**Quijano, Juan Pablo**; **Resende, Pedro**

**Actions of étale-covered groupoids.** (English) Zbl 07268491

J. Algebra 566, 222-258 (2021).

We know well [*A. Kumjian*, Pac. J. Math. 112, 141–192 (1984; Zbl 0574.46046); *A. L. T. Paterson*, Groupoids, inverse semigroups, and their operator algebras. Boston, MA: Birkhäuser (1999; Zbl 0913.22001); *J. Renault*, A groupoid approach to C*-algebras. Springer, Cham (1980; Zbl 0433.46049)] that inverse semigroups are intimately related to étale groupoids. Quantales, more specifically inverse quantal frames, were put forward as meditating objects between semigroups and groupoids in [*J. P. Quijano* and *P. Resende*, "Functoriality of groupoid quantales. II", Preprint, `arXiv:1803.01075`]. This encouraged mathematicians not only to bring the correspondence to bear on localic groupoids rather than just topological groupoids, in particular making it constructive in a topos-theoretic sense, but also, independently, to provide an alternative algebraic language with which to describe étale groupoids, with the quantales being regarded as ring-like objects, which developed naturally into a program where various constructions for étale groupoids such as actions and sheaves are translated into quantale modules [*P. Resende*, J. Pure Appl. Algebra 216, No. 1, 41–70 (2012; Zbl 1231.06020); *P. Resende* and *E. Rodrigues*, Appl. Categ. Struct. 18, No. 2, 199–217 (2010; Zbl 1200.18008)]. The correspondence between étale groupoids and their quantales were made well behaved from a viewpoint of functoriality by considering bicategories and functoriality in the form of a bi-equivalence where the morphisms (1-cells) are groupoid bi-actions and quantale bimodules [*P. Resende*, J. Pure Appl. Algebra 219, No. 8, 3089–3109 (2015; Zbl 1343.06007)]. Another possible direction of research pertains to more general groupoids.

This paper aims to recover the good behavior of actions, sheaves and functoriality of étale groupoids by restricting to a smaller class of open groupoids introduced in [*M. C. Protin* and *P. Resende*, J. Noncommut. Geom. 6, No. 2, 199–247 (2012; Zbl 1253.06019)]. Such groupoids $G$ are endowed with good pseudogroups of local bisections, leading to a certain notion of cover $J : \widehat{G} \to G$ by an étale groupoid so that they are called *étale-covered groupoids*. Dually, to such groupoids, there is the notion of *inverse-embedded quantale frame*, consisting of an inverse quantale frame $\widehat{\mathcal{O}}$ with a suitable embedding $j : \mathcal{O} \to \widehat{\mathcal{O}}$ of a non-unital quantale.

A synopsis of the paper, consisting of four sections, goes as follows. §2 fixes terminology and notation, mostly following [*J. P. Quijano* and *P. Resende*, "Functoriality of groupoid quantales. II", Preprint, `arXiv:1803.01075`; *J. P. Quijano* and *P. Resende*, Semigroup Forum 99, No. 3, 754–787 (2019; Zbl 07138719); *P. Resende*, Adv. Math. 208, No. 1, 147–209 (2007; Zbl 1116.06014); *P. Resende*, J. Pure Appl. Algebra 216, No. 1, 41–70 (2012; Zbl 1231.06020)]. §3 introduces the main definitions of the paper, namely, inverse-embedded quantales and their étale-covered groupoids. The main results are presented in §4, relating the actions of an étale-covered groupoid $G$ to the modules of the quantale $\mathcal{O}\left(\widehat{G}\right)$ which behave well with respect to the embedding $j : \mathcal{O}(G) \to \mathcal{O}\left(\widehat{G}\right)$, and leading to an equivalence between the category of $G$-actions and the category of such $\mathcal{O}\left(\widehat{G}\right)$-modules wtih extending the equivalence of categories that exists if $G$ is étale. §4 contains two applications of these results. The first application is a description of $G$-sheaves in terms of $\mathcal{O}\left(\widehat{G}\right)$-modules extending that of the étale case, whereby a $G$-sheaf $X$ is shown to correspond to an $\mathcal{O}\left(\widehat{G}\right)$-sheaf whose inner product $(-,-) : X \times X \to \mathcal{O}\left(\widehat{G}\right)$ is valued in the image $j(\mathcal{O}(G))$. The second application is an extension of the functoriality results in [*P. Resende*, J. Pure Appl. Algebra 219, No. 8, 3089–3109 (2015; Zbl 1343.06007)], ultimately yielding a biequivalence between the bicategory of étale-covered groupoids and that of inverse-embedded quantale frames.

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**MSC:**

18F10 Grothendieck topologies and Grothendieck topoi
18F20 Presheaves and sheaves, stacks, descent conditions (category-theoretic aspects)
18F70 Frames and locales, pointfree topology, Stone duality
18F75 Quantales
18N10 2-categories, bicategories, double categories
22A22 Topological groupoids (including differentiable and Lie groupoids)

**Keywords:**

localic open groupoids; groupoid quantales; actions; sheaves; bi-actions

**Full Text:** DOI

**References:**

[1] Buneci, M. R., Groupoid Categories, Perspectives in Operator Algebras and Mathematical Physics, Theta Ser. Adv. Math., vol. 8, 27-40 (2008), Theta: Theta Bucharest, MR2433025 (2010b:22007)

[2] Buss, A.; Exel, R.; Meyer, R., Inverse semigroup actions as groupoid actions, Semigroup Forum, 85, 2, 227-243 (2012), MR2969047 · Zbl 1261.22003

[3] Eklund, P.; Gutiérrez García, J.; Höhle, U.; Kortelainen, J., Semigroups in Complete Lattices, Developments in Mathematics, vol. 54 (2018), Springer: Springer Cham, Quantales, modules and related topics; With a foreword by Jimmie Lawson. MR3793185

[4] Heymans, H., Sheaves on involutive quantales: Grothendieck quantales, Fuzzy Sets Syst., 256, 117-148 (2014), MR3263495 · Zbl 1337.06009

[5] Johnstone, P. T., Stone Spaces, Cambridge Studies in Advanced Mathematics, vol. 3 (1986), Cambridge University Press: Cambridge University Press Cambridge, Reprint of the 1982 edition. MR861951 (87m:54001) · Zbl 0586.54001

[6] Joyal, A.; Tierney, M., An extension of the Galois theory of Grothendieck, Mem. Am. Math. Soc., 51, 309 (1984), vii+71. MR756176 (86d:18002) · Zbl 0541.18002

[7] Kudryavtseva, G.; Lawson, M. V., A perspective on non-commutative frame theory, Adv. Math., 311, 378-468 (2017), MR3628219 · Zbl 1423.06041

[8] Kumjian, A., On localizations and simple \(C^\ast \)-algebras, Pac. J. Math., 112, 1, 141-192 (1984), MR739145 · Zbl 0574.46046

[9] Lawson, M. V.; Lenz, D. H., Pseudogroups and their étale groupoids, Adv. Math., 244, 117-170 (2013), MR3077869 · Zbl 1311.20059

[10] Matsnev, D.; Resende, P., Étale groupoids as germ groupoids and their base extensions, Proc. Edinb. Math. Soc. (2), 53, 3, 765-785 (2010), MR2720249 · Zbl 1206.20072

[11] Paterson, A. L.T., Groupoids, Inverse Semigroups, and Their Operator Algebras, Progress in Mathematics, vol. 170 (1999), Birkhäuser Boston Inc.: Birkhäuser Boston Inc. Boston, MA, MR1724106 (2001a:22003) · Zbl 0913.22001

[12] Protin, M. C.; Resende, P., Quantales of open groupoids, J. Noncommut. Geom., 6, 2, 199-247 (2012), MR2914865 · Zbl 1253.06019

[13] Quijano, J. P.; Resende, P., Functoriality of groupoid quantales. II (2018)

[14] Quijano, J. P.; Resende, P., Effective equivalence relations and principal quantales, Semigroup Forum, 99, 3, 754-787 (2019), MR4036870 · Zbl 07138719

[15] Renault, J., A Groupoid Approach to \(C^\ast \)-Algebras, Lecture Notes in Mathematics, vol. 793 (1980), Springer: Springer Berlin, MR584266 (82h:46075) · Zbl 0433.46049

[16] Resende, P., Lectures on étale groupoids, inverse semigroups and quantales, SOCRATES Intensive Program 103466-IC-1-2003-1-BE-ERASMUS-IPUC-3 — GAMAP: Geometric and Algebraic Methods of Physics and Applications (Univ. Antwerp, 2006), available at

[17] Resende, P., Étale groupoids and their quantales, Adv. Math., 208, 1, 147-209 (2007), MR2304314 (2008c:22002) · Zbl 1116.06014

[18] Resende, P., Groupoid sheaves as quantale sheaves, J. Pure Appl. Algebra, 216, 1, 41-70 (2012), MR2826418 · Zbl 1231.06020

[19] Resende, P., Functoriality of groupoid quantales. I, J. Pure Appl. Algebra, 219, 8, 3089-3109 (2015), MR3320209 · Zbl 1343.06007

[20] Resende, P.; Rodrigues, E., Sheaves as modules, Appl. Categ. Struct., 18, 2, 199-217 (2010), (on-line 2008). MR2601963 · Zbl 1200.18008

[21] Rosenthal, K. I., Quantales and Their Applications, Pitman Research Notes in Mathematics Series, vol. 234 (1990), Longman Scientific \& Technical: Longman Scientific \& Technical Harlow, MR1088258 (92e:06028) · Zbl 0703.06007