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Structured cospans. (English) Zbl 07281441
 Theory Appl. Categ. 35, 1771-1822 (2020).

This paper aims to introduce *structured cospans* as a way to study networks with inputs and outputs and then to show that they are advantageous over *decorated cospans*.

A synopsis of the paper goes as follows. Given a functor $L : \mathbf{A} \rightarrow \mathbf{X}$, §2 defines the notion of a *structured cospan*, which is of the form

$$\begin{array}{ccc}
 & x & \\
 & \nearrow^i & \nwarrow^o \\
 L(a) & & L(b)
 \end{array}$$

where a and b are called the *input* and *output*, respectively, while x is called the *apex* with the morphisms i and o named the *legs* of the cospan. It is shown in Theorem 2.3 that for any functor $L : \mathbf{A} \rightarrow \mathbf{X}$, when \mathbf{X} has pushouts, there is a double category ${}_L\mathbf{Csp}(\mathbf{X})$ in which

- objects are objects of \mathbf{A} ,
- vertical 1-morphisms are morphisms of \mathbf{A} ,
- horizontal 1-cells are structured cospans,

• 2-morphisms are commutative diagrams

$$\begin{array}{ccccc}
 L(a) & \xrightarrow{i} & x & \xleftarrow{o} & L(b) \\
 L(\alpha) \downarrow & & f \downarrow & & \downarrow L(\beta) \\
 L(a') & \xrightarrow{i'} & x' & \xleftarrow{o'} & L(b')
 \end{array}$$

It is shown in Corollary 2.4 that for any functor $L : \mathbf{A} \rightarrow \mathbf{X}$, when \mathbf{X} has pushouts, there is a bicategory ${}_L\mathbf{Csp}(\mathbf{X})$ in which

- objects are objects of \mathbf{A} ,
- 1-morphisms are morphisms of \mathbf{A} ,
- 2-morphisms are commutative diagrams in \mathbf{X} of the form

$$\begin{array}{ccc}
 & x & \\
 & \nearrow & \nwarrow \\
 L(a) & & L(b) \\
 & \searrow & \swarrow \\
 & x' &
 \end{array}$$

It is shown in Corollary 2.5 that for any functor $L : \mathbf{A} \rightarrow \mathbf{X}$, when \mathbf{X} has pushouts, there is a category ${}_L\mathbf{Csp}(\mathbf{X})$ in which

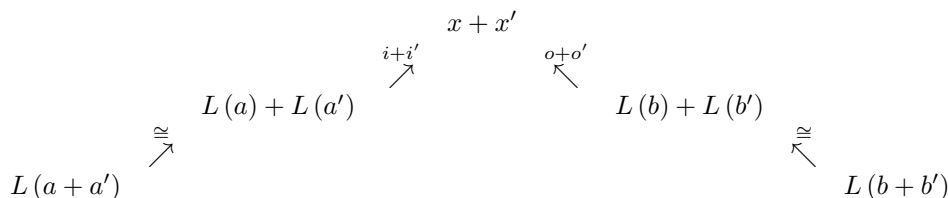
- objects are objects of \mathbf{A} ,
- morphisms are isomorphism classes of structured cospans, where two structured cospans $L(a) \rightarrow x \leftarrow L(b)$ and $L(a) \rightarrow y \leftarrow L(b)$ are said to be *isomorphic* if there is an isomorphism $f : x \rightarrow y$ such that the diagram

$$\begin{array}{ccc}
 & x & \\
 & \nearrow & \nwarrow \\
 L(a) & & L(b) \\
 & \searrow & \swarrow \\
 & y &
 \end{array}
 \quad
 \begin{array}{ccc}
 & f \downarrow & \\
 & & \\
 & &
 \end{array}$$

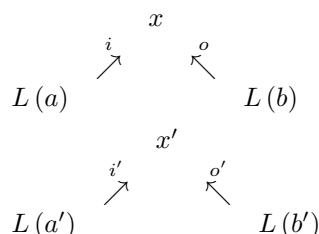
commutes.

§3 gives simple conditions under which the double category ${}_L\mathbf{Csp}(\mathbf{X})$ as well as the bicategory ${}_L\mathbf{Csp}(\mathbf{X})$ and the category ${}_L\mathbf{Csp}(\mathbf{X})$ becomes symmetric monoidal. The monoidal structure is based on the formation

of



from two cospans



via coproducts. It is shown in Theorem 3.9 that ${}_L\mathbf{Csp}(\mathbf{X})$ is a symmetric monoidal double category when \mathbf{A} and \mathbf{X} have finite colimits and L preserves them. Using [M. A. Shulman, “Constructing symmetric monoidal bicategories”, Preprint, [arXiv:1004.0993](https://arxiv.org/abs/1004.0993)], it is shown in Corollary 3.10 that ${}_L\mathbf{Csp}(\mathbf{X})$ is a symmetric monoidal bicategory when \mathbf{A} and \mathbf{X} have finite colimits and L preserves them. It is shown in Corollary 3.11 that ${}_L\mathbf{Csp}(\mathbf{X})$ is a symmetric monoidal category when \mathbf{A} and \mathbf{X} have finite colimits and L preserves them. It is shown in Theorem 3.12 that the category ${}_L\mathbf{Csp}(\mathbf{X})$ is actually a special sort of symmetric monoidal category called a *hypergraph category* [B. Fong and D. I. Spivak, “Hypergraph categories”, Preprint, [arXiv:1806.08304](https://arxiv.org/abs/1806.08304)].

§4 shows how to construct maps between structured cospan categories, bicategories or double categories.

§5 illustrates some of the advantages of structured categories over decorated ones with an example of the double category with open graphs as morphisms.

Decorated cospans have already been exploited to study electrical circuits [J. C. Baez and B. Fong, *Theory Appl. Categ.* 33, 1158–1222 (2018; [Zbl 1402.18005](https://zbmath.org/journal/TA/33/1158.html))], Markov processes [J. C. Baez et al., *J. Math. Phys.* 57, No. 3, 033301, 30 p. (2016; [Zbl 1336.60147](https://zbmath.org/journal/JMP/57/3/033301.html))] and reaction networks [J. C. Baez and B. S. Pollard, *Rev. Math. Phys.* 29, No. 9, Article ID 1750028, 41 p. (2017; [Zbl 1383.68053](https://zbmath.org/journal/RMP/29/9/1750028.html))], while structured cospans have been used to study electrical circuits [[Zbl 1400.18004](https://zbmath.org/journal/TA/33/1158.html)] and Petri nets [J. C. Baez and J. Master, “Open Petri nets”, *Math. Struct. Comput. Sci.* 30, 314–341 (2020; [doi:10.1017/S0960129520000043](https://doi.org/10.1017/S0960129520000043))]. §6 aims to show that structured cospans can take the place of decorated cospans in many of these applications.

Appendix is devoted to a brief review of double categories, referring the reader to [M. Grandis and R. Paré, *Cah. Topologie Géom. Différ. Catégoriques* 40, No. 3, 162–220 (1999; [Zbl 0939.18007](https://zbmath.org/journal/CTG/40/3/162.html)); M. Grandis and R. Paré, *Cah. Topol. Géom. Différ. Catég.* 58, No. 1, 3–48 (2017; [Zbl 1387.18013](https://zbmath.org/journal/CTG/58/1/3.html)); L. W. Hansen and M. Shulman, “Constructing symmetric monoidal bicategories functorially”, Preprint, [arXiv:1910.09240](https://arxiv.org/abs/1910.09240); M. Shulman, *Theory Appl. Categ.* 20, 650–738 (2008; [Zbl 1192.18005](https://zbmath.org/journal/TA/20/650.html)); “Constructing symmetric monoidal bicategories”, Preprint, [arXiv:1004.0993](https://arxiv.org/abs/1004.0993)] for more detailed expositions.

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MSC:

- [18B10](#) Categories of spans/cospans, relations, or partial maps
- [18M35](#) Categories of networks and processes, compositionality
- [18N10](#) 2-categories, bicategories, double categories

Keywords:

[bicategory](#); [cospan](#); [double category](#); [monoidal category](#); [network](#)

Full Text: [Link](#)

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