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Inverse scattering problem in nuclear physics—Optical model

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We consider the inverse scattering problem for the Schrödinger operator with optical potential introduced in nuclear physics to study the scattering of nucleons by nuclei. We show that the corresponding spin–orbit interaction and the complex matrix potential can be uniquely reconstructed from the scattering amplitude at fixed energy. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1753665]

I. INTRODUCTION

The optical model is an operator phenomenologically or empirically introduced in nuclear physics to study the scattering of nucleons by nuclei. The model corresponds to the Schrödinger operator with a complex potential and it was first effectively used by Feshbach, Porter, and Weisskopf[9] to reproduce with great success the experimental results on the scattering of neutrons. Since then this optical model has been improved and accepted as a fundamental tool in nuclear physics. Usually the spin–orbit interaction is included and the following form of the Hamiltonian is adopted:

$$
\begin{aligned}
&H=-\Delta+V,\\
&V=W(x)+a(x)\sigma\cdot(x\times p)+U_c(x), \quad p=-i\nabla_x,\\
&W(x)=c_1F(r;R_1,\alpha_1)+i\left\{c_2F(r;R_2,\alpha_2)-c_3\frac{\mathrm{d}}{\mathrm{d}r}F(r;R_2,\alpha_2)\right\},\\
&a(x)=c_4\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}F(r;R_3,\alpha_3),
\end{aligned}
\tag{1.1}
$$

where $F(r;R,\alpha)$ is the so-called Woods–Saxon potential having the following form:

$$F(r;R,\alpha)=\left(1+\exp\left(\frac{r-R}{\alpha}\right)\right)^{-1},$$

and $U_c(x)$ is the Coulomb interaction. Here $\sigma\cdot x=\Sigma_{i=1}^3\sigma_i x_i$ and $\sigma=(\sigma_1,\sigma_2,\sigma_3)$ is the vector of Pauli spin matrices, that is, they are the 2×2 Hermitian matrices satisfying the following commutation relations:

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$$\sigma_1\sigma_2 = i\sigma_3, \quad \sigma_2\sigma_3 = i\sigma_1, \quad \sigma_3\sigma_1 = i\sigma_2. \tag{1.2}$$

A standard representation of the Pauli matrices is

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.3}$$

The original problem of scattering nucleons should be dealt with by the N body Schrödinger operator, in which many scattering channels appear. Usually the elastic process is dominant, and in order to ignore the inelastic processes, physicists introduced the complex two-body potential so that these inelastic scattering processes are regarded to be absorbed into the background. This is the origin of the terminology *optical* in analogy to the scattering and absorption of light by materials. By suitably adjusting constants c_i (which are usually energy-dependent) and R_i, α_i, the cross section calculated by this model is known to reproduce very well the experimental data (see, e.g., Feshbach[8] or Roy and Nigam[18]).

Let us consider the operator (1.1) in $(L^2(\mathbf{R}^3))^2$, where

$$V = a(x)\sigma \cdot (x \times p) + W(x), \tag{1.4}$$

and $a(x)$, $W(x)$ satisfy the following assumptions:

(A.1) $a(x)$ is a complex-valued C^∞-function on $\mathbf{R}^3$ such that for some $\delta_0 > 0$,

$$|\partial_x^\alpha \ a(x)| \leqslant C_\alpha e^{-\delta_0|x|} \quad \forall \alpha.$$

(A.2) $W(x)$ is a 2×2-matrix valued function on $\mathbf{R}^3$ with complex entries such that for some $\delta_0 > 0$,

$$|W(x)| \leqslant C e^{-\delta_0|x|}.$$

In Sec. II, we shall show that under these assumptions, there is a discrete set $\mathcal{E}_0$ in a neighborhood of $(0,\infty)$ such that for $E \in (0,\infty)\backslash\mathcal{E}_0$, there exists a solution $\psi(x,E,\omega)$, $\omega \in S^2$, of the Schrödinger equation

$$(-\Delta + V)\psi(x,E,\omega) = E\psi(x,E,\omega)$$

having the following asymptotic expansion:

$$\psi(x,E,\omega) \sim e^{i\sqrt{E}\omega\cdot x} + \frac{e^{i\sqrt{E}r}}{r} f(E;\theta,\omega), \quad \theta = x/r, \quad r = |x| \to \infty.$$

The 2×2-matrix valued function $f(E;\theta,\omega)$ of $\theta,\omega \in S^2$ is the scattering amplitude. The main theorem of this paper is the following one.

Theorem 1.1: *For each fixed energy $E \in (0,\infty)\backslash\mathcal{E}_0$, one can uniquely reconstruct the perturbations $a(x), W(x)$ from the scattering amplitude $f(E;\theta,\omega)$.*

The Born approximation at high energies is not valid in the case considered in this paper since the perturbation is energy dependent so that it is natural to consider the fixed energy problem.

There has been considerable works in recent years in studying inverse scattering problem at fixed energy for the case of the Schrödinger equation associated to a potential, that is the two-body problem. To solve this problem one can use Faddeev's Green function[7] and the direction dependent Faddeev's Green's function (see Ref. 12 for a review and references) or the method of constructing complex geometrical optics solutions initiated by Calderón[3] and the connection to the Dirichlet-to-Neumann map (see Ref. 22 for a review and references). The problem considered here is more closely related to the case of the Schrödinger equation in the presence of a magnetic field studied in Ref. 5 or 16. An important ingredient in those articles is the reduction to the case of a lower order perturbation of the Laplacian by exponentiating with a pseudodifferential operator. We use a similar method to deal with the main difficulty in the optical model, which is the reconstruction of the spin–orbit interaction $a(x)\sigma\cdot(x\times p)$. Namely by making semi-classical analysis type arguments (Lemma 5.1) and using the commutation relations of the Pauli spin matrices, we reconstruct $a(x)$ using the complex Born approximation of the scattering amplitude. We shall also use the gauge invariance of the scattering amplitude and introduce an auxiliary magnetic field to reconstruct the complex potential $W(x)$.

For earlier results on the scattering theory for non-self-adjoint Schrödinger operators see, e.g., Refs. 15, 19, 14, and 4 from the mathematical side, and Refs. 2 and 21 from the physical side.

Some remarks on the notations. For two Banach spaces X and Y, $\mathbf{B}(X;Y)$ denotes the set of all bounded operators from X to Y. For $x\in\mathbf{R}^3$, $\langle x\rangle=(1+|x|^2)^{1/2}$. C's denote various constants.

II. RESOLVENT ESTIMATES AND THE SCATTERING AMPLITUDE

We shall derive in this section the analytic continuation of the resolvent of $-\Delta+V$ and introduce the scattering amplitude.

A. Resolvent estimates

Let $H_0=-\Delta$ in $(L^2(\mathbf{R}^3))^2$. For $a\in\mathbf{R}$, we define

$$f\in\mathcal{H}_a \Leftrightarrow \|f\|_{\mathcal{H}_a}=\|e^{a|x|}f(x)\|_{L^2(\mathbf{R}^3)}<\infty. \tag{2.1}$$

Then, by passing to the Fourier transformation and by shifting the path of $|\xi|$-integration, for any $\delta>0$ $R_0(z)=(H_0-z)^{-1}$ defined for $\mathrm{Im}\,z>0$ has an analytic continuation across $(0,\infty)$ into the region

$$\Omega_\delta=\{z;\mathrm{Im}\ \ z>0\}\cup\{z;\mathrm{Re}\,\sqrt{z}>0,\ \ 0\geqslant\mathrm{Im}\ \ \sqrt{z}>-\delta\} \tag{2.2}$$

as a $\mathbf{B}(\mathcal{H}_\delta;\mathcal{H}_{-\delta})$-valued function. We denote this operator by $R_0^{(+)}(z)$. This is actually the integral operator with kernel $e^{i\sqrt{z}|x-y|}/(4\pi|x-y|)$.

For a technical reason, which will be explained in Sec. VI, we include also a magnetic field $b(x)$. Let $H=H_0+V$, where

$$V=V(-i\nabla_x), \tag{2.3}$$

$$V(\xi)=(2b(x)\cdot\xi-i\,\mathrm{div}\,b(x)+|b(x)|^2)I+a(x)\sigma\cdot(x\times\xi)+W(x), \tag{2.4}$$

I being the 2×2 identity matrix. We shall assume that

(A.1) $a(x)\in C^\infty(\mathbf{R}^3;\mathbf{C}),b(x)\in C^\infty(\mathbf{R}^3;\mathbf{R}^3)$ *and for some* $\delta_0>0$

$$|\partial_x^\alpha a(x)|+|\partial_x^\alpha b(x)|\leqslant C_\alpha e^{-\delta_0|x|},\quad \forall\alpha.$$

(A.2) $W(x)$ *is a* $M_2(\mathbf{C})$*-valued function such that for some* $\delta_0>0$,

$$|W(x)|\leqslant Ce^{-\delta_0|x|}.$$

These assumptions imply that for $0<\delta<\delta_0/2$, $R_0^{(+)}(z)V$ is a $\mathbf{B}(\mathcal{H}_{-\delta};\mathcal{H}_{-\delta})$-valued analytic function on Ω_δ and is compact for each z. We define $\mathcal{E}_0$ to be the set

$$\mathcal{E}_0=\{z\in\Omega_\delta;-1\in\mathrm{spec}_p(R_0^{(+)}(z)V)\}. \tag{2.5}$$

Here $\mathrm{spec}_p(A)$ denotes the point spectrum of the operator A.

Lemma 2.1: (1) There exists $C>0$ *such that*

$$\mathcal{E}_0\cap\{i\tau;\tau>C\}=\emptyset.$$

(2) $\mathcal{E}_0$ *is discrete in* Ω_δ.

Proof: Suppose $i\tau\in\mathcal{E}_0$ and let u be the associated eigenvector. Since $R_0^{(+)}(i\tau)\in\mathbf{B}(L^2;H^2)$, we have $u\in H^2$. Since $(-\Delta+V-i\tau)u=0$, we then have

$$\|u\|_{H^1}^2=((1-\Delta)u,u)\leqslant C(\|u\|_{H^{1/2}}^2+\tau\|u\|_{L^2}^2),$$

which implies $\|u\|_{H^1}\leqslant C\sqrt{\tau}\|u\|_{L^2}$. Using the equation $u=-R_0^{(+)}(i\tau)Vu$, we then have

$$\|u\|_{L^2}\leqslant\frac{C}{\tau}\|u\|_{H^1}\leqslant\frac{C}{\sqrt{\tau}}\|u\|_{L^2}.$$

Therefore $u=0$ for large $\tau>0$. This proves (1). Assertion (2) follows from the analytic Fredholm theorem (Ref. 17, p. 204). □

We define

$$R(z)=(1+R_0^{(+)}(z)V)^{-1}R_0^{(+)}(z),\quad z\in\Omega_\delta\backslash\mathcal{E}_0. \tag{2.6}$$

The following theorem is easily proved.

Theorem 2.2: *(1) $R(z)$ is a $\mathbf{B}(\mathcal{H}_\delta;\mathcal{H}_{-\delta})$-valued analytic function on $\Omega_\delta\backslash\mathcal{E}_0$.*

(2) $R(z)=(H-z)^{-1}$ for $z\in\{z;\mathrm{Im}\ z>0\}\cap(\Omega_\delta\backslash\mathcal{E}_0)$.

(3) For $z\in\Omega_\delta\backslash\mathcal{E}_0$,

$$R(z)=R_0^{(+)}(z)-R_0^{(+)}(z)VR(z). \tag{2.7}$$

For $s\in\mathbf{R}$, $L^{2,s}$ is defined by

$$u\in L^{2,s}\Leftrightarrow\|u\|_s=\|(1+|x|)^s u(x)\|_{L^2(\mathbf{R}^3)}<\infty.$$

Since $R_0^{(+)}(E)=(-\Delta-E-i0)^{-1}\in\mathbf{B}(L^{2,s};L^{2,-s})$ for $s>1/2$ and $E>0$ (see, e.g., Ref. 19), it follows from (2.6) that

$$R(E)\in\mathbf{B}(L^{2,s};L^{2,-s}),\quad s>1/2,\quad E\in(0,\infty)\backslash\mathcal{E}_0. \tag{2.8}$$

In fact, let A_1 and A_2 be $R_0^{(+)}(E)V$ acting on $\mathcal{H}_{-\delta}$ and $L^{2,-s}$, respectively. Then it is easy to see that

$$-1\in\mathrm{spec}_p(A_1)\Leftrightarrow-1\in\mathrm{spec}_p(A_2). \tag{2.9}$$

B. Scattering amplitudes

Theorem 2.3: *For $E\in(0,\infty)\backslash\mathcal{E}_0$ and $\omega\in S^2$, there exists a unique solution ψ of the equation*

$$(-\Delta+V-E)\psi=0$$

such that $u=\psi-e^{i\sqrt{E}\omega\cdot x}$ satisfies the radiation condition

$$\left(\frac{\partial}{\partial r}-i\sqrt{E}\right)u\in L^{2,-\alpha},\quad 0<\alpha<1/2.$$

Such ψ is represented as

$$\psi(x,E,\omega)=e^{i\sqrt{E}\omega\cdot x}-R(E)Ve^{i\sqrt{E}\omega\cdot x}. \tag{2.10}$$

Proof: To show existence, we have only to put ψ as in (2.9). To show the uniqueness, we note that the difference of two such solutions satisfies

$$(-\Delta-E)u=-Vu,\quad \left(\frac{\partial}{\partial r}-i\sqrt{E}\right)u\in L^{2,-\alpha}.$$

Then $u=-R_0^{(+)}(E)Vu$. Hence if $u\neq 0$, $-1\in \mathrm{spec}_p(A_1)$. □

Using the resolvent equation (2.7), we have

$$\psi(x,E,\omega)\sim e^{i\sqrt{E}\omega\cdot x}+\frac{e^{i\sqrt{E}r}}{r}f(E;\theta,\omega),\quad \theta=x/r \tag{2.11}$$

as $r=|x|\to\infty$, where

$$f(E;\theta,\omega)=-\frac{1}{4\pi}\int_{\mathbf{R}^3}e^{-i\sqrt{E}\theta\cdot x}Ve^{i\sqrt{E}\omega\cdot x}\mathrm{d}x+\frac{1}{4\pi}\int_{\mathbf{R}^3}e^{-i\sqrt{E}\theta\cdot x}VR(E)Ve^{i\sqrt{E}\omega\cdot x}\mathrm{d}x. \tag{2.12}$$

We introduce the following notation for 2×2-matrices $f(x),g(x)$:

$$\langle f,g\rangle=\int_{\mathbf{R}^3}f(x)^*g(x)\mathrm{d}x. \tag{2.13}$$

Then $f(E;\theta,\omega)$ is written as

$$f(E;\theta,\omega)=-\frac{1}{4\pi}\int_{\mathbf{R}^3}e^{-i\sqrt{E}(\theta-\omega)\cdot x}V(\sqrt{E}\omega)\mathrm{d}x+\frac{1}{4\pi}\langle V^*(\sqrt{E}\theta)e^{i\sqrt{E}\theta\cdot x},R(E)V(\sqrt{E}\omega)e^{i\sqrt{E}\omega\cdot x}\rangle\rangle, \tag{2.14}$$

where

$$V^*(\xi)=2b(x)\cdot\xi+\overline{a(x)}\sigma\cdot(x\times\xi)-i\sigma\cdot(x\times\overline{\nabla a(x)})+|b(x)|^2+W(x)^*. \tag{2.15}$$

III. DIRECTION DEPENDENT GREEN OPERATORS

The aim of this section is to construct Green operators for $-\Delta+V$ depending on a direction $\gamma\in S^2$.

A. Unperturbed operator

For $\epsilon>0$, we let

$$D_\epsilon=\{z\in\mathbf{C};\mathrm{Im}\,z>0,|\mathrm{Re}\,z|<\epsilon/2\}. \tag{3.1}$$

We fix an arbitrary direction $\gamma\in S^2$.

Theorem 3.1: *(1) For any $\delta>0$ and $E>0$, there exists an $\epsilon>0$ and a $\mathbf{B}(\mathcal{H}_\delta;\mathcal{H}_{-\delta})$-valued analytic function $U_{\gamma,0}(E,z)$ defined on D_ϵ such that*

$$(-\Delta-2iz\gamma\cdot\nabla+z^2-E)U_{\gamma,0}(E,z)=I.$$

(2) When $z\to t\in(-\epsilon/2,\epsilon/2)$, $U_{\gamma,0}(E,z)$ has a boundary value $U_{\gamma,0}(E,t)$. Moreover

$$U_{\gamma,0}(E,t)\in\mathbf{B}(L^{2,s};L^{2,-s}),\quad s>1/2.$$

(3) For $\tau>0$,

$$U_{\gamma,0}(E,i\tau)f(x)=(2\pi)^{-3/2}\int_{\mathbf{R}^3}\frac{e^{ix\cdot\xi}\hat{f}(\xi)}{(\xi+i\tau\gamma)^2-E}\mathrm{d}\xi,$$

$\hat{f}$ being the Fourier transform of f. If f is rapidly decreasing, the integral is absolutely convergent.

(4) For $0<s<1$, $U_{\gamma,0}(E,i\tau)\in\mathbf{B}(L^{2,s};L^{2,s-1})$, *and*

$$\|U_{\gamma,0}(E,i\tau)\|_{\mathbf{B}(L^{2,s};L^{2,s-1})}\leqslant C_s/\tau, \quad \tau>1.$$

Proof: For the proof see Ref. 13, Theorem 2.10. The last assertion (4) is proved in the same way as in Sylvester–Uhlmann,[20] Lemma 3.1. □

We next observe a relation between this direction dependent Green operator and the $\bar{\partial}$-operator. For $\zeta\in\mathbf{C}^3$, $\mathrm{Im}\ \zeta\neq 0$, let

$$\tilde{G}(\zeta)f(x)=(2\pi)^{-3/2}\int_{\mathbf{R}^3}\frac{e^{ix\cdot\xi}\hat{f}(\xi)}{\xi^2+2\zeta\cdot\xi}d\xi. \tag{3.2}$$

For $\eta\in S^2$ such that $\eta\cdot\gamma=0$, let $p(\tau)=\sqrt{E+\tau^2}\,\eta$ and $\zeta(\tau)=p(\tau)+i\tau\gamma$. Then

$$e^{-ip(\tau)\cdot x}U_{\gamma,0}(E,i\tau)e^{ip(\tau)\cdot x}=\tilde{G}(\zeta(\tau)). \tag{3.3}$$

We also define

$$M_\gamma(\tau)=\tau\tilde{G}(\zeta(\tau)), \tag{3.4}$$

$$N_\gamma f=(2\pi)^{-3/2}\int_{\mathbf{R}^3}\frac{e^{ix\cdot\xi}\hat{f}(\xi)}{2\xi\cdot(\eta+i\gamma)}d\xi. \tag{3.5}$$

Lemma 3.2: (1) For $0<s<1$ *and* $\tau>0$,

$$M_\gamma(\tau)\in\mathbf{B}(L^{2,s};L^{2,s-1}).$$

(2) For $f\in L^{2,s}$, $0<s<1$,

$$M_\gamma(\tau)f\to N_\gamma f$$

in $L^{2,s-1}$ *as* $\tau\to\infty$.

The proof is the same as the one in Ref. 20, Proposition 3.6 and Ref. 13, Theorem 4.6.

B. Perturbed operator

Let $p=-i\nabla_x$. For $\zeta\in\mathbf{C}^3$, we let

$$H_0(\zeta)=(p+\zeta)^2, \tag{3.6}$$

and

$$H(\zeta)=(p+b(x)+\zeta)^2+a(x)\sigma\cdot(x\times(p+\zeta))+W(x)=H_0(\zeta)+V(p+\zeta), \tag{3.7}$$

where V is defined by (2.4).

Our aim is to construct a direction dependent Green operator for $H(z\gamma)-E$. The main difficulty comes from the term $2b(x)\cdot(p+z\gamma)+a(x)\sigma\cdot(x\times(p+z\gamma))$, which we are going to eliminate by introducing a suitable pseudodifferential operator (ΨDO). Namely by using the identity

$$(H(z\gamma)-E)SU_{\gamma,0}(E,z)S^{-1}=1+([H_0(z\gamma),S]+V(p+z\gamma)S)U_{\gamma,0}(E,z)S^{-1},$$

we seek S in such a way that the right-hand side is invertible.

Before entering into the technical details, we explain the idea in the background. Let us suppose for the sake of simplicity that $b(x)=0$. If S is a ΨDO belonging to $\mathcal{S}^0$ to be introduced below, the natural choice is to assume that the symbol $S(x,\xi)$ of S satisfies

$$2(\xi+z\gamma)\cdot\nabla_x S(x,\xi)+ia(x)\sigma\cdot(x\times(\xi+z\gamma))S(x,\xi)=0.$$

In general when one considers inverse scattering problems at a fixed energy for systems of partial differential equations, one often encounters the equation $\zeta\cdot\nabla_x S=B(x,\zeta)S$, where ζ is a complex vector satisfying some conditions and $B(x,\zeta)$ is a matrix coming from the lower part of the equation. This sort of Cauchy-Riemann type equation is difficult to solve for systems (see Ref. 6 for a recent review). It is trivial to solve for scalar equations like in the case of the Schrödinger equation in the presence of a magnetic field. However, in our case it is sufficient to solve the equation

$$2(\xi+z\gamma)\cdot\nabla_x\psi+ia(x)=0,$$

and put $S=e^C$, $C=\psi(x,D_x)\sigma\cdot(x\times(D_x+z\gamma))$, $D_x=-i\nabla_x$. This is due to the fact that at the level of symbols, C solves

$$2i(\xi+z\gamma)\cdot\nabla_x C=a(x)\sigma\cdot(x\times(\xi+z\gamma))=:B(x,\xi+z\gamma),$$

and the symbol of C commutes with $B(x,\xi+z\gamma)$. (See Lemma 3.3 below.) Now the above equation for ψ is just the one we encounter in considering the inverse scattering problem for the scalar Schrödinger operator in a magnetic field, and the solution plays a significant role only near the zeros of $(\xi+z\gamma)^2-E$. With these remarks in mind, let us return to the construction of the perturbed direction dependent Green operator.

For a sufficiently small $\epsilon>0$, let $\chi_0(t)\in C^\infty(\mathbf{R})$ be such that $\chi_0(t)=1$ if $|t|<\epsilon/2$, $\chi_0(t)=0$ if $|t|>\epsilon$ and let

$$\chi(\xi+i\tau\gamma)=\chi_0\left(\frac{|(\xi+i\tau\gamma)^2-E|^2}{E+\tau^2+|\xi|^2}\right). \tag{3.8}$$

Note that on the support of $\chi(\xi+i\tau\gamma)$,

$$|\xi^2+2i\tau\gamma\cdot\xi-\tau^2-E|\leqslant\epsilon(E+\tau^2+|\xi|^2). \tag{3.9}$$

Let us put

$$\varphi(x,\xi+i\tau\gamma)=-(2\pi)^{-3/2}\chi(\xi+i\tau\gamma)\int_{\mathbf{R}^3}e^{ix\cdot k}\frac{\hat{b}(k)\cdot(\xi+i\tau\gamma)}{k\cdot(\xi+i\tau\gamma)}\,dk, \tag{3.10}$$

$$\psi(x,\xi+i\tau\gamma)=-(2\pi)^{-3/2}\chi(\xi+i\tau\gamma)\int_{\mathbf{R}^3}e^{ix\cdot k}\frac{\hat{a}(k)}{2k\cdot(\xi+i\tau\gamma)}\,dk. \tag{3.11}$$

For $m\in\mathbf{R}$, let $\mathcal{S}^m$ be the class of ΨDO's with symbol $p(x,\xi;\tau)$ satisfying

$$|\partial_x^\alpha\partial_\xi^\beta\ p(x,\xi;\tau)|\leqslant C_{\alpha\beta}\langle x\rangle^{-1}(\tau+|\xi|)^{m-|\beta|}\quad\forall\alpha,\beta,\quad\tau>1.$$

We use the same notation $\mathcal{S}^m$ to denote the associated class of symbols.

Lemma 3.3: (1) $\varphi(x,\xi+i\tau\gamma)\in\mathcal{S}^0,\psi(x,\xi+i\tau\gamma)\in\mathcal{S}^{-1}$.

(2) We have

$$2i(\xi+i\tau\gamma)\cdot\nabla_x\varphi(x,\xi+i\tau\gamma)=2\chi(\xi+i\tau\gamma)b(x)\cdot(\xi+i\tau\gamma),$$

$$2i(\xi+i\tau\gamma)\cdot\nabla_x[\psi(x,\xi+i\tau\gamma)\sigma\cdot(x\times(\xi+i\tau\gamma))]=\chi(\xi+i\tau\gamma)a(x)\quad\sigma\cdot(x\times(\xi+i\tau\gamma)).$$

The assertion (2) follows from a direct computation. The assertion (1) is proved in the Appendix.

Let $\varphi_0(\tau),\psi_0(\tau)$ be ΨDO's with symbol $\varphi(x,\xi+i\tau\gamma),\psi(x,\xi+i\tau\gamma)$ and let

$$A(\tau)=\varphi_0(\tau)+\psi_0(\tau)\sigma\cdot(x\times(p+i\tau\gamma)),\quad S(\tau)=e^{A(\tau)}. \tag{3.12}$$

Lemma 3.4: Let

$$K(\tau)=([H_0(i\tau\gamma),S(\tau)]+V(p+i\tau\gamma)S(\tau))U_{\gamma,0}(E,i\tau)S(\tau)^{-1}. \tag{3.13}$$

Then for $1/2<s<1$ *and large* $\tau>0$,

$$\|K(\tau)\|_{\mathbf{B}(L^{2,s};L^{2,s})}\leqslant C_s/\tau.$$

Proof: For two operators $P_1(\tau)$ and $P_2(\tau)$ we write

$$P_1(\tau)\sim P_2(\tau), \tag{3.14}$$

if they satisfy for large $\tau>0$,

$$\|P_1(\tau)-P_2(\tau)\|_{\mathbf{B}(L^{2,s};L^{2,s})}\leqslant C_s/\tau. \tag{3.15}$$

Let $L>\sup_{-1/2<s<1,1<\tau}\|A(\tau)\|_{\mathbf{B}(L^{2,s};L^{2,s})}$. Then we have

$$e^{A(\tau)}=\frac{1}{2\pi i}\int_{|z|=L}e^z(z-A(\tau))^{-1}\mathrm{d}z. \tag{3.16}$$

Therefore we have

$$[H_0(i\tau\gamma),e^{A(\tau)}]=\frac{1}{2\pi i}\int_{|z|=L}e^z(z-A(\tau))^{-1}[H_0(i\tau\gamma),A(\tau)](z-A(\tau))^{-1}\mathrm{d}z. \tag{3.17}$$

Since $A(\tau)\in S^0$ by Lemma 3.3 (1), by the symbolic calculus we have

$$[H_0(i\tau\gamma),A(\tau)]=P_1(\tau)+P_2(\tau), \tag{3.18}$$

where $\|P_2(\tau)\|_{\mathbf{B}(L^{2,s-1};L^{2,s})}\leqslant C_s$, and $P_1(\tau)$ is the ΨDO with symbol

$$-2\chi(\xi+i\tau\gamma)b(x)\cdot(\xi+i\tau\gamma)-\chi(\xi+i\tau\gamma)a(x)\sigma\cdot(x\times(\xi+i\tau\gamma)). \tag{3.19}$$

Let $Q(\tau)$ be the ΨDO with symbol $\chi(\xi+i\tau\gamma)$. By (3.18) and (3.19), we have

$$[H_0(i\tau\gamma),A(\tau)]=-Q(\tau)R(\tau)+P_3(\tau), \tag{3.20}$$

where $\|P_3(\tau)\|_{\mathbf{B}(L^{2,s-1};L^{2,s})}\leqslant C_s$, and

$$R(\tau)=2b(x)\cdot(p+i\tau\gamma)+a(x)\sigma\cdot(x\times(p+i\tau\gamma)). \tag{3.21}$$

Let us note that

$$[Q(\tau),A(\tau)]\in\mathcal{S}^{-1}, \tag{3.22}$$

$$[R(\tau),A(\tau)]\in\mathcal{S}^{0}, \tag{3.23}$$

$$\|(1-Q(\tau))U_{\gamma,0}(E,i\tau)\|_{\mathbf{B}(L^{2,s};L^{2,s})}\leqslant C_s/\tau^2. \tag{3.24}$$

The estimate (3.24) follows from Theorem 3.1 (3). Then in view of Theorem 3.1 (4), we have

$$[H_0(i\tau\gamma),S(\tau)]U_{\gamma,0}(E,i\tau)\sim-\frac{1}{2\pi i}\int_{|z|=L}e^z(z-A(\tau))^{-1}R(\tau)Q(\tau)(z-A(\tau))^{-1}\mathrm{d}z\,U_{\gamma,0}(E,i\tau)$$

$$\sim-R(\tau)\frac{1}{2\pi i}\int_{|z|=L}e^z(z-A(\tau))^{-2}\mathrm{d}z\,U_{\gamma,0}(E,i\tau)$$

$$=-R(\tau)e^{A(\tau)}U_{\gamma,0}(E,i\tau).$$

It then follows that

$$K(\tau)\sim(-R(\tau)+V(p+i\tau\gamma))S(\tau)U_{\gamma,0}(E,i\tau)S(\tau)^{-1}\sim 0. \tag{3.25}$$

This proves the lemma. □

With the aid of Lemma 3.4, we define the modified direction dependent Green operator for large $\tau>0$ by

$$L_\gamma(\tau)=S(\tau)U_{\gamma,0}(E,i\tau)S(\tau)^{-1}(1+K(\tau))^{-1}. \tag{3.26}$$

By definition it satisfies

$$(H(i\tau\gamma)-E)L_\gamma(\tau)=I. \tag{3.27}$$

We define $\mathcal{E}_\gamma(E)$ to be the set of $z\in\overline{D_\epsilon}$ such that

$$-1\in\mathrm{spec}_p(U_{\gamma,0}(E,z)V(p+z\gamma)).$$

Lemma 3.5: (1) $\mathcal{E}_\gamma(E)\cap\{z;\mathrm{Im}\,z>0\}$ *is discrete.*
(2) $\mathcal{E}_\gamma(E)\cap\mathbf{R}$ *is a closed set of measure zero.*
(3) There exists a constant $C>0$ *such that*

$$i\tau\notin\mathcal{E}_\gamma(E)\quad\text{if}\quad\tau>C.$$

Proof: We have only to show the last assertion. The assertions (1) and (2) follow from the analytic Fredholm theorem and the well-known Riesz' theorem on boundary values of analytic functions (see, e.g., Ref. 11, p. 52). Let $K_1(\tau)=U_{\gamma,0}(E,i\tau)V(p+i\tau\gamma)$. Since $K_1(\tau)$ is compact, we have only to show that $\mathrm{Ran}(1+K_1(\tau))$ is dense in $L^{2,-s},1/2<s<1$ for large $\tau>0$. For $f\in C_0^\infty(\mathbf{R}^3)$, let $u=L_\gamma(\tau)(H_0(i\tau\gamma)-E)f$. Then we have $(H_0(i\tau\gamma)-E)(u-f+K_1(\tau)u)=0$. Since $u-f+K_1(\tau)u\in L^{2,-s},1/2<s<1$, we have $u-f+K_1(\tau)u=0$ by virtue of Theorem 2.2 of Agmon–Hörmander.[1] □

Let us define for $z\notin\mathcal{E}_\gamma(E)$,

$$U_\gamma(E,z)=(1+U_{\gamma,0}(E,z)V(p+z\gamma))^{-1}U_{\gamma,0}(E,z). \tag{3.28}$$

Theorem 3.6. *(1) As a* $\mathbf{B}(\mathcal{H}_\delta;\mathcal{H}_{-\delta})$*-valued function,* $U_\gamma(E,z)$ *is meromorphic on* D_ϵ.
(2) When $z\to t\in(-\epsilon/2,\epsilon/2)\backslash\mathcal{E}_\gamma(E)$, $U_\gamma(E,z)$ *converges to* $U_\gamma(E,t)$ *and*

$$U_\gamma(E,t)\in\mathbf{B}(L^{2,s};L^{2,-s})\quad s>1/2.$$

(3) For large $\tau>0$,

$$U_\gamma(E,i\tau)=L_\gamma(\tau).$$

Proof: We show the last assertion. We have only to show that the equation $(H(i\tau\gamma)-E)u=0,u\in L^{2,-s}$, has only a trivial solution for large $\tau>0$. Since $(H_0(i\tau\gamma)-E)u=-V(p+i\tau\gamma)u\in L^{2,s}$, we have by the uniqueness theorem of Agmon–Hörmander, $u=-U_{\gamma,0}(E,i\tau)V(p+i\tau\gamma)u$. Therefore $u=0$ by using Lemma 3.5 (3). □

IV. FADDEEV SCATTERING AMPLITUDE

The Faddeev theory, which we have rewritten in Ref. 12, is transferred without any essential change to the non-self-adjoint case. For $E>0$, let

$$\mathcal{F}_0(E)f(\omega)=(2\pi)^{-3/2}(E/4)^{1/4}\int_{\mathbf{R}^3}e^{-i\sqrt{E}\omega\cdot x}f(x)\mathrm{d}x. \tag{4.1}$$

As is well known $\mathcal{F}_0(E)\in\mathbf{B}(L^{2,s};L^2(S^2))$ if $s>1/2$. Then for $E\notin\mathcal{E}_0$, the scattering amplitude is written as, up to a constant depending only on E,

$$A(E)=\mathcal{F}_0(E)(V-VR(E)V)\mathcal{F}_0(E)^*. \tag{4.2}$$

The scattering amplitude $f(E;\theta,\omega)$ from (2.12) is the integral kernel of $A(E)$. Let for $t\in(-\epsilon/2,\epsilon/2)\setminus\mathcal{E}_\gamma(E)$,

$$R_\gamma(E,t)=e^{it\gamma\cdot x}U_\gamma(E,t)e^{-it\gamma\cdot x}. \tag{4.3}$$

Then the Faddeev scattering amplitude is defined by

$$A_\gamma(E,t)=\mathcal{F}_0(E)(V-VR_\gamma(E,t)V)\mathcal{F}_0(E)^*. \tag{4.4}$$

The following two theorems are proved in the same way as in Theorems 7.1 and 7.3 of Ref. 12.

Theorem 4.1: *Let $F_\gamma(E,t)$ be the operator of multiplication by the characteristic function of the set $\{\omega\in S^2;\gamma\cdot\omega\geqslant t/\sqrt{E}\}$. Then*

$$A_\gamma(E,t)=A(E)+2\pi iA(E)F_\gamma(E,t)A_\gamma(E,t).$$

Theorem 4.2: *Let $K=2\pi iA(E)F_\gamma(E,t)$. Then*

$$t\in\mathcal{E}_\gamma(E)\Leftrightarrow 1\in\mathrm{spec}_p(K).$$

Let us give a brief sketch of the proof of the above theorems. Let

$$T_\gamma=2\pi i\mathcal{F}_0(E)^*F_\gamma(E,t)\mathcal{F}_0(E). \tag{4.5}$$

Then we have (Ref. 12, Lemma 6.4)

$$R_\gamma=R-(1-RV)T_\gamma(1-VR_\gamma), \tag{4.6}$$

where $R_\gamma=R_\gamma(E,t),R=R(E)$. The eigenoperator $\mathcal{F}(E)$ and the Faddeev eigenoperator $\mathcal{F}_\gamma(E,t)$ are defined by

$$\mathcal{F}(E)=\mathcal{F}_0(E)(1-V^*R(E)^*), \tag{4.7}$$

$$\mathcal{F}_\gamma(E,t)=\mathcal{F}_0(E)(1-V^*R_\gamma(E,t)^*). \tag{4.8}$$

Then by the resolvent equation (4.6) we have

$$\mathcal{F}_\gamma(E,t)^*=\mathcal{F}(E)^*+(1-RV)T_\gamma V\mathcal{F}_\gamma(E,t)^*.$$

Using (4.5) we get

$$\mathcal{F}_\gamma(E,t)^*=\mathcal{F}(E)^*+2\pi i\mathcal{F}(E)^*F_\gamma(E,t)A_\gamma(E,t).$$

Multiplying this by $\mathcal{F}_0(E)V$, we obtain Theorem 4.1.

To prove Theorem 4.2 we note the following operator equation:

$$1+R_{\gamma,0}(E,t)V=(1+R_0(E+i0)V)(1-\tilde{K}), \tag{4.9}$$

$$\tilde{K}=(1-R(E)V)T_\gamma V, \tag{4.10}$$

where $R_0(z)=(-\Delta-z)^{-1}, R_{\gamma,0}(E,t)=e^{it\gamma\cdot x}U_{\gamma,0}(E,t)e^{-it\gamma\cdot x}$. In fact, this follows from the formula

$$R_{\gamma,0}(E,t)=R_0(E+i0)-T_\gamma$$

(Ref. 12, Lemma 6.3) and the resolvent equation.

Since $E\notin\mathcal{E}_0$, $1+R_0(E+i0)V$ is invertible. Therefore

$$t\in\mathcal{E}_\gamma(E)\Leftrightarrow 1\in\mathrm{spec}_p(\tilde{K}). \tag{4.11}$$

Letting

$$S_1=2\pi i(1-R(E)V)\mathcal{F}_0(E)^*F_\gamma(E,t),$$

$$S_2=\mathcal{F}_0(E)V,$$

we have

$$\tilde{K}=S_1S_2, \quad K=S_2S_1.$$

Therefore

$$1\in\mathrm{spec}_p(\tilde{K})\Leftrightarrow 1\in\mathrm{spec}_p(K). \tag{4.12}$$

This proves Theorem 4.2.

It follows from Theorems 4.1 and 4.2 that for $t\in(-\epsilon/2,\epsilon/2)\backslash\mathcal{E}_\gamma(E)$,

$$A_\gamma(E,t)=(1-K)^{-1}A(E). \tag{4.13}$$

We have thus constructed the Faddeev scattering amplitude $A_\gamma(E,t)$ from the scattering amplitude $A(E)$. The kernel of $A_\gamma(E,t)$ is written as, up to a constant depending only on E,

$$A_\gamma(E,t;\theta',\theta)=\int_{\mathbf{R}^3}e^{-i\sqrt{E}(\theta'-\theta)\cdot x}V(\sqrt{E}\theta)dx-\langle V^*(\sqrt{E}\theta')e^{i\sqrt{E}\theta'\cdot x},R_\gamma(E,t)V(\sqrt{E}\theta)e^{i\sqrt{E}\theta\cdot x}\rangle, \tag{4.14}$$

where $V^*(\xi)$ is defined by (2.15). We now put

$$\sqrt{E}\theta=\sqrt{E-t^2}\omega+t\gamma, \quad \sqrt{E}\theta'=\sqrt{E-t^2}\omega'+t\gamma, \tag{4.15}$$

where $\omega,\omega'\in S^2,\omega\cdot\gamma=\omega'\cdot\gamma=0$. Then the above kernel is rewritten as

$$\int_{\mathbf{R}^3}e^{-i\sqrt{E-t^2}(\omega'-\omega)\cdot x}V(\sqrt{E}\theta)dx-\langle V^*(\sqrt{E}\theta')e^{i\sqrt{E-t^2}\omega'\cdot x},U_\gamma(E,t)V(\sqrt{E}\theta)e^{i\sqrt{E-t^2}\omega\cdot x}\rangle, \tag{4.16}$$

which we denote by $B_\gamma(\omega',\omega;t)$. Since $U_\gamma(E,t)$ is a boundary value of a meromorphic function, $B_\gamma(\omega',\omega;t)$ is uniquely extended to a meromorphic function on D_ϵ.

V. RECONSTRUCTION OF THE SPIN–ORBIT INTERACTION

In this section we reconstruct the spin–orbit term from $B_\gamma(\omega',\omega;t)$ defined by (4.16). Our intention is to consider the case without magnetic field $b(x)$. However, since we shall use an auxiliary magnetic field in the next section, we include $b(x)$ until Lemma 5.5.

As has been noted above, $B_\gamma(\omega',\omega;t)$ is meromorphically extended to D_ϵ. By Theorem 3.6 (3), for large $\tau>0$, $B_\gamma(\omega',\omega;i\tau)$ has the following expression:

$$B_\gamma(\omega',\omega;i\tau)=\int e^{-i\sqrt{E+\tau^2}(\omega'-\omega)\cdot x}V(\sqrt{E}\theta)\mathrm{d}x$$
$$-\langle V^*(\sqrt{E}\theta')e^{i\sqrt{E+\tau^2}\omega'\cdot x},L_\gamma(\tau)V(\sqrt{E}\theta)e^{i\sqrt{E+\tau^2}\omega\cdot x}\rangle, \tag{5.1}$$

where

$$\sqrt{E}\theta=\sqrt{E+\tau^2}\omega+i\tau\gamma,\quad \sqrt{E}\theta'=\sqrt{E+\tau^2}\omega'+i\tau\gamma.$$

For $\xi\in\mathbf{R}^3$, we take $\gamma,\eta\in S^2$ such that $\xi\cdot\gamma=\xi\cdot\eta=\eta\cdot\gamma=0$, and put

$$\omega=\omega(\tau)=\left(1-\frac{|\xi|^2}{4\tau^2}\right)^{1/2}\eta-\frac{\xi}{2\tau},\quad \omega'=\omega(\tau)'=\left(1-\frac{|\xi|^2}{4\tau^2}\right)^{1/2}\eta+\frac{\xi}{2\tau},$$

$$p(\tau)=\sqrt{E+\tau^2}\omega(\tau),\quad p(\tau)'=\sqrt{E+\tau^2}\omega(\tau)',$$

$$\zeta(\tau)=p(\tau)+i\tau\gamma,\quad \zeta(\tau)'=p(\tau)'+i\tau\gamma.$$

We split $B_\gamma(\omega(\tau)',\omega(\tau);i\tau)$ into two parts:

$$B_\gamma(\omega(\tau)',\omega(\tau);i\tau)=B_\gamma^{(1)}(\tau)+B_\gamma^{(2)}(\tau), \tag{5.2}$$

$$B_\gamma^{(1)}(\tau)=\int e^{-i(p(\tau)'-p(\tau))\cdot x}V(\zeta(\tau))\mathrm{d}x, \tag{5.3}$$

$$B_\gamma^{(2)}(\tau)=-\langle V^*(\zeta(\tau)')e^{ip(\tau)'\cdot x},L_\gamma(\tau)V(\zeta(\tau))e^{ip(\tau)\cdot x}\rangle. \tag{5.4}$$

Noting that

$$p(\tau)'-p(\tau)=\xi+O(\tau^{-1}),$$

$$\zeta(\tau)/\tau=\eta+i\gamma+O(\tau^{-1}),$$

we have by (2.4)

$$\lim_{\tau\to\infty}B_\gamma^{(1)}(\tau)/\tau=\int e^{-ix\cdot\xi}\{2b(x)\cdot(\eta+i\gamma)+a(x)\sigma\cdot(x\times(\eta+i\gamma))\}\mathrm{d}x. \tag{5.5}$$

To compute $B_\gamma^{(2)}(\tau)$, we rewrite it as follows:

$$B_\gamma^{(2)}(\tau)=-\langle V^*(\zeta(\tau)')e^{i(p(\tau)'-p(\tau))\cdot x},U(\tau)^{-1}L_\gamma(\tau)U(\tau)V(\zeta(\tau))\rangle, \tag{5.6}$$

where

$$U(\tau)=e^{ip(\tau)\cdot x}. \tag{5.7}$$

By (3.3) and (3.26), the term $U(\tau)^{-1}L_\gamma(\tau)U(\tau)$ is rewritten as

$$U(\tau)^{-1}L_\gamma(\tau)U(\tau)=U(\tau)^{-1}S(\tau)U(\tau)\cdot\tilde{G}(\zeta(\tau))\cdot U(\tau)^{-1}S(\tau)^{-1}(1+K(\tau))^{-1}U(\tau).$$

Let $A(\tau)$ be from (3.12) and put

$$B(\tau)=U(\tau)^{-1}A(\tau)U(\tau). \tag{5.8}$$

Then we have by virtue of (3.12)

$$U(\tau)^{-1}S(\tau)^{-1}U(\tau)=e^{-B(\tau)}. \tag{5.9}$$

Lemma 5.1: Let

$$\Psi(x,\xi+\zeta(\tau))=\varphi(x,\xi+\zeta(\tau))+\psi(x,\xi+\zeta(\tau))\sigma\cdot(x\times(\xi+\zeta(\tau))),$$

and let $P(\tau)$ be the ΨDO with symbol $e^{-\Psi(x,\xi+\zeta(\tau))}$. Then

$$e^{-B(\tau)}-P(\tau)\in\mathcal{S}^{-1}.$$

Proof: Modulo $\mathcal{S}^{-1}$, $B(\tau)$ is a ΨDO with symbol $\Psi(x,\xi+\zeta(\tau))$. Therefore for large $|z|$, $(z-B(\tau))^{-1}$ is a ΨDO with symbol $(z-\Psi(x,\xi+\zeta(\tau)))^{-1}$ modulo $\mathcal{S}^{-1}$. Since for large M,

$$e^{-B(\tau)}=\frac{1}{2\pi i}\int_{|z|=M}e^{-z}(z-B(\tau))^{-1}\mathrm{d}z,$$

$e^{-B(\tau)}$ is a ΨDO with symbol $e^{-\Psi(x,\xi+\zeta(\tau))}$, modulo $\mathcal{S}^{-1}$. □

We put

$$\eta=e_1,\quad \gamma=e_2,\quad \eta\times\gamma=e_3, \tag{5.10}$$

$$\Psi_\infty(x)=f+g\sigma\cdot(x\times(e_1+ie_2)), \tag{5.11}$$

$$f=-(2\pi)^{-3/2}\int_{\mathbf{R}^3}e^{ix\cdot k}\frac{\hat{b}(k)\cdot(e_1+ie_2)}{k_1+ik_2}\mathrm{d}k, \tag{5.12}$$

$$g=-(2\pi)^{-3/2}\int_{\mathbf{R}^3}e^{ix\cdot k}\frac{\hat{a}(k)}{2(k_1+ik_2)}\mathrm{d}k, \tag{5.13}$$

where $k_j=k\cdot e_j, x_j=x\cdot e_j$ and $\tilde{a}(\xi_1,\xi_2,x_3)$ is the partial Fourier transform with respect to x_1,x_2 of $a(x)$. We also let

$$f_0(x)=2b(x)\cdot(e_1+ie_2)+a(x)\sigma\cdot(x\times(e_1+ie_2)). \tag{5.14}$$

Lemma 5.2:

$$\lim_{\tau\to\infty}\frac{1}{\tau}B_\gamma^{(2)}(\tau)=-\int_{\mathbf{R}^3}e^{-ix\cdot\xi}f_0e^{\Psi_\infty}N_\gamma(e^{-\Psi_\infty}f_0)\mathrm{d}x.$$

Proof: First we note

$$U(\tau)^{-1}(1+K(\tau))^{-1}U(\tau)V(\zeta(\tau))/\tau\to f_0(x). \tag{5.15}$$

Let

$$\chi_1(\xi+i\tau\gamma)=\chi_0\left(\frac{2|(\xi+i\tau\gamma)^2-E|}{E+\tau^2+|\xi|^2}\right),\quad \chi_2(\xi+i\tau\gamma)=1-\chi_1(\xi+i\tau\gamma),$$

$\chi_0(t)$ being as in (3.8), and let Q_j be the ΨDO with symbol $\chi_j(\xi+i\tau\gamma)$. On the support of $\chi_2(\xi+i\tau\gamma)$, we have

$$|(\xi+i\tau\gamma)^2-E|\geqslant\frac{\epsilon}{4}(E+\tau^2+|\xi|^2).$$

So we have

$$\|U_{\gamma,0}(E,i\tau)Q_2\|_{\mathbf{B}(L^2;L^2)}\leqslant C/\tau^2.$$

Therefore as an operator in $\mathbf{B}(L^{2,s};L^{2,s-1}),0<s<1$,

$$U_{\gamma,0}(E,i\tau)=U_{\gamma,0}(E,i\tau)Q_1+O(\tau^{-2}). \tag{5.16}$$

On the other hand, by Lemma 5.1, we have

$$\tilde{Q}_1U(\tau)^{-1}S(\tau)^{-1}U(\tau)=\tilde{Q}_1e^{-B(\tau)}\sim\tilde{Q}_1P(\tau), \tag{5.17}$$

and $\tilde{Q}_1P(\tau)$ converges strongly to $e^{-\Phi_\infty(x)}$.

Similarly $U(\tau)^{-1}S(\tau)U(\tau)$ converges strongly to $e^{\Psi_\infty(x)}$. Furthermore by Lemma 3.2,

$$\tau\tilde{G}(\zeta(\tau))\to N_\gamma.$$

These facts prove Lemma 5.2. □

Our next aim is to compute $N_\gamma(e^{-\Psi_\infty}f_0)$. Let us note that putting $\overline{\partial}=\frac{1}{2}(e_1+ie_2)\cdot\nabla_x$, we have

$$4i\overline{\partial}\Psi_\infty=f_0. \tag{5.18}$$

Since $\overline{\partial}\Psi_\infty$ and Ψ_∞ commute, we also have

$$\overline{\partial}e^{\Psi_\infty}=(\overline{\partial}\Psi_\infty)e^{\Psi_\infty}. \tag{5.19}$$

Lemma 5.3:

$$\int_{\mathbf{R}^3}e^{-ix\cdot\xi}f_0e^{\Psi_\infty}N_\gamma e^{-\Psi_\infty}f_0\mathrm{d}x=4i\int_{\mathbf{R}^3}e^{-ix\cdot\xi}(\overline{\partial}e^{\Psi_\infty})(e^{-\Psi_\infty}-1)\mathrm{d}x.$$

Proof: The left-hand side is equal to

$$\int_{\mathbf{R}^3}e^{-ix\cdot\xi}4i(\overline{\partial}\Psi_\infty)e^{\Psi_\infty}N_\gamma e^{-\Psi_\infty}4i(\overline{\partial}\Psi_\infty)\mathrm{d}x=16\int_{\mathbf{R}^3}e^{-ix\cdot\xi}(\overline{\partial}e^{\Psi_\infty})N_\gamma(\overline{\partial}e^{-\Psi_\infty})\mathrm{d}x.$$

Using

$$N_\gamma\overline{\partial}e^{-\Psi_\infty}=\frac{i}{4}(e^{-\Psi_\infty}-1),$$

which follows from Liouville's theorem, we get the lemma. □

Lemma 5.4:

$$\int_{\mathbf{R}^3}e^{-ix\cdot\xi}f_0e^{\Psi_\infty}N_\gamma e^{-\Psi_\infty}f_0\mathrm{d}x=0.$$

Proof: By integration by parts, we have

$$\int_{x_1^2+x_2^2<r^2} e^{-ix\cdot\xi}(\overline{\partial} e^{\Psi_\infty})(e^{-\Psi_\infty}-1)\mathrm{d}x_1\,\mathrm{d}x_2$$

$$=\frac{1}{2}\int_{x_1^2+x_2^2=r^2} e^{-ix\cdot\xi}(x_1+ix_2)(1-e^{\Psi_\infty})\mathrm{d}\theta-\int_{x_1^2+x_2^2<r^2} e^{-ix\cdot\xi}e^{\Psi_\infty}\overline{\partial}e^{-\Psi_\infty}\mathrm{d}x_1\,\mathrm{d}x_2\equiv(I)-(II).$$

The second term is written as

$$(II)=-\int_{x_1^2+x_2^2<r^2} e^{-ix\cdot\xi}\overline{\partial}\Psi_\infty\mathrm{d}x_1\,\mathrm{d}x_2=-\frac{1}{2}\int_{x_1^2+x_2^2=r^2} e^{-ix\cdot\xi}(x_1+ix_2)\Psi_\infty\mathrm{d}\theta.$$

On the other hand, $e^{\Psi_\infty}=1+\Psi_\infty+O(|x|^{-2})$. Therefore

$$(I)\sim-\frac{1}{2}\int_{x_1^2+x_2^2=r^2} e^{-ix\cdot\xi}(x_1+ix_2)\Psi_\infty\mathrm{d}\theta.$$

□

Using (5.5) and Lemmas 5.2, 5.4, one can compute

$$\int_{\mathbf{R}^3} e^{-ix\cdot\xi}\{2b(x)\cdot(\eta+i\gamma)+a(x)\sigma\cdot(x\times(\eta+i\gamma))\}\mathrm{d}x \tag{5.20}$$

form the scattering amplitude.

Here let us recall the following formulas for spin matrices, which are proved by using the commutation relations:

$$(\sigma\cdot\xi)(\sigma\cdot\eta)=\xi\cdot\eta+i\sigma\cdot(\xi\times\eta), \tag{5.21}$$

$$[\sigma\cdot(x\times(e_1+ie_2)),\sigma\cdot e_3]=2ix_3\sigma\cdot(e_1+ie_2). \tag{5.22}$$

We now reconstruct $a(x)$. We take $b(x)\equiv 0$. Then by (5.20) and (5.22), one can recover

$$\int e^{-ix\cdot\xi}a(x)x_3\mathrm{d}x.$$

Since $\xi=(0,0,\xi_3)$, one can recover

$$\int_{\mathbf{R}^3} a(x_1,x_2,x_3)\mathrm{d}x_1\,\mathrm{d}x_2.$$

Choosing the direction of ξ arbitrarily, one can reconstruct $a(x)$ by the inversion formula of the Radon transform (see, e.g., Ref. 10).

VI. RECONSTRUCTION OF THE COMPLEX POTENTIAL

A. Gauge invariance

In the previous section we constructed the spin–orbit term $a(x)\sigma\cdot(x\times p)$ from the scattering amplitude of the operator

$$-\Delta+V(p),\quad V(p)=a(x)\sigma\cdot(x\times p)+W(x).$$

To reconstruct $W(x)$ we shall make use of the gauge invariance.

Let $\psi(x)$ be the solution of

$$(-\Delta+V(p)-E)\psi=0 \tag{6.1}$$

having the asymptotic expansion

$$\psi \sim e^{i\sqrt{E}\omega\cdot x} + \frac{e^{i\sqrt{E}r}}{r} f(E;\theta,\omega), \quad \theta = x/r, \quad r = |x| \to \infty. \tag{6.2}$$

Let $c(x) = \exp(-|x|^2)$ and $\psi_\lambda(x) = e^{i\lambda c(x)}\psi(x)$, λ being a large parameter. Then ψ_λ satisfies

$$((p-b_\lambda)^2 + V(p-b_\lambda) - E)\psi_\lambda = 0 \tag{6.3}$$

with $b_\lambda(x) = \lambda \nabla c(x)$. Since $c(x)$ is exponentially decreasing, ψ_λ has the same asymptotic expansion as in (6.2). This means that the family of operators $\{(p-b_\lambda)^2 + V(p-b_\lambda); \lambda > 0\}$ has the same scattering amplitude. One should also note that due to the unitary equivalence, the sets of exceptional points $\mathcal{E}_0$ and $\mathcal{E}_\gamma(E)$ are independent of $\lambda > 0$.

B. Reconstruction of the complex potential

We use the same notation as in Sec. V with b replaced by $b_\lambda = \lambda \nabla c(x)$. Let

$$V_1(\xi) = 2b_\lambda \cdot \xi + a\sigma \cdot (x \times \xi),$$

$$V_2 = -i \operatorname{div} b_\lambda - |b_\lambda|^2 - a\sigma \cdot (x \times b_\lambda) + W.$$

Then

$$V(\zeta(\tau) - b_\lambda) \simeq \tau V_1(\eta + i\gamma) - V_1(\xi/2) + V_2,$$

$$V^*(\zeta(\tau)' - b_\lambda) \simeq \tau V_1^*(\eta - i\gamma) + V_1^*(\xi/2) + V_2^*.$$

Recall that we already know $a(x)$ and $b_\lambda(x)$.

We first show that up to known terms

$$B_\gamma^{(1)}(\tau) + B_\gamma^{(2)}(\tau) \simeq \int_{\mathbf{R}^3} e^{-ix\cdot\xi} W(x)\mathrm{d}x - \langle e^{-ix\cdot\xi} V_1^*(\eta - i\gamma), e^{\Psi_\infty} N_\gamma e^{-\Psi_\infty}(-V_1(\xi/2) + V_2)\rangle$$
$$- \langle e^{-ix\cdot\xi}(V_1^*(\xi/2) + V_2), e^{\Psi_\infty} N_\gamma e^{-\Psi_\infty} V_1(\eta + i\gamma)\rangle.$$

In fact,

$$B_\gamma^{(1)}(\tau) \simeq \tau \int_{\mathbf{R}^3} e^{-ix\cdot\xi} V_1(\eta + i\gamma)\mathrm{d}x - \int_{\mathbf{R}^3} e^{-ix\cdot\xi} V_1(\xi/2)\mathrm{d}x + \int_{\mathbf{R}^3} e^{-ix\cdot\xi} V_2 \mathrm{d}x.$$

Up to a known term, this is equal to $\int e^{-ix\cdot\xi} W(x)\mathrm{d}x$.

Next we note that

$$B_\gamma^{(2)}(\tau) \sim -\tau^2 \langle V_1^*(\eta - i\gamma)e^{ip(\tau)'x}, L_\gamma(\tau) V_1(\eta + i\gamma)e^{ip(\tau)x}\rangle - \tau \langle V_1^*(\eta - i\gamma)e^{ip(\tau)'x}, L_\gamma(\tau)$$
$$\times(-V_1(\xi/2) + V_2)e^{ip(\tau)x}\rangle - \tau \langle (V_1^*(\xi/2) + V_2^*)e^{ip(\tau)'x}, L_\gamma(\tau) V_1(\eta + i\gamma)e^{ip(\tau)x}\rangle.$$

Since

$$L_\gamma(\tau) = S(\tau) U_{\gamma,0}(E, i\tau) S(\tau)^{-1}(1 - K(\tau)) + O(\tau^{-3}),$$

the first term is a known term. Applying

$$L_\gamma(\tau) = S(\tau) U_{\gamma,0}(E, i\tau) S(\tau)^{-1} + O(\tau^{-2}),$$

and arguing in the same way as in the proof of Lemma 5.2, we get

$$B_\gamma^{(2)}(\tau)\simeq -\langle e^{-ix\cdot\xi}V_1^*(\eta-i\gamma), e^{\Psi_\infty}N_\gamma e^{-\Psi_\infty}(-V_1(\xi/)+V_2)\rangle - \langle e^{-ix\cdot\xi}(V_1^*(\xi/2)+V_2^*), e^{\Psi_\infty}N_\gamma e^{-\Psi_\infty}V_1(\eta+i\gamma)\rangle.$$

The right-hand side is equal to

$$-\int e^{ix\cdot\xi}(\overline{\partial} e^{\Psi_\infty})N_\gamma e^{-\Psi_\infty}(-V_1(\xi/2)+V_2)\mathrm{d}x+\int e^{ix\cdot\xi}(V_1(\xi/2)+V_2)e^{\Psi_\infty}N_\gamma(\overline{\partial} e^{-\Psi_\infty})\mathrm{d}x$$
$$=\int e^{ix\cdot\xi}(N_\gamma\overline{\partial} e^{\Psi_\infty})e^{-\Psi_\infty}(-V_1(\xi/2)+V_2)\mathrm{d}x+\int e^{ix\cdot\xi}(V_1(\xi/2)+V_2)e^{\Psi_\infty}(N_\gamma\overline{\partial} e^{-\Psi_\infty})\mathrm{d}x,$$

where we have used (7.6). Since

$$N_\gamma e^{\pm i\Psi_\infty}=\frac{i}{4}(e^{\pm i\Psi_\infty}-1),$$

this is equal to

$$\frac{i}{4}\int e^{ix\cdot\xi}(1-e^{\Psi_{-\infty}})(-V_1(\xi/2)+V_2)\mathrm{d}x+\frac{i}{4}\int e^{ix\cdot\xi}(V_1(\xi/2)+V_2)(1-e^{\Psi_\infty})\mathrm{d}x.$$

Since $\Psi_\infty=-i\lambda c(x)+g$, by the stationary phase method, the term containing $e^{\pm i\Psi_\infty}$ vanishes as $\lambda\to\infty$. Here one must note that $b_\lambda(0)=0$. What remains is

$$\frac{i}{2}\int e^{ix\cdot\xi}V_2\mathrm{d}x.$$

Up to a known term this is equal to $i/2\int e^{ix\cdot\xi}W\,\mathrm{d}x$. We have thus reconstructed $\hat{W}(\xi)$.

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APPENDIX

We summarize here basic properties of the $\overline{\partial}$-operator used in this paper. Note that we define $\overline{\partial}=(\partial/\partial x_1+i\partial/\partial x_2)/2$.

Theorem 7.1: *If $|f(z)|\leqslant C(1+|z|)^{-1-\epsilon}$ for some $C,\epsilon>0$, the solution of the equation $\overline{\partial}u=f$ satisfying $u(z)\to 0$ as $|z|\to\infty$ is unique and is given by*

$$u(z)=\frac{1}{2\pi i}\int_{\mathbf{C}}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta}=\frac{1}{\pi}\int_{\mathbf{R}^2}\frac{f(x_1-y_1,x_2-y_2)}{y_1+iy_2}\mathrm{d}y_1\mathrm{d}y_2.$$

Using the identity

$$\frac{1}{\zeta-z}=-\frac{1}{z}\sum_{k=0}^{n}\left(\frac{\zeta}{z}\right)^k+\frac{1}{\zeta-z}\left(\frac{\zeta}{z}\right)^{n+1}$$

we have if $(1+|z|)^n f(z)\in L^1(\mathbf{C})$,

$$u(z)=-\frac{1}{2\pi i}\sum_{k=0}^{n}z^{-k-1}\int_{\mathbf{C}}\zeta^k f(\zeta)\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta}+\frac{1}{2\pi i}z^{-n-1}\int_{\mathbf{C}}\frac{\zeta^{n+1}f(\zeta)}{\zeta-z}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta}.$$

In particular we have

Theorem 7.2. *If* $|f(z)|\leqslant C(1+|z|)^{-3-\epsilon}$, *the above solution satisfies*

$$(x_1+ix_2)u(z)=\frac{1}{\pi}\int_{\mathbf{R}^2}f(y_1,y_2)\mathrm{d}y_1\mathrm{d}y_2+O(|x|^{-1}).$$

From this theorem it follows that

$$u(x)=(2\pi)^{-1}\int_{\mathbf{R}^2}\frac{e^{ix\cdot k}\hat{f}(k)}{k_1+ik_2}\mathrm{d}k \tag{A1}$$

satisfies

$$(x_1+ix_2)u(x)=\frac{i}{2\pi}\int_{\mathbf{R}^2}f(y)\mathrm{d}y+O(|x|^{-1}), \tag{A2}$$

if $|f(y)|\leqslant C(1+|y|)^{-3-\epsilon}$.

For $f\in\mathcal{S}(\mathbf{R}^3)$, let

$$Nf(x)=(2\pi)^{-3/2}\int_{\mathbf{R}^3}\frac{e^{ix\cdot\xi}\hat{f}(\xi)}{2(\xi_1+i\xi_2)}\mathrm{d}\xi. \tag{A3}$$

Then we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\right)Nf=\frac{i}{4}f, \tag{A4}$$

$$Nf(x)=\frac{i}{4\pi}\int_{\mathbf{R}^2}\frac{f(y_1,y_2,x_3)}{x_1-y_1+i(x_2-y_2)}\mathrm{d}y, \tag{A5}$$

$$\int_{\mathbf{R}^3}(Nf(x))g(x)\mathrm{d}x=-\int_{\mathbf{R}^3}f(x)(Ng(x))\mathrm{d}x. \tag{A6}$$

Let us prove Lemma 3.3 (1). We first note that $|\xi^2+2i\tau\gamma\cdot\xi-\tau^2-E|\leqslant\epsilon(E+\tau^2+|\xi|^2)$ implies there exists a constant $C>0$ such that

$$C^{-1}\tau\leqslant|\xi|\leqslant C\tau,\quad |\gamma\cdot\frac{\xi}{|\xi|}|\leqslant C\epsilon, \tag{A7}$$

for large $\tau>0$. Therefore we have only to show the following lemma.

Lemma 7.3: Let $m\in\mathbf{R}$. *Suppose* $f(x,\xi;\tau)$ *satisfies*

$$|\partial_x^\alpha\partial_\xi^\beta f(x,\xi;\tau)|\leqslant C_{\alpha\beta}\langle x\rangle^{-3-|\alpha|}(\tau+|\xi|)^{m-|\beta|}\quad\forall\alpha,\beta$$

for ξ, τ *satisfying the condition (7.7). Then*

$$g(x,\xi;\tau)=(2\pi)^{-3/2}\int_{\mathbf{R}^3}e^{ix\cdot k}\frac{\hat{f}(k,\xi;\tau)}{k\cdot(\xi+i\tau\gamma)}\mathrm{d}k$$

satisfies

$$|\partial_x^\alpha\partial_\xi^\beta g(x,\xi;\tau)|\leqslant C_{\alpha\beta}\langle x\rangle^{-1}(\tau+|\xi|)^{m-1-|\beta|}\quad\forall\alpha,\beta,$$

for ξ, τ *satisfying the condition (7.7).*

Proof: We make the linear change of variables $p=Ak$, where

$$p_1 = \frac{\xi}{\tau} \cdot k,\ p_2 = \gamma \cdot k, p_3 = \left(\frac{\xi}{\tau} \times \gamma\right) \cdot k.$$

Then letting $f_A(x,\xi;\tau) = f({}^tAx,\xi;\tau)$, we have

$$g({}^tAx,\xi;\tau) = \frac{i}{2\pi\tau}\int_{\mathbf{R}^2} \frac{f_A(x_1 - y_1, x_2 - y_2, x_3, \xi;\tau)}{y_1 + iy_2}\,\mathrm{d}y,$$

whose derivative is estimated as follows:

$$|\partial_x^\alpha \partial_\xi^\beta g({}^tAx,\xi;\tau)| \leqslant C_{\alpha\beta}(\tau + |\xi|)^{m-1-|\beta|}\int_{\mathbf{R}^2} \frac{(1+|x'-y|+|x_3|)^{-3}}{|y|}\,\mathrm{d}y,$$

where $x' = (x_1, x_2)$. The integral over the region $\{|y| < |x'|/2\}$ is estimated as

$$\int_{|y|<|x'|/2} \frac{(1+|x'-y|+|x_3|)^{-3}}{|y|}\,\mathrm{d}y \leqslant C(1+|x|)^{-2}.$$

The integral over the region $\{|y| > |x'|/2\}$ is estimated as

$$\int_{|y|>|x'|/2} \frac{(1+|x'-y|+|x_3|)^{-3}}{|y|}\,\mathrm{d}y \leqslant \frac{C}{|x'|}\int_{\mathbf{R}^2}(1+|y|+|x_3|)^{-3}\mathrm{d}y \leqslant \frac{C}{|x'|(1+|x^3|)}.$$

If $|x'| > 1$, this is dominated from above by $C(1+|x|)^{-1}$. If $|x'| \leqslant 1$, we estimate in the following manner:

$$\int_{|x'|/2<|y|<1} \frac{(1+|x'-y|+|x_3|)^{-3}}{|y|}\,\mathrm{d}y \leqslant C(1+|x_3|)^{-3}\int_{|y|<1}\frac{\mathrm{d}y}{|y|} \leqslant C(1+|x_3|)^{-3} \leqslant C(1+|x|)^{-3},$$

$$\int_{|y|>1} \frac{(1+|x'-y|+|x_3|)^{-3}}{|y|}\,\mathrm{d}y \leqslant C\int_{|y|<1}(1+|x'-y|+|x_3|)^{-3}\mathrm{d}y$$
$$\leqslant C(1+|x_3|)^{-1} \leqslant C(1+|x|)^{-1}.$$

We have thus proven

$$|\partial_x^\alpha \partial_\xi^\beta g({}^tAx,\xi;\tau)| \leqslant C_{\alpha\beta}\langle x\rangle^{-1}(\tau+|\xi|)^{m-1-|\beta|}.$$

From this we can conclude the lemma. □

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