Perturbative calculation of improvement coefficients to $O(g^2a)$ for bilinear quark operators in lattice QCD

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We calculate the $O(g^2a)$ mixing coefficients of bilinear quark operators in lattice QCD using a standard perturbative evaluation of on-shell Green's functions. Our results for the plaquette gluon action are in agreement with those previously obtained with the Schrödinger functional method. The coefficients are also calculated for a class of improved gluon actions having six-link terms. [S0556-2821(98)09621-0]

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I. INTRODUCTION

Symanzik's improvement program [1] applied to on-shell quantities [2] attempts to eliminate cut-off dependence order by order by an expansion in powers of the lattice spacing a. To $O(a)$ in lattice QCD, this requires adding the $O(a)$ "clover" term to the Wilson quark action [3]. Quark operators also have to be modified by $O(a)$ counterterms, which generally involve new operators of higher dimension [4–6]. In perturbation theory, the tree-level value of the clover coefficient and those of the counterterms of quark operators can be easily determined. They are sufficient to remove terms of $O(g^2a \log a)$ in on-shell Green's functions evaluated at one-loop order, as has been explicitly demonstrated in Ref. [4]. To remove $O(g^2a)$ terms which still remain, the counterterm coefficients for quark operators have to be corrected by $O(g^2a)$ terms. For bilinear quark operators, these coefficients have been calculated in Refs. [7–9] using the Schrödinger functional technique.

In this article we analyze the $O(g^2a)$ coefficients of bilinear quark operators through a standard perturbative treatment of on-shell Green's functions of the operators. One-loop amplitudes with external quarks on the mass shell are expanded in powers of qa and m_Ra, with q the external momenta and m_R the renormalized quark mass, which leads to an alternative determination of the coefficients. Applying the procedure for the standard plaquette gluon action, we obtain results which are in agreement with those of Refs. [7–9].

Another application of our procedure is a calculation of the coefficients for gluon actions improved by an addition of six-link loop terms to the plaquette action. We treat three cases: the action [10] which is tree-level improved in Symanzik's sense to $O(a^4)$, and two types of actions [11,12] improved by a renormalization-group treatment. The results should be useful in simulations employing improved actions for gluons.

This paper is organized as follows. In Sec. II we define bilinear quark operators examined in this article, and write down renormalization relations between the renormalized and bare operators. Analysis of one-loop amplitudes are carried out in Sec. III. Numerical results for the coefficients and a comparison with previous work are made in Sec. IV. We close with some concluding remarks in Sec. V.

II. CLOVER QUARK ACTION AND BILINEAR QUARK OPERATORS

Consider the clover quark action defined by

$$\begin{aligned} S_{\text{quark}} = a^3 \sum_n \frac{1}{2} \sum_\mu & \big(\overline{\psi}_n(-r+\gamma_\mu)U_{n,\mu}\psi_{n+\hat{\mu}} \\ & + \overline{\psi}_n(-r-\gamma_\mu)U^\dagger_{n-\mu,\mu}\psi_{n-\hat{\mu}}\big) \\ & + (am_0+4r)\overline{\psi}_n\psi_n \\ & - c_{\text{SW}} a^3 \sum_n \sum_{\mu,\nu} ig\frac{r}{4}\overline{\psi}_n\sigma_{\mu\nu}P_{\mu\nu}(n)\psi_n . \end{aligned} \tag{2.1}$$

We wish to construct a renormalized bilinear quark operator of the form

$$O_R^\Gamma = (\overline{\psi}_c \Gamma \psi_c)_R, \quad \Gamma = 1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu} \tag{2.2}$$

which is improved to $O(a)$ to one-loop order of perturbation theory, i.e., on-shell matrix elements of the operator have no errors of $O(a)$, $O(g^2a \log a)$ or $O(g^2a)$ when $a \to 0$ with external momenta and the renormalized quark mass m_R kept fixed.

Our starting point is the tree-level improved operator of the form

$$O = \overline{\psi}_c \Gamma \psi_c , \tag{2.3}$$

where the rotated quark fields ψ_c and $\overline{\psi}_c$ are given by

$$\psi_c = \left[1 - \frac{ar}{4}(\gamma_\mu \vec{D}_\mu - m_0)\right]\psi, \tag{2.4}$$

$$\overline{\psi}_c=\overline{\psi}\left[1-\frac{ar}{4}(-\gamma_\mu\overleftarrow{D}_\mu-m_0)\right] \tag{2.5}$$

with the covariant derivative defined as

$$a\overrightarrow{D}_\mu\psi(x)=\frac{1}{2}[U_\mu(x)\psi(x+\hat{\mu})-U_\mu^\dagger(x-\hat{\mu})\psi(x-\hat{\mu})], \tag{2.6}$$

$$a\overline{\psi}(x)\overleftarrow{D}_\mu=\frac{1}{2}[\overline{\psi}(x+\hat{\mu})U_\mu^\dagger(x)-\overline{\psi}(x-\hat{\mu})U_\mu(x-\hat{\mu})]. \tag{2.7}$$

It has been demonstrated in Ref. [4] that this operator is on-shell improved to $O(a)$ and $O(g^2a\ \log a)$ with the tree-level value of the clover coefficient $c_{SW}=1$. It has been noted furthermore that the field rotation can be generalized by using the equation of motion $(\not{D}+m_0)\psi=0$ to

$$\psi_c=\left[1-\frac{ar}{2}(z\gamma_\mu\overrightarrow{D}_\mu-(1-z)m_0)\right]\psi, \tag{2.8}$$

$$\overline{\psi}_c=\overline{\psi}\left[1-\frac{ar}{2}(-z\gamma_\mu\overleftarrow{D}_\mu-(1-z)m_0)\right], \tag{2.9}$$

where z is a parameter. We then consider a generalized operator given by

$$O_0^\Gamma=[1+ar(1-z)m_0]\overline{\psi}\Gamma\psi+z\overline{\psi}\Gamma^{\otimes}\psi-z^2\overline{\psi}\Gamma'\psi, \tag{2.10}$$

where $\Gamma^{\otimes}$ and Γ' are $O(a)$ and $O(a^2)$ vertices defined as

$$\Gamma^{\otimes}=\frac{ar}{2}(\gamma_\mu\overleftarrow{D}_\mu\Gamma-\Gamma\gamma_\mu\overrightarrow{D}_\mu), \tag{2.11}$$

$$\Gamma'=\frac{a^2r^2}{4}\gamma_\nu\overleftarrow{D}_\nu\Gamma\gamma_\mu\overrightarrow{D}_\mu. \tag{2.12}$$

The one-loop relation expected between the bare operator (2.10) and the renormalized improved operator (2.2) has the form

$$O_0^\Gamma=Z_\Gamma^{-1}O_{\mathrm{R}}^\Gamma-g^2C_Fam_RB_\Gamma O_{\mathrm{R}}^\Gamma-g^2C_FaC_\Gamma\tilde{O}_{\mathrm{R}}^\Gamma, \tag{2.13}$$

where C_F denotes the second-order Casimir eigenvalue for the quark field, and the last two terms are needed to remove $O(g^2a)$ errors from on-shell matrix elements, with $\tilde{O}_{\mathrm{R}}^\Gamma$ a dimension 4 operator with derivative. In a previous paper [13] we have evaluated Z_Γ for a class of improved gluon actions. Our task now is to generalize the analysis to (i) check that there are no $O(g^2a\ \log a)$ errors with the operator (2.10), and (ii) determine the $O(g^2a)$ coefficients B_Γ and C_Γ. In the following we set the Wilson parameter $r=1$.

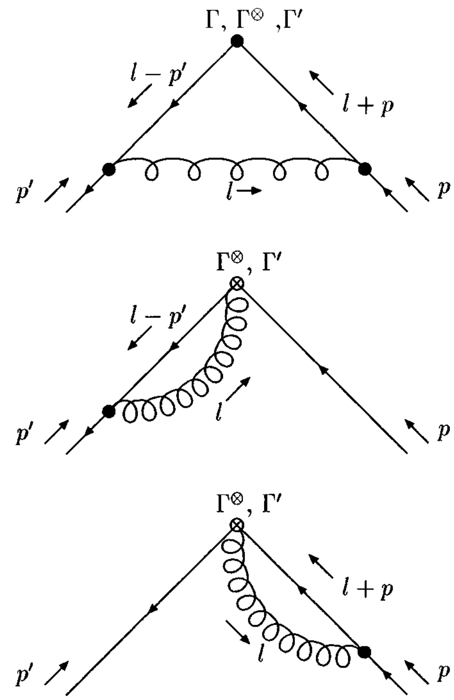

FIG. 1. Structure of one-loop diagrams for bilinear quark operator (2.10).

III. ANALYSIS OF ONE-LOOP AMPLITUDES

The structure of one-loop diagrams relevant for our analysis is depicted in Fig. 1 where we indicate our momentum assignment. We calculate the corresponding amplitudes in Feynman gauge imposing the on-shell condition to external momenta, i.e., setting $-i\not{p}'+m_R=0$ and $i\not{p}+m_R=0$ when such a factor appears to the left-most or right-most in the amplitudes. The renormalized mass m_R, as explicitly defined below, coincides with the on-shell mass to $O(a^2)$. We note that the bare quark mass m_0 enters in the field rotation (2.9) of the bare operator, where we make use of the tree-level equation of motion.

The vertex function in momentum space calculated to one-loop order has the form

$$\begin{aligned}G^\Gamma=&[1+am_0(1-z)]\Gamma\\&+za\frac{1}{2}(i\not{p}'\Gamma-\Gamma i\not{p})+g^2C_F[1+am_0(1-z)]T_\Gamma\\&+g^2C_FzT_{\Gamma^{\otimes}}-g^2C_Fz^2T_{\Gamma'},\end{aligned} \tag{3.1}$$

where Γ represents the tree level contribution, T_Γ, $T_{\Gamma^{\otimes}}$ and $T_{\Gamma'}$ are one-loop contributions from the vertices Γ, $\Gamma^{\otimes}$ and Γ'.

A problem in extracting $O(a)$ terms of the one-loop contributions is that they are infrared divergent for on-shell external momenta. We treat this problem by supplying a mass λ to the gluon propagator. The one-loop amplitudes, being functions of pa, $p'a$, m_Ra, and λa, are then finite, which we expand around $pa=p'a=0$ and $m_Ra=0$, keeping λa finite. For this procedure to be justified, infrared divergences which remain in the vertex function G_Γ after wave function

TABLE I. Dimension 4 vertex mixing with $\bar{\psi}\Gamma\psi$ at one loop order.

Γ	$q^+_\mu\tilde{\Gamma}^+_\mu$	$q^-_\mu\tilde{\Gamma}^-_\mu$
$\gamma_\mu\gamma_5$	$q^+_\mu\gamma_5$	$q^-_\nu\sigma_{\mu\nu}\gamma_5$
γ_μ	$q^+_\nu\sigma_{\mu\nu}$	q^-_μ
γ_5	$q^+_\mu\gamma_\mu\gamma_5$	$q^-_\mu\gamma_\mu\gamma_5$
1	$q^+_\mu\gamma_\mu$	$q^-_\mu\gamma_\mu$
$\sigma_{\mu\nu}$	$q^+_\mu\gamma_\nu - q^+_\nu\gamma_\mu$	$p'_\rho(\sigma_{\rho\mu}\gamma_\nu - \sigma_{\rho\nu}\gamma_\mu) - p_\rho(\gamma_\mu\sigma_{\nu\rho} - \gamma_\nu\sigma_{\mu\rho})$

renormalization should coincide with those in the continuum. We check this point explicitly below.

As a first step to extract $O(a)$ terms, we expand one-loop contributions in terms of external momenta. Under the on-shell condition the $O(a)$ term can be written in two alternative forms, i.e.,

$$T_{(\Gamma,\Gamma\otimes,\Gamma')} = V^+_{(\Gamma,\Gamma\otimes,\Gamma')}\Gamma + v^+_{(\Gamma,\Gamma\otimes,\Gamma')} iaq^+_\mu\tilde{\Gamma}^+_\mu, \tag{3.2}$$

$$= V^-_{(\Gamma,\Gamma\otimes,\Gamma')}\Gamma + v^-_{(\Gamma,\Gamma\otimes,\Gamma')} iaq^-_\mu\tilde{\Gamma}^-_\mu, \tag{3.3}$$

where $q^\pm_\mu = p'_\mu \pm p_\mu$, and $q^\pm_\mu\tilde{\Gamma}^\pm_\mu$ is a dimension-4 operator vertex with the same quantum number as Γ as listed in Table I for each Γ. The on-shell identities which relate the two forms are given by

$$-iq^+_\mu\gamma_5 + iq^-_\nu\sigma_{\mu\nu}\gamma_5 + 2m_R\gamma_\mu\gamma_5 = 0, \tag{3.4}$$

$$-i\sigma_{\mu\nu}q^+_\nu + iq^-_\mu - 2m_R\gamma_\mu = 0, \tag{3.5}$$

$$q^-_\mu\gamma_\mu\gamma_5 = q^+_\mu\gamma_\mu = 0, \tag{3.6}$$

$$q^+_\mu\gamma_\mu\gamma_5 + 2im_R\gamma_5 = 0, \tag{3.7}$$

$$q^-_\mu\gamma_\mu + 2im_R = 0, \tag{3.8}$$

$$p'_\rho(\sigma_{\rho\mu}\gamma_\nu - \sigma_{\rho\nu}\gamma_\mu) - p_\rho(\gamma_\mu\sigma_{\nu\rho} - \gamma_\nu\sigma_{\mu\rho}) + (q^+_\mu\gamma_\nu - q^+_\nu\gamma_\mu) + 4im_R\sigma_{\mu\nu} = 0. \tag{3.9}$$

We choose to work with $q^+_\mu\tilde{\Gamma}^+_\mu$ and drop the $+$ suffix. The momentum $q^+_\mu = q_\mu$ represents the momentum transfer at the operator vertex. We observe from the identities above and Table I that $q^+_\mu\tilde{\Gamma}^+_\mu$ actually vanishes by the on-shell condition for the scalar and pseudo scalar operators. Substituting the expansion (3.3) into (3.1), we obtain

$$\begin{aligned} G^\Gamma = {} & [1 + am_0(1-z) + zam_R]\Gamma + g^2C_F[1 + am_0(1-z)] \\ & \times (V_\Gamma\Gamma + v_\Gamma iaq_\mu\tilde{\Gamma}_\mu) + g^2C_F z(V_{\Gamma\otimes}\Gamma + v_{\Gamma\otimes} iaq_\mu\tilde{\Gamma}_\mu) \\ & - g^2C_F z^2(V_{\Gamma'}\Gamma + v_{\Gamma'} iaq_\mu\tilde{\Gamma}_\mu). \end{aligned} \tag{3.10}$$

As a second step, we expand the one-loop amplitudes $V_{(\Gamma,\Gamma\otimes,\Gamma')}$ in terms of the lattice spacing a multiplied by the renormalized quark mass m_R. This leads to

TABLE II. Coefficients of logarithmically divergent term.

Γ	$h_2(\Gamma)$	$h'_2(\Gamma)$
$\gamma_\mu\gamma_5$	4	-2
γ_μ	4	-2
γ_5	16	1
1	16	1
$\sigma_{\mu\nu}$	0	-3

$$V_\Gamma = \frac{h_2(\Gamma)}{4}L + V^{(0)}_\Gamma + am_R V^{(1)}_\Gamma + O(a^2), \tag{3.11}$$

$$V_{\Gamma\otimes} = h'_2(\Gamma)am_R L + V^{(0)}_{\Gamma\otimes} + am_R V^{(1)}_{\Gamma\otimes} + O(a^2), \tag{3.12}$$

$$V'_\Gamma = V^{(0)}_{\Gamma'} + am_R V^{(1)}_{\Gamma'} + O(a^2), \tag{3.13}$$

where $V^{(0,1)}_{(\Gamma,\Gamma\otimes,\Gamma')}$ are constants independent of a and g. In a similar expansion of $v_{(\Gamma,\Gamma\otimes,\Gamma')}$, we only need to keep the leading term in a, and hence they can be regarded as a constant as well. The logarithmic divergence L is defined as

$$L = -\frac{1}{16\pi^2}\log \lambda^2 a^2, \tag{3.14}$$

with λ being gluon mass to regularize the infrared divergence which appears in on-shell vertex functions, typically in terms of form

$$\int_{-\pi}^{\pi}\frac{d^4l}{(2\pi)^4}\frac{1}{l^2}\frac{1}{l^2+\lambda^2a^2}. \tag{3.15}$$

The coefficients $h_2(\Gamma)$ and $h'_2(\Gamma)$ are given in Table II.

In order to relate bare operators to renormalized operators we further need the renormalization factor for quark wave function Z_ψ and mass Z_m, which are defined as

$$\psi_0 = Z_\psi^{-1/2}\psi_R, \tag{3.16}$$

$$m_0 = Z_m^{-1}m_R + g^2C_F\frac{\Sigma_0}{a}. \tag{3.17}$$

The explicit form of these factors are obtained from the inverse full quark propagator expanded to $O(g^2a)$,

$$S_F^{-1} = i\not{p} + m_0 + \frac{1}{2}ap^2 - \Sigma(p,m_0), \tag{3.18}$$

where the one-loop correction to self energy, expanded around $p=0$ and $m_0=0$, is given by

$$\Sigma(p,m_0)=g^2C_F\Bigg[\frac{\Sigma_0}{a}+i\not p(-L+\Sigma_1)+m_0(-4L+\Sigma_2)+ap^2(-L+\sigma_1)+am_0i\not p(4L+\sigma_2)+am_0^2(2L+\sigma_3)\Bigg]+O(a^2) \tag{3.19}$$

with $\Sigma_{0,1,2}$ and $\sigma_{1,2,3}$ being constants. We fix the renormalization constants by the condition that the full quark propagator takes the form

$$S_F=\frac{Z_\psi(-i\not p+m_R+O(a))}{p^2+m_R^2+O(a^2)}, \tag{3.20}$$

which yields

$$Z_\psi^{-1}=1+g^2C_F(-L+\Sigma_1)+am(-1+g^2C_F(L+2\sigma_1+\sigma_2-3\Sigma_1+\Sigma_2)), \tag{3.21}$$

$$Z_m^{-1}=1+g^2C_F(-3L-\Sigma_1+\Sigma_2)+am\left(\frac{1}{2}+\frac{g^2C_F}{2}(-3L-2\,\sigma_1-2\sigma_2+2\sigma_3+\Sigma_1)\right). \tag{3.22}$$

Here we have defined

$$m=m_0-g^2C_F\frac{\Sigma_0}{a}, \tag{3.23}$$

and have made use of the relation $g^2m_0=g^2m+O(g^4)$.

With the renormalization constants above, the inverse full quark propagator has the form

$$S_F^{-1}=(Z_\psi^{-1}+O(a))(i\not p+m_R)+O(a)(p^2+m_R^2)+O(a^2). \tag{3.24}$$

Hence the renormalized mass m_R coincides with the on-shell mass to $O(a^2)$. The wave function renormalization factor can be rewritten in terms of the renormalized mass:

$$Z_\psi^{-1}=1+g^2C_F(-L+\Sigma_1)+am_R(-1+g^2C_F(4L-z_m+\Sigma_1^{(1)})), \tag{3.25}$$

where

$$z_m=-\Sigma_1+\Sigma_2, \tag{3.26}$$

$$\Sigma_1^{(1)}=2\sigma_1+\sigma_2-3\Sigma_1+\Sigma_2. \tag{3.27}$$

Replacing the quark mass m_0 with the renormalized mass m_R, we obtain, for the vertex,

$$\begin{aligned}G^\Gamma&=\left[1+am_R+g^2C_FL\left(\frac{h_2(\Gamma)}{4}+am_R\left(\frac{h_2(\Gamma)}{4}(1-z)+h_2'(\Gamma)z-3(1-z)\right)\right)\right]\Gamma+g^2C_F[\Sigma_0(1-z)+V_\Gamma^{(0)}+zV_{\Gamma\otimes}^{(0)}\\&\quad-z^2V_{\Gamma'}^{(0)}]\Gamma+g^2C_Fam_R[z_m(1-z)+(1-z)V_\Gamma^{(0)}+V_\Gamma^{(1)}+zV_{\Gamma\otimes}^{(1)}-z^2V_{\Gamma'}^{(1)}]\Gamma+g^2C_F(v_\Gamma+zv_{\Gamma\otimes}-z^2v_{\Gamma'})iaq_\mu\tilde\Gamma_\mu\\&=\left[1+am_R+g^2C_FL\left(\frac{h_2(\Gamma)}{4}+am_R\left(\frac{h_2(\Gamma)}{4}-3\right)\right)\right]\Gamma+g^2C_F[\Sigma_0(1-z)+V_\Gamma^{(0)}+zV_{\Gamma\otimes}^{(0)}-z^2V_{\Gamma'}^{(0)}]\Gamma\\&\quad+g^2C_Fam_R[z_m(1-z)+(1-z)V_\Gamma^{(0)}+V_\Gamma^{(1)}+zV_{\Gamma\otimes}^{(1)}-z^2V_{\Gamma'}^{(1)}]\Gamma+g^2C_F(v_\Gamma+zv_{\Gamma\otimes}-z^2v_{\Gamma'})iaq_\mu\tilde\Gamma_\mu,\end{aligned} \tag{3.28}$$

where the relation $h_2(\Gamma)/4-h_2'(\Gamma)=3$, valid for each Γ, is used for the second equality.

We now multiply the vertex function G^Γ by the quark wave function renormalization factor Z_ψ^{-1} of Eq. (3.25). The $O(a)$ and $O(g^2a\ \log a)$ terms all cancel out in the combination $Z_\psi^{-1}G^\Gamma$ for arbitrary values of the parameter z, and the result can be written as an operator identity (2.13) with the constants given by

$$Z_\Gamma^{-1}=1+g^2C_F\left(\left(\frac{h_2(\Gamma)}{4}-1\right)L+\Sigma_1+V_\Gamma^{(0)}+\Sigma_0(1-z)+zV_{\Gamma\otimes}^{(0)}-z^2V_{\Gamma'}^{(0)}\right), \tag{3.29}$$

$$B_\Gamma=-\Big(\Sigma_1+\Sigma_1^{(1)}+V_\Gamma^{(1)}-\Sigma_0(1-z)+z(-z_m-V_\Gamma^{(0)}-V_{\Gamma\otimes}^{(0)}+V_{\Gamma\otimes}^{(1)})+z^2(V_{\Gamma'}^{(0)}-V_{\Gamma'}^{(1)})\Big), \tag{3.30}$$

$$C_\Gamma=-(v_\Gamma+zv_{\Gamma\otimes}-z^2v_{\Gamma'}). \tag{3.31}$$

TABLE III. Finite part z_Γ of renormalization factor for bilinear quark operators ($z=0$) without additive mass renormalization Σ_0. Coefficients of the term $c_{SW}^n (n=0,1,2)$ in an expansion $z_\Gamma = z_\Gamma^{(0)} + c_{SW} z_\Gamma^{(1)} + c_{SW}^2 z_\Gamma^{(2)}$ are given in the column labeled (n). Errors are at most in the last digit given.

gauge action			V			A	
c_1	c_{23}	(0)	(1)	(2)	(0)	(1)	(2)
0	0	−20.618	4.745	0.543	−15.797	−0.248	2.251
−1/12	0	−16.603	4.228	0.464	−12.540	−0.198	2.021
−0.331	0	−11.099	3.326	0.336	−8.192	−0.125	1.610
−0.27	−0.04	−11.540	3.418	0.353	−8.523	−0.131	1.657
−0.252	−0.17	−10.525	3.248	0.338	−7.707	−0.117	1.58

gauge action			S			P			T	
c_1	c_{23}	(0)	(1)	(2)	(0)	(1)	(2)	(0)	(1)	(2)
0	0	−12.953	−7.738	1.380	−22.596	2.249	−2.036	−17.018	3.913	1.972
−1/12	0	−9.607	−6.836	1.367	−17.734	2.015	−1.745	−13.539	3.490	1.719
−0.331	0	−4.858	−5.301	1.266	−10.673	1.601	−1.281	−8.939	2.751	1.300
−0.27	−0.04	−5.260	−5.454	1.292	−11.292	1.644	−1.316	−9.283	2.827	1.344
−0.252	−0.17	−4.366	−5.166	1.287	−10.001	1.565	−1.212	−8.427	2.687	1.271

Let us add a remark on infrared divergence. For $v_{(\Gamma,\Gamma^\otimes,\Gamma')}$ that enter in the coefficient C_Γ, these divergences cancel out in the total contribution at each order of z. On the other hand, infrared divergences remain in $V_{(\Gamma,\Gamma^\otimes,\Gamma')}$. The divergence in the $O(g^2 a)$ term in G^Γ are, however, canceled by that of the quark wave function renormalization factor. The remaining divergence, which is of form $g^2 L$ and appears in Z_Γ, coincides with that which is present in on-shell vertex functions for the renormalized operator in the continuum.

In the calculation above we employed the bare operator (2.10) which contains the bare quark mass m_0. It is possible to replace m_0 by the subtracted mass m of Eq. (3.23). Defining

$$O_0^\Gamma = [1 + ar(1-z)m]\bar{\psi}\Gamma\psi + z\bar{\psi}\Gamma^\otimes\psi - z^2\bar{\psi}\Gamma'\psi, \tag{3.32}$$

it is straightforward to check that $O(a)$ and $O(g^2 a \log a)$ terms also cancel for this operator; Σ_0 plays no role in improving the operator. The renormalization coefficients for this operator, which is more convenient for practical use, are obtained from Eqs. (3.29)–(3.31) by eliminating Σ_0.

IV. RESULTS FOR THE ONE-LOOP COEFFICIENTS

Manipulations in the previous section have reduced the determination of one-loop coefficients to an evaluation of a number of integral constants. Working out the integrands for the integrals is a straightforward but tedious task, which we carry out by *Mathematica*. The output is converted to a FORTRAN code, and the integrals are evaluated by the Monte Carlo routine VEGAS in double precision. We employ 20 sets of 10^5 points for integration except for C_A for $z=0$ for which we use 20 sets of 10^6 points. Errors are estimated from variation of integrated values over the sets.

For the gluon action we consider the form given by

$$S_{\rm gluon} = \frac{1}{g^2}\Bigg\{ c_0 \sum_{\rm plaquette} {\rm Tr}\, U_{pl} + c_1 \sum_{\rm rectangle} {\rm Tr}\, U_{rtg} + c_2 \sum_{\rm chair} {\rm Tr}\, U_{chr} + c_3 \sum_{\rm parallelogram} {\rm Tr}\, U_{plg} \Bigg\}, \tag{4.1}$$

where the first term represents the standard plaquette term, and the remaining terms are six-link loops formed by a 1×2 rectangle, a bent 1×2 rectangle (chair) and a 3-dimensional parallelogram. The coefficients $c_0,\ldots,c_3$ satisfy the normalization condition

$$c_0 + 8c_1 + 16c_2 + 8c_3 = 1. \tag{4.2}$$

At the one-loop level, the choice of the gluon action is specified by the pair of numbers c_1 and $c_{23}=c_2+c_3$. We calculate the constants for five cases: (i) the standard plaquette action $c_1=0$, $c_{23}=0$, (ii) the tree-level improved action in the Symanzik approach $c_1=-1/12$, $c_{23}=0$ [10], and (iii) three choices suggested by an approximate renormalization-group analysis, $c_1=-0.331$, $c_{23}=0$ and $c_1=-0.27$, $c_{23}=-0.04$ by Iwasaki [12], and $c_1=-0.252$, $c_{23}=-0.17$ by Wilson [11].

Let us write $O_{\overline{\rm MS}}^\Gamma$ for the renormalized bilinear operator in the continuum in the modified minimal subtraction ($\overline{\rm MS}$) scheme, and define

$$O_{\overline{\rm MS}}^\Gamma = Z_{\overline{\rm MS}}^\Gamma O_0^\Gamma, \tag{4.3}$$

where O_0^Γ denotes the lattice bare operator and

TABLE IV. Mixing coefficients C_A, C_V, C_T for axial vector, vector and tensor currents. Coefficients of the term $z^n (n=0,1,2)$ are given in the column marked as (n). $C_T^{(1)}$ and $C_T^{(2)}$ are not calculated.

gauge action		C_A			C_V		
c_1	c_{23}	(0)	(1)	(2)	(0)	(1)	(2)
0	0	−0.005680(2)	−0.002316(3)	−0.003808(1)	−0.01226(3)	−0.01560(4)	−0.000217(9)
−1/12	0	−0.00451(1)	−0.00179(2)	−0.002781(7)	−0.01030(4)	−0.01262(4)	0.000034(8)
−0.331	0	−0.00285(1)	−0.00099(2)	−0.001579(6)	−0.00729(4)	−0.00825(4)	0.000175(7)
−0.27	−0.04	−0.00302(1)	−0.00108(2)	−0.001660(6)	−0.00757(4)	−0.00858(4)	0.000159(7)
−0.252	−0.17	−0.00281(1)	−0.00098(2)	−0.001431(6)	−0.00705(4)	−0.00772(4)	0.000153(7)

gauge action		C_T
c_1	c_{23}	(0)
0	0	−0.00898(1)
−1/12	0	−0.00741(1)
−0.331	0	−0.00508(1)
−0.27	−0.04	−0.00530(1)
−0.252	−0.17	−0.00495(1)

$$Z_{\overline{\rm MS}}^{\Gamma}=1+\frac{g^2C_F}{16\pi^2}\left(\left(\frac{h_2(\Gamma)}{4}-1\right)\times\log(\mu a)^2+z_\Gamma\right). \tag{4.4}$$

Results for the finite constant z_Γ for the lattice operator (2.10) rotated with the bare mass m_0 have already been given in a previous paper [13]. We list in Table III the values of z_Γ for the operator (3.32) defined with the subtracted mass m at $z=0$ for completeness.

Our new results for the one-loop coefficients C_Γ are given in Table IV, and those for B_Γ in Tables V and VI for the definition excluding Σ_0. Numerical values are given for the coefficients of expansion in z defined as

$$B_\Gamma=B_\Gamma^{(0)}+zB_\Gamma^{(1)}-z^2B_\Gamma^{(2)}, \tag{4.5}$$

$$C_\Gamma=C_\Gamma^{(0)}+zC_\Gamma^{(1)}-z^2C_\Gamma^{(2)}. \tag{4.6}$$

In the tensor channel only $C_T^{(0)}$ and $B_T^{(0)}$ are evaluated as necessity for the operator in this channel does not seem to warrant a CPU time-consuming calculation of integrands which are more complex than the other cases. In Tables V and VI a general trend is apparent that the coefficients are reduced by roughly a factor two for renormalization-group improved gluon actions as compared to those for the plaquette action, as already observed for z_Γ [13].

Comparison of our results with those of Refs. [7–9] obtained with the Schrödinger functional is made in the following way. The authors of these references start from a local bilinear operator

$$O_0^\Gamma=\overline{\psi}\Gamma\psi, \tag{4.7}$$

and relate it to the renormalized operator through

$$O_0^\Gamma=Z_\Gamma^{-1}(g^2,a)O_R^\Gamma-c_\Gamma iq_\mu\tilde{O}_\mu^\Gamma. \tag{4.8}$$

The renormalization factor Z_Γ is expanded in the lattice spacing a as follows:

$$Z_\Gamma(g^2,a)=Z_{0\Gamma}(g^2)(1+amb_\Gamma(g^2)). \tag{4.9}$$

where $Z_{0\Gamma}(g^2)$ does not contain terms of $O(a)$, $O(g^2a\ \log a)$, or $O(g^2a)$. The renormalization factor for the quark field ψ and the quark mass m are expanded in a similar manner.

With these definitions, the vertex function of the bare operator has the form

TABLE V. Mixing coefficients B_A, B_V, B_T for axial vector, vector and tensor currents without additive mass renormalization Σ_0. Coefficients of the term $z^n (n=0,1,2)$ are given in the column marked as (n). $B_T^{(1)}$ and $B_T^{(2)}$ are not calculated.

gauge action		B_A			B_V			B_T
c_1	c_{23}	(0)	(1)	(2)	(0)	(1)	(2)	(0)
0	0	0.1141(1)	−0.0846(1)	0.01637(3)	0.1150(2)	−0.0442(2)	0.03255(5)	0.1044(1)
−1/12	0	0.0881(1)	−0.0666(1)	0.01328(3)	0.0886(2)	−0.0353(2)	0.02551(4)	0.0795(1)
−0.331	0	0.0547(1)	−0.0419(1)	0.00867(3)	0.0550(2)	−0.0228(1)	0.01583(4)	0.0482(1)
−0.27	−0.04	0.0572(1)	−0.0438(1)	0.00909(3)	0.0575(2)	−0.0238(1)	0.01656(4)	0.0505(1)
−0.252	−0.17	0.0512(1)	−0.0393(1)	0.00827(3)	0.0514(2)	−0.0213(1)	0.01479(4)	0.0448(1)

TABLE VI. Mixing coefficients B_P, B_S for pseudoscalar and scalar density without additive mass renormalization Σ_0. Coefficients of the term $z^n (n=0,1,2)$ are given in the column marked as (n).

gauge action		B_P			B_S		
c_1	c_{23}	(0)	(1)	(2)	(0)	(1)	(2)
0	0	0.1148(1)	−0.01698(6)	0.03788(5)	0.1444(2)	−0.0535(2)	0.00864(6)
−1/12	0	0.0890(1)	−0.01378(6)	0.02872(4)	0.1144(2)	−0.0403(2)	0.00718(5)
−0.331	0	0.0561(1)	−0.00909(6)	0.01697(4)	0.0747(2)	−0.0235(1)	0.00475(4)
−0.27	−0.04	0.0586(1)	−0.00937(6)	0.01782(4)	0.0777(2)	−0.0248(2)	0.00501(4)
−0.252	−0.17	0.0527(5)	−0.00816(6)	0.01569(4)	0.0706(2)	−0.0221(2)	0.00452(4)

$$G^{\Gamma}=\Gamma+g^2C_F\left(\frac{h_2(\Gamma)}{4}L+V_{\Gamma}^{(0)}+amV_{\Gamma}^{(1)}\right)\Gamma+g^2C_F a v_{\Gamma} i q_{\mu}\tilde{\Gamma}_{\mu}. \tag{4.10}$$

The expressions for the wave function, quark mass, and quark bilinear operator renormalization factors are given by

$$Z_{\psi}^{-1/2}=\left(1+\frac{g^2C_F}{2}(-L+\Sigma_1)\right)\left[1+am\left(-\frac{1}{2}+g^2C_F\left(\sigma_1+\frac{\sigma_2}{2}-\Sigma_1+\frac{\Sigma_2}{2}\right)\right)\right], \tag{4.11}$$

$$Z_m=[1+g^2C_F(3L+(\Sigma_1-\Sigma_2))]\left[1+am\times\left(-\frac{1}{2}+g^2C_F\left(\sigma_1+\sigma_2-\sigma_3-\Sigma_1+\frac{\Sigma_2}{2}\right)\right)\right], \tag{4.12}$$

$$Z_{\Gamma}=\left[1-g^2C_F\left(\left(\frac{h_2(\Gamma)}{4}-1\right)L+\Sigma_1+V_{\Gamma}^{(0)}\right)\right][1+am(1-g^2C_F\times(\Sigma_1+\Sigma_1^{(1)}+V_{\Gamma}^{(1)}))]. \tag{4.13}$$

Making an expansion

$$b_x=b_x^{(0)}+g^2C_Fb_x^{(1)}+\cdots, \quad x=\Gamma,\psi,m, \tag{4.14}$$

$$c_{\Gamma}=g^2C_Fc_{\Gamma}^{(1)}+\cdots, \tag{4.15}$$

we find that the $O(g^2a)$ coefficients are given as

$$b_{\psi}^{(1)}=\sigma_1+\frac{\sigma_2}{2}-\Sigma_1+\frac{\Sigma_2}{2}, \tag{4.16}$$

$$b_m^{(1)}=\sigma_1+\sigma_2-\sigma_3-\Sigma_1+\frac{\Sigma_2}{2}, \tag{4.17}$$

$$b_{\Gamma}^{(1)}=-(\Sigma_1+\Sigma_1^{(1)}+V_{\Gamma}^{(1)}), \tag{4.18}$$

$$c_{\Gamma}^{(1)}=-v_{\Gamma}. \tag{4.19}$$

Comparing these expressions with Eqs. (3.30) and (3.31), we see that $c_{\Gamma}^{(1)}$ equals our $C_{\Gamma}^{(0)}$ for $z=0$ tabulated in Table IV, and $b_{\Gamma}^{(1)}$ equals our $B_{\Gamma}^{(0)}$ for $z=0$ without the Σ_0 term given in Tables V and VI. Our results for $b_{\psi}^{(1)}$ and $b_m^{(1)}$ are given in Table VII, where we also list the contribution from the wave function renormalization factor $b_0=-(\Sigma_1+\Sigma_1^{(1)})=-2b_{\psi}^{(1)}$.

In Table VIII we collect our results for the mixing coefficients for $z=0$ for the plaquette gluon action, and compare them with those of Refs. [7–9]. As we already remarked, we employ 20 sets of 10^6 points for evaluating $c_A^{(1)}$ by VEGAS in this table, with which we find a complete agreement with the result of Refs. [7–9]. Good agreement is also found for all the other coefficients obtained with 20 sets of 10^5 points. We do not pursue more precise evaluation for the latter coefficients since it would require significantly more computing power due to an increased complexity of integrands and the number of terms.

TABLE VII. Mixing coefficients $b_{\psi}^{(1)}$, $b_m^{(1)}$ for quark operator. Values for b_0 are also included.

gauge action				
c_1	c_{23}	$b_{\psi}^{(1)}$	$b_m^{(1)}$	b_0
0	0	−0.05191(3)	−0.07218(5)	0.10381(9)
−1/12	0	−0.03968(3)	−0.05722(5)	0.07937(9)
−0.331	0	−0.02430(3)	−0.03737(5)	0.04860(9)
−0.27	−0.04	−0.02543(3)	−0.03891(5)	0.05086(9)
−0.252	−0.17	−0.02262(3)	−0.03526(5)	0.04525(9)

TABLE VIII. Comparison with previous work [7–9] for the plaquette gluon action.

	$c_A^{(1)}$	$c_V^{(1)}$	$c_T^{(1)}$
Ours	$-0.005680(2)$	$-0.01226(3)$	$-0.00898(1)$
Sint-Weisz [8,9]	$-0.005680(2)$	$-0.01225(1)$	0.00896(1)

	$b_m^{(1)}$	$b_A^{(1)}$	$b_V^{(1)}$	$b_P^{(1)}$	$b_S^{(1)}$	$b_T^{(1)}$
Ours	$-0.07218(5)$	0.1141(1)	0.1150(2)	0.1148(1)	0.1444(2)	0.1044(1)
Sint-Weisz [8,9]	$-0.07217(2)$	0.11414(4)	0.11492(4)	0.11484(2)	0.14434(5)	0.10434(4)

Two more points are worthy to note: (i) The identity $b_S = -2b_m$, noted in Ref. [9] for quenched QCD, also holds with our results for the improved gluon action. (ii) It was observed in Ref. [8] that the values of $b_\Gamma^{(1)}$ are close to each other. Numerically this arises from the fact that the contribution from the wave function renormalization b_0, common to various Dirac channels Γ, dominates over the vertex contributions. Since the wave function renormalization factor is generally gauge dependent, this property of $b_\Gamma^{(1)}$ may be specific to Feynman gauge, however.

V. CONCLUDING REMARKS

In this article we have carried out a perturbative evaluation of vertex functions to determine the $O(g^2a)$ mixing coefficients of bilinear quark operators. For the standard plaquette action for gluons, our results agree with those obtained previously with the Schrödinger functional method. We have also generalized the determination to a class of improved gluon actions for use in numerical simulations employing such actions.

Our calculations are carried out by an expansion of vertex functions regarding external momenta and renormalized quark mass m_R as small in units of lattice spacing a. Hence the present work does not cover the case of heavy quark such that $m_R a > O(1)$. It has been pointed out recently in a one-loop calculation in nonrelativistic QCD [14] that the mixing coefficient $c_A^{(1)}$ for heavy-light axial vector current is large compared to the value for the light-light case treated here. In our calculation a significant cancellation is observed between terms from various diagrams contributing to $c_A^{(1)}$. To understand whether the large value of $c_A^{(1)}$ for heavy quarks results from lifting of such a cancellation requires an extension of our calculation without making an expansion in $m_R a$ [15–17].

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