Garner, Richard; Kock, Joachim; Weber, Mark
Operadic categories and décalage. (English) Zbl 07289434
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Operads originated in [*J. M. Boardman* and *R. M. Vogt*, Bull. Am. Math. Soc. 74, 1117–1122 (1968; Zbl 0165.26204)] under the name "category of operators in standard form", their modern name and modern development having begun with [*J. P. May*, The geometry of iterated loop spaces. Berlin-Heidelberg-New York: Springer-Verlag (1972; Zbl 0244.55009)]. As the use of operads grew out, their re-expression in various more abstract ways emerged [*J. C. Baez* and *J. Dolan*, Adv. Math. 135, No. 2, 145–206 (1998; Zbl 0909.18006); *A. Joyal*, Lect. Notes Math. 1234, 126–159 (1986; Zbl 0612.18002); *G. M. Kelly*, Repr. Theory Appl. Categ. 2005, No. 13, 1–13 (2005; Zbl 1082.18009)], leading to a rich profusion of operad-like structures. Various authors have proposed unifying frameworks to bring order to this proliferation, one such framework being that of *operadic categories* [*M. Batanin* and *M. Markl*, Adv. Math. 285, 1630–1687 (2015; Zbl 1360.18009)], which inspired *S. Lack* [High. Struct. 2, No. 1, 1–29 (2018; Zbl 1410.18012)] to draw a intimate correspondence between operadic categories and the *skew-monoidal categories* of *K. Szlachányi* [Adv. Math. 231, No. 3–4, 1694–1730 (2012; Zbl 1283.18006)]. This paper gives another reconfiguration of the definition of operadic category linking it to the (upper) *décalage* construction.

An *operadic category* consists of the following entities.

- a small category $\mathcal{C}$ with a chosen terminal object in each connected component;
- a cardinality functor $|-| : \mathcal{C} \to \mathcal{S}$ into the category of finite ordinals and arbitrary mappings;
- an operation assigning to every $f : Y \to X$ in $\mathcal{C}$ and $i \in |X|$ an abstract fiber $f^{-1}(i) \in \mathcal{C}$, functorially in Y.

There are two main aspects to the close relationship between operadic categories and décalage. The first connection comes from the fact that the décalage construction on categories underlies a comonad D on $\mathcal{C}$at, whose coalgebras may be identified with categories endowed with a choice of terminal object in each connected component. The second connection arises through the functorial assignment of abstract fibers $f \mapsto f^{-1}(i)$ in an operadic category with functoriality claiming that, for a fixed $X \in \mathcal{C}$ and $i \in |X|$, this assignment is the action on objects of a functor $\varphi_{X,i} : \mathcal{C}/X \to \mathcal{C}$, so that the totality of the abstract fibers is to be expressed via a single functor

$$\varphi : \sum_{\substack{X \in \mathcal{C} \\ i \in |X|}} \mathcal{C}/X \to \mathcal{C}$$

The domain of this functor is clearly related to the décalage of $\mathcal{C}$, and the authors explain it in terms of a *modified décalage* construction on categories equipped with a functor to $\mathcal{S}$ but that the operadic category is unary, where an operadic category is called *unary* if each $|X|$ is a singleton.

A synopsis of the paper consisting of seven sections goes as follows. §2 recalls Batanin and Markl's definition of operadic category [*M. Batanin* and *M. Markl*, Adv. Math. 285, 1630–1687 (2015; Zbl 1360.18009)]. §3 recalls the décalage construction, establishing the first of the two links with the notion of operadic category. §4 establishes the first main theorem characterizing unary operadic categories in terms of décalage. §5 is engaged in describing the modified décalage construction required to capture general operadic categories. §6 and §7 establish the second and third main theorems giving the characterization of lax-operadic categories and, finally, of operadic categories themselves.

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MSC:

18M60	Operads (general)
18C20	Eilenberg-Moore and Kleisli constructions for monads
18N50	Simplicial sets, simplicial objects

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