Feller semigroups and degenerate elliptic operators III

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Dedicated to Professor Francesco Altomare on the occasion of his 70th birthday

This paper is devoted to the functional analytic approach to the problem of construction of Feller semigroups in the *characteristic case* via the Fichera function. Probabilistically, our result may be stated as follows: We construct a Feller semigroup corresponding to such a diffusion phenomenon that a Markovian particle moves continuously in the interior of the state space, without reaching the boundary. We make use of the Hille–Yosida–Ray theorem that is a Feller semigroup version of the classical Hille–Yosida theorem in terms of the positive maximum principle. Our proof is based on a method of *elliptic regularizations* essentially due to Oleĭnik and Radkevič. The weak convergence of approximate solutions follows from the local sequential weak compactness of Hilbert spaces and Mazur's theorem in normed linear spaces.

1 Introduction and main results

This paper is devoted to the functional analytic approach to the problem of construction of strong Markov processes *without* boundary condition. More precisely, we construct a Feller semigroup corresponding to such a diffusion phenomenon that a Markovian particle moves continuously in the interior of the state space without reaching the boundary. This paper is inspired by the work of Altomare et al. [2], [3] and [5] (see Remark 1.3), and it is a continuation of the previous papers Taira [23] through [31] and Taira–Favini–Romanelli [32].

Let D be a bounded domain of Euclidean space $\mathbf{R}^N$, $N \geq 2$, with smooth boundary ∂D; its closure $\overline{D} = D \cup \partial D$ is an N-dimensional, compact smooth manifold with boundary. Let A be a second-order, *degenerate* elliptic differential operator with real coefficients such that

$$Au(x) = \sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{N} b^i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x). \tag{1.1}$$

Here:

(1) $a^{ij} \in C^\infty(\mathbf{R}^N)$ and $a^{ij}(x) = a^{ji}(x)$ for all $x \in \mathbf{R}^N$ and $1 \leq i, j \leq N$, and

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq 0 \quad \text{for all } x \in \mathbf{R}^N \text{ and } \xi = \sum_{j=1}^{N} \xi_j \, dx_j \in T_x^*(\mathbf{R}^N),$$

where $T_x^*(\mathbf{R}^N) = \mathbf{R}^N$ is the cotangent space of $\mathbf{R}^N$ at x.

(2) $b^i \in C^\infty(\mathbf{R}^N)$ for $1 \leq i \leq N$.

(3) $c = A1 \in C^\infty(\mathbf{R}^N)$ and $c(x) \leq 0$ on $\overline{D}$.

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Following Fichera [12], we introduce a function b, called the *Fichera function* for A, on the boundary ∂D by the formula

$$b(x') = \sum_{i=1}^{N} \left(b^i(x') - \sum_{j=1}^{N} \frac{\partial a^{ij}}{\partial x_j}(x') \right) n_i \quad \text{for all } x' \in \partial D, \tag{1.2}$$

where $\mathbf{n} = (n_1, n_2, \dots, n_N)$ is the unit inward normal to the boundary ∂D.

It is easy to verify (see [30, Section 3]) that the Fichera function b is invariantly defined on the *characteristic set*

$$\Sigma^0 = \left\{ x' \in \partial D : \sum_{i,j=1}^{N} a^{ij}(x') n_i n_j = 0 \right\}.$$

We divide the boundary ∂D into the following four disjoint subsets:

$$\Sigma_3 = \left\{ x' \in \partial D : \sum_{i,j=1}^{N} a^{ij}(x') n_i n_j > 0 \right\} = \partial D \setminus \Sigma^0, \tag{1.3a}$$

$$\Sigma_2 = \left\{ x' \in \partial D : \sum_{i,j=1}^{N} a^{ij}(x') n_i n_j = 0, \ b(x') < 0 \right\} = \left\{ x' \in \Sigma^0 : b(x') < 0 \right\}, \tag{1.3b}$$

$$\Sigma_1 = \left\{ x' \in \partial D : \sum_{i,j=1}^{N} a^{ij}(x') n_i n_j = 0, \ b(x') > 0 \right\} = \left\{ x' \in \Sigma^0 : b(x') > 0 \right\}, \tag{1.3c}$$

$$\Sigma_0 = \left\{ x' \in \partial D : \sum_{i,j=1}^{N} a^{ij}(x') n_i n_j = 0, \ b(x') = 0 \right\} = \left\{ x' \in \Sigma^0 : b(x') = 0 \right\}. \tag{1.3d}$$

Our fundamental hypothesis for the operator A is stated as follows (cf. Figure 3.1 in Section 3):

> *The boundary ∂D consist of a finite number of connected* hypersurfaces $\qquad$ (G)
> *of the sets Σ_0 and Σ_1: $\partial D = \Sigma_0 \cup \Sigma_1$.*

This hypothesis makes it possible to develop the basic machinery of Oleĭnik and Radkevič [18] with a minimum of bother and the principal ideas can be presented more concretely and explicitly.

It should be emphasized that a Markovian particle moves continuously in the interior D, and approaches the boundary portion $\Sigma_2 \cup \Sigma_3$ in *finite time* with positive probability ([21], [30]). Hence, we may impose a boundary condition only on the boundary portion $\Sigma_2 \cup \Sigma_3$ (see [18], [22]). Under hypothesis (G), we cannot impose any boundary condition on the boundary ∂D, since $\Sigma_2 = \Sigma_3 = \emptyset$.

We give a simple example of hypothesis (G) in the unit disk in the plane $\mathbf{R}^2$:

Example 1.1 Let $D = \{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 < 1\}$ be the unit disk with the boundary $\partial D = \{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 = 1\}$. For a local coordinate system $x_1 = r \cos\theta$ and $x_2 = r \sin\theta$ with $0 \leq \theta \leq 2\pi$ near the boundary $\partial D = \{r = 1\}$, we assume that the differential operator A is written in the form

$$A = \varphi(r)\Delta - \frac{\partial}{\partial r} = \varphi(r) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) - \frac{\partial}{\partial r},$$

where $\varphi(r)$ is a smooth function defined by the formula

$$\varphi(r) = \begin{cases} \exp\left[-\frac{1}{1-r^2} \right] & \text{for } r < 1, \\ 0 & \text{for } r \geq 1. \end{cases}$$

Then it is easy to see that $\Sigma_3 = \emptyset$ and that $b = 1$ on $\partial D = \{r = 1\}$. This proves that $\partial D = \Sigma_1$.

Let $C(\overline{D})$ be the space of real-valued, continuous functions f on $\overline{D}$. We equip the space $C(\overline{D})$ with the topology of uniform convergence on the whole $\overline{D}$; hence it is a Banach space with the maximum norm

$$\|f\| = \max_{x \in \overline{D}} |f(x)|.$$

A strongly continuous semigroup $\{T_t\}_{t \geq 0}$ on the space $C(\overline{D})$ is called a *Feller semigroup* on $\overline{D}$ if it is non-negative and contractive on $C(\overline{D})$:

$$f \in C(\overline{D}), 0 \leq f(x) \leq 1 \quad \text{on } \overline{D} \implies 0 \leq T_t f(x) \leq 1 \quad \text{on } \overline{D}.$$

It is known (cf. [28, Chapter 9]) that if T_t is a Feller semigroup on $\overline{D}$, then there exists a unique Markov transition function $p_t(x, \cdot)$ on $\overline{D}$ such that

$$T_t f(x) = \int_{\overline{D}} p_t(x, dy) f(y) \quad \text{for every } f \in C(\overline{D}).$$

Furthermore, the function $p_t(x, \cdot)$ is the transition function of some *strong Markov process* $\mathscr{X}$; hence the value $p_t(x, E)$ expresses the transition probability that a Markovian particle starting at position x will be found in the set E at time t.

The next theorem asserts that there exists a Feller semigroup on $\overline{D}$ corresponding to such a diffusion phenomenon that a Markovian particle moves continuously in the interior D of the state space $\overline{D}$ without reaching the boundary ∂D:

Theorem 1.2 *Assume that hypothesis (G) is satisfied. We define a linear operator $\mathscr{A}$ from the space $C(\overline{D})$ into itself as follows.*

(1) *The domain $D(\mathscr{A})$ of $\mathscr{A}$ is the space*

$$D(\mathscr{A}) = C^2(\overline{D}). \tag{1.4}$$

(2) *$\mathscr{A} u = Au$ for every $u \in D(\mathscr{A})$.*

Then the operator $\mathscr{A}$ is closable *in the space $C(\overline{D})$, and its minimal closed extension $\mathfrak{A} = \overline{\mathscr{A}}$ is the infinitesimal generator of some Feller semigroup $\{T_t\}_{t \geq 0}$ on $\overline{D}$.*

Remark 1.3 Some remarks are in order:

1° Altomare et al. [2] through [5] consider a *convex* compact domain K with not necessarily smooth boundary ∂K and a second-order differential operator V which degenerates on a subset of the boundary ∂K containing the extreme points of K. They prove that the closure A of the operator V generates a Feller semigroup $\{T_t\}_{t \geq 0}$ and further that the Feller semigroup $\{T_t\}_{t \geq 0}$ can be approximated by iterates of modified Bernstein–Schnabl operators ([4]). It should be emphasized that Theorem 1.2 coincides with [2, Theorem 4.1], [3, Theorem 4.3] and [5, Theorem 3.1] with $K := D$ if the boundary ∂K is *smooth*, as in Example 1.1.

2° Theorem 1.2 is proved by Bony–Courrège–Priouret [8] in the elliptic case (see [8, Théorème XVI]) and then by Cancelier [9] in the *non-characteristic case*: $\partial D = \Sigma_3$ (cf. [9, Théorème 7.2]).

By a version of the Hille–Yosida theorem in semigroup theory, the proof of Theorem 1.2 is reduced to the study of the homogeneous Dirichlet problem in the theory of partial differential equations. However, if hypothesis (G) is satisfied, the proof of Theorem 1.2 is reduced to the study of the equation

$$(\mathscr{A} - \lambda) u = f \quad \text{in } D \tag{1.5}$$

without any boundary condition. In this way, the essential step in the proof is the following existence and uniqueness theorem for the equation (1.5) in the framework of Hölder spaces:

Theorem 1.4 *Assume that hypothesis (G) is satisfied. For each integer $m \geq 2$, there exists a constant $\lambda_{m+1} > 0$ such that if $\lambda \geq \lambda_{m+1}$, the equation*

$$(A - \lambda) u = f \quad \text{in } D \tag{$\star$}_\lambda$$

has a unique solution u in the Hölder space $C^{m+\theta}(\overline{D})$ for any function $f \in C^{m+\theta}(\overline{D})$ with $0 < \theta < 1$. Furthermore, the solution u satisfies the inequality

$$\|u\|_{C^{m+\theta}(\overline{D})} \leq C_{m+\theta}(\lambda) \|f\|_{C^{m+\theta}(\overline{D})}, \tag{1.6}$$

where $C_{m+\theta}(\lambda) > 0$ is a constant independent of f.

Rephrased, Theorem 1.4 asserts that if hypothesis (G) is satisfied, the equation $(\star)_\lambda$ has a solution with *loss of two derivatives* compared with the elliptic case.

1.1 Background of the paper

In this subsection we consider a second-order, boundary condition L with real coefficients such that, in terms of local coordinates $(x_1, x_2, \ldots, x_{N-1})$ on the boundary portion $\Sigma_2 \cup \Sigma_3$,

$$Lu(x') = L_0 u(x') + \gamma(x') u(x') + \mu(x') \frac{\partial u}{\partial \mathbf{n}}(x') - \delta(x') Au(x') \tag{1.7}$$

$$:= \left(\sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2 u}{\partial x_i \partial x_j}(x') + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial u}{\partial x_i}(x') \right)$$

$$+ \gamma(x') u(x') + \mu(x') \frac{\partial u}{\partial \mathbf{n}}(x') - \delta(x') Au(x').$$

Here:

(1) The α^{ij} are the components of a C^∞ symmetric, contravariant tensor of type $\binom{2}{0}$ on $\Sigma_2 \cup \Sigma_3$ and

$$\sum_{i,j=1}^{N-1} \alpha^{ij}(x') \xi_i \xi_j \geq 0 \quad \text{for all } x' \in \Sigma_2 \cup \Sigma_3 \text{ and } \xi' = \sum_{j=1}^{N-1} \xi_j \, dx_j \in T_{x'}^*(\Sigma_2 \cup \Sigma_3),$$

where $T_{x'}^*(\Sigma_2 \cup \Sigma_3)$ is the cotangent space of $\Sigma_2 \cup \Sigma_3$ at x'.

(2) $\beta^i \in C^\infty(\Sigma_2 \cup \Sigma_3)$ for $1 \leq i \leq N-1$.

(3) $\gamma \in C^\infty(\Sigma_2 \cup \Sigma_3)$ and $\gamma(x') \leq 0$ on $\Sigma_2 \cup \Sigma_3$.

(4) $\mu \in C^\infty(\Sigma_2 \cup \Sigma_3)$ and $\mu(x') \geq 0$ on $\Sigma_2 \cup \Sigma_3$.

(5) $\delta \in C^\infty(\Sigma_2 \cup \Sigma_3)$ and $\delta(x') \geq 0$ on $\Sigma_2 \cup \Sigma_3$.

(6) $\mathbf{n} = (n_1, \ldots, n_N)$ is the unit inward normal to $\Sigma_2 \cup \Sigma_3$.

The boundary condition L defined by formula (1.7) is called a second-order *Ventcel' (Wentzell) boundary condition* ([35]). The four terms of L are supposed to correspond to the diffusion along the boundary, the absorption phenomenon, the reflection phenomenon and the viscosity phenomenon, respectively (cf. [11]).

We say that the boundary condition L is *transversal* on the boundary portion $\Sigma_2 \cup \Sigma_3$ if it satisfies the condition:

$$\mu(x') + \delta(x') > 0 \quad \text{on } \Sigma_2 \cup \Sigma_3. \tag{1.8}$$

Probabilistically, the transversality condition (1.8) implies that either reflection or viscosity phenomenon occurs at each point of $\Sigma_2 \cup \Sigma_3$.

We are interested in the following problem:

Problem 1.5 For analytic data (A, L) given by formulas (1.1) and (1.7), construct a Feller semigroup $\{T_t\}_{t\geq 0}$ on $\overline{D}$.

Our functional analytic approach to Markov processes is distinguished by the extensive use of the ideas and techniques characteristic of the recent developments in the theory of elliptic boundary value problems which may be considered as a modern theory of the classical potential theory ([14], [18], [34], [30]).

Remark 1.6 Problem 1.5 was studied by Taira [24, Theorem 1], [25, Main Theorem] and [26, Main Theorem] under the following hypothesis (cf. Figure 3.1 in Section 3):

The boundary ∂D consist of a finite number of connected hypersurfaces   (H)

of the sets Σ_0, Σ_1, Σ_2 and Σ_3: $\partial D = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, but $\Sigma_2 \cup \Sigma_3 \neq \emptyset$.

It should be emphasized that hypothesis (H) excludes hypothesis (G), but includes the two cases where $\partial D = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ and $\partial D = \Sigma_0 \cup \Sigma_1 \cup \Sigma_3$.

We give an overview of general results on generation theorems for Feller semigroups studied mainly by the author via the theory of pseudo-differential operators in Table 1.1 below ([23], [24], [25], [26], [27], [32]).

Degeneracy of type Σ_i on the operator A	Ventcel' condition L of the form (1.7)	Domain $D(\mathfrak{A})$ or $D(\mathscr{A})$	studied by
$\partial D = \Sigma_3$ [hypothesis (H)] ($\Sigma_0 = \Sigma_1 = \Sigma_2 = \emptyset$)	$\mu + \delta > 0$ on $\partial D = \Sigma_3$	$D(\mathfrak{A}) = \{u \in C(\overline{D}) :$ $Au \in C(\overline{D}), Lu = 0 \text{ on } \partial D\}$	[23, Theorem 2.2]
$\partial D =$ $\Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ [hypothesis (H)]	$\mu = \delta = 0, \gamma = 1$ on $\Sigma_2 \cup \Sigma_3$ (Dirichlet case)	$D(\mathscr{A}) = \{u \in C^2(\overline{D}),$ $u = Au = 0 \text{ on } \Sigma_2 \cup \Sigma_3\},$ $\mathfrak{A} = \overline{\mathscr{A}}$	[24, Theorem 1]
$\partial D =$ $\Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ [hypothesis (H)]	$\mu + \delta > 0$ on $\Sigma_2 \cup \Sigma_3$	$D(\mathfrak{A}) = \{u \in C(\overline{D}) :$ $Au \in C(\overline{D}), Lu = 0 \text{ on } \Sigma_2 \cup \Sigma_3\}$	[25, Main Theorem] [26, Main Theorem]
$\partial D = \Sigma_2 \cup \Sigma_3$ [hypothesis (H)] ($\Sigma_0 = \Sigma_1 = \emptyset$)	$\mu = \gamma = 0, \delta = 1$ on $\partial D = \Sigma_2 \cup \Sigma_3$ ($L_0 = 0$)	$D(\mathfrak{A}) = \{u \in C(\overline{D}) :$ $Au \in C(\overline{D}), Au = 0 \text{ on } \partial D\}$	[32, Theorem 1]
$\partial D = \Sigma_0 \cup \Sigma_1$ [hypothesis (G)]	null	$D(\mathscr{A}) = C^2(\overline{D}),$ $\mathfrak{A} = \overline{\mathscr{A}}$	the present paper

Table 1.1 An overview of generation theorems for Feller semigroups by degenerate elliptic operators

1.2 Outline of the paper

The rest of this paper is organized as follows. Section 2 provides a brief description of basic definitions and results about Feller semigroups that forms a functional analytic background for the proof of Theorem 1.2. Our proof of Theorem 1.2 is based on a Feller semigroup version of the classical Hille–Yosida theorem (Theorem 2.1) in terms of the *positive maximum principle* (PMP).

In Section 3, we study the equation

$$Au = f \quad \text{in } D \tag{$\star$}$$

under hypothesis (G) in the framework of spaces of bounded measurable functions, and prove an existence and uniqueness theorem of weak solutions of the equation ($\star$) (Theorem 3.3), by using a method of *elliptic regularizations* just as in Oleĭnik–Radkevič [18], Cancelier [9] and Taira [24]. The proof of Theorem 3.3 is based on the local sequential weak compactness theorem of Hilbert spaces and Mazur's theorem in normed linear

spaces ([36, Chapter V]). It is hypothesis (G) that makes it possible to develop the basic machinery of Oleĭnik–Radkevič [18] with a minimum of bother and the principal ideas can be presented more concretely and explicitly.

Moreover, we study the equation

$$(A - \lambda) u = f \quad \text{in } D \tag{$\star$}_\lambda$$

in the framework of Sobolev spaces where λ is a real spectral parameter. By applying Theorem 3.3 with

$$A := A - \lambda, \quad c := (A - \lambda)1 = c - \lambda,$$

we can obtain that if hypothesis (G) is satisfied and $\lambda > 0$, then the equation $(\star)_\lambda$ has a unique weak solution $u \in L^\infty(D)$ for any function $f \in L^\infty(D)$ (Theorem 3.5).

In Section 4, we prove two regularity theorems (Theorems 4.1 and 4.7) for the weak solutions of the equation $(\star)_\lambda$ constructed in Theorem 3.5 in the framework of Sobolev spaces when the spectral parameter λ tends to $+\infty$. We remark that $W^{0,\infty}(D) = L^\infty(D)$. This section is the heart of the subject. In the proof, the *uniform estimates* for approximate solutions of the equation $(\star)_\lambda$ play an essential role (Lemma 4.3 and Lemma 4.9). Furthermore, we make use of Sobolev imbedding theorems into Lipschitz spaces ([1], [17], [37]) and also the Ascoli–Arzelà theorem. Theorem 1.4 follows from Theorems 4.1 and 4.7 by a well-known *real interpolation argument* due to Lions–Peetre [16] (Theorem 4.12). Indeed, the Hölder space $C^{m+\theta}(\overline{D})$ is a real interpolation space between the Sobolev spaces $W^{m,\infty}(D) = C^{m-1,1}(\overline{D})$ and $W^{m+1,\infty}(D) = C^{m,1}(\overline{D})$ (see formula (4.68)).

It should be emphasized that the uniform estimates for approximate solutions of our method of elliptic regularizations are obtained from the *weak maximum principle* for second-order, *degenerate* elliptic differential operators due to Bony [7], Oleĭnik–Radkevič [18, Chapter III, Section 1], Troianiello [34, Subsection 3.7.2] and also Taira [31, Section 8].

The last Section 5 is devoted to the proof of Theorem 1.2. We verify all the conditions of the Hille–Yosida–Ray theorem (see Theorem 2.1) for the operator $\mathscr{A}$ defined by formula (1.4), by making use of Theorem 1.4 for $m := 2$ or Theorem 4.12 (Theorem 5.1 and Lemma 5.2) and the positive maximum principle (Claim 5.1).

2 Feller semigroups

This section provides a brief description of the basic definitions and results about Feller semigroups, which forms a functional analytic background for the proof of Theorem 1.2. For detailed study of this subject, the reader might be referred to Dynkin [10], Lamperti [15] and Taira [28].

Let K be a *compact* metric space and let $C(K)$ be the space of real-valued, bounded continuous functions on K. The space $C(K)$ is a Banach space with the maximum norm

$$\|f\| = \max_{x \in K} |f(x)|.$$

A family $\{T_t\}_{t \geq 0}$ of bounded linear operators acting on $C(K)$ is called a *Feller semigroup* on K if it satisfies the following three conditions:

(i) $T_{t+s} = T_t \cdot T_s$ for all $t, s \geq 0$ (the semigroup property) where $T_0 = I = $ the identity.

(ii) The family $\{T_t\}$ is strongly continuous in t for $t \geq 0$:

$$\lim_{s \downarrow 0} \|T_{t+s}f - T_t f\| = 0 \quad \text{for every } f \in C(K).$$

(iii) The family $\{T_t\}$ is non-negative and contractive on $C(K)$:

$$f \in C(K), \, 0 \leq f(x) \leq 1 \quad \text{on } K \implies 0 \leq T_t f(x) \leq 1 \quad \text{on } K.$$

If $\{T_t\}_{t \geq 0}$ is a Feller semigroup on K, then we define its *infinitesimal generator* $\mathfrak{A}$ by the formula

$$\mathfrak{A}u = \lim_{t \downarrow 0} \frac{T_t u - u}{t}, \tag{2.1}$$

provided that the limit (2.1) exists in the space $C(K)$.

We conclude this section by giving useful criteria in terms of the *positive maximum principle* (cf. [19, Theorem 1.2]; [8, Théorème de Hille–Yosida–Ray]; [28, Theorem 9.50]):

Theorem 2.1 (Hille–Yosida–Ray) *Let B be a linear operator from the space $C(K)$ into itself. We assume that*

(α) *The domain $D(B)$ of B is* dense *in the space $C(K)$.*

(β) *If $u \in D(B)$ and $\max_{x \in K} u(x) > 0$, then there exists a point x_0 of K such that the* positive maximum principle

$$\begin{cases} u(x_0) = \max_K u, \\ Bu(x_0) \leq 0 \end{cases} \tag{PMP}$$

holds true.

(γ) *For some $\alpha_0 \geq 0$, the range $R(\alpha_0 I - B)$ of $\alpha_0 I - B$ is* dense *in the space $C(K)$.*

Then the operator B is closable *in the space $C(K)$, and its minimal closed extension $\overline{B}$ is the infinitesimal generator of some Feller semigroup $\{T_t\}_{t \geq 0}$ on K.*

3 Existence and uniqueness theorem for the equation $(\star)$

In this section, we will study the equation $(\star)$ in the framework of spaces of bounded measurable functions, and prove an existence and uniqueness theorem of weak solutions, by using a method of *elliptic regularizations* as in Oleĭnik–Radkevič [18] and also as in Cancelier [9].

Basic definitions and facts about function spaces such as Hölder spaces, Sobolev spaces and Besov spaces can be found in Adams–Fournier [1], Bergh–Löfström [6], Gilbarg–Trudinger [14] and Triebel [33].

3.1 Definition of weak solutions of the homogeneous Dirichlet problem

Let D be a bounded domain of $\mathbf{R}^N$, $N \geq 2$, with smooth boundary ∂D. In this subsection we assume that the degenerate elliptic operator A given by formula (1.1) satisfies the following hypothesis (F) (see Figure 3.1):

> *The boundary ∂D consist of a finite number of connected* hypersurfaces (F)
> *of the sets Σ_0, Σ_1, Σ_2 and Σ_3: $\partial D = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$.*

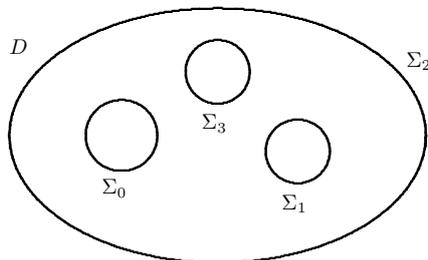We remark that hypothesis (F) includes both hypotheses (G) and (H).

Fig. 3.1 The degeneracy of type Σ_i on the differential operator A

Let A^* be the formal adjoint differential operator of A:

$$A^* v(x) = \sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{N} \left(2 \sum_{j=1}^{N} \frac{\partial a^{ij}}{\partial x_j}(x) - b^i(x) \right) \frac{\partial v}{\partial x_i}(x)$$

$$+ \left(\sum_{i,j=1}^{N} \frac{\partial^2 a^{ij}}{\partial x_i \partial x_j}(x) - \sum_{i=1}^{N} \frac{\partial b^i}{\partial x_i}(x) + c(x) \right) v(x).$$

Then it is easy to see that the *Fichera function* b^* for the operator A^* is given by the formula (cf. formula (1.2))

$$b^*(x') = -\sum_{i=1}^{N} \left(b^i(x') - \sum_{j=1}^{N} \frac{\partial a^{ij}}{\partial x_j}(x') \right) n_i = -b(x') \quad \text{for all } x' \in \partial D. \tag{3.1}$$

Following Oleĭnik–Radkevič [18, Chapter I], we consider the homogeneous Dirichlet problem for the operator A under hypothesis (H): For a given function $f \in L^\infty(D)$, find a function $u \in L^\infty(D)$ such that

$$\begin{cases} Au = f & \text{in } D, \\ u = 0 & \text{on } \Sigma_2 \cup \Sigma_3. \end{cases} \tag{D}$$

Now we give the precise definition of a weak solution of the homogeneous Dirichlet problem (D) (see [18, p. 30, Definition]):

Definition 3.1 A function $u \in L^\infty(D)$ is called a *weak solution* of the homogeneous Dirichlet problem (D) if we have the formula

$$\int_D u \cdot A^* v \, dx = \int_D f \cdot v \, dx$$

for every function $v \in C^2(\overline{D})$ satisfying the condition $v = 0$ on $\Sigma_1 \cup \Sigma_3$.

Our definition of a weak solution may be justified by using the following Green formula for the operators A and A^* (see [18, formula (1.1.14)], [24, Theorem 2.2]):

Theorem 3.2 (Green) *Assume that hypothesis (F) is satisfied. For all functions u and v in $C^2(\overline{D})$, we have the formula*

$$\int_D (Au \cdot v - u \cdot A^* v) \, dx = - \int_{\Sigma_3} \left(\frac{\partial u}{\partial \nu} v - u \frac{\partial v}{\partial \nu} \right) d\sigma - \int_{\partial D \setminus \Sigma_0} b(x') u \cdot v \, d\sigma. \tag{3.2}$$

Here $\partial/\partial\nu$ is the conormal derivative associated with the operator A:

$$\frac{\partial}{\partial \nu} = \sum_{i,j=1}^{N} a^{ij}(x') n_j \frac{\partial}{\partial x_i},$$

and b is the Fichera function for A and $d\sigma$ is the surface element of ∂D.

3.2   Existence and uniqueness of weak solutions of the equation $(\star)$

We recall that we cannot impose any boundary condition on the boundary ∂D under hypothesis (G), since $\Sigma_2 = \Sigma_3 = \emptyset$.

In this subsection, following [18, Theorem 1.5.1] and [24, Theorem 2.3] we will prove an existence and uniqueness theorem for weak solutions of the equation

$$Au = f \quad \text{in } D. \tag{$\star$}$$

More precisely, we will prove that there exists a unique function $u \in L^\infty(D)$ such that (see Definition 3.1)

$$\int_D u \cdot A^* v \, dx = \int_D f \cdot v \, dx \quad \text{for all } v \in C^2(\overline{D}) \text{ satisfying } v = 0 \text{ on } \Sigma_1. \tag{3.3}$$

Theorem 3.3 *Assume that hypothesis (G) is satisfied and further that*

$$c(x) = (A1)(x) < 0 \quad \text{on } \overline{D}. \tag{3.4}$$

Then, for any function $f \in L^\infty(D)$ there exists a weak solution $u \in L^\infty(D)$ of the equation $(\star)$. Furthermore, the weak solution u satisfies the inequality

$$\|u\|_{L^\infty(D)} \leq \frac{1}{c_0} \|f\|_{L^\infty(D)}, \tag{3.5}$$

where

$$c_0 := \min_{x \in \overline{D}} (-c(x)) > 0. \tag{3.6}$$

Steps	Elliptic regularizations	Data	Solutions
(I)	Dirichlet problem $(D)_\varepsilon$	$f \in C^1(\overline{D})$	$u_\varepsilon \in C^{2+\theta}(\overline{D})$
(II)	Equation $(\star)$ $Au = f$	$f \in C^1(\overline{D})$	$u = w - \lim_{\varepsilon_k \downarrow 0} u_{\varepsilon_k} \in L^\infty(D)$ $u_{\varepsilon_k} \in C^{2+\theta}(\overline{D})$
(III-1)	Equation $(\star)_n$ $Au_n = f_n$	$f_n \in C^1(\overline{D})$	$u_n = w - \lim_{\varepsilon_k \downarrow 0} u_{\varepsilon_k,n} \in L^\infty(D)$ $u_{\varepsilon_k,n} \in C^{2+\theta}(\overline{D})$
(III-2)	Equation $(\star)$ $Au = f$	$f \in L^\infty(D)$ $f = \lim_{n \to \infty} f_n$ $f_n \in C^1(\overline{D})$	$u = w - \lim_{n' \to \infty} u_{n'} \in L^\infty(D)$ $u_n = w - \lim_{\varepsilon_k \downarrow 0} u_{\varepsilon_k,n} \in L^\infty(D)$

Table 3.1 An overview of the proof of Theorem 3.3

Proof. The proof of Theorem 3.3 is based on a method of *elliptic regularizations* and the *weak maximum principle*, just as in Oleĭnik–Radkevič [18]. The proof is divided into three steps (see Table 3.1).

Step (I): First, let $f \in C^1(\overline{D})$, f arbitrary. We will show that the equation

$$Au = f \quad \text{in } D \tag{$\star$}$$

has a weak solution $u \in L^2(D)$. More precisely, we will show that the formula

$$\int_D u \cdot A^* v \, dx = \int_D f \cdot v \, dx \tag{3.3}$$

holds true for all $v \in C^2(\overline{D})$ satisfying $v = 0$ on Σ_1 (see Definition 3.1).

Following Oleĭnik–Radkevič [18], we consider the *homogeneous Dirichlet problem* for a second-order, elliptic differential operator

$$A_\varepsilon := \varepsilon \Delta + A,$$

where $\varepsilon > 0$ and Δ is the usual Laplacian. More precisely, we consider the Dirichlet problem

$$\begin{cases} A_\varepsilon u_\varepsilon = f & \text{in } D, \\ u_\varepsilon = 0 & \text{on } \partial D, \end{cases} \tag{$D)_\varepsilon$}$$

It is known (see [14, Theorem 6.14]) that the Dirichlet problem $(D)_\varepsilon$ has a unique solution $u_\varepsilon \in C^{2+\theta}(\overline{D})$ for all $0 < \theta < 1$, since $f \in C^1(\overline{D})$.

The weak convergence of approximate solutions u_ε as $\varepsilon \downarrow 0$ follows from the weak maximum principle and the local sequential weak compactness of Hilbert spaces. Indeed, since we have, by condition (3.4) and formulas (3.5) and (3.6),

$$(A_\varepsilon 1)(x) = (A1)(x) = c(x) \leq -c_0 \quad \text{on } \overline{D},$$

by applying the *weak maximum principle* to the elliptic differential operator A_ε we obtain that

$$\|u_\varepsilon\|_{C(\overline{D})} = \max_{x \in \overline{D}} |u_\varepsilon(x)| \leq \frac{1}{c_0} \max_{x \in \overline{D}} |f(x)| = \frac{1}{c_0} \|f\|_{C(\overline{D})} \quad \text{for all } \varepsilon > 0. \tag{3.7}$$

Hence, if $|D|$ is the volume of the domain D, it follows from the uniform estimate (3.7) that

$$\|u_\varepsilon\|_{L^2(D)} \leq \sqrt{|D|}\, \|u_\varepsilon\|_{L^\infty(D)} \leq \frac{\sqrt{|D|}}{c_0}\, \|f\|_{L^\infty(D)} \quad \text{for all } \varepsilon > 0. \tag{3.8}$$

That is, the solutions $\{u_\varepsilon\}$ are norm bounded in the Hilbert space $L^2(D)$ for all $\varepsilon > 0$. However, it is known ([36, p. 126, Theorem 1]) that the unit ball in a Hilbert space is *sequentially weakly compact*.

Therefore, we can find a subsequence $\{u_{\varepsilon_k}\}_{k=1}^{\infty}$ of $\{u_\varepsilon\}$ that converges *weakly* to some function u in $L^2(D)$, as $\varepsilon_k \downarrow 0$:

$$u = w - \lim_{\varepsilon_k \downarrow 0} u_{\varepsilon_k} \quad \text{in } L^2(D). \tag{3.9}$$

Step (II): Next, we will show that, for any function $f \in C^1(\overline{D})$ the weak limit function $u \in L^2(D)$, defined by formula (3.9), satisfies the inequality

$$\|u\|_{L^\infty(D)} \leq \frac{1}{c_0}\, \|f\|_{L^\infty(D)}, \tag{3.10}$$

and is a weak solution of the equation

$$Au = f \quad \text{in } D. \tag{$\star$}$$

It should be emphasized that the inequality (3.10) implies the *uniqueness* of weak solutions of the equation $(\star)$.

Substep (II-1): In order to estimate the solutions $\{u_\varepsilon\}$ of the Dirichlet problem $(D)_\varepsilon$, we need the following lemma (see [24, Lemma 2.4]):

Lemma 3.4 *Assume that hypothesis (F) and condition (3.4) are satisfied. For each $g \in C^\theta(\overline{D})$ with $0 < \theta < 1$, let $v_\varepsilon \in C^{2+\theta}(\overline{D})$ be a unique solution of the homogeneous Dirichlet problem for the elliptic differential operator $A_\varepsilon = \varepsilon\Delta + A$:*

$$\begin{cases} A_\varepsilon v_\varepsilon = g & in\ D, \\ v_\varepsilon = 0 & on\ \partial D. \end{cases} \tag{$D)_\varepsilon$}$$

Then the solution v_ε satisfies the estimates

$$\max_{x' \in \Sigma_3} \left| \operatorname{grad} v_\varepsilon(x') \right| \leq M\, \|g\|_{C(\overline{D})}, \tag{3.11a}$$

$$\max_{x' \in \Sigma_2} \left| \operatorname{grad} v_\varepsilon(x') \right| \leq M\, \|g\|_{C(\overline{D})}, \tag{3.11b}$$

$$\max_{x' \in \Sigma_0} \left| \operatorname{grad} v_\varepsilon(x') \right| \leq \frac{C}{\sqrt{\varepsilon}}\, \|g\|_{C(\overline{D})}, \tag{3.11c}$$

where $M > 0$ and $C > 0$ are constants independent *of $\varepsilon > 0$ and $0 < \theta < 1$.*

Substep (II-2): Now, by applying Lemma 3.4 with $g := f$ and $v_\varepsilon := u_\varepsilon$ to the solutions $u_{\varepsilon,n}$ of the homogeneous Dirichlet problem $(D)_\varepsilon$ under hypothesis (G), we obtain from estimate (3.11c) that

$$\max_{x' \in \Sigma_0} \left| \operatorname{grad} u_\varepsilon(x') \right| \leq \frac{C}{\sqrt{\varepsilon}}\, \|f\|_{C(\overline{D})}. \tag{3.12}$$

Then, by using Green's formula (3.2) for the differential operators $A_\varepsilon = \varepsilon\Delta + A$ and $A_\varepsilon^* = \varepsilon\Delta + A^*$, we have, for all $v \in C^2(\overline{D})$ satisfying $v = 0$ on the set Σ_1,

$$\int_D f \cdot v\, dx = \int_D A_\varepsilon u_\varepsilon \cdot v\, dx \tag{3.13}$$

$$= \int_D u_\varepsilon \cdot A_\varepsilon^* v\, dx - \int_{\Sigma_1} b(x')\, u_\varepsilon \cdot v\, d\sigma - \varepsilon \left(\int_{\partial D} \left(\frac{\partial u_\varepsilon}{\partial \mathbf{n}} v - u_\varepsilon \frac{\partial v}{\partial \mathbf{n}} \right) d\sigma \right)$$

$$= \varepsilon \int_D u_\varepsilon \cdot \Delta v \, dx + \int_D u_\varepsilon \cdot A^* v \, dx - \varepsilon \int_{\Sigma_0} \frac{\partial u_\varepsilon}{\partial \mathbf{n}} v \, d\sigma.$$

However, we can let $\varepsilon_k \downarrow 0$ in formula (3.13) with $\varepsilon := \varepsilon_k$ to obtain the desired formula (3.3). Indeed, by estimate (3.12) it follows that the last term of the right-hand side of formula (3.13) tends to zero as $\varepsilon_k \downarrow 0$.

Therefore, we have proved that the weak limit function u is a weak solution of the equation ($\star$).

Substep (II-3): On the other hand, it is easy to verify that the set

$$K = \left\{ w \in L^2(D) : \|w\|_{L^\infty(D)} \le \frac{1}{c_0} \|f\|_{L^\infty(D)} \right\}$$

is convex and strongly closed in the Hilbert space $L^2(D)$, since L^2 convergence of a sequence implies almost everywhere convergence of some subsequence (see [13, Lemma 2.5.2, Theorems 2.4.3 and 2.3.1]). Thus, it follows that the convex set K is *weakly closed* in $L^2(D)$, by applying Mazur's theorem (see [36, p. 125, Theorem 11]).

Hence, by combining the uniform estimate (3.7) and assertion (3.9) we obtain that

$$u = w - \lim_{\varepsilon_k \downarrow 0} u_{\varepsilon_k} \in K,$$

since we have the assertion

$$\left\{ u_{\varepsilon_k} \right\} \subset K \quad \text{for all } \varepsilon_k.$$

Summing up, we have proved that the weak limit function $u \in L^\infty(D)$, defined by formula (3.9), is a unique weak solution of the equation ($\star$) for any function $f \in C^1(\overline{D})$, and satisfies the inequality (3.10).

Step (III): Finally, let $f \in L^\infty(D)$, f arbitrary. By using Friedrichs mollifiers (see [34, Subsection 1.3.2]), we can find a sequence $\{f_n\}_{n=1}^\infty$ in the space $C^1(\overline{D})$ such that

$$\max_{x \in \overline{D}} |f_n(x)| \le \|f\|_{L^\infty(D)}, \tag{3.14a}$$

$$f_n \longrightarrow f \quad \text{in } L^2(D) \text{ as } n \to \infty. \tag{3.14b}$$

Substep (III-1): Let $u_{\varepsilon,n} \in C^{2+\theta}(\overline{D})$ be a unique solution of the homogeneous Dirichlet problem

$$\begin{cases} (\varepsilon \Delta + A) u_{\varepsilon,n} = f_n & \text{in } D, \\ u_{\varepsilon,n} = 0 & \text{on } \partial D. \end{cases} \tag{$D)_{\varepsilon,n}$}$$

Then, by Step (II) we find that the equation

$$A u_n = f_n \quad \text{in } D \tag{$\star)_n$}$$

has a unique weak solution

$$u_n = w - \lim_{\varepsilon_k \downarrow 0} u_{\varepsilon_k, n} \in L^\infty(D),$$

which satisfies the *uniform estimates* (cf. estimate (3.8) and inequalities (3.10) and (3.14a))

$$\|u_n\|_{L^2(D)} \le \frac{\sqrt{|D|}}{c_0} \|f_n\|_{L^\infty(D)} \le \frac{\sqrt{|D|}}{c_0} \|f\|_{L^\infty(D)} \quad \text{for all } n \in \mathbf{N} \tag{3.15}$$

and

$$\|u_n\|_{L^\infty(D)} \le \frac{1}{c_0} \|f_n\|_{L^\infty(D)} \le \frac{1}{c_0} \|f\|_{L^\infty(D)} \quad \text{for all } n \in \mathbf{N}. \tag{3.16}$$

Substep (III-2): First, since the unit ball in a Hilbert space is *sequentially weakly compact*, by using the uniform estimate (3.15) we can find a subsequence $\{u_{n'}\}$ of the sequence $\{u_n\}$ that converges *weakly* to some function u in $L^2(D)$ as $n' \to \infty$:

$$u = w - \lim_{n' \to \infty} u_{n'} \quad \text{in } L^2(D). \tag{3.17}$$

Hence, by letting $n' \to \infty$ in the equation $(\star)_{n'}$ we obtain from assertion (3.17) that the weak limit function u satisfies the desired formula (3.3) (see Substep (II-2)). That is, the weak limit function $u \in L^2(D)$ is a weak solution of the equation $(\star)$.

Secondly, it follows from an application of Mazur's theorem ([36, p. 125, Theorem 11]) that the set K is *weakly closed* in $L^2(D)$. Therefore, by combining the uniform estimate (3.16) and assertion (3.17) we find that $u \in K$, since $\{u_n\} \subset K$ for all $n \in \mathbf{N}$ (see Substep (II-3)). That is, the weak limit function $u \in L^\infty(D)$ satisfies the desired inequality (3.5).

The proof of Theorem 3.3 is now complete. $\square$

By applying Theorem 3.3 with

$$A := A - \lambda, \quad c := (A - \lambda)1 = c - \lambda,$$

we can obtain the following:

Theorem 3.5 *Assume that hypothesis (G) is satisfied and further that*

$$c(x) = (A1)(x) \leq 0 \quad on \, \overline{D}.$$

If $\lambda > 0$, the equation

$$(A - \lambda)u = f \quad in \, D \tag{$\star$}_\lambda$$

has a unique weak solution $u \in L^\infty(D)$ for any function $f \in L^\infty(D)$. Furthermore, the weak solution u satisfies the inequality

$$\|u\|_{L^\infty(D)} \leq \frac{1}{\lambda + c_0} \|f\|_{L^\infty(D)}, \tag{3.18}$$

where

$$c_0 := \min_{x \in \overline{D}} (-c(x)) \geq 0.$$

Rephrased, Theorem 3.5 asserts that if hypothesis (G) is satisfied, the equation $(\star)_\lambda$ has a solution with *loss of two derivatives* compared with the elliptic case (cf. [18, Theorem 1.8.1]; [9, Théorème 4.4], [24, Theorem 2.3]).

4 Regularity theorems of the equation $(\star)_\lambda$

In this section, we study the equation

$$(A - \lambda)u = f \quad in \, D \tag{$\star$}_\lambda$$

in the framework of Sobolev spaces when the spectral parameter λ tends to $+\infty$. More precisely, we will prove two *higher regularity* theorems for weak solutions of the equation $(\star)_\lambda$ constructed in Theorem 3.5 in the framework of the Sobolev spaces $W^{m,\infty}(D)$ when the spectral parameter λ tends to $+\infty$ (Theorem 4.1 for $\lambda \geq \lambda_1$ and Theorem 4.7 for $\lambda \geq \lambda_m$ with each integer $m \geq 2$). We remark that $W^{0,\infty}(D) = L^\infty(D)$. This section is the heart of the subject.

4.1 Lipschitz continuity of weak solutions of the equation $(\star)_\lambda$

In this subsection we will prove that if hypothesis (G) is satisfied, the equation $(\star)_\lambda$ has a solution with *loss of two derivatives* compared with the elliptic case (cf. [18, Theorem 1.8.1]; [9, Théorème 4.4], [24, Theorem 3.1]). More precisely, we will prove a regularity theorem for the equation $(\star)_\lambda$ in the Sobolev space $W^{1,\infty}(D)$ for $\lambda \geq \lambda_1$ (Theorem 4.1). This gives a sufficient condition for the *Lipschitz continuity* for weak solutions of the equation $(\star)_\lambda$ (Remark 4.2):

Theorem 4.1 *Assume that hypothesis (G) is satisfied. There exists a constant $\lambda_1 > 0$ such that if $\lambda \geq \lambda_1$, the equation $(\star)_\lambda$ has a unique weak solution $u \in W^{1,\infty}(D)$ for any function $f \in W^{1,\infty}(D)$. More precisely, we have the formula*

$$\int_D f \cdot v \, dx = \int_D u \cdot (A^* - \lambda) v \, dx \tag{4.1}$$

for all $v \in C^2(\overline{D})$ satisfying the condition $v = 0$ on the set Σ_1 (see Definition 3.1).

Furthermore, the weak solution u satisfies the inequality (cf. inequality (3.18))

$$\|u\|_{W^{1,\infty}(D)} \leq C_1(\lambda) \|f\|_{W^{1,\infty}(D)}, \tag{4.2}$$

where $C_1(\lambda) > 0$ is a constant independent of f.

Remark 4.2 We find from Adams–Fournier [1, Lemma 4.28], Malý–Ziemer [17, Corollary 1.73] and Ziemer [37, Theorem 2.2.1] that the weak solution $u \in W^{1,\infty}(D)$ of the equation $(\star)_\lambda$ coincides almost everywhere with a Lipschitz continuous function in D and further that there exists a constant $K_1 > 0$ such that

$$|u(x) - u(y)| \leq K_1 \|u\|_{W^{1,\infty}(D)} |x - y| \quad \text{for almost all } x, y \in D.$$

That is, $W^{1,\infty}(D) = C^{0,1}(\overline{D})$.

Steps	Elliptic regularizations	Data	Solutions
(I)	Dirichlet problem $(\widetilde{D})_{\varepsilon,\lambda}$	$\widetilde{f} \in C^2(\overline{\Omega})$	$\widetilde{u}_\varepsilon \in C^{3+\theta}(\overline{\Omega})$
(II-4)		$\widetilde{f} \in C^2(\overline{\Omega})$	$\widetilde{u} = \lim_{\varepsilon_{k'}\downarrow 0} \widetilde{u}_{\varepsilon_{k'}}$ in $C(\overline{\Omega})$ $\widetilde{u} = \lim_{\varepsilon_{k'}\downarrow 0} \widetilde{u}_{\varepsilon_{k'}} \in W^{1,\infty}(\Omega)$ $\widetilde{u}_{\varepsilon_{k'}} \in C^{3+\theta}(\overline{\Omega})$
(III-1)	Dirichlet problem $(\widetilde{D})_\lambda$	$\widetilde{f} \in C^2(\overline{\Omega})$	$\widetilde{u} = \lim_{\varepsilon_{k'}\downarrow 0} \widetilde{u}_{\varepsilon_{k'}} \in W^{1,\infty}(\Omega)$ $\widetilde{u}_{\varepsilon_{k'}} \in C^{3+\theta}(\overline{\Omega})$
(III-2)	Equation $(\star)_\lambda$ $(A - \lambda)u = f$	$f \in C^2(\overline{D})$ $\widetilde{f} \in C^2(\overline{\Omega})$ $\widetilde{f}\big|_D = f$	$u = \widetilde{u}\big|_D \in W^{1,\infty}(D)$ $\widetilde{u} = \lim_{\varepsilon_{k'}\downarrow 0} \widetilde{u}_{\varepsilon_{k'}} \in W^{1,\infty}(\Omega)$
(IV-1)	Equation $(\star)_{\lambda,n}$ $(A - \lambda)u_n = f_n$	$f_n \in C^2(\overline{D})$ $\widetilde{f_n} \in C^2(\overline{\Omega})$ $\widetilde{f_n}\big|_D = f_n$	$u_n = \widetilde{u}_n\big|_D \in W^{1,\infty}(D)$ $\widetilde{u}_n = \lim_{\varepsilon_{k'}\downarrow 0} \widetilde{u}_{\varepsilon_{k'},n} \in W^{1,\infty}(\Omega)$
(IV-2)	Equation $(\star)_\lambda$ $(A - \lambda)u = f$	$f \in W^{1,\infty}(D)$ $f = \lim_{n\to\infty} f_n$ $f_n \in C^2(\overline{D})$	$u = \lim_{n'\to\infty} u_{n'}$ in $C(\overline{D})$ $u \in W^{1,\infty}(D)$ $u_n = \widetilde{u}_n\big|_D \in W^{1,\infty}(D)$

Table 4.1 An overview of the proof of Theorem 4.1, where $W^{1,\infty}(D) = C^{0,1}(\overline{D})$

Proof. The proof of Theorem 4.1 is based on a method of *elliptic regularizations* and the *weak maximum principle*, just as in Subsection 3.2. The proof is divided into four steps (see Table 4.1).

Step (I): First, we modify the domain D and the differential operator A so that the boundary is of type Σ_3. By hypothesis (G), we can choose a bounded domain Ω with smooth boundary $\partial\Omega$ such that (see Figure 4.1)

$$\overline{D} = D \cup (\Sigma_0 \cup \Sigma_1) \subset \Omega.$$

Without loss of generality, we may assume that

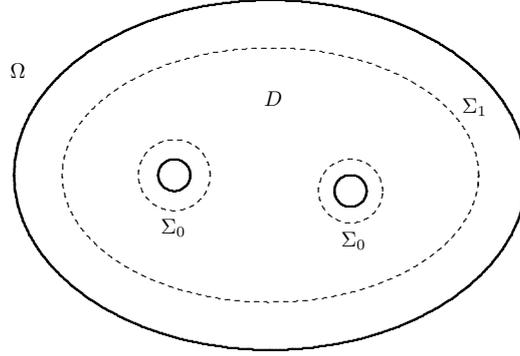$$c(x) = (A1)(x) \leq 0 \quad \text{on } \overline{\Omega}.$$

Fig. 4.1 The modified domain Ω and the original domain D

Now we take a function $a \in C^{\infty}(\overline{\Omega})$ such that

$$\begin{cases} a(x) = 0 & \text{in } D, \\ a(x) > 0 & \text{in } \overline{\Omega} \setminus \overline{D}, \end{cases}$$

and consider the homogeneous Dirichlet problem for an *elliptic* differential operator $\varepsilon \Delta + A + a(x)\Delta - \lambda$ for $\varepsilon > 0$ and $\lambda > 0$:

$$\begin{cases} (\varepsilon \Delta + A + a(x)\Delta - \lambda)\widetilde{u}_{\varepsilon} = \widetilde{f} & \text{in } \Omega, \\ \widetilde{u}_{\varepsilon} = 0 & \text{on } \partial\Omega. \end{cases} \qquad (\widetilde{D})_{\varepsilon,\lambda}$$

Then the boundary $\partial\Omega$ of the modified domain Ω is of type Σ_3 for the differential operator $A + a(x)\Delta$, since we have the condition

$$\sum_{i,j=1}^{N} a^{ij}(x')n_i n_j + a(x') \sum_{i=1}^{N} n_i^2 > 0 \quad \text{on } \partial\Omega. \tag{4.3}$$

Let $\widetilde{f} \in C^2(\overline{\Omega})$, $\widetilde{f}$ arbitrary. Then it is known (see [14, Theorem 6.19]) that the homogeneous Dirichlet problem $(\widetilde{D})_{\varepsilon,\lambda}$ has a unique solution $\widetilde{u}_{\varepsilon} \in C^{3+\theta}(\overline{\Omega})$ for all $0 < \theta < 1$, since $\widetilde{f} \in C^2(\overline{\Omega})$. Furthermore, it follows from an application of the *weak maximum principle* that

$$\|\widetilde{u}_{\varepsilon}\|_{C(\overline{\Omega})} = \max_{x \in \overline{\Omega}} |\widetilde{u}_{\varepsilon}(x)| \leq \frac{1}{\lambda} \max_{x \in \overline{\Omega}} \left|\widetilde{f}(x)\right| = \frac{1}{\lambda} \left\|\widetilde{f}\right\|_{C(\overline{\Omega})} \quad \text{for all } \varepsilon > 0, \tag{4.4}$$

since we have the inequality (cf. inequality (3.4) and formula (3.6))

$$(\varepsilon \Delta + A + a(x)\Delta - \lambda)\,1(x) = c(x) - \lambda \leq -\lambda \quad \text{on } \overline{\Omega}.$$

Step (II): In Substep (II-3), by using the uniform estimate (4.4) for the solutions $\{\widetilde{u}_{\varepsilon}\}$, we will show that there exists a subsequence $\{\widetilde{u}_{\varepsilon_{k'}}\}$ which converges *uniformly* on $\overline{\Omega}$ to a function $\widetilde{u} \in W^{1,\infty}(\Omega)$, as $\varepsilon_{k'} \downarrow 0$ (see assertions (4.17))

$$\widetilde{u} = \lim_{\varepsilon_{k'} \downarrow 0} \widetilde{u}_{\varepsilon_{k'}} \in W^{1,\infty}(\Omega),$$

The uniform convergence of the subsequence $\{\widetilde{u}_{\varepsilon_{k'}}\}$ is based on the Ascoli–Arzelà theorem.

In Substep (II-4), the proof of the assertion $\widetilde{u} \in W^{1,\infty}(\Omega)$ will be carried out by using the following lemma for the homogeneous Dirichlet problem in the *non-characteristic case* (cf. [18, Lemma 1.8.1], [24, Lemma 3.4]):

Lemma 4.3 *Assume that the boundary $\partial\Omega$ is of type Σ_3 for the differential operator A ($\partial\Omega = \Sigma_3$). Then there exists a constant $\gamma_1 > 0$ such that if $\lambda \geq \gamma_1$, the homogeneous Dirichlet problem*

$$\begin{cases} (\varepsilon\Delta + A - \lambda)\, u_\varepsilon = f & in\ \Omega, \\ u_\varepsilon = 0 & on\ \partial\Omega \end{cases} \qquad (D)_{\varepsilon,\lambda}$$

has a unique solution $u_\varepsilon \in C^{3+\theta}(\overline{\Omega})$ for any function $f \in C^{1+\theta}(\overline{\Omega})$ with $0 < \theta < 1$. Moreover, the solution u_ε satisfies the uniform estimate

$$\|u_\varepsilon\|_{C^1(\overline{\Omega})} \leq M_1(\lambda)\, \|f\|_{C^1(\overline{\Omega})} \quad for\ all\ \varepsilon > 0, \qquad (4.5)$$

where $M_1(\lambda) > 0$ is a constant independent *of $\varepsilon > 0$ and $0 < \theta < 1$.*

P r o o f. First, it follows from an application of the *weak maximum principle* for the elliptic differential operator $\varepsilon\Delta + A - \lambda$ that the solution $u_\varepsilon \in C^{3+\theta}(\overline{\Omega})$ satisfies the uniform estimate

$$\|u_\varepsilon\|_{C(\overline{\Omega})} \leq \frac{1}{\lambda}\, \|f\|_{C(\overline{\Omega})} \quad for\ all\ \varepsilon > 0, \qquad (4.6)$$

since we have the inequality (cf. inequality (3.4) and formula (3.6))

$$(\varepsilon\Delta + A - \lambda)\, 1(x) = c(x) - \lambda \leq -\lambda \quad on\ \overline{\Omega}.$$

Thus, in order to prove the uniform estimate (4.5), it suffices to show that

$$\max_{x \in \overline{\Omega}} |\mathrm{grad}\, u_\varepsilon(x)| \leq M(\lambda) \|f\|_{C^1(\overline{\Omega})} \quad for\ all\ \varepsilon > 0, \qquad (4.7)$$

where $M(\lambda) > 0$ is a constant independent of $\varepsilon > 0$ and $0 < \theta < 1$. Indeed, we then have the desired uniform estimate (4.5)

$$\|u_\varepsilon\|_{C^1(\overline{\Omega})} \leq \|u_\varepsilon\|_{C(\overline{\Omega})} + \max_{x \in \overline{\Omega}} |\mathrm{grad}\, u_\varepsilon(x)| \leq \frac{1}{\lambda}\, \|f\|_{C(\overline{\Omega})} + M(\lambda) \|f\|_{C^1(\overline{\Omega})}$$

$$\leq M_1(\lambda) \|f\|_{C^1(\overline{\Omega})} \quad for\ all\ \varepsilon > 0,$$

with

$$M_1(\lambda) := \frac{1}{\lambda} + M(\lambda).$$

Substep (II-1): In order to prove the uniform estimate (4.7), following Cancelier [9, p. 1694] we consider a continuous function associated with a second-order, *degenerate* elliptic differential operator A of the form (1.1) such that

$$(A1)(x) \leq 0 \quad on\ \overline{\Omega}.$$

Definition 4.4 We define a bilinear form $B_A(\cdot, \cdot)$ on the product space $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ by the formula

$$B_A(\varphi, \psi) := A(\varphi \cdot \psi) - A\varphi \cdot \psi - \varphi \cdot A\psi \qquad (4.8)$$

$$= 2\sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial\varphi}{\partial x_i} \frac{\partial\psi}{\partial x_j} - (A1)(x)\varphi \cdot \psi \quad for\ all\ \varphi,\ \psi \in C^2(\overline{\Omega}).$$

We remark that the function $B_A(\varphi, \varphi)$ is continuous and non-negative on $\overline{\Omega}$. Indeed, we have, for all $\varphi \in C^2(\overline{\Omega})$,

$$B_A(\varphi, \varphi)(x) = 2\sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial\varphi}{\partial x_i}(x) \frac{\partial\varphi}{\partial x_j}(x) - (A1)(x) \cdot \varphi(x)^2 \geq 0 \quad on\ \overline{\Omega},$$

since the matrix $\left(a^{ij}(x)\right)$ is non-negative definite and $(A1)(x) \leq 0$ on $\overline{\Omega}$.

The next result may be proved just as in the proof of Cancelier [9, Théorème 4.1] (cf. [24, Lemma 3.2]):

Lemma 4.5 *If* $\varphi \in C^3(\overline{\Omega})$, *we let*

$$p_1(x) = \sum_{i=1}^{N} \left| \frac{\partial \varphi}{\partial x_i}(x) \right|^2 \quad \text{for } x \in \overline{\Omega},$$

and

$$R_1(x) = (A p_1)(x) - \sum_{\ell=1}^{N} B_A \left(\frac{\partial \varphi}{\partial x_\ell}, \frac{\partial \varphi}{\partial x_\ell} \right)(x) \quad \text{for } x \in \overline{\Omega}.$$

Then, for each $\eta > 0$ *there exist constants* $\beta_0(\eta) > 0$ *and* $\beta_1(\eta) > 0$ *such that we have, for all* $x \in \overline{\Omega}$,

$$|R_1(x)| \leq \eta \sum_{\ell=1}^{N} B_A \left(\frac{\partial \varphi}{\partial x_\ell}, \frac{\partial \varphi}{\partial x_\ell} \right)(x) + \beta_0(\eta) \|\varphi\|_{C(\overline{\Omega})}^2 + \beta_1(\eta) \|\varphi\|_{C^1(\overline{\Omega})}^2 + \frac{1}{2} \|A\varphi\|_{C^1(\overline{\Omega})}^2.$$

Remark 4.6 When the differential operator A is replaced by a family $\{A + \varepsilon\Delta - \lambda\}$ of perturbed differential operators for $0 \leq \varepsilon \leq 1$ and $\lambda \geq 0$, then the constants $\beta_0(\eta)$ and $\beta_1(\eta)$ are *independent* of ε and λ.

Substep (II-2) (End of Proof of Lemma 4.3): We let

$$p_1^\varepsilon(x) = \sum_{i=1}^{N} \left| \frac{\partial u_\varepsilon}{\partial x_i}(x) \right|^2 \quad \text{for } x \in \overline{\Omega}.$$

(i) First, we assume that the function p_1^ε attains its positive maximum at an *interior point* x_0 of Ω:

$$p_1^\varepsilon(x_0) = \max_{x \in \overline{\Omega}} \sum_{i=1}^{N} \left| \frac{\partial u_\varepsilon}{\partial x_i}(x) \right|^2.$$

Then, since the matrix $\left(a^{ij}(x) \right)$ is non-negative definite, we obtain that

$$((\varepsilon\Delta + A) p_1^\varepsilon)(x_0) \leq (A1)(x_0) p_1^\varepsilon(x_0) = c(x_0) p_1^\varepsilon(x_0). \tag{4.9}$$

However, it follows from an application of Lemma 4.5 with

$$\eta := \frac{1}{2}, \quad A := \varepsilon\Delta + A - \lambda, \quad \varphi := u_\varepsilon \in C^{3+\theta}(\overline{\Omega})$$

that

$$(\varepsilon\Delta + A - \lambda) p_1^\varepsilon(x) = \sum_{\ell=1}^{N} B_{\varepsilon\Delta+A-\lambda} \left(\frac{\partial u_\varepsilon}{\partial x_\ell}, \frac{\partial u_\varepsilon}{\partial x_\ell} \right)(x) + R_1^\varepsilon(x) \quad \text{for all } x \in \overline{\Omega},$$

where the error term R_1^ε satisfies the estimate

$$|R_1^\varepsilon(x)| \leq \frac{1}{2} \sum_{\ell=1}^{N} B_{\varepsilon\Delta+A-\lambda} \left(\frac{\partial u_\varepsilon}{\partial x_\ell}, \frac{\partial u_\varepsilon}{\partial x_\ell} \right)(x) \tag{4.10}$$

$$+ \beta_0(1/2) \|u_\varepsilon\|_{C(\overline{\Omega})}^2 + \beta_1(1/2) \|u_\varepsilon\|_{C^1(\overline{\Omega})}^2 + \frac{1}{2} \|f\|_{C^1(\overline{\Omega})}^2 \quad \text{for all } x \in \overline{\Omega}.$$

Here we remark (cf. Remark 4.6) that the constants $\beta_0(1/2)$ and $\beta_1(1/2)$ in inequality (4.10) are independent of $\varepsilon > 0$ and $\lambda > 0$.

Hence, since $c(x_0) \leq 0$ we obtain from inequalities (4.9), (4.10) and (4.6) that

$$\lambda p_1^\varepsilon(x_0) \leq (\lambda - c(x_0)) p_1^\varepsilon(x_0) \leq ((\lambda - \varepsilon\Delta - A) p_1^\varepsilon)(x_0) \tag{4.11}$$

$$= -\left(((\varepsilon\Delta + A - \lambda) p_1^\varepsilon)(x_0) - \sum_{\ell=1}^{N} B_{\varepsilon\Delta+A-\lambda} \left(\frac{\partial u_\varepsilon}{\partial x_\ell}, \frac{\partial u_\varepsilon}{\partial x_\ell} \right)(x_0) \right)$$

$$- \sum_{\ell=1}^{N} B_{\varepsilon\Delta+A-\lambda} \left(\frac{\partial u_\varepsilon}{\partial x_\ell}, \frac{\partial u_\varepsilon}{\partial x_\ell} \right) (x_0)$$

$$= -R_1^\varepsilon(x_0) - \sum_{\ell=1}^{N} B_{\varepsilon\Delta+A-\lambda} \left(\frac{\partial u_\varepsilon}{\partial x_\ell}, \frac{\partial u_\varepsilon}{\partial x_\ell} \right) (x_0)$$

$$\leq -\frac{1}{2} \sum_{\ell=1}^{N} B_{\varepsilon\Delta+A-\lambda} \left(\frac{\partial u_\varepsilon}{\partial x_\ell}, \frac{\partial u_\varepsilon}{\partial x_\ell} \right) (x_0)$$

$$+ \beta_0(1/2) \|u_\varepsilon\|_{C(\overline{\Omega})}^2 + \beta_1(1/2) \|u_\varepsilon\|_{C^1(\overline{\Omega})}^2 + \frac{1}{2} \|f\|_{C^1(\overline{\Omega})}^2$$

$$\leq \beta_0(1/2) \|u_\varepsilon\|_{C(\overline{\Omega})}^2 + \beta_1(1/2) \left(\|u_\varepsilon\|_{C(\overline{\Omega})}^2 + p_1^\varepsilon(x_0) \right) + \frac{1}{2} \|f\|_{C^1(\overline{\Omega})}^2$$

$$\leq \frac{1}{\lambda^2} \left(\beta_0(1/2) + \beta_1(1/2) \right) \|f\|_{C(\overline{\Omega})}^2 + \beta_1(1/2) \, p_1^\varepsilon(x_0) + \frac{1}{2} \|f\|_{C^1(\overline{\Omega})}^2.$$

Therefore, if $\lambda > 0$ is so large that

$$\lambda \geq \gamma_1 := 2\beta_1(1/2),$$

then it follows from inequality (4.11) that

$$\max_{x \in \overline{\Omega}} p_1^\varepsilon(x) = p_1^\varepsilon(x_0) \leq C(\lambda) \|f\|_{C^1(\overline{\Omega})}^2,$$

with

$$C(\lambda) := \frac{1}{\lambda} \left(1 + 2\left(\beta_0(1/2) + \beta_1(1/2) \right) \right).$$

Thus, we have proved that

$$\max_{x \in \overline{\Omega}} p_1^\varepsilon(x) \leq C(\lambda) \|f\|_{C^1(\overline{\Omega})}^2 + \max_{x' \in \partial D} p_1^\varepsilon(x'), \tag{4.12}$$

or equivalently

$$\max_{x \in \overline{\Omega}} |\mathrm{grad}\, u_\varepsilon(x)| \leq \sqrt{C(\lambda)} \|f\|_{C^1(\overline{\Omega})} + \max_{x' \in \partial\Omega} |\mathrm{grad}\, u_\varepsilon(x')|. \tag{4.12$'$}$$

(ii) On the other hand, it follows from an application of Lemma 3.4 with

$$D := \Omega, \quad \Sigma_3 := \partial\Omega, \quad A := A - \lambda, \quad g := f, \quad v_\varepsilon := u_\varepsilon$$

that we have, by estimate (3.11a),

$$\max_{x' \in \partial\Omega} |\mathrm{grad}\, u_\varepsilon(x')| \leq M_0(\lambda) \|f\|_{C(\overline{\Omega})} \quad \text{for all } \varepsilon > 0, \tag{4.13}$$

where $M_0(\lambda) > 0$ is a constant *independent* of $\varepsilon > 0$ and $0 < \theta < 1$.

Therefore, the desired estimate (4.7) (and hence estimate (4.5)) follows by combining estimates (4.12$'$) and (4.13):

$$\max_{x \in \overline{\Omega}} |\mathrm{grad}\, u_\varepsilon(x')| \leq \sqrt{C(\lambda)} \|f\|_{C^1(\overline{\Omega})} + M_0(\lambda) \|f\|_{C(\overline{\Omega})} \leq M(\lambda) \|f\|_{C^1(\overline{\Omega})} \quad \text{for all } \varepsilon > 0,$$

with

$$M(\lambda) := \max \left\{ \sqrt{C(\lambda)}, M_0(\lambda) \right\}.$$

The proof of Lemma 4.3 is complete. $\qquad\square$

Substep (II-3): Let $\widetilde{f} \in C^2(\overline{\Omega})$, $\widetilde{f}$ arbitrary. Now we can prove that there exists a subsequence $\{\widetilde{u}_{\varepsilon_k}\}$ of the sequence $\{\widetilde{u}_{\varepsilon}\}$ which converges uniformly on $\overline{\Omega}$ to a function $\widetilde{u} \in C(\overline{\Omega})$, as $\varepsilon_k \downarrow 0$.

We recall (see condition (4.3)) that the boundary $\partial\Omega$ is of type Σ_3 for the differential operator $A + a(x)\Delta$. Hence, by applying Lemma 4.3 with

$$A := A + a(x)\Delta, \quad f := \widetilde{f} \in C^2(\overline{\Omega}), \quad u_{\varepsilon} := \widetilde{u}_{\varepsilon}, \quad \gamma_1 := \lambda_1,$$

we can find a constant $\lambda_1 > 0$ such that if $\lambda \geq \lambda_1$, then the unique solution $\widetilde{u}_{\varepsilon} \in C^{3+\theta}(\overline{\Omega})$ of the homogeneous Dirichlet problem

$$\begin{cases} (\varepsilon\Delta + A + a(x)\Delta - \lambda)\widetilde{u}_{\varepsilon} = \widetilde{f} & \text{in } \Omega, \\ \widetilde{u}_{\varepsilon} = 0 & \text{on } \partial\Omega \end{cases} \tag{$\widetilde{D}$}_{\varepsilon,\lambda}$$

satisfies the *uniform estimate*

$$\|\widetilde{u}_{\varepsilon}\|_{C^1(\overline{\Omega})} \leq \widetilde{M}_1(\lambda) \left\|\widetilde{f}\right\|_{C^1(\overline{\Omega})} \quad \text{for all } \varepsilon > 0, \tag{4.14}$$

where $\widetilde{M}_1(\lambda) > 0$ is a constant *independent* of $\varepsilon > 0$ and $0 < \theta < 1$. We find from the uniform estimate (4.14) that the sequence $\{\widetilde{u}_{\varepsilon}\}$ is *uniformly bounded* and *equicontinuous* on $\overline{\Omega}$. Hence, by virtue of the Ascoli–Arzelà theorem we can choose a subsequence $\{\widetilde{u}_{\varepsilon_k}\}$ of $\{\widetilde{u}_{\varepsilon}\}$ that converges *uniformly* to a function $\widetilde{u} \in C(\overline{\Omega})$, as $\varepsilon_k \downarrow 0$:

$$\widetilde{u} = \lim_{\varepsilon_k \downarrow 0} \widetilde{u}_{\varepsilon_k} \quad \text{in } C(\overline{\Omega}). \tag{4.15}$$

Then it follows from the uniform estimate (4.4) that the limit function $\widetilde{u}$ satisfies the inequality

$$\|\widetilde{u}\|_{C(\overline{\Omega})} \leq \frac{1}{\lambda} \left\|\widetilde{f}\right\|_{C(\overline{\Omega})}. \tag{4.16}$$

Substep (II-4): Moreover, we can choose a subsequence $\{\widetilde{u}_{\varepsilon_{k'}}\}$ of the sequence $\{\widetilde{u}_{\varepsilon_k}\}$ that converges uniformly to the function $\widetilde{u} \in C(\overline{\Omega})$, as $\varepsilon_{k'} \downarrow 0$, such that

$$\widetilde{u} = \lim_{\varepsilon_{k'} \downarrow 0} \widetilde{u}_{\varepsilon_{k'}} \quad \text{in } C(\overline{\Omega}), \tag{4.17a}$$

$$\widetilde{u} = \lim_{\varepsilon_{k'} \downarrow 0} \widetilde{u}_{\varepsilon_{k'}} \in W^{1,\infty}(\Omega), \tag{4.17b}$$

$$\|\widetilde{u}\|_{W^{1,\infty}(\Omega)} = \|\widetilde{u}\|_{L^{\infty}(\Omega)} + \|\operatorname{grad}\widetilde{u}\|_{L^{\infty}(\Omega)} \leq \widetilde{C}_1(\lambda) \left\|\widetilde{f}\right\|_{C^1(\overline{\Omega})}, \tag{4.17c}$$

with some constant $\widetilde{C}_1(\lambda) > 0$.

Indeed, by using the uniform estimate (4.14) (and inequality (3.8)) we obtain that the first partial derivative $\{\partial_j \widetilde{u}_{\varepsilon_k}\}$ satisfies the uniform estimate

$$\left\|\partial_j \widetilde{u}_{\varepsilon_k}\right\|_{L^2(\Omega)} \leq \sqrt{|\Omega|} \left\|\partial_j \widetilde{u}_{\varepsilon_k}\right\|_{L^{\infty}(\Omega)} \leq \sqrt{|\Omega|} \left\|\widetilde{u}_{\varepsilon_k}\right\|_{C^1(\overline{\Omega})} \quad \text{for all } \varepsilon_k > 0 \text{ and } 1 \leq j \leq N.$$

This proves that the sequence $\{\partial_j \widetilde{u}_{\varepsilon_k}\}$ is norm bounded in the Hilbert space $L^2(\Omega)$ for all $\varepsilon_k > 0$ and $1 \leq j \leq N$.

Since the unit ball in a Hilbert space is *sequentially weakly compact* (cf. [36, p. 126, Theorem 1]), we can find a subsequence $\{\partial_j \widetilde{u}_{\varepsilon_{k'}}\}$ of the sequence $\{\partial_j \widetilde{u}_{\varepsilon_k}\}$ and a function $\widetilde{w}_j \in L^2(\Omega)$ such that $\partial_j \widetilde{u}_{\varepsilon_{k'}}$ converges *weakly* to $\widetilde{w}_j$ in $L^2(\Omega)$ as $\varepsilon_{k'} \downarrow 0$, for all $1 \leq j \leq N$:

$$\widetilde{w}_j = w - \lim_{\varepsilon_{k'} \downarrow 0} \partial_j \widetilde{u}_{\varepsilon_{k'}} \quad \text{in } L^2(\Omega) \text{ for all } 1 \leq j \leq N. \tag{4.18}$$

Then we have, by assertion (4.15) or (4.17a),

$$\partial_j \widetilde{u} = \lim_{\varepsilon_{k'} \downarrow 0} \partial_j \widetilde{u}_{\varepsilon_{k'}} = \widetilde{w}_j \in L^2(\Omega) \quad \text{in the sense of } \textit{distributions}, \text{ for all } 1 \leq j \leq N. \tag{4.19}$$

On the other hand, it is easy to verify that the set

$$\widetilde{K} = \left\{ \widetilde{v} \in L^2(\Omega) : \|\widetilde{v}\|_{L^\infty(\Omega)} \leq \widetilde{M}_1(\lambda) \left\| \widetilde{f} \right\|_{C^1(\overline{\Omega})} \right\}$$

is convex and strongly closed in $L^2(\Omega)$. Thus, it follows from an application of Mazur's theorem that the convex set $\widetilde{K}$ is *weakly closed* in $L^2(\Omega)$ (see [36, p. 125, Theorem 11]). However, we have, by inequality (4.14),

$$\partial_j \widetilde{u}_{\varepsilon_{k'}} \in \widetilde{K} \quad \text{for all } 1 \leq j \leq N.$$

Hence, we find from assertions (4.18) and (4.19) that

$$\partial_j \widetilde{u} = \widetilde{w}_j = w - \lim_{\varepsilon_{k'} \downarrow 0} \partial_j \widetilde{u}_{\varepsilon_{k'}} \in \widetilde{K} \quad \text{for all } 1 \leq j \leq N,$$

that is,

$$\left\| \partial_j \widetilde{u} \right\|_{L^\infty(\Omega)} \leq \widetilde{M}_1(\lambda) \left\| \widetilde{f} \right\|_{C^1(\overline{\Omega})} \quad \text{for all } 1 \leq j \leq N. \tag{4.20}$$

By combining assertion (4.15) and inequalities (4.16) and (4.20), we have proved that there exists a subsequence $\{\widetilde{u}_{\varepsilon_{k'}}\}$ of $\{\widetilde{u}_{\varepsilon_k}\}$ such that the limit function

$$\widetilde{u} = \lim_{\varepsilon_{k'} \downarrow 0} \widetilde{u}_{\varepsilon_{k'}} \quad \text{in } C(\overline{\Omega})$$

satisfies the desired assertions (4.17b) and (4.17c) with

$$\widetilde{C}_1(\lambda) := \frac{1}{\lambda} + N \widetilde{M}_1(\lambda).$$

Step (III): Let $f \in C^2(\overline{D})$, f arbitrary. We will show that there exists a weak solution $u \in W^{1,\infty}(D)$ of the equation $(\star)_\lambda$ which satisfies the inequality

$$\|u\|_{W^{1,\infty}(D)} \leq \widetilde{C}_1(\lambda) \|f\|_{C^1(\overline{D})}. \tag{4.21}$$

Substep (III-1): By using the Seeley extension operator ([1, Theorems 5.21 and 5.22], [20, Theorem], [29, Theorem 4.21]), we can construct an extension $\widetilde{f}$ of f to the domain Ω in such a way that

$$\widetilde{f} \in C^2(\overline{\Omega}), \tag{4.22a}$$

$$\left\| \widetilde{f} \right\|_{C^1(\overline{\Omega})} \leq \|f\|_{C^1(\overline{D})}. \tag{4.22b}$$

Then the homogeneous Dirichlet problem $(\widetilde{D})_{\varepsilon,\lambda}$ has a unique solution $\widetilde{u}_\varepsilon \in C^{3+\theta}(\overline{\Omega})$. Hence, we obtain from Green's formula (3.2) that

$$\int_\Omega \widetilde{f} \cdot \widetilde{v} \, dx = \int_\Omega \varepsilon \Delta \widetilde{u}_\varepsilon \cdot \widetilde{v} \, dx + \int_\Omega a(x) \Delta \widetilde{u}_\varepsilon \cdot \widetilde{v} \, dx + \int_\Omega (A - \lambda) \widetilde{u}_\varepsilon \cdot \widetilde{v} \, dx \tag{4.23}$$

$$= \varepsilon \int_\Omega \widetilde{u}_\varepsilon \cdot \Delta \widetilde{v} \, dx + \int_\Omega \widetilde{u}_\varepsilon \cdot \Delta \left(a(x) \widetilde{v} \right) dx + \int_\Omega \widetilde{u}_\varepsilon \cdot (A^* - \lambda) \widetilde{v} \, dx$$

for all $\widetilde{v} \in C^2(\overline{\Omega})$ satisfying $\widetilde{v} = 0$ on $\partial\Omega$, since $a(x') \widetilde{v}(x') = 0$ on $\partial\Omega$.

However, we recall (see assertions (4.17)) that the subsequence $\{\widetilde{u}_{\varepsilon_{k'}}\}_{k'=1}^\infty$ converges uniformly to the function $\widetilde{u} \in W^{1,\infty}(\Omega)$, as $\varepsilon_{k'} \downarrow 0$:

$$\widetilde{u} = \lim_{\varepsilon_{k'} \downarrow 0} \widetilde{u}_{\varepsilon_{k'}} \in W^{1,\infty}(\Omega).$$

By letting $\varepsilon_{k'} \downarrow 0$ in formula (4.23) with $\varepsilon := \varepsilon_{k'}$, we obtain that

$$\int_\Omega \widetilde{f} \cdot \widetilde{v} \, dx = \int_\Omega \widetilde{u} \cdot \Delta \left(a(x) \widetilde{v} \right) dx + \int_\Omega \widetilde{u} \cdot (A^* - \lambda) \widetilde{v} \, dx \tag{4.24}$$

for all $\widetilde{v} \in C^2(\overline{\Omega})$ satisfying the condition $\widetilde{v} = 0$ on $\partial\Omega$ (see Definition 3.1).

Therefore, we have proved that, for any function $\widetilde{f} \in C^2(\overline{\Omega})$ the homogeneous Dirichlet problem

$$\begin{cases} (A + a(x)\Delta - \lambda)\widetilde{u} = \widetilde{f} & \text{in } \Omega, \\ \widetilde{u} = 0 & \text{on } \partial\Omega \end{cases} \qquad (\widetilde{D})_\lambda$$

has a unique weak solution $\widetilde{u} \in W^{1,\infty}(\Omega)$ which satisfies inequality (4.17c).

Substep (III-2): If $\widetilde{f} \in C^2(\overline{\Omega})$ is a Seeley extension of f, then the homogeneous Dirichlet problem $(\widetilde{D})_\lambda$ has a unique weak solution

$$\widetilde{u} = \lim_{\varepsilon_{k'} \downarrow 0} \widetilde{u}_{\varepsilon_{k'}} \in W^{1,\infty}(\Omega),$$

as is shown in Substep (III-1). In this substep, we will show that the *restriction*

$$u := \widetilde{u}|_D \in W^{1,\infty}(D)$$

is a weak solution of the desired equation

$$(A - \lambda)u = \widetilde{f}|_D = f \quad \text{in } D. \qquad (\star)_\lambda$$

We remark that the desired inequality (4.21) follows from inequalities (4.17c) and inequality (4.22b), since we have the inequality

$$\|u\|_{W^{1,\infty}(D)} \le \|\widetilde{u}\|_{W^{1,\infty}(\Omega)} \le \widetilde{C}_1(\lambda) \left\|\widetilde{f}\right\|_{C^1(\overline{\Omega})} \le \widetilde{C}_1(\lambda) \|f\|_{C^1(\overline{D})}.$$

Step (1): We study formula (4.1) for all $v \in C^2(\overline{D})$ satisfying the condition $v = 0$ on the set Σ_1 (see Definition 3.1). To do so, we replace the function v by a function $v\phi_\delta$ for $\delta > 0$ sufficiently small, where $\phi_\delta \in C^\infty(\overline{\Omega})$ such that $0 \le \phi_\delta \le 1$ on $\overline{\Omega}$ and that $\phi_\delta = 0$ in the δ-neighborhood G_δ of $\Sigma_0 \cup \Sigma_1$ and in $\Omega \setminus D$ and further that $\phi_\delta = 1$ in D outside the 2δ-neighborhood of $\Sigma_0 \cup \Sigma_1$ (see formula (4.26) below), just as in the proof of [18, Theorem 1.8.1].

More precisely, by hypothesis (G) we can introduce a local coordinate system

$$y = (y_1, y_2, \ldots, y_N)$$

in a tubular neighborhood of the boundary $\partial\Omega$ such that

$$\Omega = \{y_N > 0\}, \quad \partial\Omega = \{y_N = 0\}.$$

Assume that, in terms of this coordinate system, the adjoint differential operator A^* of A is of the form

$$A^*v = \sum_{i,j=1}^{N} \alpha^{ij}(y) \frac{\partial^2 v}{\partial y_i \partial y_j} + \sum_{i=1}^{N} \beta^i(y) \frac{\partial v}{\partial y_i} + c^*(y)v. \qquad (4.25)$$

If $\delta > 0$ is sufficiently small, we choose a function $\phi_\delta \in C^\infty(\overline{\Omega})$ such that $0 \le \phi_\delta(y) \le 1$ on $\overline{\Omega}$ and further that

$$\phi_\delta(y) = \begin{cases} 0 & \text{in the } \delta\text{-neighborhood } G_\delta \text{ of } \Sigma_0 \cup \Sigma_1 \text{ and in } \Omega \setminus D, \\ 1 & \text{in } D \text{ outside the } 2\delta\text{-neighborhood } G_{2\delta} \text{ of } \Sigma_0 \cup \Sigma_1. \end{cases} \qquad (4.26)$$

We may assume that the function $\phi_\delta(y)$ depends only on the variable y_N and that we have, as $\delta \downarrow 0$,

$$\frac{\partial \phi_\delta}{\partial y_N} = O\left(\delta^{-1}\right) \text{ and } \frac{\partial^2 \phi_\delta}{\partial y_N^2} = O\left(\delta^{-2}\right) \quad \text{in a tubular neighborhood of } \partial\Omega.$$

Now let $v \in C^2(\overline{D})$, v arbitrary, such that $v = 0$ on Σ_1. Then it follows that the function $v\phi_\delta$ belongs to $C^2(\overline{D})$ and satisfies the condition

$$v(x')\phi_\delta(x') = 0 \quad \text{on } \partial\Omega.$$

Thus, by applying formula (4.24) to the function $\widetilde{v} := v\phi_\delta \in C^2(\overline{\Omega})$ we obtain from condition (4.26) that

$$\int_D f \cdot v\phi_\delta \, dx = \int_D u \cdot (A^* - \lambda)(v\phi_\delta) \, dx, \tag{4.27}$$

since $a(x)v(x)\phi_\delta(x) = 0$ in Ω and $u = \widetilde{u}|_D$.

Step (2): We will show that formula (4.27) tends to formula (4.1) as $\delta \downarrow 0$.

(i) First, by using the Lebesgue dominated convergence theorem we obtain from condition (4.26) that the left-hand side of formula (4.26) tends to the left-hand side of formula (4.1) as $\delta \downarrow 0$:

$$\lim_{\delta \downarrow 0} \int_D f \cdot v\phi_\delta \, dx = \int_D f \cdot v \, dx. \tag{4.28}$$

(ii) Secondly, we can rewrite the right-hand side of formula (4.27) in the following form:

$$\int_D u \cdot (A^* - \lambda)(v\phi_\delta) \, dx = \int_D u \cdot ((A^* - \lambda)v) \phi_\delta \, dx \tag{4.29}$$

$$+ \int_D uv \cdot (A^*\phi_\delta - c^*(y)\phi_\delta) \, dx + 2 \int_D u \cdot \left(\sum_{i,j=1}^N \alpha^{ij}(y) \frac{\partial v}{\partial y_i} \frac{\partial \phi_\delta}{\partial y_j} \right) dx$$

$$:= I_1^\delta + I_2^\delta + I_3^\delta.$$

Indeed, by formula (4.25) it suffices to note that

$$A^*(v\phi_\delta) = \sum_{i,j=1}^N \alpha^{ij}(y) \frac{\partial^2}{\partial y_i \partial y_j}(v\phi_\delta) + \sum_{i=1}^N \beta^i(y) \frac{\partial}{\partial y_i}(v\phi_\delta) + c^*(y)(v\phi_\delta)$$

$$= \left(\sum_{i,j=1}^N \alpha^{ij}(y) \frac{\partial^2 v}{\partial y_i \partial y_j} \right) \phi_\delta + 2 \sum_{i,j=1}^N \alpha^{ij}(y) \frac{\partial v}{\partial y_i} \frac{\partial \phi_\delta}{\partial y_j} + \left(\sum_{i=1}^N \beta^i(y) \frac{\partial v}{\partial y_i} \right) \phi_\delta$$

$$+ (c^*(y)v) \phi_\delta + v \left(\sum_{i,j=1}^N \alpha^{ij}(y) \frac{\partial^2 \phi_\delta}{\partial y_i \partial y_j} \right) + v \left(\sum_{i=1}^N \beta^i(y) \frac{\partial \phi_\delta}{\partial y_i} \right)$$

$$= A^*v \cdot \phi_\delta + v \cdot (A^*\phi_\delta - c^*(y)\phi_\delta) + 2 \sum_{i,j=1}^N \alpha^{ij}(y) \frac{\partial v}{\partial y_i} \frac{\partial \phi_\delta}{\partial y_j},$$

since $\alpha^{ij}(y) = \alpha^{ji}(y)$ for $1 \leq i, j \leq N$.

Now we will calculate the limit of the three terms I_1^δ, I_2^δ and I_3^δ as $\delta \downarrow 0$.

(ii-a) For the term I_1^δ, we have, by the Lebesgue dominated convergence theorem,

$$\lim_{\delta \downarrow 0} I_1^\delta = \int_D u \cdot (A^* - \lambda) v \, dx, \tag{4.30}$$

just as in assertion (4.29).

(ii-b) For the terms I_2^δ and I_3^δ, we remark that the integrals I_2^δ and I_3^δ are taken over the 2δ-neighborhood $G_{2\delta}$ of the set

$$\partial D = \Sigma_0 \cup \Sigma_1$$

where the functions $\frac{\partial \phi_\delta}{\partial x_i}$ and $\frac{\partial^2 \phi_\delta}{\partial x_i \partial x_j}$ may be different from zero. Thus, by passing to the local coordinate system $(y_1, y_2, \ldots, y_N)$ we obtain from formula (4.25) of A^* that

$$I_2^\delta = \int_{G_{2\delta}} \left(\alpha^{NN}(y) \frac{\partial^2 \phi_\delta}{\partial y_N^2} + \beta^N(y) \frac{\partial \phi_\delta}{\partial y_N} \right) v(y)u(y)\kappa(y) \, dy,$$

$$I_3^\delta = 2 \int_{G_{2\delta}} \alpha^{NN}(y) \frac{\partial v}{\partial y_N} \frac{\partial \phi_\delta}{\partial y_N} u(y)\kappa(y) \, dy + 2 \sum_{i=1}^{N-1} \int_{G_{2\delta}} \alpha^{iN}(y) \frac{\partial v}{\partial y_i} \frac{\partial \phi_\delta}{\partial y_N} u(y)\kappa(y) \, dy,$$

since the function $\phi_\delta(y)$ depends *only* on the variable y_N. Here $\kappa(y)$ is some smooth density function.

First, we consider the limit of the term I_2^δ as $\delta \downarrow 0$: Since we have the three assertions

- $\alpha^{NN}(y) = O(\delta^2)$ near the set $\Sigma_0 \cup \Sigma_1$,

- $\dfrac{\partial^2 \phi_\delta}{\partial y_N^2} = O(\delta^{-2})$ near the set $\Sigma_0 \cup \Sigma_1$,

- the measure $|G_{2\delta}|$ of $G_{2\delta}$ is of order δ,

it follows that

$$\lim_{\delta\downarrow 0} \int_{G_{2\delta}} \alpha^{NN}(y) \frac{\partial^2 \phi_\delta}{\partial y_N^2} v(y)u(y)\kappa(y)\,dy = \lim_{\delta\downarrow 0} O(\delta) = 0. \tag{4.31}$$

On the other hand, we remark that $v = 0$ on the set Σ_1 and further that the function $\beta^N(y)$ coincides with the Fichera function $b^*(y) = -b(y)$ for the operator A^* on the set Σ_0 (see formulas (1.3d) and (3.1)). This implies that

- $v(y) = O(\delta)$ near Σ_1,
- $\beta^N(y) = O(\delta)$ near Σ_0.

Hence, we have the assertion

$$\lim_{\delta\downarrow 0} \int_{G_{2\delta}} \beta^N(y) \frac{\partial \phi_\delta}{\partial y_N} v(y)u(y)\kappa(y)\,dy = \lim_{\delta\downarrow 0} O(\delta) = 0, \tag{4.32}$$

since $\frac{\partial \phi_\delta}{\partial y_N} = O(\delta^{-1})$ and $|G_{2\delta}| = O(\delta)$ as $\delta \downarrow 0$.

Therefore, we obtain from formulas (4.31) and (4.32) that

$$\lim_{\delta\downarrow 0} I_2^\delta = 0. \tag{4.33}$$

(ii-c) Next we consider the limit of the term I_3^δ as $\delta \downarrow 0$: Since we have the assertions

- $\alpha^{NN}(y) = O(\delta^2)$ near $\Sigma_0 \cup \Sigma_1$,

- $\dfrac{\partial \phi_\delta}{\partial y_N} = O(\delta^{-1})$ near $\Sigma_0 \cup \Sigma_1$,

it follows that

$$\lim_{\delta\downarrow 0} \int_{G_{2\delta}} \alpha^{NN}(y) \frac{\partial v}{\partial y_N} \frac{\partial \phi_\delta}{\partial y_N} u(y)\kappa(y)\,dy = \lim_{\delta\downarrow 0} O(\delta^2) = 0. \tag{4.34}$$

Furthermore, since the matrix $\left(\alpha^{ij}(x)\right)$ is non-negative definite, we find that

$$\alpha^{iN}(y') = 0 \quad \text{on } \Sigma_0 \cup \Sigma_1 \text{ for } 1 \le i \le N-1,$$

and so

- $\alpha^{iN}(y) = O(\delta)$ near $\Sigma_0 \cup \Sigma_1$ for $1 \le i \le N-1$.

Thus, we have the assertion

$$\lim_{\delta\downarrow 0} \sum_{i=1}^{N-1} \int_{G_{2\delta}} \alpha^{iN}(y) \frac{\partial v}{\partial y_i} \frac{\partial \phi_\delta}{\partial y_N} u(y)\kappa(y)\,dy = \lim_{\delta\downarrow 0} O(\delta) = 0, \tag{4.35}$$

since $\frac{\partial \phi_\delta}{\partial y_N} = O(\delta^{-1})$ and $|G_{2\delta}| = O(\delta)$ as $\delta \downarrow 0$.

Hence, we obtain from formulas (4.34) and (4.35) that

$$\lim_{\delta\downarrow 0} I_3^{\delta} = 0. \tag{4.36}$$

(ii-d) Summing up, we obtain from formulas (4.29), (4.30), (4.33) and (4.36) that the right-hand side of formula (4.27) tends to the right-hand side of formula (4.1) as $\delta \downarrow 0$:

$$\lim_{\delta\downarrow 0} \int_D u \cdot (A^* - \lambda)(v\phi_\delta)\,dx = \lim_{\delta\downarrow 0} I_1^{\delta} + \lim_{\delta\downarrow 0} I_2^{\delta} + \lim_{\delta\downarrow 0} I_3^{\delta} = \int_D u \cdot (A^* - \lambda)v\,dx. \tag{4.37}$$

(iii) Therefore, the desired formula (4.1) follows from formula (4.27) by combining formulas (4.28) and (4.37), provided that $f \in C^2(\overline{D})$.

Step (IV): Finally, let $f \in W^{1,\infty}(D) = C^{0,1}(\overline{D})$, f arbitrary. We will prove that the equation $(\star)_\lambda$ has a weak solution $u \in W^{1,\infty}(D) = C^{0,1}(\overline{D})$ which satisfies the desired inequality (4.2).

Substep (IV-1): By using Friedrichs mollifiers (see [34, Subsection 1.3.2]) and Remark 4.2, we can find a sequence $\{f_n\}_{n=1}^{\infty}$ in $C^2(\overline{D})$ such that

$$\|f_n\|_{C^1(\overline{D})} \le \|f\|_{W^{1,\infty}(D)}, \tag{4.38a}$$

$$f_n \longrightarrow f \quad \text{in } C(\overline{D}) \text{ as } n \to \infty. \tag{4.38b}$$

Let $\widetilde{f_n}$ be a Seeley extension of f_n to the domain Ω such that

$$\widetilde{f_n} \in C^2(\overline{\Omega}),$$

$$\left\|\widetilde{f_n}\right\|_{C^1(\overline{\Omega})} \le \|f_n\|_{C^1(\overline{D})} \le \|f\|_{W^{1,\infty}(D)}.$$

Let $\widetilde{u}_{\varepsilon,n} \in C^{3+\theta}(\overline{\Omega})$ be a unique solution of the homogeneous Dirichlet problem

$$\begin{cases} (\varepsilon\Delta + A + a(x)\Delta - \lambda)\widetilde{u}_{\varepsilon,n} = \widetilde{f_n} & \text{in } \Omega, \\ \widetilde{u}_{\varepsilon,n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\widetilde{D}$}_{\varepsilon,\lambda,n}$$

Then, by Substep (III-1) we find that the homogeneous Dirichlet problem

$$\begin{cases} (A + a(x)\Delta - \lambda)\widetilde{u}_n = \widetilde{f_n} & \text{in } \Omega, \\ \widetilde{u}_n = 0 & \text{on } \partial\Omega \end{cases} \tag{$\widetilde{D}$}_{\lambda,n}$$

has a unique weak solution

$$\widetilde{u}_n = \lim_{\varepsilon_{k'}\downarrow 0} \widetilde{u}_{\varepsilon_{k'},n} \in W^{1,\infty}(\Omega).$$

Hence, it follows from an application of Step (III-2) with $\widetilde{f} := \widetilde{f_n}$ that the *restriction*

$$u_n = \widetilde{u}_n|_D \in W^{1,\infty}(D)$$

is a weak solution of the equation

$$(A - \lambda)u_n = \widetilde{f_n}|_D = f_n \quad \text{in } D. \tag{$\star$}_{\lambda,n}$$

Furthermore, it follows from estimate (4.17c) with $\widetilde{f} := \widetilde{f_n}$ and $\widetilde{u} := \widetilde{u}_n$ and inequality (4.38a) that the solution u_n satisfies the *uniform estimate*

$$\|u_n\|_{W^{1,\infty}(D)} \le \|\widetilde{u}_n\|_{W^{1,\infty}(\Omega)} \le \widetilde{C}_1(\lambda)\left\|\widetilde{f_n}\right\|_{C^1(\overline{\Omega})} \le \widetilde{C}_1(\lambda)\|f_n\|_{C^1(\overline{D})} \tag{4.39}$$

$$\le \widetilde{C}_1(\lambda)\|f\|_{W^{1,\infty}(D)} \quad \text{for all } n \in \mathbf{N}.$$

Substep (IV-2): By using a Sobolev imbedding theorem (see Remark 4.2), we find from the uniform estimate (4.39) that the sequence $\{u_n\}_{n=1}^{\infty}$ is *uniformly bounded* and *equicontinuous* on the closure $\overline{D}$. Thus, by virtue of the Ascoli–Arzelà theorem we can choose a subsequence $\{u_{n'}\}$ of the sequence $\{u_n\}$ that converges uniformly to a function u in $C(\overline{D})$ as $n' \to \infty$:

$$u = \lim_{n' \to \infty} u_{n'} \quad \text{in } C(\overline{D}). \tag{4.40}$$

Therefore, by using the equation $(\star)_{\lambda,n}$ with $u_n := u_{n'}$, $f_n := f_{n'}$ and assertions (4.40) and (4.38b) we obtain that

$$\int_D u \cdot (A^* - \lambda) v \, dx = \lim_{n' \to \infty} \int_D u_{n'} \cdot (A^* - \lambda) v \, dx = \lim_{n' \to \infty} \int_D f_{n'} \cdot v \, dx$$
$$= \int_D f \cdot v \, dx \quad \text{for all } v \in C^2(\overline{D}) \text{ satisfying the condition } v = 0 \text{ on } \Sigma_1.$$

This proves that the limit function $u \in C(\overline{D})$ is a weak solution of the equation $(\star)_{\lambda}$ (see Definition 3.1 with $A := A - \lambda$):

$$(A - \lambda) u = f \quad \text{in } D. \tag{$\star$}_{\lambda}$$

Moreover, just as in the proof of Step (II) (cf. inequalities (4.16) and (4.20)) we find from the uniform estimate (4.39) that

$$\begin{cases} u \in W^{1,\infty}(D), \\ \|u\|_{W^{1,\infty}(D)} \leq C_1(\lambda) \|f\|_{W^{1,\infty}(D)}, \end{cases} \tag{4.2}$$

where the constant $C_1(\lambda) = \widetilde{C}_1(\lambda)$ is independent of f.

The proof of Theorem 4.1 is now complete. $\qquad\square$

4.2 Higher regularity of weak solutions of the equation $(\star)_{\lambda}$

In this subsection we will prove a *higher regularity* theorem of weak solutions of the equation $(\star)_{\lambda}$ in the Sobolev space $W^{m,\infty}(D)$ for $\lambda \geq \lambda_m$ with each integer $m \geq 2$ (Theorem 4.7 and Remark 4.8), generalizing Theorem 4.1 for $\lambda \geq \lambda_1$:

Theorem 4.7 *Assume that hypothesis (G) is satisfied. For each integer $m \geq 2$, we can find a constant $\lambda_m > 0$ such that if $\lambda \geq \lambda_m$, the equation $(\star)_{\lambda}$ has a weak solution $u \in W^{m,\infty}(D)$ for any function $f \in W^{m,\infty}(D)$. More precisely, we have the formula*

$$\int_D f \cdot v \, dx = \int_D u \cdot (A^* - \lambda) v \, dx \tag{4.1}$$

for all $v \in C^2(\overline{D})$ satisfying the condition $v = 0$ on the set Σ_1 (see Definition 3.1).

Furthermore, the weak solution u satisfies the inequality

$$\|u\|_{W^{m,\infty}(D)} \leq C_m(\lambda) \|f\|_{W^{m,\infty}(D)}, \tag{4.41}$$

where $C_m(\lambda) > 0$ is a constant independent of f.

Remark 4.8 We find from Adams–Fournier [1, Theorem 4.12, Part II], Malý–Ziemer [17, Corollary 1.73] and Ziemer [37, Theorem 2.2.1] that the weak solution $u \in W^{m,\infty}(D)$ of the equation $(\star)_{\lambda}$ coincides almost everywhere with a function of $C^{m-1}(\overline{D})$ in D such that the derivatives $\partial^{\alpha} u$ for $|\alpha| = m - 1$ are Lipschitz continuous almost everywhere in D and further that there exists a constant $K_m > 0$ such that

$$\sum_{|\alpha|=m-1} |\partial^{\alpha} u(x) - \partial^{\alpha} u(y)| \leq K_m \|u\|_{W^{m,\infty}(D)} |x - y| \quad \text{for almost all } x, y \in D.$$

That is, $W^{m,\infty}(D) = C^{m-1,1}(\overline{D})$.

Steps	Elliptic regularizations	Data	Solutions
(I)	Dirichlet problem $(\widetilde{D})_{\varepsilon,\lambda}$	$\widetilde{f} \in C^3(\overline{\Omega})$	$\widetilde{u}_\varepsilon \in C^{4+\theta}(\overline{\Omega})$
(III)	Dirichlet problem $(\widetilde{D})_\lambda$	$\widetilde{f} \in C^3(\overline{\Omega})$	$\widetilde{u} = \lim_{\varepsilon_k \downarrow 0} \widetilde{u}_{\varepsilon_k}$ in $C^1(\overline{\Omega})$ $\widetilde{u} = \lim_{\varepsilon_k \downarrow 0} \widetilde{u}_{\varepsilon_k} \in W^{2,\infty}(\Omega)$ $\widetilde{u}_{\varepsilon_k} \in C^{4+\theta}(\overline{\Omega})$
(IV)	Equation $(\star)_\lambda$ $(A-\lambda)u = f$	$f \in C^3(\overline{D})$ $\widetilde{f} \in C^3(\overline{\Omega})$ $\widetilde{f}\big|_D = f$	$u = \widetilde{u}|_D \in W^{2,\infty}(D)$ $\widetilde{u} = \lim_{\varepsilon_k \downarrow 0} \widetilde{u}_{\varepsilon_k} \in W^{2,\infty}(\Omega)$
(V-1)	Equation $(\star)_{\lambda,n}$ $(A-\lambda)u_n = f_n$	$f_n \in C^3(\overline{D})$ $\widetilde{f_n} \in C^3(\overline{\Omega})$ $\widetilde{f_n}\big|_D = f_n$	$u_n = \widetilde{u}_n|_D \in W^{2,\infty}(D)$ $\widetilde{u}_n = \lim_{\varepsilon_k \downarrow 0} \widetilde{u}_{\varepsilon_k,n} \in W^{2,\infty}(\Omega)$
(V-2)	Equation $(\star)_\lambda$ $(A-\lambda)u = f$	$f \in W^{2,\infty}(D)$ $f = \lim_{n\to\infty} f_n$ $f_n \in C^3(\overline{D})$	$u = \lim_{n'\to\infty} u_{n'}$ in $C^1(\overline{D})$ $u \in W^{2,\infty}(D)$ $u_n \in W^{2,\infty}(D)$

Table 4.2 An overview of the proof of Theorem 4.7 for $m = 2$, where $W^{2,\infty}(D) = C^{1,1}(\overline{D})$

P r o o f. We prove Theorem 4.7 only in the case $m = 2$. The general case may be proved by induction on m for $m \geq 2$ (see the proof of [18, Theorem 1.9.1]).

The proof is based on a method of *elliptic regularizations* and the *weak maximum principle*, just as in Subsection 4.1. The proof is divided into five steps (see Table 4.2).

Step (I): We modify the domain D and the differential operator A so that the boundary is of type Σ_3, just as in the proof of Theorem 4.1 (see Figure 4.1):

$$\overline{D} = D \cup (\Sigma_0 \cup \Sigma_1) \subset \Omega,$$
$$(A1)(x) = c(x) \leq 0 \quad \text{on } \overline{\Omega}.$$

We take a function $a \in C^\infty(\overline{\Omega})$ such that

$$\begin{cases} a(x) = 0 & \text{in } D, \\ a(x) > 0 & \text{in } \overline{\Omega} \setminus \overline{D}, \end{cases}$$

and consider the homogeneous Dirichlet problem $(\widetilde{D})_{\varepsilon,\lambda}$ for the *elliptic* differential operator

$$\varepsilon\Delta + A + a(x)\Delta - \lambda, \quad \varepsilon > 0.$$

We recall (condition (4.3)) that the boundary $\partial\Omega$ of the modified domain Ω is of type Σ_3 for the differential operator $A + a(x)\Delta$.

First, let $f \in C^3(\overline{D})$, f arbitrary. We will show that there exists a weak solution $u \in W^{2,\infty}(D)$ of the equation $(\star)_\lambda$ which satisfies inequality (4.41) for $m = 2$.

By using the Seeley extension operator, we can find an extension $\widetilde{f}$ of f to the domain Ω such that

$$\widetilde{f} \in C^3(\overline{\Omega}), \tag{4.42a}$$

$$\left\|\widetilde{f}\right\|_{C^2(\overline{\Omega})} \leq \|f\|_{C^2(\overline{D})}. \tag{4.42b}$$

Then it is known (cf. [14, Theorem 6.19]) that the homogeneous Dirichlet problem

$$\begin{cases} (\varepsilon\Delta + A + a(x)\Delta - \lambda)\widetilde{u}_\varepsilon = \widetilde{f} & \text{in } \Omega, \\ \widetilde{u}_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{$\widetilde{D})_{\varepsilon,\lambda}$}$$

has a unique solution $\widetilde{u}_\varepsilon \in C^{4+\theta}(\overline{\Omega})$ for all $0 < \theta < 1$, since $\widetilde{f} \in C^3(\overline{\Omega})$. We remark (see the uniform estimate (4.4)) that the solution $\widetilde{u}_\varepsilon$ satisfies the uniform estimate

$$\|\widetilde{u}_\varepsilon\|_{C(\overline{\Omega})} \leq \frac{1}{\lambda} \left\| \widetilde{f} \right\|_{C(\overline{\Omega})} \leq \frac{1}{\lambda} \|f\|_{C(\overline{D})} \quad \text{for all } \varepsilon > 0. \tag{4.43}$$

Step (II): In Step (III), by using the uniform estimate (4.43), we will show that there exists a subsequence $\{\widetilde{u}_{\varepsilon_k}\}_{k=1}^\infty$ of the sequence $\{\widetilde{u}_\varepsilon\}$ which, together with all their derivatives of order ≤ 2, converges weakly to some function $\widetilde{u} \in W^{2,\infty}(\Omega)$, as $\varepsilon_k \downarrow 0$ (see assertions (4.59)):

$$\widetilde{u} = \lim_{\varepsilon_k \downarrow 0} \widetilde{u}_{\varepsilon_k} \in W^{2,\infty}(\Omega).$$

The proof of the assertion $\widetilde{u} \in W^{2,\infty}(\Omega)$ is based on the following lemma for the homogeneous Dirichlet problem in the *non-characteristic case*, analogous to Lemma 4.3 (cf. [18, Lemma 1.8.1], [24, Lemma 3.7]):

Lemma 4.9 *Assume that the boundary $\partial\Omega$ is of type Σ_3 for the differential operator A ($\partial\Omega = \Sigma_3$). Then there exists a constant $\gamma_2 > 0$ such that if $\lambda \geq \gamma_2$, the homogeneous Dirichlet problem*

$$\begin{cases} (\varepsilon\Delta + A - \lambda)u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{D}_{\varepsilon,\lambda}$$

has a unique solution $u_\varepsilon \in C^{4+\theta}(\overline{\Omega})$ for any function $f \in C^{2+\theta}(\overline{\Omega})$ with $0 < \theta < 1$. Moreover, the solution u_ε satisfies the uniform estimate

$$\|u_\varepsilon\|_{C^2(\overline{\Omega})} \leq M_2(\lambda) \|f\|_{C^2(\overline{\Omega})} \quad \text{for all } \varepsilon > 0, \tag{4.44}$$

where $M_2(\lambda) > 0$ is a constant independent *of $\varepsilon > 0$ and $0 < \theta < 1$.*

Proof. We recall that

$$\|u_\varepsilon\|_{C^1(\overline{\Omega})} \leq M_1(\lambda) \|f\|_{C^1(\overline{\Omega})} \quad \text{for all } \varepsilon > 0. \tag{4.5}$$

Thus, in order to prove estimate (4.44), it suffices to show that

$$\left(\max_{x \in \overline{\Omega}} \sum_{i,j=1}^N \left| \frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}(x) \right|^2 \right)^{1/2} \leq M(\lambda) \left(\|u_\varepsilon\|_{C^1(\overline{\Omega})} + \|f\|_{C^2(\overline{\Omega})} \right), \tag{4.45}$$

where $M(\lambda) > 0$ is a constant independent of $\varepsilon > 0$ and $0 < \theta < 1$. Indeed, we then have the desired estimate

$$\begin{aligned} \|u_\varepsilon\|_{C^2(\overline{\Omega})} &\leq \|u_\varepsilon\|_{C^1(\overline{\Omega})} + \left(\max_{x \in \overline{\Omega}} \sum_{i,j=1}^N \left| \frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}(x) \right|^2 \right)^{1/2} \\ &\leq \|u_\varepsilon\|_{C^1(\overline{\Omega})} + M(\lambda) \left(\|u_\varepsilon\|_{C^1(\overline{\Omega})} + \|f\|_{C^2(\overline{\Omega})} \right) \\ &\leq M_2(\lambda) \|f\|_{C^2(\overline{\Omega})} \quad \text{for all } \varepsilon > 0, \end{aligned}$$

with

$$M_2(\lambda) := \max\left\{ M_1(\lambda)(1 + M(\lambda)), M(\lambda) \right\}.$$

Substep (II-1): In order to prove the uniform estimate (4.43), we make use of the following lemma, analogous to Lemma 4.5 (cf. [9, Théorème 4.1], [24, Lemma 3.6]):

Lemma 4.10 *If $\varphi \in C^4(\overline{\Omega})$, we let*

$$p_2(x) = \sum_{i,j=1}^N \left| \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x) \right|^2 \quad \text{for } x \in \overline{\Omega},$$

and

$$R_2(x) = (Ap_2)(x) - \sum_{\ell,m=1}^{N} B_A \left(\frac{\partial^2 \varphi}{\partial x_\ell \partial x_m}, \frac{\partial^2 \varphi}{\partial x_\ell \partial x_m} \right)(x) \quad for \ x \in \overline{D},$$

where the function $B_A(\cdot, \cdot)$ is defined by formula (4.8). Then, for each $\eta > 0$ there exist constants $\beta_1(\eta) > 0$ and $\beta_2(\eta) > 0$ such that we have, for all $x \in \overline{\Omega}$,

$$|R_2(x)| \leq \eta \sum_{\ell,m=1}^{N} B_A \left(\frac{\partial^2 \varphi}{\partial x_\ell \partial x_m}, \frac{\partial^2 \varphi}{\partial x_\ell \partial x_m} \right)(x) + \beta_1(\eta) \|\varphi\|_{C^1(\overline{\Omega})}^2 + \beta_2(\eta) \|\varphi\|_{C^2(\overline{\Omega})}^2 + \frac{1}{2} \|A\varphi\|_{C^2(\overline{\Omega})}^2.$$

Remark 4.11 When the differential operator A is replaced by a family $\{A + \varepsilon\Delta - \lambda I\}$ of perturbed differential operators for $0 \leq \varepsilon \leq 1$ and $\lambda \geq 0$, then the constants $\beta_1(\eta)$ and $\beta_2(\eta)$ are *independent* of ε and λ.

Substep (II-2) (End of Proof of Lemma 4.9): We let

$$p_2^\varepsilon(x) = \sum_{i,j=1}^{N} \left| \frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}(x) \right|^2 \quad for \ x \in \overline{\Omega}.$$

(i) First, we assume that the function p_2^ε attains its positive maximum at an i*interior point x_0 of Ω:*

$$p_2^\varepsilon(x_0) = \max_{x \in \overline{\Omega}} \sum_{i,j=1}^{N} \left| \frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}(x) \right|^2.$$

Then, since the matrix $\left(a^{ij}(x)\right)$ is non-negative definite, we obtain that

$$((\varepsilon\Delta + A) p_2^\varepsilon)(x_0) \leq (A1)(x_0) p_2^\varepsilon(x_0) = c(x_0) p_2^\varepsilon(x_0). \tag{4.46}$$

However, it follows from an application of Lemma 4.10 with

$$\eta := \frac{1}{2}, \quad A := \varepsilon\Delta + A - \lambda I, \quad \varphi := u_\varepsilon \in C^{4+\theta}(\overline{\Omega}),$$

that

$$(\varepsilon\Delta + A - \lambda) p_2^\varepsilon(x) = \sum_{\ell,m=1}^{N} B_{\varepsilon\Delta + A - \lambda} \left(\frac{\partial^2 u_\varepsilon}{\partial x_\ell \partial x_m}, \frac{\partial^2 u_\varepsilon}{\partial x_\ell \partial x_m} \right)(x) + R_2^\varepsilon(x) \quad for \ all \ x \in \overline{\Omega},$$

where the error term R_2^ε satisfies the inequality

$$|R_2^\varepsilon(x)| \leq \frac{1}{2} \sum_{\ell,m=1}^{N} B_{\varepsilon\Delta + A - \lambda} \left(\frac{\partial^2 u_\varepsilon}{\partial x_\ell \partial x_m}, \frac{\partial^2 u_\varepsilon}{\partial x_\ell \partial x_m} \right)(x) \tag{4.47}$$

$$+ \beta_1(1/2) \|u_\varepsilon\|_{C^1(\overline{\Omega})}^2 + \beta_2(1/2) \|u_\varepsilon\|_{C^2(\overline{\Omega})}^2 + \frac{1}{2} \|f\|_{C^2(\overline{\Omega})}^2 \quad for \ all \ x \in \overline{\Omega}.$$

Here we recall (see Remark 4.11) that the constants $\beta_1(1/2)$ and $\beta_2(1/2)$ in inequality (4.47) are independent of $\varepsilon > 0$ and $\lambda > 0$.

Hence, we obtain from inequalities (4.46), (4.47) and (4.5) that

$$\lambda p_2^\varepsilon(x_0) \tag{4.48}$$

$$\leq (\lambda - c(x_0)) p_2^\varepsilon(x_0) \leq ((\lambda - \varepsilon\Delta - A) p_2^\varepsilon)(x_0)$$

$$= -\left(((\varepsilon\Delta + A - \lambda) p_2^\varepsilon)(x_0) - \sum_{\ell,m=1}^{N} B_{\varepsilon\Delta + A - \lambda} \left(\frac{\partial^2 u_\varepsilon}{\partial x_\ell \partial x_m}, \frac{\partial^2 u_\varepsilon}{\partial x_\ell \partial x_m} \right)(x_0) \right)$$

$$- \sum_{\ell,m=1}^{N} B_{\varepsilon\Delta + A - \lambda} \left(\frac{\partial^2 u_\varepsilon}{\partial x_\ell \partial x_m}, \frac{\partial^2 u_\varepsilon}{\partial x_\ell \partial x_m} \right)(x_0)$$

$$= -R_2^\varepsilon(x_0) - \sum_{\ell,m=1}^{N} B_{\varepsilon\Delta+A-\lambda}\left(\frac{\partial^2 u_\varepsilon}{\partial x_\ell \partial x_m}, \frac{\partial^2 u_\varepsilon}{\partial x_\ell \partial x_m}\right)(x_0)$$

$$\le -\frac{1}{2}\sum_{\ell,m=1}^{N} B_{\varepsilon\Delta+A-\lambda}\left(\frac{\partial^2 u_\varepsilon}{\partial x_\ell \partial x_m}, \frac{\partial^2 u_\varepsilon}{\partial x_\ell \partial x_m}\right)(x_0)$$

$$+ \beta_1(1/2)\|u_\varepsilon\|_{C^1(\overline{\Omega})}^2 + \beta_2(1/2)\|u_\varepsilon\|_{C^2(\overline{\Omega})}^2 + \frac{1}{2}\|f\|_{C^2(\overline{\Omega})}^2$$

$$\le \beta_1(1/2)\|u_\varepsilon\|_{C^1(\overline{\Omega})}^2 + \beta_2(1/2)\left(\|u_\varepsilon\|_{C^1(\overline{\Omega})}^2 + p_2^\varepsilon(x_0)\right) + \frac{1}{2}\|f\|_{C^2(\overline{\Omega})}^2$$

$$\le (\beta_1(1/2)+\beta_2(1/2))M_1(\lambda)^2\|f\|_{C^1(\overline{\Omega})}^2 + \beta_2(1/2)p_2^\varepsilon(x_0) + \frac{1}{2}\|f\|_{C^2(\overline{\Omega})}^2.$$

Therefore, if $\lambda > 0$ is so large that

$$\lambda \ge \gamma_2 := 2\beta_2(1/2),$$

then it follows from inequality (4.48) that

$$\max_{x\in\overline{\Omega}} p_2^\varepsilon(x) = p_2^\varepsilon(x_0) \le C(\lambda)\|f\|_{C^2(\overline{\Omega})}^2 \quad \text{for all } \varepsilon > 0, \tag{4.49}$$

with

$$C(\lambda) := \frac{1}{\lambda}\left(1 + 2\left(\beta_1(1/2)+\beta_2(1/2)\right)M_1(\lambda)^2\right).$$

(ii) Next we assume that the function p_2^ε attains its positive maximum at a *boundary point* x_0' of $\partial\Omega = \Sigma_3$, and let

$$q_2^\varepsilon = \sqrt{p_2^\varepsilon(x_0')} = \left(\max_{x\in\overline{\Omega}}\sum_{i,j=1}^{N}\left|\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}(x)\right|^2\right)^{1/2} = \left(\max_{x'\in\partial\Omega}\sum_{i,j=1}^{N}\left|\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}(x')\right|^2\right)^{1/2}.$$

(ii-a) Since we have the assertions

$$\frac{\partial u_\varepsilon}{\partial x_j} = 0 \quad \text{on } \partial\Omega \text{ for } 1 \le j \le N-1,$$

we can prove the following claim just as in the proof of [24, assertion (3.40)]:

Claim 4.1 *For* every $\eta > 0$, *there exists a constant* $M_0(\eta) > 0$, *independent of* $\varepsilon > 0$, *such that*

$$\max_{x'\in\partial\Omega}\left|\frac{\partial^2 u_\varepsilon}{\partial x_j \partial x_N}(x')\right| \le \eta\left\|(\varepsilon\Delta+A-\lambda)\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\right\|_{C(\overline{\Omega})} + M_0(\eta)\left\|\frac{\partial u_\varepsilon}{\partial x_j}\right\|_{C(\overline{\Omega})} \tag{4.50}$$

$$\text{for all } 1 \le j \le N-1.$$

However, by a direct calculation we find that

$$(\varepsilon\Delta+A-\lambda)\left(\frac{\partial u_\varepsilon}{\partial x_j}\right) \tag{4.51}$$

$$= \frac{\partial}{\partial x_j}\left((\varepsilon\Delta+A-\lambda)u_\varepsilon\right) + \left[\varepsilon\Delta+A-\lambda, \frac{\partial}{\partial x_j}\right]u_\varepsilon = \frac{\partial f}{\partial x_j} + \left[\varepsilon\Delta+A-\lambda, \frac{\partial}{\partial x_j}\right]u_\varepsilon$$

$$= \frac{\partial f}{\partial x_j} - \left(\sum_{\ell,m=1}^{N}\frac{\partial a^{\ell m}}{\partial x_j}\frac{\partial^2 u_\varepsilon}{\partial x_\ell \partial x_m} + \sum_{\ell=1}^{N}\frac{\partial b^\ell}{\partial x_j}\frac{\partial u_\varepsilon}{\partial x_\ell} + \frac{\partial c}{\partial x_j}u_\varepsilon\right) - \varepsilon\left(\sum_{\ell,m=1}^{N}\frac{\partial \mu^{\ell m}}{\partial x_j}\frac{\partial^2 u_\varepsilon}{\partial x_\ell \partial x_m} + \sum_{\ell=1}^{N}\frac{\partial v^\ell}{\partial x_j}\frac{\partial u_\varepsilon}{\partial x_\ell}\right)$$

where $[P,Q] = PQ - QP$ is the commutator of the differential operators P and Q, and

$$\Delta w = \sum_{\ell,m=1}^{N}\mu^{\ell m}(x)\frac{\partial^2 w}{\partial x_\ell \partial x_m} + \sum_{\ell=1}^{N}v^\ell(x)\frac{\partial w}{\partial x_\ell}, \tag{4.52a}$$

$$Aw = \sum_{\ell,m=1}^{N} a^{\ell m}(x) \frac{\partial^2 w}{\partial x_\ell \partial x_m} + \sum_{\ell=1}^{N} b^\ell(x) \frac{\partial w}{\partial x_\ell} + c(x)w. \tag{4.52b}$$

Hence, we have, by formula (4.51),

$$\left\| (\varepsilon \Delta + A - \lambda) \left(\frac{\partial u_\varepsilon}{\partial x_j} \right) \right\|_{C(\overline{\Omega})} \leq \|f\|_{C^1(\overline{\Omega})} + C \left(q_2^\varepsilon + \|u_\varepsilon\|_{C^1(\overline{\Omega})} \right) \quad \text{for } 1 \leq j \leq N-1, \tag{4.53}$$

with a constant $C > 0$ independent of $\varepsilon > 0$ and $0 < \theta < 1$.

Therefore, by combining inequality (4.50) with

$$\eta := \frac{1}{2(N-1)C}$$

and inequality (4.53) we obtain that

$$\left(\max_{x' \in \partial \Omega} \sum_{j=1}^{N-1} \left| \frac{\partial^2 u_\varepsilon}{\partial x_j \partial x_N}(x') \right|^2 \right)^{1/2} \leq \frac{1}{2} q_2^\varepsilon + C' \left(\|u_\varepsilon\|_{C^1(\overline{\Omega})} + \|f\|_{C^1(\overline{\Omega})} \right), \tag{4.54}$$

where

$$C' := \max \left\{ \frac{1}{2C}, \frac{1}{2} + M_0 \left(\frac{1}{2(N-1)C} \right) \right\}.$$

(ii-b) In order to estimate the term

$$\frac{\partial^2 u_\varepsilon}{\partial y_N^2}(x_0'),$$

we choose a local coordinate system $(y_1, y_2, \ldots, y_N)$ in a neighborhood of $x_0' \in \partial \Omega = \Sigma_3$ such that

$$x_0' = 0,$$
$$\Omega = \{y_N > 0\}, \quad \partial \Omega = \{y_N = 0\}.$$

By using formulas (4.52) for the differential operators Δ and A, we may assume that the equation

$$(\varepsilon \Delta + A - \lambda) u_\varepsilon = f \quad \text{in terms of the coordinate system } (y_1, y_2, \ldots, y_N)$$

is of the form

$$(\varepsilon \Delta + A - \lambda) u_\varepsilon = \left(\alpha^{NN}(y) + \varepsilon \mu^{NN}(y) \right) \frac{\partial^2 u_\varepsilon}{\partial y_N^2} + \sum_{i,j=1}^{N-1} \left(\alpha^{ij}(y) + \varepsilon \mu^{ij}(y) \right) \frac{\partial^2 u_\varepsilon}{\partial y_i \partial y_j}$$

$$+ \left(\beta^N(y) + \varepsilon \nu^N(y) \right) \frac{\partial u_\varepsilon}{\partial y_N} + \sum_{i=1}^{N-1} \left(\beta^i(y) + \varepsilon \nu^i(y) \right) \frac{\partial u_\varepsilon}{\partial y_i} + (c(y) - \lambda) u_\varepsilon$$

$$= f.$$

Since $\alpha^{NN}(0) > 0$ (by $0 = x_0' \in \Sigma_3$) and $\mu^{NN}(0) > 0$ and since $u_\varepsilon = 0$ on $\partial \Omega$, it follows that

$$\frac{\partial^2 u_\varepsilon}{\partial y_N^2}(0) = \frac{1}{\alpha^{NN}(0) + \varepsilon \mu^{NN}(0)} \left(f(0) - \left(\beta^N(0) + \varepsilon \nu^N(0) \right) \frac{\partial u_\varepsilon}{\partial y_N}(0) \right).$$

Hence, we have the inequality

$$\left| \frac{\partial^2 u_\varepsilon}{\partial y_N^2}(0) \right| \leq C'' \left(\|u_\varepsilon\|_{C^1(\overline{D})} + \|f\|_{C(\overline{D})} \right), \tag{4.55}$$

with a constant $C'' > 0$ independent of $\varepsilon > 0$ and $0 < \theta < 1$.

(ii-c) Finally, we remark that

$$\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j} = 0 \quad \text{on } \partial\Omega \text{ for all } 1 \le i, j \le N - 1. \tag{4.56}$$

Therefore, by combining estimates (4.54) and (4.55) and formula (4.56) we find that

$$q_2^\varepsilon = \left(\max_{x' \in \partial\Omega} \sum_{i,j=1}^{N} \left| \frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}(x') \right|^2 \right)^{1/2} \le \frac{1}{2} q_2^\varepsilon + \left(C' + C'' \right) \left(\|u_\varepsilon\|_{C^1(\overline{\Omega})} + \|f\|_{C^1(\overline{\Omega})} \right),$$

so that

$$q_2^\varepsilon \le 2 \left(C' + C'' \right) \left(\|u_\varepsilon\|_{C^1(\overline{\Omega})} + \|f\|_{C^1(\overline{\Omega})} \right). \tag{4.57}$$

(iii) The desired estimate (4.45) (and hence estimate (4.44)) follows by combining estimates (4.49) and (4.57):

$$\left(\max_{x \in \overline{\Omega}} \sum_{i,j=1}^{N} \left| \frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}(x) \right|^2 \right)^{1/2} \le \sqrt{C(\lambda)} \|f\|_{C^2(\overline{\Omega})} + 2 \left(C' + C'' \right) \left(\|u_\varepsilon\|_{C^1(\overline{\Omega})} + \|f\|_{C^1(\overline{\Omega})} \right)$$

$$\le 2 \left(C' + C'' \right) \|u_\varepsilon\|_{C^1(\overline{\Omega})} + \left(\sqrt{C(\lambda)} + 2 \left(C' + C'' \right) \right) \|f\|_{C^2(\overline{\Omega})}$$

$$\le M(\lambda) \left(\|u_\varepsilon\|_{C^1(\overline{\Omega})} + \|f\|_{C^2(\overline{\Omega})} \right),$$

with

$$M(\lambda) := \sqrt{C(\lambda)} + 2 \left(C' + C'' \right).$$

The proof of Lemma 4.9 is complete. $\square$

Step (III): Now, by applying Lemma 4.9 with

$$A := A + a(x)\Delta,$$

we can find a constant $\lambda_2 > 0$ such that if $\lambda \ge \lambda_2$, the homogeneous Dirichlet problem

$$\begin{cases} (\varepsilon\Delta + A + a(x)\Delta - \lambda)\widetilde{u}_\varepsilon = \widetilde{f} & \text{in } \Omega, \\ \widetilde{u}_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{$\widetilde{D}$}_{\varepsilon,\lambda}$$

has a unique solution $\widetilde{u}_\varepsilon \in C^{4+\theta}(\overline{\Omega})$ for all $0 < \theta < 1$, provided that $\widetilde{f} \in C^3(\overline{\Omega})$. Moreover, the solution $\widetilde{u}_\varepsilon$ satisfies the *uniform estimate* (cf. estimate (4.16) and inequality (4.42b))

$$\|\widetilde{u}_\varepsilon\|_{C^2(\overline{\Omega})} \le \widetilde{M}_2(\lambda) \left\| \widetilde{f} \right\|_{C^2(\overline{\Omega})} \le \widetilde{M}_2(\lambda) \|f\|_{C^2(\overline{D})} \quad \text{for all } \varepsilon > 0, \tag{4.58}$$

where $\widetilde{M}_2(\lambda) > 0$ is a constant independent of $\varepsilon > 0$ and $0 < \theta < 1$.

Therefore, by arguing as in the proof of Theorem 4.1 we can choose a subsequence $\{\widetilde{u}_{\varepsilon_k}\}_{k=1}^\infty$ of the sequence $\{\widetilde{u}_\varepsilon\}$ which, together with all its derivatives of order ≤ 2, converges weakly to some function $\widetilde{u}$ in the Hilbert space $L^2(\Omega)$ as $\varepsilon_k \downarrow 0$. More precisely, we have the following assertions (cf. assertions (4.15) and (4.18) and inequality (4.20)):

$$\widetilde{u} = \lim_{\varepsilon_k \downarrow 0} \widetilde{u}_{\varepsilon_k} \quad \text{in } C^1(\overline{\Omega}). \tag{4.59a}$$

$$\widetilde{w}_{ij} = w - \lim_{\varepsilon_k \downarrow 0} \partial_i \partial_j \left(\widetilde{u}_{\varepsilon_k} \right) = \partial_i \partial_j \widetilde{u} \quad \text{in } L^2(\Omega), \text{ for all } 1 \le i, j \le N. \tag{4.59b}$$

$$\left\| \widetilde{w}_{ij} \right\|_{L^\infty(\Omega)} \le \widetilde{M}_2(\lambda) \|f\|_{C^2(\overline{D})} \quad \text{for all } 1 \le i, j \le N. \tag{4.59c}$$

Thus, by passing to the limit in the homogeneous Dirichlet problem $(\widetilde{D})_{\varepsilon_k,\lambda}$ as $\varepsilon_k \downarrow 0$ we obtain from assertions (4.59) and the uniform estimate (4.58) that

$$\widetilde{u} = \lim_{\varepsilon_k \downarrow 0} u_{\varepsilon_k} \in W^{2,\infty}(\Omega),$$

and further (see Substep (III-1) of the proof of Theorem 4.1) that the limit function $\widetilde{u}$ is a weak solution of the homogeneous Dirichlet problem

$$\begin{cases} (A + a(x)\Delta - \lambda)\widetilde{u} = \widetilde{f} & \text{in } \Omega, \\ \widetilde{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\widetilde{D}$}_\lambda$$

Moreover, the weak solution $\widetilde{u}$ satisfies the inequality

$$\|\widetilde{u}\|_{W^{2,\infty}(\Omega)} \le \widetilde{C}_2(\lambda) \|f\|_{C^2(\overline{D})}, \tag{4.60}$$

where $\widetilde{C}_2(\lambda) > 0$ is a constant *independent* of $\varepsilon > 0$ and $0 < \theta < 1$.

Step (IV): Let $f \in C^3(\overline{D})$, f arbitrary. If $\widetilde{f} \in C^3(\overline{\Omega})$ is a Seeley extension of f to Ω, by arguing as in Step (III) of the proof of Theorem 4.1 we find that the *restriction*

$$u := \widetilde{u}|_D \in W^{2,\infty}(D)$$

satisfies the equation

$$(A - \lambda)u = \widetilde{f}|_D = f \quad \text{in } D, \tag{$\star$}_\lambda$$

since $a(x) = 0$ in D.

Therefore, we obtain from inequality (4.60) that, for any function $f \in C^3(\overline{D})$ there exists a weak solution $u \in W^{2,\infty}(D)$ of the equation $(\star)_\lambda$ which satisfies the inequality

$$\|u\|_{W^{2,\infty}(D)} \le \|\widetilde{u}\|_{W^{2,\infty}(\Omega)} \le \widetilde{C}_2(\lambda) \|f\|_{C^2(\overline{D})}. \tag{4.61}$$

Step (V): Finally, let $f \in W^{2,\infty}(D) = C^{1,1}(\overline{D})$, f arbitrary. We will show that the equation $(\star)_\lambda$ has a weak solution $u \in W^{2,\infty}(D) = C^{1,1}(\overline{D})$ which satisfies inequality (4.41) for $m = 2$.

Substep (V-1): By using Friedrichs mollifiers ([34, Subsection 1.3.2]) and Remark 4.8 with $m := 2$, we can find a sequence $\{\widetilde{f}_n\}_{n=1}^\infty$ in $C^3(\overline{\Omega})$ such that

$$\left\|\widetilde{f}_n\right\|_{C^2(\overline{\Omega})} \le \|f\|_{W^{2,\infty}(D)}, \tag{4.62a}$$

$$\widetilde{f}_n \longrightarrow f \quad \text{in } C^1(\overline{D}) \text{ as } n \to \infty. \tag{4.62b}$$

Let $\widetilde{u}_{\varepsilon,n} \in C^{4+\theta}(\overline{\Omega})$ be a unique solution of the homogeneous Dirichlet problem

$$\begin{cases} (\varepsilon\Delta + A + a(x)\Delta - \lambda)\widetilde{u}_{\varepsilon,n} = \widetilde{f}_n & \text{in } \Omega, \\ \widetilde{u}_{\varepsilon,n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\widetilde{D}$}_{\varepsilon,\lambda,n}$$

Then, by Step (III) we find that the homogeneous Dirichlet problem

$$\begin{cases} (A + a(x)\Delta - \lambda)\widetilde{u}_n = \widetilde{f}_n & \text{in } \Omega, \\ \widetilde{u}_n = 0 & \text{on } \partial\Omega \end{cases} \tag{$\widetilde{D}$}_{\lambda,n}$$

has a unique weak solution

$$\widetilde{u}_n = \lim_{\varepsilon_k \downarrow 0} \widetilde{u}_{\varepsilon_k,n} \in W^{2,\infty}(\Omega).$$

Moreover, by Step (IV) it follows that the *restriction*

$$u_n := \widetilde{u}_n|_D \in W^{2,\infty}(D)$$

satisfies the equation

$$(A - \lambda) u_n = \widetilde{f}_n|_D = f_n \quad \text{in } D. \tag{$\star$}_{\lambda,n}$$

Moreover, we find from inequality (4.61) with $u := u_n$ and $f := f_n$ and inequality (4.62a) that the solution u_n satisfies the *uniform estimate*

$$\|u_n\|_{W^{2,\infty}(D)} \leq \widetilde{C}_2(\lambda) \|f_n\|_{C^2(\overline{D})} \leq \widetilde{C}_2(\lambda) \left\|\widetilde{f}_n\right\|_{C^2(\overline{\Omega})} \leq \widetilde{C}_2(\lambda) \|f\|_{W^{2,\infty}(D)} \quad \text{for all } n \in \mathbf{N}. \tag{4.63}$$

Step (V-2): By virtue of the uniform estimate (4.63) and Remark 4.8 with $m := 2$, we can choose a subsequence $\{u_{n'}\}$ of the sequence $\{u_n\}$ that converges uniformly to a function u in $C(\overline{D})$ as $n' \to \infty$ (cf. assertion (4.40)):

$$u = \lim_{n' \to \infty} u_{n'} \quad \text{in } C^1(\overline{D}). \tag{4.64}$$

Just as in Substep (IV-2) of the proof of Theorem 4.1, it follows from assertions (4.62b) and (4.64) that the limit function $u \in W^{2,\infty}(D)$ is a weak solution of the equation

$$(A - \lambda) u = f \quad \text{in } D, \tag{$\star$}_{\lambda}$$

and further that the weak solution u satisfies the desired inequality (4.41) for $m = 2$:

$$\|u\|_{W^{2,\infty}(D)} \leq C_2(\lambda) \|f\|_{W^{2,\infty}(D)},$$

where the constant $C_2(\lambda) = \widetilde{C}_2(\lambda)$ is independent of f.

The proof of Theorem 4.7 (when $m = 2$) is now complete. $\qquad\square$

4.3 Proof of Theorem 1.4

The existence and uniqueness theorem for the equation (1.5) follows from Theorem 4.7 by a well-known real interpolation argument due to Lions–Peetre [16] (cf. [6, Theorem 6.4.5], [33, Theorem 2.4.2]). Indeed, it suffices to note that the Hölder space $C^{m+\theta}(\overline{D})$ is a *real interpolation space* between the Sobolev spaces $W^{m,\infty}(D)$ and $W^{m+1,\infty}(D)$ (see formula (4.68) below).

More precisely, we can obtain the following existence and uniqueness theorem for the equation $(\star)_{\lambda}$ in the framework of the Hölder space $C^{2+\theta}(\overline{D})$, which proves Theorem 1.4 for $m := 2$ (cf. [24, Theorem 3.8]):

Theorem 4.12 *Assume that hypothesis (G) is satisfied. There exists a constant $\lambda_3 > 0$ such that if $\lambda \geq \lambda_3$, the equation*

$$(A - \lambda) u = f \quad \text{in } D \tag{$\star$}_{\lambda}$$

has a unique solution $u \in C^{2+\theta}(\overline{D})$ for any function $f \in C^{2+\theta}(\overline{D})$ with $0 < \theta < 1$. Moreover, the solution u satisfies inequality (1.6) for $m = 2$ with

$$C_{2+\theta}(\lambda) = C_2(\lambda)^{1-\theta} C_3(\lambda)^{\theta} \quad \text{for } 0 < \theta < 1. \tag{4.65}$$

P r o o f. The proof is divided into three steps.

Step (1): First, by using Theorem 4.7 for $m = 2$ and $m = 3$ we obtain the following two inequalities for all $\lambda \geq \lambda_3$:

$$\|(A - \lambda)^{-1} f\|_{2,\infty} \leq C_2(\lambda) \|f\|_{2,\infty} \quad \text{for all } f \in W^{2,\infty}(D). \tag{4.66}$$

$$\|(A - \lambda)^{-1} f\|_{3,\infty} \leq C_3(\lambda) \|f\|_{3,\infty} \quad \text{for all } f \in W^{3,\infty}(D). \tag{4.67}$$
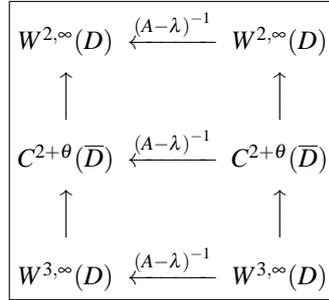
$$
\begin{array}{ccc}
W^{2,\infty}(D) & \xleftarrow{\ (A-\lambda)^{-1}\ } & W^{2,\infty}(D) \\
\uparrow & & \uparrow \\
C^{2+\theta}(\overline{D}) & \xleftarrow{\ (A-\lambda)^{-1}\ } & C^{2+\theta}(\overline{D}) \\
\uparrow & & \uparrow \\
W^{3,\infty}(D) & \xleftarrow{\ (A-\lambda)^{-1}\ } & W^{3,\infty}(D)
\end{array}
$$

Fig. 4.2 Proof of formula (4.65) for $\lambda \geq \lambda_3$

Here, without loss of generality we may assume that $\lambda_3 \geq \lambda_2$.

The idea of our proof may be visualized by Figure 4.2 below.

Step (2): We define a *real interpolation space*

$$
\left(W^{2,\infty}(D), W^{3,\infty}(D)\right)_{\theta,\infty} \quad \text{for } 0 < \theta < 1,
$$

between the Sobolev spaces $W^{2,\infty}(D)$ and $W^{3,\infty}(D)$ as follows:

$$
\left(W^{2,\infty}(D), W^{3,\infty}(D)\right)_{\theta,\infty} = \left\{ u \in W^{2,\infty}(D) : \|u\|_{\theta,\infty} := \sup_{t>0} \frac{K(t,u)}{t^\theta} < \infty \right\},
$$

where

$$
K(t,u) = \inf\left\{ \|u_0\|_{k,\infty} + t\|u_1\|_{k+1,\infty} : u = u_0 + u_1,\ u_0 \in W^{k,\infty}(M),\ u_1 \in W^{k+1,\infty}(M) \right\}.
$$

Then it is known that the Hölder space $C^{2+\theta}(\overline{D})$ coincides with the interpolation space

$$
C^{2+\theta}(\overline{D}) = \left(W^{2,\infty}(D), W^{3,\infty}(D)\right)_{\theta,\infty} \quad \text{for } 0 < \theta < 1, \tag{4.68}
$$

with the norm $\|\cdot\|_{\theta,\infty}$.

Indeed, it suffices to note the following three assertions (a), (b) and (c):

(a) For any integer $m \in \mathbf{N}$, we have the inclusions (see Remarks 4.2 and 4.8)

$$
C^m(\overline{D}) \subset W^{m,\infty}(D) = C^{m-1,1}(\overline{D}) \subset B^m_{\infty,\infty}(D).
$$

(b) For any $0 < \theta < 1$, we have the formulas (see Bergh–Löfström [6, Theorem 6.4.5]; Triebel [33, Theorem 2.4.2])

$$
\left(B^2_{\infty,\infty}(D), B^3_{\infty,\infty}(D)\right)_{\theta,\infty} = B^{2+\theta}_{\infty,\infty}(D) = C^{2+\theta}(\overline{D}),
$$
$$
\left(C^2(\overline{D}), C^3(\overline{D})\right)_{\theta,\infty} = C^{2+\theta}(\overline{D}).
$$

(c) For any $0 < \theta < 1$, we have the inclusions

$$
C^{2+\theta}(\overline{D}) = \left(C^2(\overline{D}), C^3(\overline{D})\right)_{\theta,\infty} \subset \left(W^{2,\infty}(D), W^{3,\infty}(D)\right)_{\theta,\infty} \subset \left(B^2_{\infty,\infty}(D), B^3_{\infty,\infty}(D)\right)_{\theta,\infty} = C^{2+\theta}(\overline{D}).
$$

Step (3): Therefore, by an interpolation argument we obtain from inequalities (4.66) and (4.67) that

$$
\|u\|_{C^{2+\theta}(\overline{D})} = \|(A-\lambda)^{-1} f\|_{C^{2+\theta}(\overline{D})} \leq C_3(\lambda)^\theta C_2(\lambda)^{1-\theta} \|f\|_{C^{2+\theta}(\overline{D})} \quad \text{for all } f \in C^{2+\theta}(\overline{D}).
$$

This proves the desired formula (4.65).

The proof of Theorem 4.12 is complete. $\qquad\square$

5   Proof of Theorem 1.2

In this last section we will prove the generation theorem of a Feller semigroup on $\overline{D}$ (Theorem 1.2). Our proof is based on the Hille–Yosida–Ray theorem (Theorem 2.1) which is a Feller semigroup version of the classical Hille–Yosida theorem in terms of the *positive maximum principle*. We will verify three conditions (α), (β) and (γ) of Theorem 2.1 with $C(K) := C(\overline{D})$ and $B := \mathscr{A}$ under hypothesis (G).

5.1   Green operators G_α

By using Theorem 4.12 with

$$\lambda := \alpha, \quad \lambda_3 := \alpha_0,$$

we can obtain following fundamental theorem (cf. [24, Theorem 4.1] under hypothesis (H)):

Theorem 5.1 *Assume that hypothesis (G) is satisfied. There exists a constant $\alpha_0 > 0$ such that if $\alpha \geq \alpha_0$, the equation*

$$(\alpha - A)u = f \quad in \ D \tag{5.1}$$

has a unique solution $u \in C^{2+\theta}(\overline{D})$ for any function $f \in C^{2+\theta}(\overline{D})$ with $0 < \theta < 1$. Moreover, the solution u satisfies the inequality

$$\max_{x \in \overline{D}} |u(x)| \leq \frac{1}{\alpha} \max_{x \in \overline{D}} |f(x)|. \tag{5.2}$$

Theorem 5.1 asserts that if $\alpha \geq \alpha_0$, the equation (5.1) has a unique solution $u \in C^{2+\theta}(\overline{D})$ for any function $f \in C^{2+\theta}(\overline{D})$ with $0 < \theta < 1$. Therefore, we can introduce a linear operator

$$G_\alpha : C^{2+\theta}(\overline{D}) \longrightarrow C^{2+\theta}(\overline{D})$$

as follows: For each $f \in C^{2+\theta}(\overline{D})$, the function $u = G_\alpha f \in C^{2+\theta}(\overline{D})$ is the unique solution of the equation

$$(\alpha - A)u = (\alpha - A)(G_\alpha f) = f \quad in \ D. \tag{5.3}$$

The operator G_α is called the *Green operator* for the equation (5.1).

Then we have the following fundamental result for the Green operators G_α (cf. [24, Lemma 4.2]):

Lemma 5.2 *The operator G_α for $\alpha \geq \alpha_0$, considered from $C(\overline{D})$ into itself, is non-negative and continuous with norm*

$$\|G_\alpha\| = \|G_\alpha 1\| = \max_{x \in \overline{D}} G_\alpha 1(x) \leq \frac{1}{\alpha} \quad for \ all \ \alpha \geq \alpha_0. \tag{5.4}$$

P r o o f.   First, we prove the *non-negativity* of the Green operator G_α for all $\alpha \geq \alpha_0$. To do so, by virtue of Friedrichs mollifiers ([34, Subsection 1.3.2]) we may assume that

$$f \in C^3(\overline{D}) \quad and \quad f(x) \geq 0 \quad on \ \overline{D},$$

since the space $C^3(\overline{D})$ is dense in $C(\overline{D})$.

We modify the domain D and the differential operator A just as in the proof of Theorem 4.1 (see Figure 4.1), and consider the homogeneous Dirichlet problem $(\widetilde{D})_{\varepsilon,\lambda}$ for the elliptic differential operator

$$\varepsilon \Delta + A + a(x)\Delta - \lambda, \quad \varepsilon > 0,$$

where the boundary $\partial \Omega$ of the modified domain Ω is of type Σ_3 for the differential operator $A + a(x)\Delta$. By using the Seeley extension operator ([1, Theorems 5.21 and 5.22], [20, Theorem], [29, Theorem 4.21]), we can construct an extension $\widetilde{f}$ of f to the domain Ω in such a way that

$$\widetilde{f} \in C^3(\overline{\Omega}) \quad and \quad \widetilde{f}(x) \geq 0 \quad on \ \overline{\Omega}.$$

Then, by Step (III) of the proof of Theorem 4.7 with $\lambda := \alpha$ it follows that the homogeneous Dirichlet problem

$$\begin{cases} (A + a(x)\Delta - \alpha)\widetilde{v} = \widetilde{f} & \text{in } \Omega, \\ \widetilde{v} = 0 & \text{on } \partial\Omega \end{cases} \qquad (\widetilde{D})_\alpha$$

has a *unique* weak solution $\widetilde{v} \in W^{2,\infty}(\Omega)$. Furthermore, since $a(x) = 0$ in D, we find from Step (III) of the proof of Theorem 4.1 that the *restriction*

$$v = \widetilde{v}|_D \in W^{2,\infty}(D)$$

satisfies the equation

$$(A - \alpha)v = \widetilde{f}|_D = f \quad \text{in } D.$$

Since $f \in C^3(\overline{D})$, it follows from the equation (5.3) that

$$v = -G_\alpha f \in C^{2+\theta}(\overline{D}). \qquad (5.5)$$

On the other hand, since we have the condition

$$\widetilde{f}(x) \geq 0 \quad \text{on } \overline{\Omega},$$

by applying the *weak maximum principle* for the *degenerate* elliptic differential operator $A + a(x)\Delta - \alpha$ (see [7], [18, Chapter III, Section 1], [34, Subsection 3.7.2], [31, Section 8]) we obtain that

$$\widetilde{v}(x) \leq 0 \quad \text{on } \overline{\Omega}. \qquad (5.6)$$

Therefore, by combining assertions (5.5) and (5.6) we have proved that

$$(G_\alpha f)(x) = -v(x) = -\widetilde{v}(x) \geq 0 \quad \text{on } \overline{D}.$$

This proves the non-negativity of G_α for all $\alpha \geq \alpha_0$.

Finally, the desired inequality (5.4) follows from inequality (5.2) by taking $f = 1$.

The proof of Lemma 5.2 is complete. $\qquad \square$

5.2 End of proof of Theorem 1.2

We recall that $\mathscr{A}$ is a linear operator from the space $C(\overline{D})$ into itself defined by the following:

(1) The domain $D(\mathscr{A})$ of $\mathscr{A}$ is the space

$$D(\mathscr{A}) = C^2(\overline{D}). \qquad (1.4)$$

(2) $\mathscr{A}u = Au$ for every $u \in D(\mathscr{A})$.

We will verify three conditions (α), (β) and (γ) of Theorem 2.1 with $C(K) := C(\overline{D})$ and $B := \mathscr{A}$ under hypothesis (G). The proof is divided into three steps.

Step (I): First, we verify condition (α), that is, the *density* of the domain $D(\mathscr{A})$ in the space $C(\overline{D})$.

Now let $f \in C^3(\overline{D})$, f arbitrary. Then we obtain that

$$G_\alpha f \in C^2(\overline{D}) = D(\mathscr{A}) \quad \text{for all } \alpha \geq \alpha_0.$$

However, it follows from an application of the *uniqueness theorem* for the equation (5.1) that

$$f - \alpha G_\alpha f = G_\alpha\left((1 - A)f\right) - G_\alpha f.$$

Indeed, it suffices to note that the both sides satisfy the same equation

$$(\alpha - A)u = -Af \quad \text{in } D.$$

In view of inequality (5.4), we have the inequality

$$\|f - \alpha G_\alpha f\| \le \frac{1}{\alpha} \|(1-A)f\| + \frac{1}{\alpha} \|f\| \quad \text{for all } \alpha \ge \alpha_0.$$

Therefore, we have the assertion

$$\lim_{\alpha \to +\infty} \|f - \alpha G_\alpha f\| = 0.$$

This verifies condition (α), since the space $C^3(\overline{D})$ is dense in the space $C(\overline{D})$.

Step (II): Next, in order to verify condition (β), we assume that

$$u \in D(\mathscr{A}) = C^2(\overline{D}) \quad \text{and} \quad \max_{x \in \overline{D}} u(x) > 0.$$

Then we consider the following two cases:

(i) There exists an *interior point* x_0 of D such that

$$u(x_0) = \max_{x \in \overline{D}} u(x) > 0.$$

(ii) There exists a *boundary point* x_0' of $\partial D = \Sigma_0 \cup \Sigma_1$ such that

$$u(x_0') = \max_{x \in \overline{D}} u(x) > 0.$$

Case (i): In this case, we have the assertion

$$\mathscr{A} u(x_0) = Au(x_0) = \sum_{i,j=1}^{N} a^{ij}(x_0) \frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) + c(x_0) u(x_0) \le 0,$$

since the matrix $\left(a^{ij}(x)\right)$ is non-negative definite and $c(x) \le 0$ in D.

Case (ii): We choose a local coordinate system $(y_1, y_2, \ldots, y_N)$ in a neighborhood of $x_0' \in \Sigma_0 \cup \Sigma_1$ such that

$$x_0' = 0,$$
$$D = \{y_N > 0\}, \quad \partial D = \{y_N = 0\},$$

and assume that, in terms of this coordinate system, the operator A is of the form

$$Au = \alpha^{NN}(y) \frac{\partial^2 u}{\partial y_N^2} + \beta^N(y) \frac{\partial u}{\partial y_N} + \sum_{i,j=1}^{N-1} \alpha^{ij}(y) \frac{\partial^2 u}{\partial y_i \partial y_j} + \sum_{i=1}^{N-1} \beta^i(y) \frac{\partial u}{\partial y_i} + c(y)u. \tag{5.7}$$

Here:

(ii-a) $\alpha^{NN}(0) = 0$ and $\beta^N(0) > 0$ if $x_0' = 0 \in \Sigma_1$.

(ii-b) $\alpha^{NN}(0) = 0$ and $\beta^N(0) = 0$ if $x_0' = 0 \in \Sigma_0$.

Then we have the assertions

$$\begin{cases} u(0) > 0, \\ \frac{\partial u}{\partial y_i}(0) = 0 \quad \text{for } 1 \le i \le N-1, \\ \frac{\partial u}{\partial y_N}(0) \le 0, \end{cases}$$

and also

$$\sum_{i,j=1}^{N-1} \alpha^{ij}(0) \frac{\partial^2 u}{\partial y_i \partial y_j}(0) \le 0, \quad c(0)u(0) \le 0.$$

Hence, it follows from formula (5.7) and conditions (ii) that

$$\mathscr{A} u(x_0') = Au(x_0') \le \begin{cases} \beta^N(0) \frac{\partial u}{\partial y_N}(0) + c(0)u(0) \le 0 & \text{if } x_0' = 0 \in \Sigma_1, \\ c(0)u(0) \le 0 & \text{if } x_0' = 0 \in \Sigma_0. \end{cases}$$

Therefore, we have proved the following *positive maximum principle* (cf. [24, p. 425, Claim]):

Claim 5.1 *If $u \in D(\mathscr{A})$ and $\max_{\overline{D}} u > 0$, then there exists a point $x \in \overline{D} = D \cup \partial D$ such that the positive maximum principle*

$$\begin{cases} u(x) = \max_{\overline{D}} u, \\ \mathscr{A} u(x) \leq 0 \end{cases} \tag{PMP}$$

holds true.

This claim verifies condition (β).

Step (III): It remains to verify condition (γ). By Theorem 5.1, we find that if $\alpha \geq \alpha_0$, then the range $R(\alpha I - \mathscr{A})$ contains a *dense* subspace $C^{2+\theta}(\overline{D})$ of $C(\overline{D})$. This implies that the range $R(\alpha I - \mathscr{A})$ is dense in the space $C(\overline{D})$, for all $\alpha \geq \alpha_0$.

Summing up, we have proved that the operator $\mathscr{A}$ satisfies three conditions (α), (β) and (γ) of Theorem 2.1 with $C(K) := C(\overline{D})$ and $B := \mathscr{A}$ for all $\alpha \geq \alpha_0$. Therefore, it follows from an application of the same theorem that the operator $\mathscr{A}$ is *closable* in the space $C(\overline{D})$, and its minimal closed extension $\mathfrak{A} = \overline{\mathscr{A}}$ is the infinitesimal generator of some Feller semigroup $\{T_t\}_{t \geq 0}$ on $\overline{D}$.

The proof of Theorem 1.2 is now complete. $\qquad \square$

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