle\pi^+\pi^0|Q_+^{(4)}(k^*)|K^+\rangle\\ &=C_+^{(4)}(\mu)D(\mu,k^*,q^*)\langle\pi^+\pi^0|Q_+^{lattice}|K^+\rangle,\end{aligned} \tag{22}$$

where

$$D(\mu,k^*,q^*)=U^{(4)}(\mu,k^*)U^{(0)}(k^*,q^*)Z(q^*). \tag{23}$$

For the renormalization group evolution in the continuum we follow Buchalla *et al.* [18]. In particular we use their $C_+^{(4)}(2\ \ {\rm GeV})=0.859$ with $\Lambda^{(4)}_{\overline{\rm MS}}=215$ MeV.

TABLE II. Values of $D(2\ \text{GeV},k^*,q^*)$ for $\Lambda_{\overline{MS}}^{(4)}=215$ MeV and $\Lambda_{\overline{MS}}^{(0)}=293$ MeV.

k^*(GeV)	$q^*=1/a$	$q^*=\pi/a$
0.700	0.759038	0.830913
1.000	0.761556	0.833670
1.500	0.765198	0.837657
2.000	0.768126	0.840863

The value of the matching scale k^* is not known. The variation of $D(\mu,k^*,q^*)$ with respect to the scale k^*, however, arises from the difference of the Λ parameter and the anomalous dimension of Q_+ for $N_f=0$ and 4, and so it is expected to be small. The values of $D(\mu,k^*,q^*)$ for several k^* are tabulated for $q^*=1/a$ and π/a in Table II. We observe that the dependence on k^* is indeed very small, and we set $k^*=1$ GeV in the following analysis.

Let us note that the difference of $D(\mu,k^*,q^*)$ for $q^*=1/a$ and π/a is about 10%. This is the largest systematic error in our operator matching procedure other than the assumption of the matching scale k^*, and it is comparable to our statistical errors.

IV. RESULTS FOR THE $K^+\to\pi^+\pi^0$ AMPLITUDE

A. Decay amplitude with tree-level CHPT

As in the previous work [2,4] we take degenerate strange and up-down quarks, and assume all external mesons at rest. The amplitude obtained with this kinematics is clearly unphysical, having an energy injection at the weak operator. In order to relate the lattice result to the physical amplitude information is needed on the dependence of the amplitude on the K and π masses away from the physical point.

Earlier calculations have used chiral perturbation theory (CHPT) at tree level for this purpose. The operator Q_+ is decomposed under chiral $SU(3)_L$ into terms belonging to $[\mathbf{8},\Delta I=1/2]$, $[\mathbf{27},\Delta I=1/2]$ and $[\mathbf{27},\Delta I=3/2]$. The $[\mathbf{27},\Delta I=3/2]$ part of Q_+, which contributes to $K^+\to\pi^+\pi^0$, is given by

$$\frac{1}{3}Q_4=\frac{1}{3}[2Q_+-\bar{s}\gamma_\mu(1-\gamma_5)d\ \ \bar{d}\gamma_\mu(1-\gamma_5)d]. \qquad (24)$$

In general the **27** operator in QCD can be described by operators in CHPT as

$$\mathcal{O}_{27}^{\text{QCD}}=\alpha_{27}\cdot R_{kl}^{ij}\cdot(\Sigma\partial_\mu\Sigma^\dagger)_{ik}(\Sigma\partial_\mu\Sigma^\dagger)_{jl}, \qquad (25)$$

where, for Q_4, the nonvanishing components of the tensor R_{kl}^{ij} are $R_{31}^{21}=R_{13}^{12}=R_{31}^{12}=R_{31}^{21}=\frac{1}{2}$ and $R_{32}^{22}=R_{23}^{22}=-\frac{1}{2}$, and the pseudoscalar meson field is given by

$$\Sigma=e^{i\pi/f} \qquad (26)$$

for the full theory, or

$$\Sigma=e^{i\pi/f}e^{i\eta'/\sqrt{3}f} \qquad (27)$$

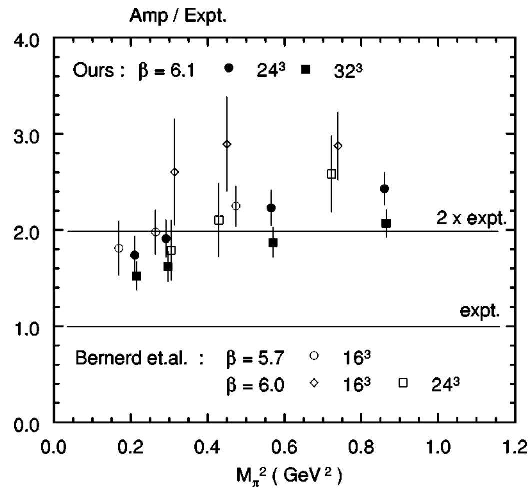

FIG. 3. Comparison of our results for $C_+\langle\pi^+\pi^0|Q_+|K^+\rangle$ normalized by the experimental value obtained with the tree-level CHPT relation (28) for $q^*=\pi/a$ at $\beta=6.1$ with those of previous work [2] at $\beta=6.0$ and 5.7. Results are plotted as a function of lattice meson mass M_π^2. Traditional $\sqrt{2\kappa}$ normalization is employed for quark fields and tadpole improvement is not applied in the renormalization factor.

for the quenched theory. At tree level of CHPT one obtains the formula connecting the physical amplitude and that calculated on the lattice [2]:

$$\langle\pi^+\pi^0|Q_+|K^+\rangle_{phys} = \frac{m_K^2-m_\pi^2}{2M_\pi^2}\left(\frac{\alpha_{27}}{\alpha_{27}^q}\right)\left(\frac{f_q}{f}\right)^3\langle\pi^+\pi^0|Q_+|K^+\rangle_{lattice}, \qquad (28)$$

where $m_K=497$ MeV and $m_\pi=136$ MeV are physical masses, and M_π is the degenerate K and π masses on the lattice. We emphasize that the constant α_{27} and the tree-level decay constant f may take different values in the full and quenched theories. We denote the constants in quenched theory with superscript q.

In Fig. 3 we compare the decay amplitude $C_+\langle\pi^+\pi^0|Q_+|K^+\rangle$ of the previous work at $\beta=5.7$ and 6.0 [2] with ours at $\beta=6.1$. Here, as a working hypothesis, we set α_{27} and f to be equal in full and quenched theories. For the sake of comparison, our data are analyzed in a manner parallel to that in Ref. [2] as much as possible, i.e., employing the traditional $\sqrt{2\kappa}$ normalization for quark fields, no tadpole improvement in the renormalization factor, and applying the tree-level relation (28). The matching factor (23) with $q^*=\pi/a$ is applied in our results for consistency. Since the normalization adopted in Ref. [2] for comparison with experiment differs from ours, we plot the results divided by the experimental value. In view of various differences in the simulation parameters and details of analysis procedures, the values from the two studies are taken to be consistent, both being larger than experiment roughly by a factor of two.

Let us note that our results, which attain errors of about 10%, show a clear dependence on the lattice meson mass M_π. The presence of finite-size effects is also evident, ex-

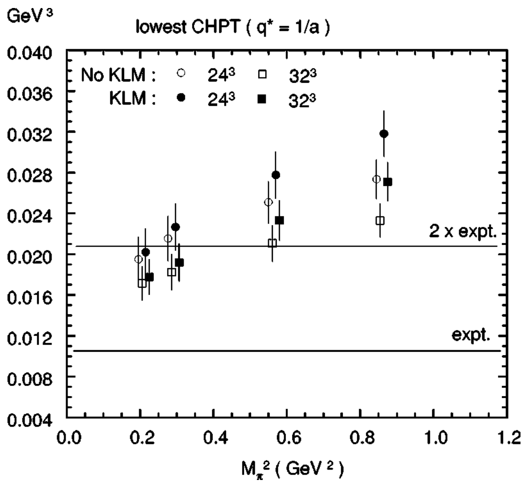

FIG. 4. Decay amplitude $C_+\langle\pi^+\pi^0|Q_+|K^+\rangle$ for $q^*=1/a$ as a function of lattice meson mass M_π^2 for tree-level CHPT. Circles and squared refer to decay amplitude obtained on a 24^3 and 32^3 lattice. Open symbols correspond to the traditional $\sqrt{2\kappa/2\kappa_c}/2$ normalization factor of Wilson quark fields, while filled symbols are for the KLM normalization.

hibiting a decrease between 24^3 and 32^3 spatial sizes. We observe that these features were present in the previous simulations when examined in the light of our results, but they were not evident at the time because of large statistical errors of $20-30\,\%$.

In Fig. 4 we show how the use of the KLM normalization affects the meson mass dependence of the decay amplitude (plotted with filled symbols) as compared to the conventional normalization $\sqrt{2\kappa/2\kappa_c}/2$ (open symbols). While the amplitudes for small M_π change only slightly, those for larger M_π increase by about 20%, which is beyond the statistical error by a factor of two. A significant meson mass dependence and finite-size effects observed in our data show that tree-level CHPT is inadequate to extract the physical amplitude from lattice calculations.

B. Decay amplitude with one-loop CHPT

Recently Golterman and Leung have carried out a one-loop calculation of CHPT for the decay amplitude in full and quenched QCD for degenerate and non-degenerate K and π mesons [9]. Their formula also includes finite-size correction terms. Combining with the one-loop formula calculated for the physical point [19], we analyze how our results are changed by one-loop effects of CHPT.

Let us denote by $\langle\pi^+\pi^0|Q_+|K^+\rangle_{phys}$ the physical amplitude in the full theory with non-degenerate K and π mesons of mass m_K and m_π, and by $\langle\pi^+\pi^0|Q_+|K^+\rangle_{lattice}$ the amplitude in the quenched theory with degenerate K and π mesons of mass M_π. According to Golterman and Leung,

$$\langle\pi^+\pi^0|Q_+|K^+\rangle_{phys} = \frac{m_K^2-m_\pi^2}{2M_\pi^2}\left(\frac{\alpha_{27}}{\alpha_{27}^q}\right)\left(\frac{f_q}{f}\right)^3 Y\langle\pi^+\pi^0|Q_+|K^+\rangle_{lattice}, \tag{29}$$

where α and f are defined in Eqs. (25)–(27), and the factor Y is given by

$$Y=\frac{1+\dfrac{m_\pi^2}{(4\pi f_\pi)^2}[U+d]}{1+\dfrac{M_\pi^2}{(4\pi F_\pi)^2}\left[-3\log\left(\dfrac{M_\pi}{\Lambda^q}\right)^2+F(M_\pi aL)+d_q\right]}. \tag{30}$$

The numerator of Y represents the one-loop effect in the full theory, and the denominator is the corresponding effect in the quenched theory. The dimensionless constants d and d_q are the contact term coefficients arising from the $O(p^4)$ terms of the chiral Lagrangian. f_π and F_π are the one-loop corrected decay constants in the full and quenched theories, which differ from the tree-level values f and f_q. In the numerator of Y, U is a complicated function of physical K and π masses, the decay constant f_π and f_K, and the cutoff of CHPT for the full theory Λ^{cont}, and a numerical approximation is

$$U=A+B\log\left(\frac{m_\pi}{\Lambda^{cont}}\right)^2, \tag{31}$$

where $A=-104.73$ and $B=-29.57$ for $m_\pi=136$ MeV, $m_K=497$ MeV, $f_\pi=132$ MeV, and $f_K=160$ MeV. In the denominator of Eq. (30), Λ^q is the cutoff of CHPT for quenched QCD, and $F(M_\pi aL)$ represents finite-size corrections for a spatial size L which takes the form

$$F(M_\pi aL)=\frac{17.827}{M_\pi aL}+\frac{12\pi^2}{(M_\pi aL)^3}. \tag{32}$$

We set α_{27} and f to be equal in the quenched and full theories as in the analysis with the tree-level CHPT. We initially ignore the effects of $O(p^4)$ terms of the chiral Lagrangian d and d_q. We leave Λ^{cont} and Λ^q to be different, however, and examine the dependence of the results on these cutoffs.

In Fig. 5 we plot the one-loop corrected decay amplitude for $\Lambda^q=770$ MeV and 1 GeV for the choice $\Lambda^{cont}=770$ MeV. We set $f_\pi=F_\pi=132$ MeV in Eq. (30), and the finite-size corrections $F(M_\pi aL)$ are taken into account. The results of a similar analysis for the choice $\Lambda^{cont}=1$ GeV are plotted in Fig. 6.

An important feature observed in Figs. 5 and 6 is that the size dependence seen with the tree-level analysis in Fig. 4 is removed after finite-size corrections at the one-loop level. At the same time, the amplitude decreases by $30-40\,\%$ over the range of meson mass covered in our simulation.

Another noteworthy feature in Figs. 5 and 6 is that a sizable lattice meson mass dependence still remains in the amplitude, and that the magnitude of the slope depends sensitively on the choice of Λ^q. This feature can be understood as arising from the $O(p^4)$ coupling constants in the quenched theory, i.e., d_q in Eq. (30), which was ignored above. If we denote our present results by $\langle\pi^+\pi^0|Q_+|K^+\rangle_{ours}$ we find from Eq. (30) that

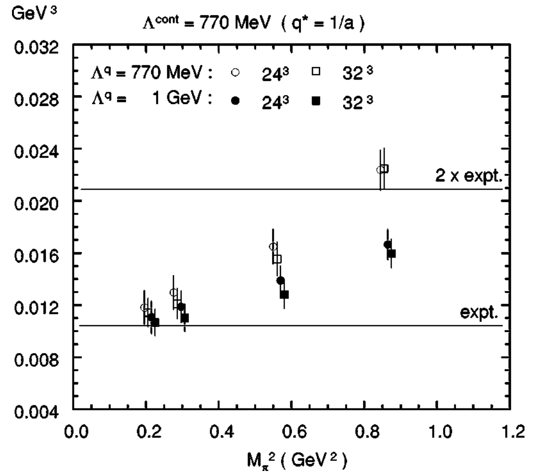

FIG. 5. Decay amplitude $C_+\langle\pi^+\pi^0|Q_+|K^+\rangle$ for $q^*=1/a$ obtained with one-loop CHPT for $\Lambda^{cont}=770$ MeV plotted as a function of M_π^2. Circles and squares refer to data for 24^3 and 32^3 spatial sizes. Open symbols are for $\Lambda^q=770$ MeV and filled symbols for $\Lambda^q=1$ GeV.

$$\langle\pi^+\pi^0|Q_+|K^+\rangle_{ours}=\langle\pi^+\pi^0|Q_+|K^+\rangle_{phys}\frac{1+\dfrac{M_\pi^2}{(4\pi F_\pi)^2}d_q}{1+\dfrac{m_\pi^2}{(4\pi f_\pi)^2}d} \qquad (33)$$

showing the presence of a term linear in M_π^2. We note furthermore that d_q actually depends on Λ^q: $d_q=d_q(\Lambda^q)$. Since the total $O(p^4)$ correction in the denominator of Eq. (30) should be independent of the cutoff Λ^q, d_q for different values of Λ^q varies according to

$$d_q(\Lambda^q)=d_q(\Lambda'^q)-3\log\left(\frac{\Lambda^q}{\Lambda'^q}\right)^2 . \qquad (34)$$

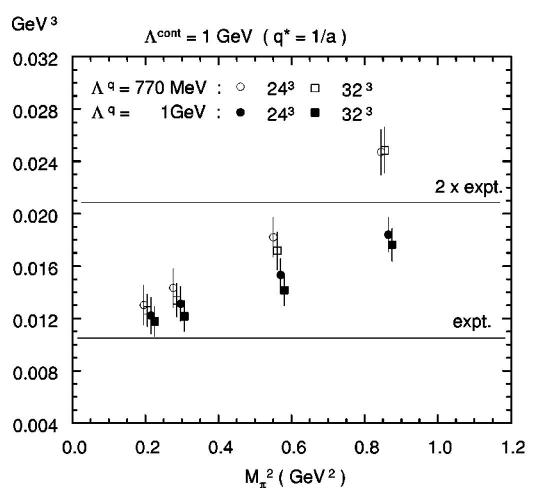

FIG. 6. Decay amplitude $C_+\langle\pi^+\pi^0|Q_+|K^+\rangle$ for $q^*=1/a$ as a function of M_π^2 obtained with one-loop CHPT for $\Lambda^{cont}=1$ GeV. Meaning of symbols are the same as in Fig. 5.

To compare these relations with our results, we fit $\langle\pi^+\pi^0|Q_+|K^+\rangle_{ours}$ as a function of M_π^2 to the form (33). Employing data for which the value of M_π does not exceed the cutoff, we find $d_q(1\ \text{GeV})/(4\pi)^2\approx 0.015$ and $d_q(770\ \text{MeV})/(4\pi)^2\approx 0.025$. The difference $d_q(1\ \text{GeV})/(4\pi)^2-d_q(770\ \text{MeV})/(4\pi)^2\approx -0.01$ is in good agreement with the value $-3\log(1\ \text{GeV}/770\ \text{MeV})^2/(4\pi)^2=-0.0099$ expected from Eq. (34). These results show that the uncertainties associated with d_q can be removed by a chiral extrapolation of our amplitude to the chiral limit $M_\pi=0$.

A further consequence of Eq. (33) is that a correction due to the $O(p^4)$ term d in the full theory remains even after taking the limit $M_\pi\to 0$ in our results. In order to examine the magnitude of this uncertainty, we use an estimate $d(\Lambda^{cont})/(4\pi)^2=0.003(14)$ at $\Lambda^{cont}=m_\eta$ from a phenomenological analysis [19]. In view of the formula

$$d(\Lambda^{cont})=d(\Lambda'^{cont})-29.57\log\left(\frac{\Lambda^{cont}}{\Lambda'^{cont}}\right)^2 \qquad (35)$$

obtained from Eqs. (30) and (31), this leads to a value $d(770\ \text{MeV})/(4\pi)^2\approx -0.12$ and $d(1\ \text{GeV})/(4\pi)^2\approx -0.22$. These values imply that the physical decay amplitude is 10% lower than our results for $\Lambda^{cont}=770$ MeV, and 20% for $\Lambda^{cont}=1$ GeV. This provides an explanation of a discrepancy of about 10% observed in Figs. 5 and 6 between the values of $\langle\pi^+\pi^0|Q_+|K^+\rangle_{ours}$ calculated with $\Lambda^{cont}=770$ MeV and 1 GeV. Let us add a remark that the values of d estimated above for $\Lambda^{cont}=770$ MeV and 1GeV is an order of magnitude larger compared to those of d_q for the quenched theory.

We find from this analysis that including the correction due to the $O(p^4)$ coupling constants is possible if an accurate value of d is known from phenomenological studies. Since this is not yet the case [19], we shall not pursue this point further here, leaving the correction as a source of uncertainty in our final results.

The amplitude obtained from $\langle\pi^+\pi^0|Q_+|K^+\rangle_{ours}$ by a chiral extrapolation to the limit $M_\pi=0$ is listed in Table III for several choices of the cutoff and the operator matching point q^*. In the results in Table III, the systematic error due to the matching scale q^* is about 10%. Statistical errors are larger (about 20%), mainly due to a linear extrapolation to the chiral limit. Within these uncertainties and that of 10–20% due to the d term discussed above, the values in Table III are consistent with the experiment 10.4×10^{-3} GeV^3.

V. B_K FROM THE $K^+\to\pi^+\pi^0$ AMPLITUDE

The $\Delta S=2$ four-quark operator defined by

$$O_{\Delta S=2}=\bar{s}\gamma_\mu(1-\gamma_5)d\ \ \bar{s}\gamma_\mu(1-\gamma_5)d \qquad (36)$$

belongs to the same **27** representation as the operator Q_4 which is the $[\mathbf{27},\Delta I=3/2]$ part of Q_+. As a consequence one can obtain the K meson B parameter B_K from the K^+

TABLE III. Results of linear extrapolation of $C_+\langle\pi^+\pi^0|Q_+|K^+\rangle$ to $M_\pi^2=0$. For $\Lambda^q=770$ MeV fits are made with three points with smaller M_π as M_π of the fourth point exceeds the cutoff. Statistical and extrapolation errors are combined. The experimental values is 10.4×10^{-3} GeV^3.

		$C_+\langle\pi^+\pi^0\|Q_+\|K^+\rangle(\times10^{-3}\text{GeV}^3)$			
Λ^{cont}	Λ^q	24^3		32^3	
(GeV)	(GeV)	$q^*=1/a$	$q^*=\pi/a$	$q^*=1/a$	$q^*=\pi/a$
0.77	0.77	9.3(1.9)	10.2(2.1)	8.9(1.7)	9.7(1.9)
0.77	1.0	9.4(1.3)	10.3(1.4)	8.8(1.1)	9.6(1.2)
1.0	0.77	10.3(2.1)	11.3(2.3)	9.8(1.9)	10.7(2.1)
1.0	1.0	10.4(1.4)	11.4(1.5)	9.7(1.2)	10.6(1.3)

$\to\pi^+\pi^0$ amplitude using CHPT. The nonvanishing component of the tensor R_{kl}^{ij} in Eq. (25) for this operator is given by $R_{33}^{22}=1$.

The one-loop relation in CHPT for quenched QCD for the unphysical degenerate case has been obtained by Golterman and Leung [9],

$$B_K=\frac{1}{\frac{8}{3}F_K}\frac{3\langle\pi^+\pi^0|Q_+|K^+\rangle}{\frac{3}{\sqrt{2}}M_\pi^2(1+R+d_q)} \tag{37}$$

with the one-loop correction R given by

$$R=\frac{M_\pi^2}{(4\pi F_\pi)^2}\left[3\log\left(\frac{M_\pi}{\Lambda^q}\right)^2+F(M_\pi aL)\right]. \tag{38}$$

Here F_K and F_π denote the K and π meson decay constants in the quenched theory, and the other notations are the same as those in Eqs. (30)–(32).

Our procedure for calculating B_K from Eq. (37) is essentially the same as for the $K^+\to\pi^+\pi^0$ amplitude including the operator matching procedure, although the coefficient function C_+ is absent in the present case. In Figs. 7 and 8 we plot $B_K(2\ \text{GeV})$ obtained from the $K^+\to\pi^+\pi^0$ decay amplitude with tree and one-loop CHPT. We set $F_K=160$ MeV and $F_\pi=132$ MeV in Eqs. (37) and (38). The one-loop CHPT effect and the cutoff dependence for a small M_π^2 region are small compared with those for the decay amplitude. At the physical K meson mass $M_\pi^2=0.246$ GeV^2 B_K takes almost the same value for different choices of Λ^q and the lattice size. In Table IV the average of the two data points with the smallest M_π is tabulated. Our results, $B_K=0.581(56)-0.663(67)$ are consistent with the JLQCD value $B_K(2\ \text{GeV})=0.68(11)$ [20] obtained at the same coupling constant $\beta=6.1$ through a calculation of the $K^0-\overline{K^0}$ matrix element of the $\Delta S=2$ operator employing chiral Ward identities for determining the mixing coefficients.

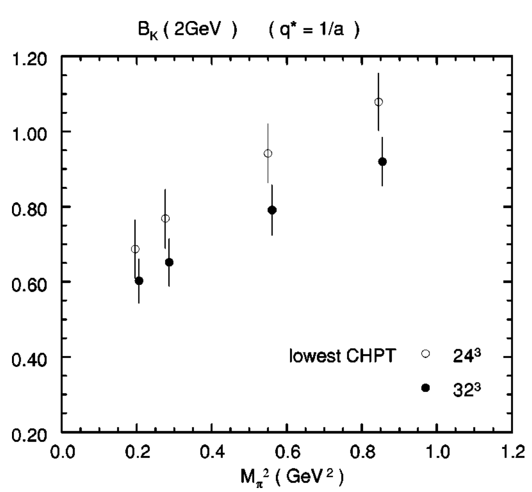

FIG. 7. $B_K(2\ \text{GeV})$ for $q^*=1/a$ obtained from $K^+\to\pi^+\pi^0$ decay amplitude as a function of M_π^2 obtained with tree level CHPT. Circles and squares refer to data for 24^3 and 32^3 spatial sizes.

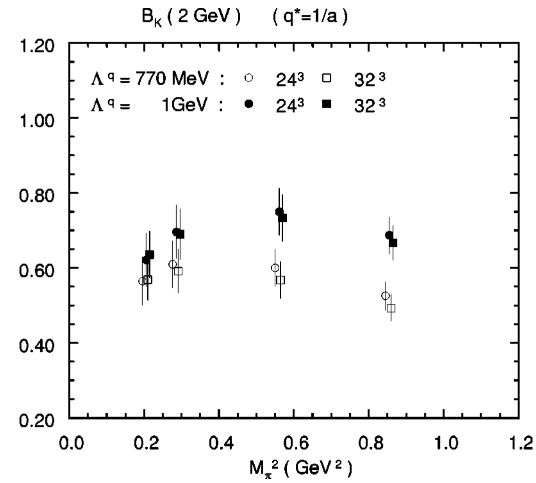

FIG. 8. $B_K(2\ \text{GeV})$ for $q^*=1/a$ obtained from $K^+\to\pi^+\pi^0$ decay amplitude as a function of M_π^2 obtained with one-loop CHPT for $\Lambda^q=770$ MeV and 1 GeV. Circles and squares refer to data for 24^3 and 32^3 spatial sizes. Open symbols are for $\Lambda^q=770$ MeV and filled symbols for $\Lambda^q=1$ GeV.

A direct calculation of B_K with the Wilson quark action has the complication that the operator mixing problem of the $\Delta S=2$ operator has to be solved nonperturbatively, which causes large statistical errors. In contrast, the Q_+ operator does not mix with other operators as mentioned in Sec. III B. Therefore, statistical errors of B_K obtained from the $K^+\to\pi^+\pi^0$ amplitude is smaller. Theoretical uncertainties associated with the use of CHPT, however, are large in this approach that offsets the advantage of the present method. In any case, our calculation, albeit with a significant error, pro-

TABLE IV. $B_K(2\ \text{GeV})$ at physical K meson mass $M_\pi=496$ MeV obtained from the $K^+\to\pi^+\pi^0$ amplitude. The row ‘‘tree’’ refers to the result with the lowest CHPT and others are obtained by one-loop CHPT for $\Lambda^q=0.77$ GeV and 1 GeV.

	Tree	$\Lambda^q=0.77$ GeV	$\Lambda^q=1$ GeV
24^3	0.728(78)	0.587(64)	0.659(71)
32^3	0.627(63)	0.581(58)	0.663(67)

vides an independent check for B_K for the Wilson quark action obtained with the chiral Ward identity procedure, and also supports the validity of CHPT.

VI. CONCLUSIONS

In this article we have reported results of a study of the $K^+ \to \pi^+ \pi^0$ decay amplitude in quenched lattice QCD. With a set of high statistics simulations we have found that the results show sizable finite-size effects, which, however, are consistent with those predicted by a recent one-loop calculation of CHPT. We have furthermore seen that a meson mass dependence which remains after inclusion of the one-loop corrections of CHPT in the prediction for the decay amplitude is due to effects of the $O(p^4)$ contact terms in the quenched theory. Making an extrapolation to the chiral limit to remove these effects, we have found $8.9(1.7)\times 10^{-3} - 11.4(1.5)\times 10^{-3}$ GeV^3 for the physical value of the decay amplitude, depending on the choice of the cutoff parameter of CHPT. These values are consistent with experiment (10.4×10^{-3} GeV^3).

The present result may be compared to those of the previous studies [2,4] which gave decay amplitudes roughly twice larger than experiment. Our smaller value originates from the two effects, one-loop corrections as also noted by Golterman and Leung in their reanalysis of the old results, and a decrease of the amplitude toward smaller values of M_π.

As a further application of the one-loop formula, we have calculated the B_K parameter, and found that it is consistent with a recent direct calculation for $K^0 - \overline{K}^0$ mixing.

The encouraging results we have obtained, however, should be taken with several reservations. The value of the $K^+ \to \pi^+ \pi^0$ decay amplitude estimated in the chiral limit suffers from uncertainties of 10–20 % due to the $O(p^4)$ contact terms of the full theory, because the phenomenological estimate available is not very accurate. A sizable finite-size correction of $30-40$ %, while consistent with the one-loop prediction of CHPT, raises the question whether ignoring higher order corrections can be justified. Furthermore, various constants of CHPT, in particular the coefficient α_{27}, may differ between the quenched and full theories, and we have no way of estimating or correcting the difference. Reliability of CHPT for calculating unphysical amplitudes could also be an issue. Reducing these sources of uncertainties, especially those related to quenching and better controlling finite-size effects would require a difficult task of carrying out simulations in full QCD on a physically large lattice.

ACKNOWLEDGMENTS

We thank Maarten Golterman for informative correspondence on his results for the decay amplitude from chiral perturbation theory. This work was supported by the Supercomputer Project (No. 97-15) of High Energy Accelerator Research Organization (KEK), and also in part by the Grants-in-Aid of the Ministry of Education (Nos. 08640349, 08640350, 08640404, 09246206, 09304029, 09740226).

[1] M. B. Gavela, L. Maiani, S. Petrarca, G. Martinelli, and O. Pene, Phys. Lett. B **211**, 139 (1988); G. Martinelli, Nucl. Phys. B (Proc. Suppl.) **17**, 523 (1990).
[2] C. Bernard and A. Soni, Nucl. Phys. B (Proc. Suppl.) **9**, 155 (1989); C. Bernard and A. Soni, *ibid.* **17**, 495 (1990); C. Bernard, in *From Actions to Answers*, Proceedings of the 1989 TASI School, edited by T. De Grand and D. Toussaint (World Scientific, Singapore, 1990).
[3] G. Kilcup, Nucl. Phys. B (Proc. Suppl.) **17**, 533 (1990); S. R. Sharpe, *ibid.* **20**, 429 (1991).
[4] M. B. Gavela, L. Maiani, S. Petrarca, F. Rapuano, G. Martinelli, P. Pene, and C. Sachrajda, Nucl. Phys. **B306**, 677 (1988).
[5] JLQCD Collaboration (unpublished).
[6] L. Miani and M. Testa, Phys. Lett. B **245**, 585 (1990).
[7] See, e.g., G. P. Lepage, Nucl. Phys. B (Proc. Suppl.) **26**, 45 (1992); P. Mackenzie, *ibid.* **34**, 35 (1994); A. Kronfeld, *ibid.* **34**, 415 (1994) and references cited in theirs.
[8] G. P. Lepage and P. B. Mackenzie, Phys. Rev. D **48**, 2250 (1993).
[9] M. F. L. Golterman and K. C. Leung, Phys. Rev. D **56**, 2950 (1997).
[10] JLQCD Collaboration, S. Aoki *et al.*, Nucl. Phys. B (Proc. Suppl.) **53**, 355 (1997); in preparation.
[11] M. Lüscher, Commun. Math. Phys. **105**, 153 (1986).
[12] S. R. Sharpe, R. Gupta, and G. W. Kilcup, Nucl. Phys. **B383**, 309 (1992).
[13] R. Gupta, A. Patel, and S. R. Sharpe, Phys. Rev. D **48**, 388 (1993).
[14] M. Fukugita, Y. Kuramashi, H. Mino, M. Okawa, and A. Ukawa, Phys. Rev. D **52**, 3003 (1995).
[15] G. Martinelli, Phys. Lett. **141B**, 395 (1984).
[16] C. Bernard, T. Draper, and A. Soni, Phys. Rev. D **36**, 3224 (1987).
[17] A. Buras, M. Jamin, and P. H. Weisz, Nucl. Phys. **B347**, 491 (1990); M. Ciuchini, E. Franco, G. Martinelli, and L. Reina, *ibid.* **B415**, 403 (1994).
[18] G. Buchalla, A. J. Buras, and M. E. Lautenbacher, Rev. Mod. Phys. **68**, 1125 (1996).
[19] J. Bijnens, H. Sonoda, and M. B. Wise, Phys. Rev. Lett. **53**, 2367 (1984); J. Kambor, J. Missimer, and D. Wyler, Phys. Lett. B **261**, 496 (1991).
[20] JLQCD Collaboration, S. Aoki *et al.*, Nucl. Phys. B (Proc. Suppl.) **53**, 349 (1997); hep-lat/9705035.