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An extension of Quillen's theorem B. (English) Zbl 07226705
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Quillen's Theorem B is one of the first results in [*D. Quillen*, Lect. Notes Math. 341, 85–147 (1973; [Zbl 0292.18004](#))], playing an important role in the foundations of algebraic K-theory. This paper establishes a very general version of Theorem B over an arbitrary site, for actions of a presheaf of simplicial categories on another simplicial presheaf (Theorem 5.1), which claims that if the action is by weak equivalences in some further left Bousfield localization of one of the standard model structures on simplicial presheaves, then the fiber and the homotopy fiber of the action become equivalent in this localization.

Theorem 5.1 has expected applications such as a version of the group completion theorem for actions of presheaves of simplicial monoids, such as the classifying space of a coproduct $\bigsqcup_n BGL_n(R)$ for a sheaf of rings R (Examples 6.11 and 6.14). When R is a sheaf of commutative rings on a site, the theorem means that the associated projective space \mathbb{P}^∞ is \mathbb{A}^1 -homotopy equivalent to its group completion $\Omega B(\mathbb{P}^\infty)$ (Example 6.10). It is expected that the theorem may have further applications when applied to specific sites such as the Nisnevich topology for \mathbb{A}^1 -homotopy theory [*F. Morel* and *V. Voevodsky*, Publ. Math., Inst. Hautes Étud. Sci. 90, 45–143 (1999; [Zbl 0983.14007](#))]. One can recover, as another special case of the theorem, a version of Puppe's theorem for homotopy cartesian morphisms between diagrams of simplicial presheaves over a site (Example 6.2), which reduces, in the particular case of simplicial sets, to a variant of Puppe's theorem for Bousfield localization [*W. Chachólski* et al., Contemp. Math. 399, 55–72 (2006; [Zbl 1117.55014](#))]. When applied to a left exact localization of simplicial presheaves, the theorem gives what is sometimes referred to as Rezk descent for ∞ -toposes [*C. Rezk*, "Fibrations and homotopy colimits of simplicial sheaves", Preprint, [arXiv:math/9811038](#)]. By recasting the theorem in terms of an equivalence of model categories, one gets a generalization of a result by *J. F. Jardine* [*K-Theory* 37, No. 3, 291–309 (2006; [Zbl 1106.14008](#))].

Reviewer: [Hirokazu Nishimura \(Tsukuba\)](#)

MSC:

- [18F20](#) Presheaves and sheaves, stacks, descent conditions (category-theoretic aspects)
- [55U35](#) Abstract and axiomatic homotopy theory in algebraic topology
- [55U10](#) Simplicial sets and complexes in algebraic topology

Keywords:

Quillen's theorem B; group completion; simplicial presheaf; Nisnevich site; Bousfield localization; Rezk descent

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