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Involutive category theory. (English) Zbl 07283178

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This is the first book on involutive categories and involutive operads with applications to GNS construction and algebraic quantum field theory. Some of the materials, including parts of Chapters 2, 4, 7 and 8, are based upon [*B. Jacobs*, Found. Phys. 42, No. 7, 874–895 (2012; Zbl 1259.81032); *M. Benini* et al., Theory Appl. Categ. 34, 13–57 (2019; Zbl 1406.18005)].

A synopsis of the book, consisting of eight chapter, goes as follows. The last section of each chapter is devoted to exercises and notes. Chapter 1 fixes some notations and recalls briefly some basic category theory.

Chapter 2 presents the definitions and basic properties of involutive categories, involutive functors, involutive natural transformations, involutive adjunctions, and involutive objects in involutive categories. The central result is Theorem 2.3.8 claiming that an adjoint of an involutive functor should also be an involutive functor such that the adjunction is an involutive adjunction, which implies that an involutive functor that is also an equivalence of categories is automatically an involutive equivalence.

Chapter 3, consisting of seven sections, aims to study the coherence of involutive categories and related topics. §3.1 shows that the forgetful functor

$$U : \mathsf{ICat}^{\mathrm{st}} \to \mathsf{Cat}$$

from the category $\mathsf{ICat}^{\mathrm{st}}$ of small strict involutive categories and strict involutive functors to Cat admits both a left adjoint $\mathcal{F}^{\mathrm{si}}$ and a right adjoint given by categorical coproducts and products, respectively. §3.2 considers the category ICat_0 of small involutive categories and strict involutive functors, constructing an explicit left adjoint $\mathcal{F}^{\mathrm{icat}}$ of the forgetful functor

$$U : \mathsf{ICat}_0 \to \mathsf{Cat}$$

§3.3 shows that there is a unique strict involutive equivalence

$$\mathcal{F}^{\mathrm{icat}}(\mathsf{C}) \xrightarrow{\sim} \mathcal{F}^{\mathrm{si}}(\mathsf{C})$$

for any small category C. §3.4 observes that the category ICat_0 is isomorphic to the category of algebras of the monad defined by the adjunction $\mathcal{F}^{\mathrm{icat}} \dashv U$. §3.5 demonstrates that the category ICat_0 is locally finitely presentable. §3.6 shows, by exploiting the local finite presentability of , that the forgetful functor

$$U : \mathsf{ICat}_0 \to \mathsf{Cat}$$

admits a right adjoint.

Chapter 4 develops the involutive versions of (symmetric) monoidal categories, (commutative) monoids, and algebras over a monad. In an involutive symmetric monoidal category, there is a notion of a reversing involutive monoid besides involutive commutative monoids.

Chapter 5, consisting of seven sections, aims to study coherence of involutive monoidal categories. §5.1 and §5.2 gives explicit constructions of the free involutive monoidal categories and of the free involutive strict monoidal categories generated by a category, observing that they are equivalent via a strict involutive strict monoidal functor. §5.3 shows that in a small involutive monoidal category, every formal diagram is commutative. §5.4 shows (Theorem 5.4.1) that every involutive monoidal category is to be strictified to an involutive strict monoidal cacategory via an involutive adjoint equivalence involving involutive strong monoidal functors. §5.5 and §5.6 deal with explicit constructions of the free involutive (strict) monoidal category generated by an involutive category.

Chapter 6, consisting of seven sections, aims to study coherence of involutive symmetric monoidal categories. most of the constructions being adapted from the previous chapter. §6.1 and §6.2 gives explicit constructions of the free involutive symmetric monoidal categories and of the free involutive strict symmetric monoidal categories generated by a category, observing that they are equivalent via a strict involutive strict symmetric monoidal functor. §6.3 shows that in a small involutive symmetric monoidal category, every formal diagram is commutative. §6.4 shows (Theorem 6.4.1) that every involutive symmetric monoidal category is to be strictified to an involutive strict symmetric monoidal cacategory via an involutive adjoint equivalence involving involutive strong symmetric monoidal functors. §6.5 and §6.6 deal with explicit constructions of the free involutive (strict) symmetric monoidal category generated by an involutive category.

Chapter 7, consisting of four sections, addresses a categorical analogue of the Gelfand-Naimark-Segal (GNS) construction [*I. Gel'fand* and *M. Naĭmark*, Mat. Sb., Nov. Ser. 12(54), 197–213 (1943; Zbl 0060.27006); *I. E. Segal*, Bull. Am. Math. Soc. 53, 73–88 (1947; Zbl 0031.36001)]. Another categorical interpretation of the GNS construction can be seen in [*A. J. Parzygnat*, Appl. Categ. Struct. 26, No. 6, 1123–1157 (2018; Zbl 1408.81017)]. While for a C^*-algebra A, the GNS construction gives a correspondence between states on A and cyclic *-representations of A, the categorical GNS correspondence (Theorem 7.2.7) is an explicit isomorphism between the category of quantum probability objects and the category of monoid inner product objects, where a quantum probability object is a categorical abstraction of a complex unital *-algebra endowed with a state on over $\mathbb{C}$, and a monoid inner product object is a categorical abstraction of a complex unital *-algebra endowed with a Hermitian form abiding by the equality

$$(ba, c) = (a, b^*c)$$

§7.1 defines the category of monoid inner product objects, the main result being an explicit description of the free-forgetful adjunction between the category of reversing involutive monoids and the category of monoid inner product objects. §7.2 discusses quantum probability objects and the categorical GNS construction. The proofs of some statements in the first two sections require checking that some large diagrams are commutative, which is relegated to §7.3.

Operad theory is a versatile parlance for studying algebraic structures with multiple inputs and a single output. An operad is a bookkeeping gadget for keeping track of operations with multiple inputs and a single output. Chapter 8, consisting of eight sections, extends operads to their involutive analogues. §8.1 recalls some basic aspects of operads. §8.2 recalls the monad associated to an operad, defining an algebra over an operad as an algebra of the associated monad. §8.3 defines involutive operads as the involutive monoids in an involutive monoidal category of symmetric sequences. §8.4 shows that each involutive operad yields an involutive monad. §8.5 defines the involutive algebras of this involutive monad as those for the involutive operad, observing that an involutive algebra over an involutive operad has an underlying algebra over the underlying operad an involutive structure, together with a compatibility axiom between the two structures. §8.6 observes that each involutive symmetric monoidal functor induces an involutive functor between the categories of involutive operads, allowing one to change the base involutive symmetric monoidal category when considering involutive operads. §8.7 observes that the AQFT (Algebraic Qauntum Field Theory) involutive operads are all obtained from the induced involutive functor in the previous section.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18-02 Research exposition (monographs, survey articles) pertaining to category theory
18Mxx Monoidal categories and operads

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