Hosseini, S. N.; **Shir Ali Nasab, A. R.**; **Tholen, W.**

Fraction, restriction, and range categories from stable systems of morphisms. (English)

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The formation of the *category* $\mathcal{C}\left[\mathcal{S}^{-1}\right]$ *of fractions* with respect to a sufficiently well-behaved class $\mathcal{S}$ of morphisms in $\mathcal{C}$, which is a fundamental device in homotopy theory, was first given in [*P. Gabriel* and *M. Zisman*, Calculus of fractions and homotopy theory. Berlin-Heidelberg-New York: Springer-Verlag (1967; Zbl 0186.56802)] . The construction is characterized by its localizing functor

$$\mathcal{C}\to\mathcal{C}\left[\mathcal{S}^{-1}\right]$$

which is universal with respect to the property of turning morphisms in $\mathcal{S}$ into isomorphisms. The question of the size of the "homs" of $\mathcal{C}\left[\mathcal{S}^{-1}\right]$ is highly delicate.

This paper aims, assuming that $\mathcal{S}$ contain all isomorphisms, is closed under composition, and is stable under pullbacks in $\mathcal{C}$, to take a stepwise approach to the formation of $\mathcal{C}\left[\mathcal{S}^{-1}\right]$, considering separately the two processes of transforming every morphism in $\mathcal{S}$ into a retraction and into a section before amalgamating them to obtain the category of fractions.

A synopsis of the paper consisting of ten sections goes as follows. §2 is concerned with span categories $\mathsf{Span}(\mathcal{C},\mathcal{S})$ and their quotients. §3 forms the *S-retractable span category* $\mathsf{Retr}(\mathcal{C},\mathcal{S})$ of $\mathcal{C}$, while §4 forms the *S-sectional span category* $\mathsf{Sect}(\mathcal{C},\mathcal{S})$ of $\mathcal{C}$. §5 shows how to amalgamate the two constructions to obtain the category $\mathcal{C}\left[\mathcal{S}^{-1}\right]$, performing and characterizing these constructions strictly at the ordinary category level. The 2-categorical structure of $\mathsf{Span}(\mathcal{C},\mathcal{S})$ [*J. Bénabou*, Lect. Notes Math. 47, 1–77 (1967; Zbl 1375.18001); *C. Hermida*, Adv. Math. 151, No. 2, 164–225 (2000; Zbl 0960.18004)] is alluded to in §10, where it is indicated how the constructions of $\mathsf{Retr}(\mathcal{C},\mathcal{S})$ and $\mathsf{Sect}(\mathcal{C},\mathcal{S})$ are naturally motivated.

§6 elaborates on how to obtain the *S-partial map category* $\mathsf{Par}(\mathcal{C},\mathcal{S})$ as a quotient category of $\mathsf{Sect}(\mathcal{C},\mathcal{S})$, which is a restriction category. Under a fairly mild additional hypothesis on S holding in particular under the weak left cancellation condition $(s, s\cdot t\in\mathcal{S}\Longrightarrow t\in\mathcal{S})$, $\mathsf{Par}(\mathcal{C},\mathcal{S})$ is a localization of $\mathsf{Sect}(\mathcal{C},\mathcal{S})$ making $\mathsf{Retr}(\mathcal{C},\mathcal{S})=\mathcal{C}\left[\mathcal{S}^{-1}\right]$ its quotient category.§8 presents the construction of the *$\mathcal{S}$-partial map range category* $\mathsf{RaPar}(\mathcal{C},\mathcal{S})$, completing the quotient construction in the paper and yielding the commutative diagram

$$\begin{array}{ccccccc}\mathcal{C} & \to & \mathsf{Span}(\mathcal{C},\mathcal{S}) & \to & \mathsf{Sect}(\mathcal{C},\mathcal{S}) & \to & \mathsf{Par}(\mathcal{C},\mathcal{S})\\ & & \downarrow & & \downarrow & & \\ & & \mathsf{Retr}(\mathcal{C},\mathcal{S}) & \to & \mathcal{C}\left[\mathcal{S}^{-1}\right] & & \end{array}$$

Extending a key result in [*J. R. B. Cockett* and *S. Lack*, Theor. Comput. Sci. 270, No. 1–2, 223–259 (2002; Zbl 0988.18003)] , §7 provides a setting which presents

$$(\mathcal{C},\mathcal{S})\longmapsto\mathsf{Par}(\mathcal{C},\mathcal{S})$$

as the left adjoint to the formation of the category $\mathsf{Total}(\mathcal{X})$ for every split restriction category $\mathcal{X}$. Extending one of the principal results in [*J. R. B. Cockett* et al., Theory Appl. Categ. 26, 412–452 (2012; Zbl 1252.18003)] , §9 provides a setting which presents

$$(\mathcal{C},\mathcal{S})\longmapsto\mathsf{RaPar}(\mathcal{C},\mathcal{S})$$

as the left adjoint to the formation of the category $\mathsf{Total}(\mathcal{X})$ for every split range category $\mathcal{X}$.

To conclude this review, I will become an adversary, giving some complaints on the paper.

- The authors give their earlier version of the paper [arXiv:1903.00081] as the eighth item in References, but the title is wrong.

  Abandoning monomorphisms:partial maps, fractions, factorizations $\Longrightarrow$ Fraction, restriction, and range categories

- In §1,

$$\text{See Sections 3 and 2, respectively} \Longrightarrow \text{See Sections 4 and 3, respectively}$$

$$\text{In Section 4} \Longrightarrow \text{In Section 5}$$

$$\text{That is why, in Section 5} \Longrightarrow \text{That is why, in Section 6}$$

This paper is written by three authors in collaboration, two living in Iran and the other living in Canada. It comes to my mind that everyone's business is no-one's business.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18A99 General theory of categories and functors
18B99 Special categories
18A32 Factorization systems, substructures, quotient structures, congruences, amalgams

Keywords:

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