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SEMILINEAR PARABOLIC BOUNDARY VALUE PROBLEMS IN COMBUSTION THEORY

Kazuaki Taira

Dedicated to Professor Takaaki Nishida on the occasion of his 60th birthday

Abstract. This paper is devoted to the analytic semigroup approach to semilinear parabolic initial boundary value problems arising in combustion theory which obey a general Arrhenius equation and Newtonian cooling. We prove a global existence and uniqueness theorem of positive solutions by using the theory of analytic semigroups in the topology of uniform convergence. Moreover, we study the asymptotic stability of maximal and minimal positive solutions when there are multiple steady-state solutions.

1. Introduction and Results

Let D be a bounded domain of Euclidean space $\mathbf{R}^N$, $N \geq 2$, with smooth boundary ∂D; its closure $\overline{D} = D \cup \partial D$ is an N-dimensional, compact smooth manifold with boundary. We let

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{N} a^{ij}(x) \frac{\partial u}{\partial x_j}(x) \right) + c(x)u(x)$$

be a second-order, *elliptic* differential operator with real coefficients such that:

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(1) $a^{ij}(x) \in C^\infty(\overline{D})$ and $a^{ij}(x) = a^{ji}(x)$ for all $x \in \overline{D}$ and $1 \leq i, j \leq N$, and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2 \quad \text{for all } (x,\xi) \in \overline{D} \times \mathbf{R}^N.$$

(2) $c(x) \in C^\infty(\overline{D})$ and $c(x) \geq 0$ in D.

In this paper we consider the following semilinear parabolic initial boundary value problem stimulated by *solid fuel models* in combustion theory: Given a function $u_0(x)$ defined in D, find a function $u(x,t)$ in $D \times [0,T)$ such that

$$\text{(1.1)} \quad \begin{cases} \left(\dfrac{\partial}{\partial t} + A\right) u = \lambda(1+\varepsilon u)^m \exp\left[\frac{u}{1+\varepsilon u}\right] & \text{in } D \times (0,T), \\ Bu = \dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 & \text{on } \partial D \times (0,T), \\ u|_{t=0} = u_0 & \text{in } D. \end{cases}$$

Here:

(1) $\lambda > 0$ and $\varepsilon > 0$ are parameters.
(2) m is a numerical exponent with $0 \leq m < 1$.
(3) $b(x') \in C^\infty(\partial D)$ and $b(x') \geq 0$ on ∂D.
(4) $\partial/\partial\boldsymbol{\nu}$ is the conormal derivative associated with the operator A

$$\frac{\partial}{\partial \boldsymbol{\nu}} = \sum_{i,j=1}^{N} a^{ij}(x')\, n_j \frac{\partial}{\partial x_i},$$

where $\mathbf{n} = (n_1, n_2, \dots, n_N)$ is the unit exterior normal to the boundary ∂D (see Figure 1.1).

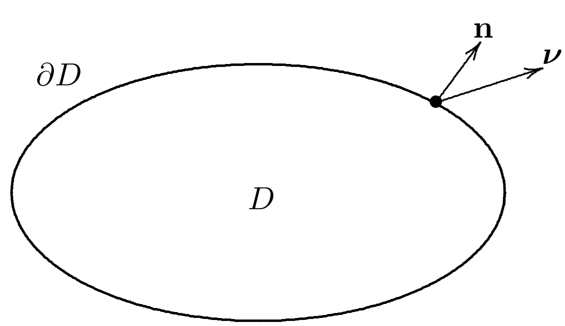

Figure 1.1

The nonlinear term

$$f(t) := (1+\varepsilon t)^m \exp\left[\frac{t}{1+\varepsilon t}\right] \tag{1.2}$$

describes the temperature dependence of reaction rate for exothermic reactions obeying the *Arrhenius equation* in circumstances in which heat flow is purely conductive, and the parameter ε is a dimensionless inverse measure of the Arrhenius activation energy or a dimensionless ambient temperature. The exponent m is the exponent of the temperature dependence of the pre-exponential factor in Arrhenius expression; the two cases $m = 0$ and $m = 1/2$ correspond to the simple Arrhenius rate law and the bimolecular rate law, respectively. The equation

$$\left(\frac{\partial}{\partial t} + A\right) u - \lambda f(u) = 0 \quad \text{in } D \times (0,T)$$

represents heat evolution with reactant consumption ignored, where the function $u(x,t)$ is a dimensionless temperature excess of a combustible material and the parameter λ, called the *Frank-Kamenetskii parameter* in combustion theory, is a dimensionless rate of heat production.

On the other hand, the Robin boundary condition

$$Bu = \frac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 \quad \text{on } \partial D \times (0,T)$$

represents the exchange of heat at the surface of the reactant by *Newtonian cooling*. The boundary condition B is called the adiabatic condition if $b(x') \equiv 0$ on ∂D.

In a reacting material undergoing an exothermic reaction in which reactant consumption is neglected, heat is being produced in accordance with Arrhenius rate law and Newtonian cooling. Thermal explosions occur when the reactions produce heat too rapidly for a stable balance between heat production and heat loss to be preserved. For detailed studies of thermal explosions, the reader might be referred to Aris [3], Bebernes–Eberly [4], Boddington–Gray–Wake [6] and Warnatz–Maas–Dibble [24].

This paper will extend substantially the previous work Taira–Umezu [21].

Our first main result is the following global existence and uniqueness theorem of positive solutions for the semilinear parabolic problem (1.1):

Theorem 1. *Let $u_0(x)$ be an arbitrary non-negative function in $C^2(\overline{D})$ which satisfies the boundary condition $Bu_0 = 0$ on ∂D. Then problem (1.1) has a unique non-negative,* global *solution*

$$u(x,t) \in C^{1,0}(\overline{D} \times [0,\infty)) \cap C^{2,1}(\overline{D} \times (0,\infty)).$$

Here $C^{1,0}(\overline{D}\times[0,\infty))$ denotes the space of continuous functions on $\overline{D}\times[0,\infty)$ which are continuously differentiable with respect to x, and $C^{2,1}(\overline{D}\times(0,\infty))$ denotes the space of continuously differentiable functions on $\overline{D}\times(0,\infty)$, twice with respect to x and once with respect to t.

To study problem (1.1) from the point of view of *stability analysis*, we consider the following semilinear elliptic boundary value problem:

$$(1.3) \qquad \begin{cases} Av = \lambda(1+\varepsilon v)^m \exp\left[\dfrac{v}{1+\varepsilon v}\right] & \text{in } D, \\ Bv = \dfrac{\partial v}{\partial \boldsymbol{\nu}} + b(x')v = 0 & \text{on } \partial D. \end{cases}$$

In the simple Arrhenius law case $m = 0$, problem (1.3) has been studied by many authors (see Brown–Ibrahim–Shivaji [7], Cohen [8], Cohen–Laetsch [9], Dancer [10], Pao [12], Parter [13], Tam [22], Wiebers [25], [26] and Williams–Leggett [27]).

To formulate our existence and multiplicity theorem of positive solutions of problem (1.3), we introduce a function $\nu(t)$ by the formula

$$\nu(t) := \frac{t}{f(t)} = \frac{t}{(1+\varepsilon t)^m \exp\left[t/(1+\varepsilon t)\right]} \quad \text{for all } t \geq 0.$$

It is easy to see that if the parameter ε satisfies the condition

$$(1.4) \qquad \varepsilon \geq \left(\frac{1}{1+\sqrt{1-m}}\right)^2,$$

then the function $\nu(t)$ is increasing for all $t \geq 0$. On the other hand, we find that if the parameter ε satisfies the condition

$$(1.5) \qquad 0 < \varepsilon < \left(\frac{1}{1+\sqrt{1-m}}\right)^2,$$

then the function $\nu(t)$ has a unique local maximum at $t = t_1(\varepsilon)$

$$t_1(\varepsilon) = \frac{1 + (m-2)\varepsilon - \sqrt{m^2\varepsilon^2 + 2(m-2)\varepsilon + 1}}{2(1-m)\varepsilon^2},$$

and has a unique local minimum at $t = t_2(\varepsilon)$

$$t_2(\varepsilon) = \frac{1 + (m-2)\varepsilon + \sqrt{m^2\varepsilon^2 + 2(m-2)\varepsilon + 1}}{2(1-m)\varepsilon^2}.$$

It should be noticed that, by a direct computation, the local maximum $\nu(t_1(\varepsilon))$ is positive near $\varepsilon = 0$, while the local minimum $\nu(t_2(\varepsilon))$ tends to 0 as $\varepsilon \downarrow 0$. The graph of the function $\nu(t)$ is shown in Figure 1.2.

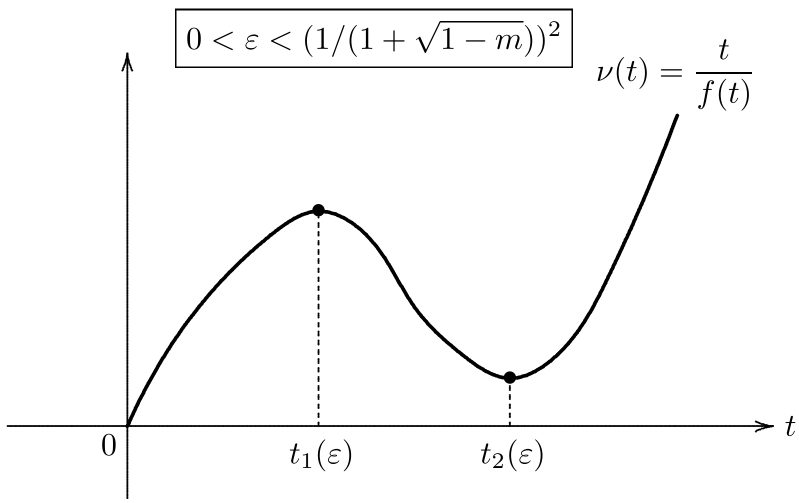

Figure 1.2

For technical reasons, we consider the case where $c(x) \geq 0$ in D (including the case where $c(x) \equiv 0$ in D). This makes it possible to develop our basic machinery with a minimum of bother and the principal ideas can be presented concretely and explicitly. In fact, it follows from an application of Taira [19, Lemma 2.7 and Theorem 0] that the linearized problem

$$\begin{cases} A\phi = 1 & \text{in } D, \\ B\phi = 0 & \text{on } \partial D, \end{cases} \tag{1.6}$$

has a unique positive solution $\phi(x) \in C^\infty(\overline{D})$, and further that the first eigenvalue λ_1 of the linearized eigenvalue problem

$$\begin{cases} A\varphi = \mu\varphi & \text{in } D, \\ B\varphi = 0 & \text{on } \partial D \end{cases} \tag{1.7}$$

is strictly positive: $\lambda_1 > 0$.

Our starting point is the following existence theorem of positive solutions for the semilinear elliptic problem (1.3) due to Taira [20]:

Theorem 2. *(i) Problem (1.3) has* at least one *positive solution $v(\lambda)$ for every $\lambda > 0$.*

(ii) If the parameter ε satisfies condition (1.4), then problem (1.3) has a unique *positive solution $v(\lambda)$ for every $\lambda > 0$.*

(iii) Assume that the parameter ε satisfies condition (1.5).

(iii-a) Problem (1.3) has a unique *positive solution $v(\lambda)$ for all small λ satisfying the condition*

$$\begin{aligned} 0 < \lambda < & \frac{\lambda_1}{m+1+\sqrt{1+2m(1-m)}} \left(\frac{1}{1+\sqrt{1+2m(1-m)}} \right)^{1-m} \\ & \times \exp\left[1 + \sqrt{1+2m(1-m)} - \frac{1}{\varepsilon} \right] \varepsilon^{m-2}. \end{aligned} \tag{1.8}$$

(iii-b) There exists a constant $\Lambda > 0$, independent of ε, such that problem (1.3) has a unique *positive solution $v(\lambda)$ for all large λ greater than Λ.*

(iii-c) There exists a positive constant β, independent of ε, such that if $\varepsilon > 0$ is so small that

$$\frac{\nu(t_2(\varepsilon))}{\beta} < \frac{\nu(t_1(\varepsilon))}{\|\phi\|_\infty},$$

then problem (1.3) has at least three *distinct positive solutions $v_1(\lambda)$, $v_2(\lambda)$, $v_3(\lambda)$ for all λ satisfying the condition*

$$\frac{\nu(t_2(\varepsilon))}{\beta} < \lambda < \frac{\nu(t_1(\varepsilon))}{\|\phi\|_\infty}.$$

Here

$$\|\phi\|_\infty = \max_{x \in \overline{D}} \phi(x).$$

The positive solution sets for $\varepsilon \geq (1/(1+\sqrt{1-m})^2$ and for $0 < \varepsilon \ll (1/(1+\sqrt{1-m}))^2$ in Theorem 1 may be represented respectively as in Figures 1.3 and 1.4.

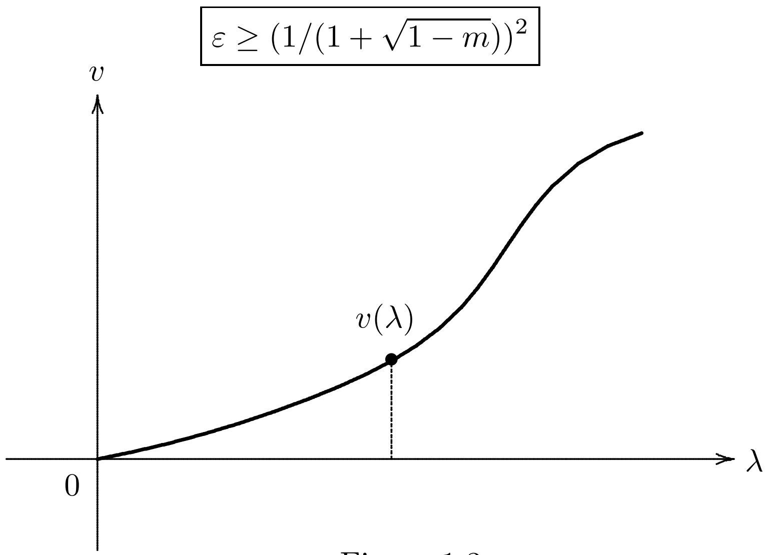

Figure 1.3

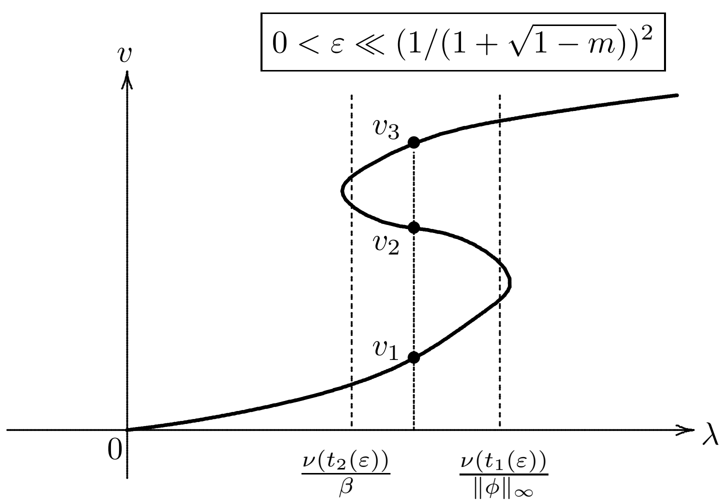

Figure 1.4

Remark 1.1. The difficulty in the proof of Theorem 2 is to construct a suitable pair of super- and sub-solutions of problem (1.3). In fact, a simple application of the maximum principle shows that the solutions $v(\lambda)$ of problem (1.3) satisfy the estimates

$$\lambda\phi(x) \leq v(\lambda) \leq \lambda C_m \phi(x) \quad \text{on } \overline{D}, \tag{1.9}$$

where C_m is a positive constant that is the unique solution of the equation (see Figure 1.5)

$$C_m = (1 + \lambda\varepsilon \|\phi\|_\infty C_m)^m \; e^{1/\varepsilon}. \tag{1.10}$$

We remark that estimates (1.9) are an immediate consequence of Lemma 3.1 with $u(x,t) := v(\lambda)(x)$, $v(x,t) := \lambda\phi(x)$ and $w(x,t) := \lambda C_m \phi(x)$, just as in the proof of Claim 3.1.

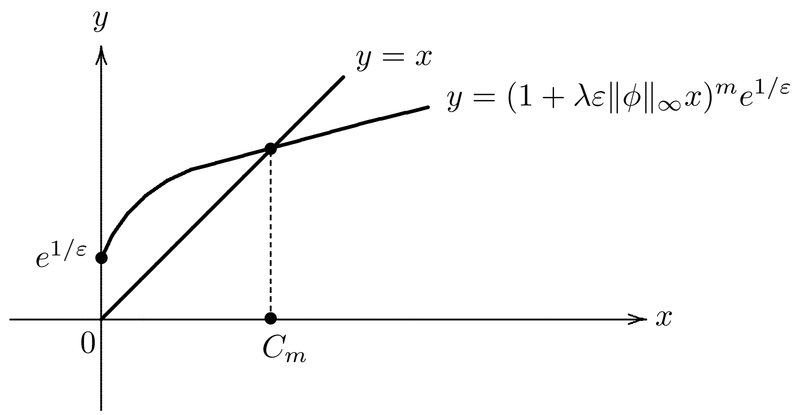

Figure 1.5

By virtue of Theorem 1, we can define two positive numbers μ_I and μ_E by the formulas

$$\mu_I = \inf \{\mu > 0 : \text{ problem (1.3) is uniquely solvable for each } \lambda > \mu\},$$
$$\mu_E = \sup \{\mu > 0 : \text{ problem (1.3) is uniquely solvable for each } 0 < \lambda < \mu\}.$$

Then certain physical conclusions may be drawn (cf. Bebernes–Eberly [4], Warnatz–Maas–Dibble [24]). If the system is in a state corresponding to a point on the lower branch and if λ is slowly increased, then the solution can be expected to change smoothly until the point μ_I is reached. Rapid transition to the upper branch will then presumably occur, corresponding to *ignition*. A subsequent slow decrease in λ is likewise anticipated to produce a smooth decrease in burning rate until *extinction* occurs at the point μ_E. It should be emphasized that the minimal positive solution $\underline{v}(\lambda)$ is continuous for $\lambda > \mu_I$ but is not continuous at $\lambda = \mu_I$, while the maximal positive solution $\overline{v}(\lambda)$ is continuous for $0 < \lambda < \mu_E$ but is not continuous at $\lambda = \mu_E$. Here we recall that a positive solution $\overline{v}(\lambda)$ of problem (1.3) is said to be *maximal* if $v(\lambda) \leq \overline{v}(\lambda)$ on $\overline{D}$ for any positive solution $v(\lambda)$ of problem (1.3). Similarly, a positive solution $\underline{v}(\lambda)$ of problem (1.3) is said to be *minimal* if $\underline{v}(\lambda) \leq v(\lambda)$ on $\overline{D}$ for any positive solution $v(\lambda)$ of problem (1.3).

The situation may be represented schematically by Figure 1.6 (cf. [5, Figure 1]).

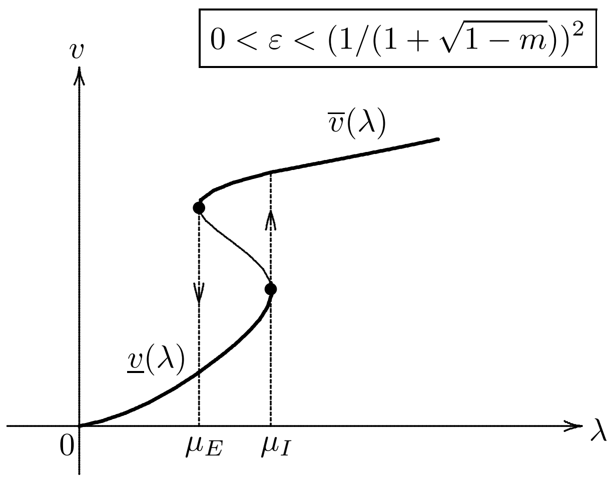

Figure 1.6

The next theorem asserts that the solution curve is asymptotically stable for all $\lambda > 0$ if the Arrhenius activation energy E is so low that $\varepsilon \geq (1/(1+\sqrt{1-m}))^2$:

Theorem 3. *If the parameter ε satisfies condition (1.4), then a positive solution $v(\lambda)$ of problem (1.3) is* asymptotically stable *for each $\lambda > 0$; more precisely, if $u_0(x)$ is a non-negative, initial function in $C^2(\overline{D})$ which satisfies the boundary condition $Bu_0 = 0$ on ∂D, then the global solution $u(\cdot, t)$ of* *problem (1.1) converges uniformly to the steady-state solution* $v(\lambda)$ *as* $t \to +\infty$.

The next theorem asserts that the solution curve is asymptotically stable for $\lambda > 0$ sufficiently small and sufficiently large if the Arrhenius activation energy E is so high that

$$0 < \varepsilon < (1/(1 + \sqrt{1 - m}))^2.$$

Theorem 4. *Let* $u_0(x)$ *be an arbitrary non-negative, initial function in* $C^2(\overline{D})$ *which satisfies the boundary condition* $Bu_0 = 0$ *on* ∂D. *If the parameter* ε *satisfies condition (1.5), then a solution* $v(\lambda)$ *of problem (1.3) is* asymptotically stable *for all* $0 < \lambda < \mu_E$ *and for all* $\lambda > \mu_I$.

The situation of Theorems 3 and 4 may be represented schematically by Figures 1.7 and 1.8, respectively.

The rest of this paper is organized as follows. In Section 2 we present a brief description of the theory of analytic semigroups which forms a functional analytic background for the proof of main results. This section is adapted from Henry [11], Pazy [14] and also Taira [18]. The material in this section is given for completeness, to minimize the necessity of consulting many references. Section 3 is devoted to the proof of Theorem 1. In the proof of Theorem 1 we make good use of a generation theorem for analytic semigroups in the topology of uniform convergence. It should be emphasized that the nonlinear term $f(t)$ is defined only for $t \geq 0$. To apply the theory of analytic semigroups, we modify the function $f(t)$ as a continuously differentiable function $\tilde{f}(t)$ on $\mathbf{R}$. However, it follows from an application of the maximum principle that all positive solutions of the new equation are solutions of the original equation. Hence, this change introduces no extra positive solutions. In Section 4 we prove Theorems 3 and 4, by using Sattinger's stability theorem. Namely, if an initial function in the parabolic problem (1.1) is a super-solution or a sub-solution of the elliptic problem (1.3), then it can be shown that the time-dependent solution is monotone in time and converges to a steady-state solution. This monotone convergence property gives a close relationship between the stability and the uniqueness property of a steady-state solution. This is a basic tool for determining the stability property of the maximal and minimal solutions when there are multiple steady-state solutions. In Section 5 we prove (local) asymptotic stability theorems for maximal and minimal positive solutions of problem

(1.3) in terms of the *size* of initial functions $u_0(x)$ with respect to the function $\phi(x)$. In Appendix we estimate the important constant β in part (iii-c) of Theorem 2 in terms of the function $\phi(x)$.

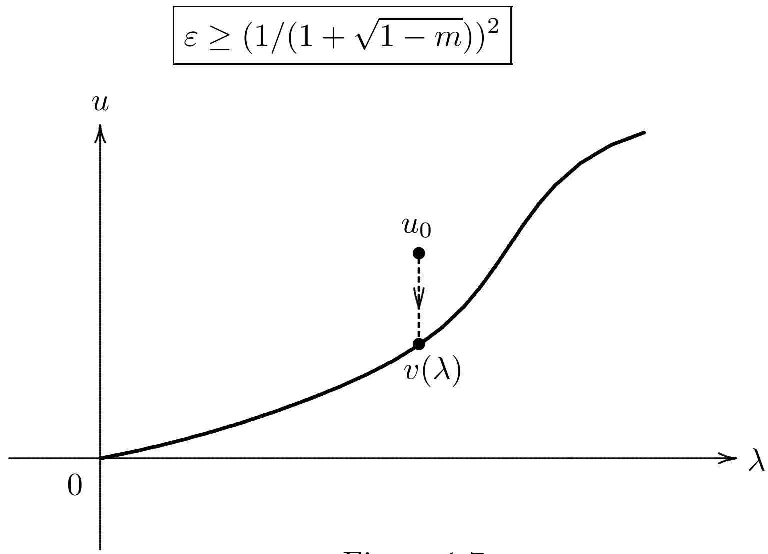

Figure 1.7

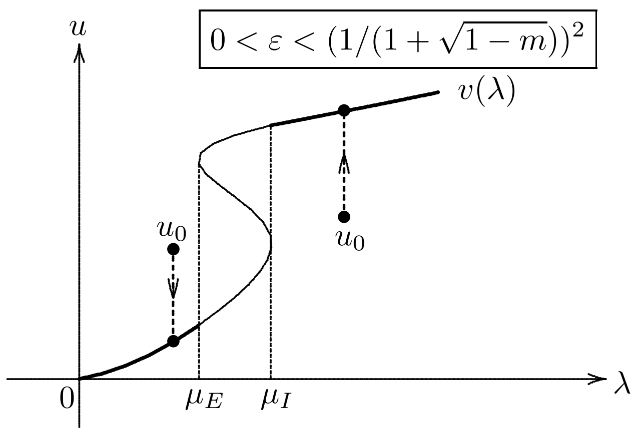

Figure 1.8

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2. Theory of Analytic Semigroups

This section is devoted to a review of standard topic from the theory of analytic semigroups which forms a functional analytic background for the proof of Theorem 1 in Section 3. For more leisurely treatments of analytic semigroups, the reader is referred to Henry [11], Pazy [14] and Yosida [28].

2.1 Generation of analytic semigroups.

Let E be a Banach space over the real number field $\mathbf{R}$ or the complex number field $\mathbf{C}$, and let $A : E \to E$ be a *densely defined*, closed linear operator with domain $D(A)$. Assume that the operator A satisfies the following two conditions:

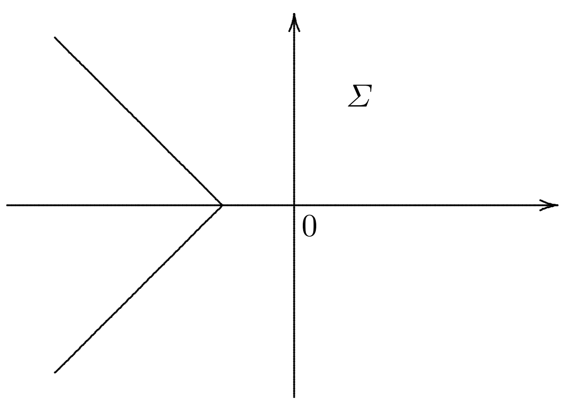

Figure 2.1

(1) The resolvent set of A contains the region Σ as in Figure 2.1.

(2) There exists a constant $M > 0$ such that the resolvent $R(\lambda) = (A - \lambda I)^{-1}$ satisfies the estimate

$$\|R(\lambda)\| \leq \frac{M}{1 + |\lambda|} \quad \text{for all } \lambda \in \Sigma. \tag{2.1}$$

The next theorem asserts that the operator A generates an *analytic* semigroup in some sector containing the positive real axis (see Henry [11], Pazy [14], Yosida [28]):

Theorem 2.1. *If the operator A satisfies condition (2.1), then it generates a semigroup $U(z)$ on E which is* analytic *in some sector $\Delta_\omega = \{z = t+is : z \neq 0, |\arg z| < \omega\}$ with $0 < \omega < \pi/2$, and enjoys the following three properties:*

(a) The operators $AU(z)$ and $\frac{dU}{dz}(z)$ are bounded operators on E for each $z \in \Delta_\omega$, and satisfy the relation

$$\frac{dU}{dz}(z) = AU(z) \quad \text{for all } z \in \Delta_\omega.$$

(b) For each $0 < \varepsilon < \omega/2$, there exist constants $M_0(\varepsilon) > 0$ and $M_1(\varepsilon) > 0$ such that

$$\|U(z)\| \leq M_0(\varepsilon) \quad \text{for all } z \in \Delta_\omega^{2\varepsilon},$$

$$\|AU(z)\| \leq \frac{M_1(\varepsilon)}{|z|}, \quad \text{for all } z \in \Delta_\omega^{2\varepsilon},$$

where

$$\Delta_\omega^{2\varepsilon} = \{z \in \mathbf{C} : z \neq 0, |\arg z| \leq \omega - 2\varepsilon\}.$$

(c) For each $x \in E$, we have, as $z \to 0$, $z \in \Delta_\omega^{2\varepsilon}$,

$$U(z)x \longrightarrow x \quad \text{in } E.$$

Now, if $0 < \alpha < 1$, we can define the negative fractional power $(-A)^{-\alpha}$ of $-A$ by the formula

$$(-A)^{-\alpha} = \frac{\sin \alpha\pi}{\pi} \int_0^\infty s^{-\alpha}(-A + sI)^{-1}\, ds,$$

and also the positive fractional power $(-A)^\alpha$ by the formula

$$(-A)^\alpha = \text{the inverse of } (-A)^{-\alpha}.$$

The operator $(-A)^\alpha$ is a closed linear, invertible operator with domain $D((-A)^\alpha) \supset D(A)$.

We let

$$E_\alpha = \text{the space } D((-A)^\alpha) \text{ endowed with the graph norm } \|\cdot\|_\alpha \text{ of } (-A)^\alpha,$$

where

$$\|x\|_\alpha = \left(\|x\|^2 + \|(-A)^\alpha x\|^2\right)^{1/2} \quad \text{for all } x \in D((-A)^\alpha).$$

Then we have the following assertions (see Taira [18, Proposition 1.17]):

Proposition 2.2. *(i) The space E_α is a Banach space.*
(ii) The graph norm $\|x\|_\alpha$ is equivalent to the norm $\|(-A)^\alpha x\|$.
(iii) If $0 < \alpha < \beta < 1$, then we have $E_\beta \subset E_\alpha$ with continuous injection.

Moreover, the next theorem states some useful relationships between the fractional powers $(-A)^\alpha$, $0 < \alpha \leq 1$, and the semigroup $U(t)$ (see Henry [11, Theorem 1.4.3], Pazy [14, Chapter 2, Theorem 6.13], Taira [18, Theorem 1.12]):

Theorem 2.3. *Let $0 < \alpha \leq 1$. For all $t > 0$, we have the following four assertions:*

(a) $U(t) : E \to D((-A)^\alpha)$.
(b) $U(t)(-A)^\alpha x = (-A)^\alpha U(t)x$ for all $x \in D((-A)^\alpha)$.
(c) $\|(-A)^\alpha U(t)\| \leq M_\alpha \, t^{-\alpha} \, e^{-\delta t}$.
(d) $\|U(t)x - x\| \leq \frac{M_{1-\alpha}}{\alpha} \, t^\alpha \, \|(-A)^\alpha x\|$ for all $x \in D((-A)^\alpha)$.
Here $\delta > 0$ and $M_\alpha > 0$ are constants independent of t.

2.2 The abstract Cauchy problem.

Let $f(t)$ be a function defined on an interval $[0, T)$ taking values in E. First, we consider the following *linear* Cauchy problem: Given $x_0 \in E$, find a function $u(t)$ such that

$$\begin{cases} \dfrac{du}{dt} = Au(t) + f(t) \quad \text{for all } 0 < t < T, \\ u(0) = x_0. \end{cases} \tag{2.2}$$

A function $u(t) : [0, T) \to E$ is called a *solution* of problem (2.2) if it satisfies the following three conditions:

(1) $u(t) \in C([0, T); E) \cap C^1((0, T); E)$ and $u(0) = x_0$.
(2) $u(t) \in D(A)$ for all $0 < t < T$.
(3) $\frac{du}{dt} = Au(t) + f(t)$ for all $0 < t < T$.

Here $C([0, T); E)$ denotes the space of continuous functions on $[0, T)$ taking values in E, and $C^1((0, T); E)$ denotes the space of continuously differentiable functions on $(0, T)$ taking values in E, respectively.

The next theorem gives an explicit formula for the solutions of problem (2.2) (see Taira [18, Theorem 1.15]):

Theorem 2.4. *If the function $f(t)$ is continuous on the interval $[0,T)$, then a solution $u(t)$ of problem (2.2), if it exists, is given by the formula*

$$u(t) = U(t)x_0 + \int_0^t U(t-s)f(s)\,ds \quad \text{for all } 0 < t < T. \tag{2.3}$$

The next theorem states that the function $u(t)$, defined by formula (2.3), is a solution of problem (2.2) (see Henry [11, Theorem 3.2.2], Pazy [14, Chapter 4, Corollary 3.3], Taira [18, Theorem 1.16]):

Theorem 2.5. *Assume that the function $f(t)$ is locally Hölder continuous on $(0,T)$ and satisfies the condition*

$$\int_0^T \|f(s)\|\,ds < \infty.$$

Then, for every $x_0 \in E$, the function $u(t)$, defined by formula (2.3), belongs to the space $C([0,T];E) \cap C^1((0,T);E)$, and is a unique solution of problem (2.2).

Secondly, we consider the *semilinear* case. Let $F(t,x)$ be a function defined on an open subset $\mathcal{U}$ of $[0,\infty) \times E_\alpha$, $0 < \alpha < 1$, taking values in E. Given $(t_0, x_0) \in \mathcal{U}$, find a function $u(t)$ such that

$$\begin{cases} \dfrac{du}{dt} = Au(t) + F(t, u(t)) & \text{for all } t_0 < t < t_1, \\ u(t_0) = x_0. \end{cases} \tag{2.4}$$

We assume that $F(t,x)$ is locally Hölder continuous in t and locally Lipschitz continuous in x. More precisely, for each point (t,x) of $\mathcal{U}$, there exist a neighborhood $\mathcal{V}$ of (t,x) in $\mathcal{U}$, constants $L = L(t,x,\mathcal{V}) > 0$ and $0 < \gamma \leq 1$ such that

$$\|F(s_1,y_1) - F(s_2,y_2)\| \leq L\left(|s_1 - s_2|^\gamma + \|y_1 - y_2\|_\alpha\right) \quad \text{for all } (s_1,y_1),(s_2,y_2) \in \mathcal{V}. \tag{2.5}$$

A function $u(t) : [t_0, t_1) \to E$ is called a *solution* of problem (2.4) if it satisfies the following three conditions:

(1) $u(t) \in C([t_0,t_1);E_\alpha) \cap C^1((t_0,t_1);E)$ and $u(t_0) = x_0$.

(2) $u(t) \in D(A)$ and $(t, u(t)) \in \mathcal{U}$ for all $t_0 < t < t_1$.
(3) $\frac{du}{dt} = Au(t) + F(t, u(t))$ for all $t_0 < t < t_1$.

Here $C([t_0, t_1); E_\alpha)$ denotes the space of continuous functions on $[t_0, t_1)$ taking values in E_α, and $C^1((t_0, t_1); E)$ denotes the space of continuously differentiable functions on (t_0, t_1) taking values in E, respectively.

After these preparations, we can state a *local* existence and uniqueness theorem for the semilinear Cauchy problem (2.4) (see Henry [11, Theorem 3.3.3], Pazy [14, Chapter 6, Theorem 3.1], Taira [18, Theorem 1.18]):

Theorem 2.6. *Assume that the function $F(t, x)$ satisfies condition (2.5). Then, for every $(t_0, x_0) \in \mathcal{U}$, there exists a constant $t_1 = t_1(t_0, x_0) > t_0$ such that problem (2.4) has a unique local solution $u(t)$ in the space $C([t_0, t_1]; E_\alpha) \cap C^1((t_0, t_1); E)$.*

3. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. To do this, we make use of the super-sub-solution method (see Pao [12], Sattinger [16]). More precisely, by constructing a suitable pair of super- and sub-solutions of problem (1.1) we can show that problem (1.1) has a unique non-negative, global solution $u(x, t)$ for any non-negative initial function $u_0(x) \in C^2(\overline{D})$ which satisfies the boundary condition $Bu_0 = 0$ on ∂D.

A non-negative function $w(x, t) \in C^{1,0}(\overline{D} \times [0, T]) \cap C^{2,1}(\overline{D} \times (0, T])$ is called a *super-solution* of problem (1.1) if it satisfies the conditions

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + A\right) w(x, t) - \lambda f(w) \geq 0 & \text{in } D \times (0, T), \\ Bw(x', t) \geq 0 & \text{on } \partial D \times [0, T), \\ w(x, 0) \geq u_0(x) & \text{in } D. \end{cases}$$

Similarly a non-negative function $v(x, t) \in C^{1,0}(\overline{D} \times [0, T]) \cap C^{2,1}(\overline{D} \times (0, T])$ is called a *sub-solution* of problem (1.1) if it satisfies the conditions

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + A\right) v(x, t) - \lambda f(v) \leq 0 & \text{in } D \times (0, T), \\ Bv(x', t) \leq 0 & \text{on } \partial D \times [0, T), \\ v(x, 0) \leq u_0(x) & \text{in } D. \end{cases}$$

3.1 The comparison principle.

We start with the following comparison principle for semilinear parabolic equations (cf. Amann [2, Lemma 4.4], Sattinger [16, Theorem 2.5.2]):

Lemma 3.1. *Let $u(x,t)$ be a positive solution of the initial boundary value problem*

$$(1.1)\qquad \begin{cases} \left(\dfrac{\partial}{\partial t} + A\right) u(x,t) - \lambda f(u) = 0 & \text{in } D \times (0,T), \\ Bu(x',t) = 0 & \text{on } \partial D \times [0,T), \\ u(x,0) = u_0(x) & \text{in } D. \end{cases}$$

If $v(x,t)$ and $w(x,t)$ are super- and sub-solutions of problem (1.1), respectively, then it follows that

$$v(x,t) \le u(x,t) \le w(x,t) \quad \text{in } D \times (0,T).$$

Proof. Let

$$\Phi(x,t) = u(x,t) - w(x,t),$$

and assume, to the contrary, that the set

$$\begin{aligned} \gamma &= \{(x,t) \in D \times (0,T) : \Phi(x,t) > 0\} \\ &= \{(x,t) \in D \times (0,T) : u(x,t) > w(x,t)\} \end{aligned}$$

is non-empty.

First, we have, by the mean value theorem,

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + A\right) \Phi(x,t) - \lambda \int_0^1 f'(\theta u(x,t) + (1-\theta) w(x,t))\, d\theta \cdot \Phi(x,t) \\ &= \lambda f(u) - \left(\frac{\partial w}{\partial t} + Aw(x,t)\right) - \lambda (f(u) - f(w)) \\ &= \lambda f(w) - \left(\frac{\partial w}{\partial t} + Aw(x,t)\right) \\ &\le 0 \quad \text{in } D \times (0,T). \end{aligned}$$

Here we notice that

$$f'(\xi) = \frac{m\varepsilon(1+\varepsilon\xi) + 1}{(1+\varepsilon\xi)^{2-m}} \exp\left[\frac{\xi}{1+\varepsilon\xi}\right],$$

so that

$$0 < f'(\xi) \leq \left(m+1+\sqrt{1+2m(1-m)}\right)\left(1+\sqrt{1+2m(1-m)}\right)^{1-m} \times \exp\left[\frac{1}{\varepsilon} - (1+\sqrt{1+2m(1-m)})\right]\varepsilon^{2-m} \quad \text{for all } \xi \geq 0.$$

On the other hand, it follows that

$$B\Phi(x',t) = Bu(x',t) - Bw(x',t) = -Bw(x',t) \leq 0 \quad \text{on } \partial D \times (0,T),$$
$$\Phi(x,0) \leq 0 \quad \text{in } D.$$

Now we may assume that there exists a point $(x'_0, t_0) \in \partial D \times (0,T)$ such that

$$\Phi(x'_0, t_0) = \sup_{\overline{D}\times(0,T)} \Phi(x,t) > 0. \tag{3.1}$$

Thus, applying the boundary point lemma (see [15, Chapter 3, Section 3, Theorem 7] and Remark (ii)) to our situation we obtain that

$$\frac{\partial \Phi}{\partial \boldsymbol{\nu}}(x'_0, t_0) > 0. \tag{3.2}$$

Hence it follows from assertions (3.1) and (3.2) that

$$B\Phi(x'_0, t_0) = \frac{\partial \Phi}{\partial \boldsymbol{\nu}}(x'_0, t_0) + b(x'_0)\Phi(x'_0, t_0) > 0, \quad (x'_0, t_0) \in \partial D \times (0,T).$$

However, this contradicts the boundary condition

$$B\Phi(x',t) \leq 0 \quad \text{on } \partial D \times (0,T).$$

Therefore, we have proved that the set γ is empty, that is,

$$u(x,t) \leq w(x,t) \quad \text{in } D \times (0,T).$$

Similarly, we can prove that

$$v(x,t) \leq u(x,t) \quad \text{in } D \times (0,T).$$

The proof of Lemma 3.1 is complete. □

3.3 Generation theorems for analytic semigroups.

First, we consider the elliptic problem (1.3) in the framework of Sobolev spaces of L^p style. We define the Sobolev space

$$W^{2,p}(D) = \text{the space of functions } v \in L^p(D) \text{ whose derivatives } D^\alpha v, \quad |\alpha| \leq 2, \text{ in the sense of distributions are in } L^p(D),$$

and associate with problem (1.3) an unbounded linear operator $\mathcal{A}$ from the Banach space $L^p(D)$ into itself as follows:

(a) The domain of definition $D(\mathcal{A})$ is the space

$$D(\mathcal{A}) = \left\{ v \in W^{2,p}(D) : Bv = \frac{\partial v}{\partial \boldsymbol{\nu}} + b(x')v = 0 \right\}.$$

(b) $\mathcal{A}v = -Av$ for all $v \in D(\mathcal{A})$.

Here Av and Bv are taken in the sense of *distributions*.

Our starting point is the following generation theorem for analytic semigroups in the L^p topology (see Pazy [14, Chapter 7, Theorem 3.5], Taira [18, Theorem 2]):

Theorem 3.2. *(i) The resolvent set of* $\mathcal{A}$ *contains the set* $\Sigma = \{\zeta \in \mathbf{C} : \mathrm{Im}\,\zeta \neq 0\} \cup \{\zeta \in \mathbf{R} : \zeta > -\lambda_1\}$ *where* λ_1 *is the first eigenvalue of the Robin eigenvalue problem (1.7) (see Figure 3.1 below).*

(ii) The operator $\mathcal{A}$ *generates a semigroup* $e^{z\mathcal{A}}$ *on the space* $L^p(D)$ *which is* analytic *in the half-plane* $\{z = t + is : t > 0\}$.

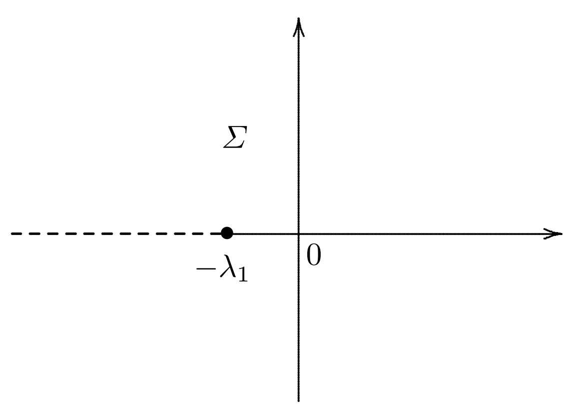

Figure 3.1

Secondly, we consider the elliptic problem (1.3) in the framework of the Banach space $C(\overline{D})$. We introduce a linear operator $\mathfrak{A}$ from $C(\overline{D})$ into itself as follows:

(a) The domain of definition $D(\mathfrak{A})$ is the set

$$D(\mathfrak{A}) = \left\{ v \in C(\overline{D}) : Av \in C(\overline{D}),\ Bv = \frac{\partial v}{\partial \boldsymbol{\nu}} + b(x')v = 0 \right\}.$$

(b) $\mathfrak{A}v = -Av$ for all $v \in D(\mathfrak{A})$.

Here Av and Bv are taken in the sense of *distributions*.

Then we have the following generation theorem for analytic semigroups in the topology of uniform convergence (see Taira [17, Theorem 3]):

Theorem 3.3. *(i) The resolvent set of $\mathfrak{A}$ contains the set Σ.*

(ii) The operator $\mathfrak{A}$ generates a contraction semigroup $e^{z\mathfrak{A}}$ on the space $C(\overline{D})$ which is analytic *in the half-plane $\{z = t + is : t > 0\}$. Moreover, the operators $e^{t\mathfrak{A}}$ are* non-negative *and* contractive *on the space $C(\overline{D})$:*

$$v \in C(\overline{D}),\ 0 \leq v \leq 1 \quad \text{on } \overline{D} \implies 0 \leq e^{t\mathfrak{A}}v \leq 1 \quad \text{on } \overline{D}. \tag{3.3}$$

It should be noticed that the following two commutative diagrams hold true for the operators $\mathcal{A}$, $\mathfrak{A}$ and the semigroups $e^{t\mathcal{A}}$, $e^{t\mathfrak{A}}$, respectively:

$$\begin{array}{ccc} D(\mathcal{A}) & \xrightarrow{\mathcal{A}} & L^p(D) \\ \uparrow & & \downarrow \\ D(\mathfrak{A}) & \xrightarrow{\mathfrak{A}} & C(\overline{D}) \end{array}$$

$$\begin{array}{ccc} L^p(D) & \xrightarrow{e^{t\mathcal{A}}} & L^p(D) \\ \uparrow & & \downarrow \\ C(\overline{D}) & \xrightarrow{e^{t\mathfrak{A}}} & C(\overline{D}) \end{array}$$

By using the operator $\mathcal{A}$, we can formulate problem (1.1) as an *abstract Cauchy problem* in $L^p(D)$ in the following form:

$$\begin{cases} \dfrac{du}{dt} = \mathcal{A}u(t) + \lambda F(u(t)) \quad \text{for all } 0 < t < T, \\ u|_{t=0} = u_0. \end{cases} \tag{3.4}$$

Here $u(t) = u(\cdot, t)$ and $F(u)$ is the *Nemytskii operator* defined by the formula

$$F(u(t)) = f(u(\cdot,t)) = (1 + \varepsilon u(\cdot,t))^m \exp\left[\frac{u(\cdot,t)}{1+\varepsilon u(\cdot,t)}\right].$$

If $0 < \alpha < 1$, we can define the negative fractional power $(-\mathcal{A})^{-\alpha}$ of $-\mathcal{A}$ by the formula

$$(-\mathcal{A})^{-\alpha} = \frac{\sin \alpha\pi}{\pi} \int_0^\infty s^{-\alpha}(-\mathcal{A} + sI)^{-1}\, ds,$$

and also the positive fractional power $(-\mathcal{A})^{\alpha}$ by the formula

$$(-\mathcal{A})^{\alpha} = \text{the inverse of } (-\mathcal{A})^{-\alpha}.$$

We let

$$\mathcal{X} = L^p(D), \quad p > N,$$

and

$$\mathcal{X}_\alpha = \text{the domain } D((-\mathcal{A})^\alpha) \text{ endowed with the graph norm } \|\cdot\|_\alpha \text{ of } (-\mathcal{A})^\alpha,$$

where

$$\|v\|_\alpha = \left(\|v\|^2 + \|(-\mathcal{A})^\alpha v\|^2\right)^{1/2} \quad \text{for all } v \in D((-\mathcal{A})^\alpha).$$

It is worth pointing out here that

$$D(\mathcal{A}) \subset \mathcal{X}_\alpha \subset C^{1+\theta}(\overline{D}), \quad 0 \le \theta < 2\alpha - \frac{N}{p} - 1,$$

if we take $p > N$ and $(1/2) + (N/2p) < \alpha < 1$ (see Henry [11, Theorem 1.6.1], Pazy [14, Chapter 8, Theorem 4.3], Taira [18, Theorem 7.1]):

Our main result is the following *global* existence and uniqueness theorem of positive solutions for the semilinear Cauchy problem (3.4):

Theorem 3.4. *Let $p > N$ and $(1/2) + (N/2p) < \alpha < 1$. For every non-negative function $u_0 \in D(\mathcal{A})$, the Cauchy problem (3.4) has a unique non-negative,* global *solution $u(t) \in C([0,\infty); \mathcal{X}_\alpha) \cap C^1((0,\infty); \mathcal{X})$.*

The proof of Theorem 3.4 will be given in the next subsection due to its length.

End of Proof of Theorem 1. By using the Schauder theory for linear parabolic differential equations just as in the proof of Amann [2, Lemma 4.2], we can prove that every solution $u(t) \in C([0,\infty); \mathcal{X}_\alpha) \cap C^1((0,\infty); \mathcal{X})$ of problem (3.4) belongs to the space

$$C^{1,0}(\overline{D} \times [0,\infty)) \cap C^{2,1}(\overline{D} \times (0,\infty))$$

if $p > N$ and $(1/2) + (N/2p) < \alpha < 1$. □

3.3 Proof of Theorem 3.4.

To apply Theorem 2.6 to the semilinear Cauchy problem (3.4), we modify the nonlinear term $f(\xi)$ as follows:

$$\tilde{f}(\xi) = \begin{cases} (1+\varepsilon\xi)^m \exp\left[\frac{\xi}{1+\varepsilon\xi}\right] & \text{if } \xi \geq 0, \\ (1+m\varepsilon)\xi + 1 & \text{if } \xi < 0. \end{cases}$$

Namely, we retain $f(\xi)$ in $\xi \geq 0$ and continue $f(\xi)$ into $\xi < 0$ by its tangent line at $\xi = 0$. It is easy to verify that the function $\tilde{f}(\xi)$ is continuously differentiable on the whole line $\mathbf{R}$, and satisfies the condition

$$\begin{aligned} 0 &< \tilde{f}'(\xi) \\ &\leq \left(m+1+\sqrt{1+2m(1-m)}\right)\left(1+\sqrt{1+2m(1-m)}\right)^{1-m} \\ &\quad \times \exp\left[\frac{1}{\varepsilon} - (1+\sqrt{1+2m(1-m)})\right]\varepsilon^{2-m} \quad \text{for all } \xi \in \mathbf{R}. \end{aligned}$$

Thus we find that Theorem 2.6 holds true for the modified nonlinear term $\tilde{F}(u(t)) = \tilde{f}(u(t))$. However, it follows from an application of the comparison principle (see the proof of Claim 3.1) that all positive solutions $u(x,t)$ of the new equation satisfy the estimates

$$0 \leq u(x,t) \leq \lambda C_m \phi(x) \quad \text{on } \overline{D} \times [0,T),$$

and hence they are solutions of the original equation. Therefore, this change introduces *no* extra positive solutions and it will be convenient in our later work. We state once and for all that throughout the remainder of the paper we are replacing $f(\xi)$ by $\tilde{f}(\xi)$.

Now let $u_0(x)$ be an element of $D(\mathcal{A})$ such that

$$u_0(x) \geq 0 \quad \text{in } D, \tag{3.5}$$

and assume, to the contrary, that

There exists a finite *time T such that the interval $[0, T)$ is the maximal interval of existence of solutions of problem (1.11).*

Step I: First, we prove an *a priori* estimate for the uniform norm of the solution $u(x, t)$:

Claim 3.1. *There exists a constant $\kappa > 0$ such that*

$$\|u(\cdot, t)\|_\infty \leq \kappa\lambda\|\phi\|_\infty \quad \textit{for all } 0 \leq t < T. \tag{3.6}$$

Proof. Since the functions $u_0(x)$ and $\phi(x)$ belong to the domain $D(\mathcal{A})$, we can choose a constant $\kappa_0 > 0$ such that

$$0 \leq u_0(x) \leq \kappa_0 \lambda e^{1/\varepsilon} \phi(x) \quad \text{on } \overline{D}.$$

If we let

$$\kappa = \max\{\kappa_0, C_m\},$$

where C_m is the unique solution of equation (1.10), then it follows that the function

$$w(x) = \kappa\lambda\phi(x)$$

is a super-solution of problem (1.1). Indeed, it suffices to note that

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + A\right) w &= Aw \\
&= \kappa\lambda \\
&\geq \lambda \left(1 + \lambda\varepsilon\|\phi\|_\infty \kappa\right)^m e^{1/\varepsilon} \\
&\geq \lambda \left(1 + \lambda\varepsilon\kappa\phi(x)\right)^m \exp\left[\frac{\lambda\kappa\phi(x)}{1 + \lambda\varepsilon\kappa\phi(x)}\right]
\end{aligned}$$

$$= \lambda f(w) \quad \text{in } D \times (0,T),$$

and that

$$Bw = \kappa\lambda\, B\phi = 0 \quad \text{on } \partial D \times (0,T).$$

Therefore, applying Lemma 3.1 with $v(x,t) := 0$ and $w(x,t) := w(x)$ we obtain that

$$0 \leq u(x,t) \leq \kappa\lambda\phi(x) \quad \text{on } \overline{D} \times [0,T).$$

This proves that the function $u(t) = u(\cdot,t)$ satisfies the *a priori* estimate (3.6). □

Step II: Secondly, we prove an *a priori* estimate for the α-norm of the solution $u(t) = u(\cdot,t)$:

Claim 3.2. *There exists a constant $C_1 = C_1(u_0,T) > 0$ such that*

$$\|u(t)\|_\alpha \leq C_1 \quad \textit{for all } 0 \leq t < T. \tag{3.7}$$

Proof. It follows from an application of Theorem 2.4 that the solution $u(t)$ can be expressed in the form

$$u(t) = T(t)u_0 + \lambda \int_0^t T(t-s)F(u(s))\,ds, \tag{3.8}$$

where

$$\begin{aligned} T(t) &= e^{t\mathcal{A}}, \\ F(u(t)) &= f(u(t)). \end{aligned}$$

Hence we have, by condition (3.5),

$$u(t) \geq 0 \quad \text{for all } 0 \leq t < T. \tag{3.9}$$

Indeed, it suffices to note the following three facts:

(1) The operators $T(t)$, restricted to $C(\overline{D})$, are non-negative and contractive (see assertion (3.3)):

$$v \in C(\overline{D}),\ 0 \leq v \leq 1 \implies 0 \leq T(t)v \leq 1. \tag{3.10}$$

(2) The nonlinear term $F(u)$ satisfies the condition

$$u \geq 0 \implies F(u) \geq 0. \tag{3.11}$$

(3) By assertions (3.10) and (3.11), we find from the proof of Theorem 2.6 (the method of successive approximation) that

$$u_0 \geq 0 \implies u(t) = T(t)u_0 + \lambda \int_0^t T(t-s)F(u(s))\,ds \geq 0.$$

By applying the closed operator $(-\mathcal{A})^\alpha$ to the both sides of formula (3.8), we obtain from part (b) of Theorem 2.3 that

$$\begin{aligned}(-\mathcal{A})^\alpha u(t) &= (-\mathcal{A})^\alpha T(t)u_0 + \lambda \int_0^t (-\mathcal{A})^\alpha T(t-s)F(u(s))\,ds \\ &= T(t)(-\mathcal{A})^\alpha u_0 + \lambda \int_0^t (-\mathcal{A})^\alpha T(t-s)F(u(s))\,ds.\end{aligned}$$

Hence we have, by part (c) of Theorem 2.3, assertion (3.9) and estimate (3.6),

$$\begin{aligned}\|u(t)\|_\alpha &\leq \|T(t)(-\mathcal{A})^\alpha u_0\| \\ &\quad + \lambda \int_0^t \|(-\mathcal{A})^\alpha T(t-s)\| \cdot \|F(u(s))\|\,ds, \\ &\leq \|u_0\|_\alpha + \lambda c(\alpha)(1+\varepsilon\lambda\kappa\|\phi\|_\infty)^m\, e^{1/\varepsilon} \int_0^t \frac{1}{(t-s)^\alpha}\,ds \\ &\leq \|u_0\|_\alpha + \lambda c(\alpha)(1+\varepsilon\lambda\kappa\|\phi\|_\infty)^m\, e^{1/\varepsilon} \frac{T^{1-\alpha}}{1-\alpha} \quad \text{for all } 0 \leq t < T.\end{aligned}$$

This proves assertion (3.7). □

Step III: Furthermore, we need the following estimate:

Claim 3.3. *Let t_1 be sufficiently close to T. Then, for any $\beta \in (\alpha, 1)$ there exists a constant $C_2 = C_2(u_0, t_1, T) > 0$ such that*

$$\|u(t)\|_\beta \leq C_2 \quad \textit{for all } t_1 \leq t < T. \tag{3.12}$$

Proof. By formula (3.8), we have, for $t_1 \leq t < T$,

$$\begin{aligned}\|u(t)\|_\beta &\leq \|(-\mathcal{A})^{\beta-\alpha} T(t)(-\mathcal{A})^\alpha u_0\| \\ &\quad + \lambda \int_0^t \|(-\mathcal{A})^\beta T(t-s)\| \cdot \|f(u(s))\| ds \\ &\leq c(\alpha,\beta)\, \frac{1}{t^{\beta-\alpha}}\, \|u_0\|_\alpha + \lambda c(\beta) \cdot \sup_{0 \leq s \leq T} \|F(u(s))\| \cdot \frac{t^{1-\beta}}{1-\beta} \\ &\leq \frac{c(\alpha,\beta)}{t_1^{\beta-\alpha}}\, \|u_0\|_\alpha + \lambda c(\beta)(1+\varepsilon\lambda\kappa\|\phi\|_\infty)^m\, e^{1/\varepsilon}\, \frac{T^{1-\beta}}{1-\beta}.\end{aligned}$$

This proves Claim 3.3. □

Step IV: Finally, we can prove the following claim:

Claim 3.4. *The limit* $\lim_{t\uparrow T} u(t)$ *exists in the space* $\mathcal{X}_\alpha$.

Proof. Since we have, for $t_1 \leq \tau < t < T$,

$$\begin{aligned}u(t) - u(\tau) &= T(t-\tau)u(\tau) + \lambda \int_\tau^t T(t-s)F(u(s))\, ds - u(\tau) \\ &= [T(t-\tau) - T(0)]\, u(\tau) + \lambda \int_\tau^t T(t-s)F(u(s))\, ds,\end{aligned}$$

it follows that

$$\begin{aligned}\|u(t) - u(\tau)\|_\alpha &\leq \|(-\mathcal{A})^\alpha\, [T(t-\tau) - T(0)]\, u(\tau)\| \\ &\quad + \lambda \int_\tau^t \|(-\mathcal{A})^\alpha T(t-s)F(u(s))\|\, ds.\end{aligned} \tag{3.13}$$

However, for $\alpha < \beta < 1$ and $0 < \gamma < \beta - \alpha$, applying part (d) of Theorem 2.3 we obtain that

$$\begin{aligned}&\left\|(-\mathcal{A})^\alpha\, [T(t-\tau) - T(0)]\, (-\mathcal{A})^{-\beta}\right\| \\ &= \left\|(-\mathcal{A})^{-(\beta-\alpha-\gamma)} \cdot (-\mathcal{A})^{-\gamma}\, [T(t-\tau) - T(0)]\right\| \\ &\leq c(\alpha,\beta,\gamma)|t-\tau|^\gamma.\end{aligned}$$

Hence we have

$$\begin{aligned} (3.14) \qquad & \|(-\mathcal{A})^{\alpha}\,[T(t-\tau)-T(0)]\,u(\tau)\| \\ & \le \|(-\mathcal{A})^{\alpha}\,[T(t-\tau)-T(0)]\,(-\mathcal{A})^{-\beta}\| \cdot \|(-\mathcal{A})^{\beta}u(\tau)\| \\ & \le c(\alpha,\beta,\gamma)|t-\tau|^{\gamma}\|u(\tau)\|_{\beta}. \end{aligned}$$

On the other hand, it follows from part (c) of Theorem 2.3 and estimate (3.6) that

$$\begin{aligned} (3.15) \qquad & \int_{\tau}^{t} \|(-\mathcal{A})^{\alpha}T(t-s)F(u(s))\|\,ds \\ & \le c(\alpha)(1+\varepsilon\lambda\kappa\|\phi\|_{\infty})^{m}\,e^{1/\varepsilon}\int_{\tau}^{t}\frac{1}{(t-s)^{\alpha}}\,ds \\ & = \frac{c(\alpha)}{1-\alpha}(1+\varepsilon\lambda\kappa\|\phi\|_{\infty})^{m}\,e^{1/\varepsilon}(t-\tau)^{1-\alpha}. \end{aligned}$$

Thus, carrying inequalities (3.14) and (3.15) into the right-hand side of inequality (3.13) we obtain from estimate (3.12) that, for $t_1 \le \tau < t < T$,

$$\begin{aligned} (3.16) \qquad \|u(t)-u(\tau)\|_{\alpha} & \le c(\alpha,\beta,\gamma)\cdot|t-\tau|^{\gamma}\cdot\sup_{t_1\le s<T}\|u(s)\|_{\beta} \\ & \qquad + \lambda\,\frac{c(\alpha)}{1-\alpha}(1+\varepsilon\lambda\kappa\|\phi\|_{\infty})^{m}\,e^{1/\varepsilon}\,|t-\tau|^{1-\alpha} \\ & \le c(\alpha,\beta,\gamma)C_2|t-\tau|^{\gamma} \\ & \qquad + \lambda\,\frac{c(\alpha)}{1-\alpha}(1+\varepsilon\lambda\kappa\|\phi\|_{\infty})^{m}\,e^{1/\varepsilon}\,|t-\tau|^{1-\alpha}. \end{aligned}$$

Therefore, we find from estimate (3.16) that the limit $\lim_{t\uparrow T} u(t)$ exists in the Banach space $\mathcal{X}_{\alpha}$. □

Step V: By Claim 3.4, we can apply Theorem 2.6 to extend the solution $u(t)$ beyond the maximal time T. This contradiction proves Theorem 3.4. □

4. Proof of Theorems 3 and 4

In this section we prove Theorems 3 and 4, by using Sattinger's stability theorem. The idea of proof is stated as follows. If the initial function $u_0(x)$

in the parabolic problem (1.1) is a super-solution or a sub-solution of the corresponding elliptic problem (1.3), then it can be shown that the time-dependent solution $u(\cdot, t)$ is monotone in t and converges to a steady-state solution $v(\lambda)$. This monotone convergence property gives a close relationship between the stability and the uniqueness property of positive solutions of the elliptic problem (1.3).

Now we formulate the stability of solutions of problem (1.3) in terms of super-solutions and sub-solutions.

A non-negative function $\varphi(x) \in C^2(\overline{D})$ is called a *super-solution* of problem (1.3) if it satisfies the conditions

$$\begin{cases} A\varphi - \lambda f(\varphi) \geq 0 & \text{in } D, \\ B\varphi \geq 0 & \text{on } \partial D. \end{cases}$$

Similarly, a non-negative function $\psi(x) \in C^2(\overline{D})$ is called a *sub-solution* of problem (1.3) if it satisfies the conditions

$$\begin{cases} A\psi - \lambda f(\psi) \leq 0 & \text{in } D, \\ B\psi \leq 0 & \text{on } \partial D. \end{cases}$$

The next theorem is a basic tool for determining the stability property of the maximal and minimal solutions when there are multiple steady-state solutions (see Pao [12, Chapter 5, Theorem 4.4], Sattinger [16, Theorem 2.6.2]):

Theorem 4.1. *Let $\widehat{v}(x)$ be a positive solution of problem (1.3) and let $\varphi(x)$, $\psi(x)$ be respectively sub- and super-solutions of problem (1.3) such that*

$$\psi(x) \leq \widehat{v}(x) \leq \varphi(x) \quad \text{on } \overline{D}.$$

Then all solutions $u(x,t)$ of the parabolic problem (1.1) with initial values $u_0 \in [\psi, \varphi]$ converge uniformly to $\widehat{v}(x)$ as $t \to +\infty$ if and only if the uniqueness *of positive solutions of the elliptic problem (1.3) holds true in the order interval*

$$[\psi, \varphi] = \left\{ v \in C^2(\overline{D}) : \psi(x) \leq v(x) \leq \varphi(x) \ \text{ on } \overline{D} \right\}.$$

4.1 Proof of Theorem 3. Let $u_0(x)$ be an arbitrary, non-negative function in the domain $D(\mathcal{A})$. Then, just as in the proof of Claim 3.1 we can construct a super-solution $\lambda C\phi(x)$ such that

$$0 \leq u_0(x) \leq \lambda C\phi(x) \quad \text{on } \overline{D},$$

if we take $C \geq C_m$, where C_m is the unique solution of equation (1.10). It is clear that the function $\psi(x) \equiv 0$ is a sub-solution of problem (1.3).

Therefore, Theorem 3 follows from an application of Theorem 4.1 with

$$\widehat{v}(x) := v(x), \quad \psi(x) := 0, \quad \varphi(x) := \lambda C\phi(x).$$

Indeed, it suffices to note that the uniqueness of positive solutions of problem (1.3) holds true for each $\lambda > 0$ if the parameter ε satisfies condition (1.4). □

4.2 Proof of Theorem 4. If either $0 < \lambda < \mu_E$ or $\lambda > \mu_I$, then, by the comparison principle (see Lemma 3.1) it follows that a unique positive solution $v(x)$ of problem (1.3) satisfies the inequalities

$$0 \leq v(x) \leq \lambda C_m\, \phi(x) \quad \text{on } \overline{D}.$$

Theorem 4 is an immediate consequence of Theorem 4.1 with

$$\widehat{v}(x) := v(x), \quad \psi(x) := 0, \quad \varphi(x) := \lambda C_m\, \phi(x).$$

The Proof of Theorem 4.1 is complete. □

5. Concluding Remarks

The purpose of this section is to study the (local) asymptotic stability of maximal and minimal positive solutions of problem (1.3) in terms of the *size* of initial values $u_0(x)$ of problem (1.1) with respect to the solution $\phi(x)$ of problem (1.6).

5.1 Stability theorems for maximal and minimal positive solutions.

The stability of positive solutions of problem (1.3) with $m = 0$ was studied by Wiebers [25]. He considered the linearized eigenvalue problem at a positive solution $v(\lambda)$ of problem (1.3) with $m = 0$

$$\begin{cases} Aw - \lambda f'(v(\lambda))w = \mu w & \text{in } D, \\ Bw = \dfrac{\partial w}{\partial \boldsymbol{\nu}} + b(x')w = 0 & \text{on } \partial D, \end{cases}$$

and proved that the first eigenvalue $\mu_1(v(\lambda))$ is positive if $v(\lambda)$ is unique (see [25, Theorems 2.6 and 2.9]), and further that the first eigenvalues $\mu_1(\overline{v}(\lambda))$ and $\mu_1(\underline{v}(\lambda))$ are both non-negative (see [25, Corollary 1.4 and Proposition 1.2]).

Our first result asserts the (local) asymptotic stability of minimal positive solutions of problem (1.3) in the case where λ is sufficiently small (see Pao [12, Chapter 5, Theorem 4.3]):

Theorem 5. *Let* $0 < \varepsilon < (1/(1+\sqrt{1-m}))^2$. *Then the minimal positive solution* $\underline{v}(\lambda)$ *of problem (1.3) is* asymptotically stable *if* λ *is so small that*

$$0 < \lambda \leq \frac{\nu(t_1(\varepsilon))}{\max_{\overline{D}} \phi}. \tag{5.1}$$

More precisely, any global solution $u(x,t)$ *of problem (1.1) with an initial value* $u_0(x) \in D(\mathcal{A})$, *which satisfies the condition*

$$0 \leq u_0(x) \leq \frac{t_1(\varepsilon)}{\max_{\overline{D}} \phi} \phi(x) \quad \text{on } \overline{D},$$

converges uniformly to the minimal steady-state solution $\underline{v}(\lambda)$ *as* $t \to +\infty$.

Secondly, we study the asymptotic stability of maximal positive solutions of problem (1.3) in the case where λ is sufficiently large. To do this, we notice that

$$\min_{\overline{D}} \phi > 0.$$

Indeed, it follows from a simple application of the strong maximum principle and the boundary point lemma (see Taira [19, Lemma 2.7]) that the solution $\phi(x)$ of the Robin problem (1.6) is strictly positive on $\overline{D}$.

Then we have the following (local) asymptotic stability of maximal positive solutions of problem (1.3) for λ sufficiently large (cf. Pao [12, Chapter 5, Theorem 4.3]):

Theorem 6. $0 < \varepsilon < (1/(1+\sqrt{1-m}))^2$. *Then the maximal positive solution* $\overline{v}(\lambda)$ *of problem (1.3) is* asymptotically stable *if* λ *is so large that*

$$\lambda \geq \frac{\nu(t_2(\varepsilon))}{\min_{\overline{D}} \phi}.$$

More precisely, any global solution $u(x,t)$ of problem (1.1) with an initial value $u_0(x) \in D(\mathcal{A})$, which satisfies the condition

$$u_0(x) \geq \frac{t_2(\varepsilon)}{\min_{\overline{D}} \phi} \phi(x) \quad on\ \overline{D},$$

converges uniformly to the maximal steady-state solution $\overline{v}(\lambda)$ as $t \to +\infty$.

Finally, we consider the case where the Arrhenius activation energy is so high that

$$0 < \varepsilon \ll (1/(1+\sqrt{1-m}))^2$$

as in part (iii) of Theorem 1. By combining Theorems 5 and 6, we can obtain the following (local) asymptotic stability theorem for maximal and minimal positive solutions of problem (1.3) for λ in an interval as in Figure 1.6:

Theorem 7. *If $\varepsilon > 0$ is so small that*

$$\frac{\nu(t_2(\varepsilon))}{\min_{\overline{D}} \phi} < \frac{\nu(t_1(\varepsilon))}{\max_{\overline{D}} \phi}, \tag{5.2}$$

then, for each λ such that

$$\frac{\nu(t_2(\varepsilon))}{\min_{\overline{D}} \phi} \leq \lambda \leq \frac{\nu(t_1(\varepsilon))}{\max_{\overline{D}} \phi},$$

any global solution $u(x,t)$ of problem (1.1) with an initial value $u_0(x) \in D(\mathcal{A})$, which satisfies the condition

$$0 \leq u_0(x) \leq \frac{t_1(\varepsilon)}{\max_{\overline{D}} \phi} \phi(x) \quad on\ \overline{D},$$

converges uniformly to the minimal steady-state solution $\underline{v}(\lambda)$ as $t \to +\infty$.

On the other hand, any global solutions $u(x,t)$ of problem (1.1) with an initial value $u_0(x) \in D(\mathcal{A})$, which satisfies the condition

$$u_0(x) \geq \frac{t_2(\varepsilon)}{\min_{\overline{D}} \phi} \phi(x) \quad on\ \overline{D},$$

converges uniformly to the maximal steady-state solution $\overline{v}(\lambda)$ as $t \to +\infty$.

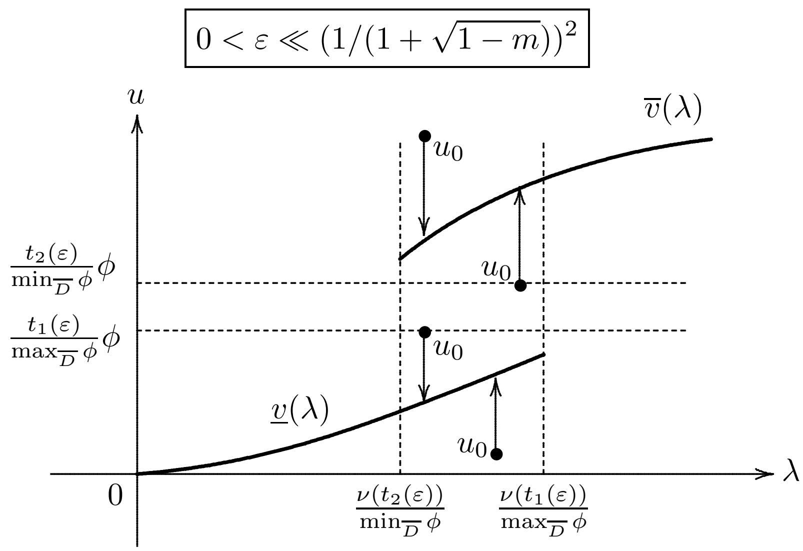

Figure 5.1

Remark 5.1. For condition (5.2), we notice that (see estimate (A.2)):

$$\frac{\nu(t_2(\varepsilon))}{\beta} \leq \frac{\nu(t_2(\varepsilon))}{\min_{\overline{D}} \phi}.$$

The situation of Theorems 5, 6 and 7 may be represented schematically by Figure 5.1.

5.2 Proof of Theorem 5. To construct a super-solution of problem (1.3), we let

$$\varphi(x) = \frac{t_1(\varepsilon)}{\max_{\overline{D}} \phi}\phi(x).$$

If λ is so small that

$$0 < \lambda \leq \frac{\nu(t_1(\varepsilon))}{\max_{\overline{D}} \phi}, \tag{5.1}$$

then we have

$$B\varphi = \frac{t_1(\varepsilon)}{\max_{\overline{D}} \phi}\, B\phi = 0 \quad \text{on } \partial D,$$

and also

$$\begin{aligned} A\varphi - \lambda f(\varphi) &\geq \frac{t_1(\varepsilon)}{\max_{\overline{D}} \phi} - \frac{\nu(t_1(\varepsilon))}{\max_{\overline{D}} \phi} f\left(\frac{t_1(\varepsilon)}{\max_{\overline{D}} \phi}\phi\right) \\ &= \frac{t_1(\varepsilon)}{\max_{\overline{D}} \phi}\left(1 - \frac{f\left(\frac{t_1(\varepsilon)}{\max_{\overline{D}} \phi}\phi\right)}{f(t_1(\varepsilon))}\right) \\ &\geq 0 \quad \text{in } D, \end{aligned}$$

since the function $f(t)$ is strictly increasing for all $t \geq 0$. This implies that $\varphi(x)$ is a super-solution of (1.3) if λ satisfies condition (5.1). Hence, by using the super-sub-solution method we can construct a positive solution $\widehat{v}(\lambda)$ of problem (1.3) such that

$$0 \leq \widehat{v}(\lambda) \leq \frac{t_1(\varepsilon)}{\max_{\overline{D}} \phi}\phi(x) \quad \text{on } \overline{D}.$$

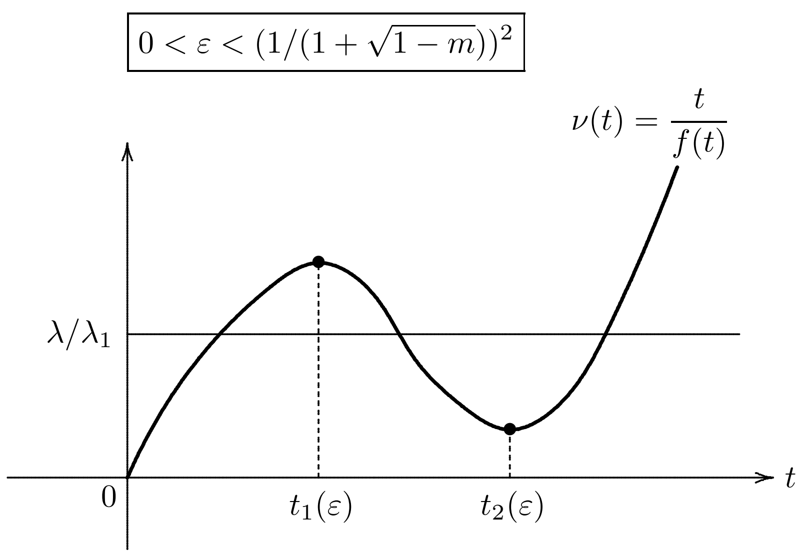

Figure 5.2

Now it should be noticed that the scalar equation, called the *Semenov approximation* in combustion theory,

$$\lambda = \lambda_1 \nu(t) = \lambda_1 \frac{t}{f(t)}$$

has at most one solution in the interval $[0, t_1(\varepsilon)]$ if λ satisfies the condition

$$0 \leq \lambda/\lambda_1 \leq \nu(t_1(\varepsilon))$$

(see Figure 5.2).

A similar assertion holds true for problem (1.3). In fact, we have the following uniqueness result for problem (1.3) (cf. the proof of Taira [19, Theorem 5]):

Lemma 5.1. *Let* $0 < \varepsilon < (1/(1+\sqrt{1-m}))^2$. *Then problem (1.3) has* at most one *positive solution in the order interval*

$$[0, t_1(\varepsilon)] = \{v \in C^2(\overline{D}) : 0 \leq v(x) \leq t_1(\varepsilon) \text{ on } \overline{D}\}$$

for $\lambda > 0$.

Proof. Assume that $v_i(x)$, $i = 1, 2$, are two positive solutions of problem (1.3)

$$\begin{cases} Av_i = \lambda f(v_i) & \text{in } D, \\ Bv_i = \dfrac{\partial v_i}{\partial \boldsymbol{\nu}} + b(x')v_i = 0 & \text{on } \partial D \end{cases}$$

which satisfy the conditions

$$0 \leq v_1(x),\, v_2(x) \leq t_1(\varepsilon) \quad \text{on } \overline{D}.$$

Then we have, by Green's formula,

(5.3)

$$\begin{aligned} &\lambda \int_D \left(\frac{f(v_1)}{v_1} - \frac{f(v_2)}{v_2} \right) \left(v_1^2 - v_2^2 \right) dx \\ &= \int_D \left(\frac{Av_1}{v_1} - \frac{Av_2}{v_2} \right) \left(v_1^2 - v_2^2 \right) dx \end{aligned}$$

$$
\begin{aligned}
&= -\int_D \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\sum_{j=1}^N a^{ij}(x)\frac{\partial v_1}{\partial x_j}\right) v_1\,dx \\
&\quad + \int_D \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\sum_{j=1}^N a^{ij}(x)\frac{\partial v_1}{\partial x_j}\right)\left(\frac{v_2^2}{v_1}\right) dx \\
&\quad - \int_D \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\sum_{j=1}^N a^{ij}(x)\frac{\partial v_2}{\partial x_j}\right) v_2\,dx \\
&\quad + \int_D \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\sum_{j=1}^N a^{ij}(x)\frac{\partial v_2}{\partial x_j}\right)\left(\frac{v_1^2}{v_2}\right) dx \\
&= \int_D \sum_{i,j=1}^N a^{ij}(x)\frac{\partial v_1}{\partial x_i}\frac{\partial v_1}{\partial x_j}\,dx - \int_D \sum_{i,j=1}^N a^{ij}(x)\frac{\partial v_1}{\partial x_i}\frac{\partial}{\partial x_j}\left(\frac{v_2^2}{v_1}\right) dx \\
&\quad - \int_{\partial D} \frac{\partial v_1}{\partial \boldsymbol{\nu}} v_1\,d\sigma + \int_{\partial D} \frac{\partial v_1}{\partial \boldsymbol{\nu}}\left(\frac{v_2^2}{v_1}\right) d\sigma \\
&\quad + \int_D \sum_{i,j=1}^N a^{ij}(x)\frac{\partial v_2}{\partial x_i}\frac{\partial v_2}{\partial x_j}\,dx - \int_D \sum_{i,j=1}^N a^{ij}(x)\frac{\partial v_2}{\partial x_i}\frac{\partial}{\partial x_j}\left(\frac{v_1^2}{v_2}\right) dx \\
&\quad - \int_{\partial D} \frac{\partial v_2}{\partial \boldsymbol{\nu}} v_2\,d\sigma + \int_{\partial D} \frac{\partial v_2}{\partial \boldsymbol{\nu}}\left(\frac{v_1^2}{v_2}\right) d\sigma,
\end{aligned}
$$

where $d\sigma$ is the surface element of ∂D. However, we find that the four integrals over ∂D in the last equality of formula (5.3) vanish, since the solutions $v_1(x)$ and $v_2(x)$ satisfy the boundary conditions

$$
\begin{pmatrix} \dfrac{\partial v_1}{\partial \boldsymbol{\nu}} & v_1 \\ \dfrac{\partial v_2}{\partial \boldsymbol{\nu}} & v_2 \end{pmatrix}\begin{pmatrix} 1 \\ b(x') \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on } \partial D.
$$

Therefore, it follows from formula (5.3) that

$$
\begin{aligned}
&\lambda \int_D \left(\frac{f(v_1)}{v_1} - \frac{f(v_2)}{v_2}\right)\left(v_1^2 - v_2^2\right) dx \\
&= \int_D \sum_{i,j=1}^N a^{ij}(x)\left(\frac{\partial v_1}{\partial x_i} - \frac{v_2}{v_1}\frac{\partial v_1}{\partial x_i}\right)\left(\frac{\partial v_1}{\partial x_j} - \frac{v_2}{v_1}\frac{\partial v_1}{\partial x_j}\right) dx
\end{aligned}
$$

$$+\int_D \sum_{i,j=1}^N a^{ij}(x)\left(\frac{\partial v_2}{\partial x_i} - \frac{v_1}{v_2}\frac{\partial v_2}{\partial x_i}\right)\left(\frac{\partial v_2}{\partial x_j} - \frac{v_1}{v_2}\frac{\partial v_2}{\partial x_j}\right)dx$$
$$\geq 0.$$

This implies that $v_1(x) \equiv v_2(x)$ on $\overline{D}$, since the function $f(t)/t$ is strictly decreasing for all $0 < t \leq t_1(\varepsilon)$.

The proof of Lemma 5.1 is complete. □

Lemma 5.1 asserts that the positive solution $\widehat{v}(x)$ is unique in the order interval

$$\left[0, \frac{t_1(\varepsilon)}{\max_{\overline{D}} \phi}\phi\right],$$

so that it is minimal. Therefore, it follows from an application of Theorem 4.1 with

$$\widehat{v}(x) := \underline{v}(x), \quad \psi(x) := 0, \quad \varphi(x) := (t_1(\varepsilon)/\max_{\overline{D}} \phi)\phi(x)$$

that the minimal positive solution $\underline{v}(x) = \widehat{v}(x)$ is asymptotically stable.

The proof of Theorem 5 is now complete. □

5.3 Proof of Theorem 6. The proof is essentially the same as that of Theorem 5. Indeed, it suffices to verify the following (see Figure 5.1):

(a) The function $(t_2(\varepsilon)/\min_{\overline{D}} \phi)\phi(x)$ is a sub-solution of problem (1.3) for every $\lambda \geq \nu(t_2(\varepsilon))/\min_{\overline{D}} \phi$.

(b) The function $\lambda C\phi(x)$ is a super-solution of problem (1.3) if $C \geq C_m$.

(c) There exists at most one positive solution $v(x) \in C^2(\overline{D})$ of problem (1.3) in the order interval

$$\left[\frac{t_2(\varepsilon)}{\min_{\overline{D}} \phi}\phi, \lambda C\phi\right], \quad C \geq C_m.$$

Therefore, Theorem 6 follows from an application of Theorem 4.1 with

$$\widehat{v}(x) := \overline{v}(x), \quad \psi(x) := \frac{t_2(\varepsilon)}{\min_{\overline{D}} \phi}\phi(x), \quad \varphi(x) := \lambda C\phi(x).$$

The proof of Theorem 6 is complete. □

Appendix: Estimate of the constant β

In this appendix we estimate the constant β in part (iii-c) of Theorem 2 in terms of the function $\phi(x)$.

First, we make the precise definition of the constant β in Theorem 1 (see Wiebers [26]). For a relatively compact subdomain Ω of D with smooth boundary, we consider the following linear boundary value problem:

$$\begin{cases} Aw = \chi_\Omega & \text{in } D, \\ Bw = \dfrac{\partial w}{\partial \boldsymbol{\nu}} + b(x')w = 0 & \text{on } \partial D. \end{cases} \tag{A.1}$$

Here $\chi_\Omega(x)$ is the characteristic function of Ω in D. It is known (see Agmon–Douglis–Nirenberg [1], Taylor [23]) that problem (A.1) is uniquely solvable in the framework of L^p spaces. Moreover, we can show that the solution $w_\Omega(x)$ belongs to $C^1(\overline{D})$ and is positive everywhere in D. Then the constant β is defined by the formula

$$\beta = \sup_{\Omega \Subset D} \inf_{x \in \Omega} w_\Omega(x).$$

The next lemma gives an estimate for the constant β in terms of the function $\phi(x)$:

Lemma A.1. *The constant β can be estimated as*

$$\min_{\overline{D}} \phi \leq \beta \leq \max_{\overline{D}} \phi. \tag{A.2}$$

Proof. If $\varphi_1(x)$ is the eigenfunction corresponding to the first eigenvalue λ_1 of problem (1.7) and if $\max_{\overline{D}} \varphi_1 = 1$, then it follows from Wiebers [25, Lemma 5.1] that

$$\lambda_1 \min_{\overline{D}} \varphi_1 \leq \frac{1}{\max_{\overline{D}} \phi} \leq \lambda_1 \leq \frac{1}{\beta} \leq \frac{\lambda_1}{\min_{\overline{D}} \varphi_1}.$$

Hence we have only to prove that

$$\beta \geq \min_{\overline{D}} \phi. \tag{A.3}$$

To do this, we choose a sequence $\{\Omega_j\}$ of relatively compact subdomains of D, with smooth boundary, such that $\Omega_j \uparrow D$ as $j \to \infty$. If $w_{\Omega_j}(x)$ is a unique solution of problem (A.1) with $\Omega := \Omega_j$, then it follows from an application of L^p theory of linear elliptic boundary value problems (see Taira [18, Theorem 1]) that

$$w_{\Omega_j}(x) \longrightarrow \phi(x) \quad \text{in } C(\overline{D}) \text{ as } j \to \infty$$

if $p > N$.

Now, since $w_{\Omega_j}(x)$ and $\phi(x)$ are both strictly positive on $\overline{D}$, we obtain that

$$\text{(A.4)} \qquad \begin{cases} \dfrac{1}{\inf_{\Omega_j} w_{\Omega_j}} = \sup_{\Omega_j} \dfrac{1}{w_{\Omega_j}} = \sup_{\Omega_j} \dfrac{\chi_{\Omega_j}}{w_{\Omega_j}} = \sup_D \dfrac{\chi_{\Omega_j}}{w_{\Omega_j}}, \\ \dfrac{1}{\min_{\overline{D}} \phi} = \dfrac{1}{\inf_D \phi} = \sup_D \dfrac{1}{\phi}. \end{cases}$$

However, it is easy to verify that

$$\frac{\chi_{\Omega_j}(x)}{w_{\Omega_j}(x)} \longrightarrow \frac{1}{\phi(x)} \quad \text{in } L^\infty(D) \text{ as } j \to \infty,$$

so that

$$\sup_D \frac{\chi_{\Omega_j}}{w_{\Omega_j}} \longrightarrow \sup_D \frac{1}{\phi} \quad \text{as } j \to \infty.$$

In view of formulas (A.4), this implies that

$$\text{(A.5)} \qquad \inf_{\Omega_j} w_{\Omega_j} \longrightarrow \min_{\overline{D}} \phi \quad \text{as } j \to \infty.$$

Therefore, the desired inequality (A.3) follows from assertion (A.5), since we have

$$\beta = \sup_{\Omega \Subset D} \inf_{\Omega} w_\Omega \geq \inf_{\Omega_j} w_{\Omega_j}. \quad \square$$

Example A.1. If $c(x)$ is a positive constant c and $b(x') \equiv 0$ on ∂D, then we find from assertion (A.2) that

$$\beta = \frac{1}{c},$$

since $\phi(x) = 1/c$ in D (see Wiebers [26, Theorem 5.4]).

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Institute of Mathematics, University of Tsukuba, Tsukuba, Ibaraki 305–8571, Japan

E-mail address: taira@math.tsukuba.ac.jp