Burtscher, Annegret; **Ketterer, Christian**; **McCann, Robert J.**; **Woolgar, Eric**

Inscribed radius bounds for lower Ricci bounded metric measure spaces with mean convex boundary. (English) Zbl 07302816

SIGMA, Symmetry Integrability Geom. Methods Appl. 16, Paper 131, 29 p. (2020).

A. Kasue [J. Math. Soc. Japan 35, 117–131 (1983; Zbl 0494.53039)] established a sharp estimate for the inscribed radius, or inradius denoted InRad, of a smooth n-dimensional Riemannian manifold M with nonnegative Ricci curvature and smooth boundary ∂M whose mean curvature is bounded from below by $n-1$. Exactly speaking, he concluded that

$$\mathrm{InRad}_M \leq 1$$

The result was rediscovered by [*M. M. C. Li*, J. Geom. Anal. 24, No. 3, 1490–1496 (2014; Zbl 1303.53053)], being extended to weighted Riemannian manifolds with Bakry-Émery curvature bounds in [*H. Li* and *Y. Wei*, J. Geom. Anal. 25, No. 1, 421–435 (2015; Zbl 1320.53075); *H. Li* and *Y. Wei*, Int. Math. Res. Not. 2015, No. 11, 3651–3668 (2015; Zbl 1317.53065); *Y. Sakurai*, Tohoku Math. J. (2) 71, No. 1, 69–109 (2019; Zbl 1422.53029)]. These results are to be seen either as a manifold-with-boundary analogue of Bonnet and Myers' diameter bound or as a Riemannian analogue of the Hawking singularity theorem [*S. W. Hawking*, Proc. R. Soc. Lond., Ser. A 294, 511–521 (1966; Zbl 0139.45803)], whose generalization to a nonsmooth setting is of paramount interest [*M. Graf*, Commun. Math. Phys. 378, No. 2, 1417–1450 (2020; Zbl 1445.53052); *M. Kunzinger* et al., Classical Quantum Gravity 32, No. 7, Article ID 075012, 19 p. (2015; Zbl 1328.83123); *Y. Lu* et al., "Geometry of weighted Lorentz-Finsler manifolds. I: Singularity theorems", Preprint, arXiv:1908.03832].

This paper generalizes Kasue and Li's estimate to subsets Ω of a possibly nonsmooth space X abiding by a curvature dimension condition $\mathrm{CD}(K,N)$ with $K \in \mathbb{R}$ and $N > 1$, provided the topological boundary $\partial\Omega$ has a lower bound on its inner mean curvature in the sense of [*C. Ketterer*, Proc. Am. Math. Soc. 148, No. 9, 4041–4056 (2020; Zbl 1444.53028)]. The authors' result not only covers Kasue's theorem but also holds for a large class of domains in Alexandrov spaces or in Finsler manifolds. Kasue as well as Li was able to establish a rigidity result analogous to Cheng's theorem [*S.-Y. Cheng*, Math. Z. 143, 289–297 (1975; Zbl 0329.53035)] in the Bonnet-Myers context [*S. B. Myers*, Duke Math. J. 8, 401–404 (1941; JFM 67.0673.01); *S. B. Myers*, Duke Math. J. 8, 401–404 (1941; Zbl 0025.22704)], namely that, among smooth manifolds, their inscribed radious bound is obtained exactly by the Euclidean unit ball. In the nonsmooth case there are also truncated cones attaining maximal inradius. The authors establish, under an additional hypothesis known as RCD, that these are the only nonsmooth oprimizers provided Ω is compact and its interior is connected.

Independently and almost simultaneously, *F. Cavalletti* and *A. Mondino* [Commun. Contemp. Math. 19, No. 6, Article ID 1750007, 27 p. (2017; Zbl 1376.53064); Invent. Math. 208, No. 3, 803–849 (2017; Zbl 1375.53053); Anal. PDE 13, 2091–2147 (2020); "Optimal transport in Lorentzian synthetic spaces, synthetic timelike Ricci curvature lower bounds and applications", Preprint, arXiv:2004.08934] have proposed a synthetic new framework for Lorentzian geometry in which an analogue of the Hawking result is established.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

51K10 Synthetic differential geometry
53C21 Methods of global Riemannian geometry, including PDE methods; curvature restrictions
30L99 Analysis on metric spaces
83C75 Space-time singularities, cosmic censorship, etc.

Keywords:

curvature-dimension condition; synthetic mean curvature; optimal transport; comparison geometry; diameter bounds; singularity theorems; inscribed radius; inradius bounds; rigidity; measure contraction

property

Full Text: DOI

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