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Decomposing the tube category. (English) Zbl 07194257
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The *tube algebra* of a monoidal category $\mathcal{C}$ was introduced in [*A. Ocneanu*, in: Subfactors. Proceedings of the Taniguchi symposium on operator algebras, Kyuzeso, Japan, July 6–10, 1993. 32nd Taniguchi international symposium. Singapore: World Scientific. 39–63 (1994; Zbl 0927.46032)] within the realm of operator algebras. Connections between the tube algebra and the Drinfeld center $Z(\mathcal{C})$ have inspired much research, culminating in [Zbl 1415.46042] which claims that the category of representations of the tube algebra is equivalent to $Z(\mathcal{C})$. If $\mathcal{C}$ is a modular tensor category (MTC), then $Z(\mathcal{C})$ is equivalent to the category $\mathcal{C}\boxtimes\overline{\mathcal{C}}$, where $\boxtimes$ is the Deligne tensor product and $\overline{\mathcal{C}}$ is obtained from $\mathcal{C}$ by equipping it with the opposite braiding [*M. Müger*, Proc. Lond. Math. Soc. (3) 87, No. 2, 291–308 (2003; Zbl 1037.18005)].

This paper focuses alternatively on the *tube category* $\mathcal{TC}$, which is a multiobject version of the tube algebra, being Morita equivalent to it. To put it another way, the category of representations

$$\mathcal{RTC} := \mathrm{Fun}\left(\mathcal{TC}^{\mathrm{op}}, \underline{\mathrm{Vect}}\right)$$

is equivalent to that of the tube algebra, where $\underline{\mathrm{Vect}}$ denotes the category of finite-dimensional vector spaces. Therefore we have

$$\mathcal{RTC} \simeq Z(\mathcal{C}) \simeq \mathcal{C}\boxtimes\overline{\mathcal{C}}$$

which gives, as $\{I\boxtimes J\}_{I,J\in\mathrm{Irr}(\mathcal{C})}$ forms a complete set of simple objects in $\mathcal{C}\boxtimes\overline{\mathcal{C}}$, a complete set $\{F_{IJ}\}_{I,J\in\mathrm{Irr}(\mathcal{C})}$ of irreducible functors in $\mathcal{RTC}$. The so-called Yoneda embedding, which maps an object X in $\mathcal{TC}$ to $X^{\#} = \mathrm{Hom}_{\mathcal{TC}}(-,X)$ in $\mathcal{RTC}$, gives rise to the decomposition

$$\mathrm{Hom}_{\mathcal{TC}}(X,Y) = \mathrm{Hom}_{\mathcal{RTC}}\left(X^{\#},Y^{\#}\right) = \oplus_{I,J}\mathrm{Hom}_{\mathcal{RTC}}\left(X^{\#},F_{IJ}\right)\otimes\mathrm{Hom}_{\mathcal{RTC}}\left(F_{IJ},Y^{\#}\right)$$

Since, for fixed $I,J\in\mathrm{Irr}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{RTC}}\left(X^{\#},F_{IJ}\right)$ is to be put down at F_{IJ} via the canonical Yoneda map while $\mathrm{Hom}_{\mathcal{RTC}}\left(F_{IJ},Y^{\#}\right)$ is to be identified with $\mathrm{Hom}_{\mathcal{RTC}}\left(Y^{\#},F_{IJ}\right)^{*}$ via the perfect pairing given by composition which is to be regarded as $F_{IJ}(Y)^{*}$ via Yoneda again, the (I,J) summand is to be looked upon as $F_{IJ}(Y)^{*}\otimes F_{IJ}(X)$ so that we have a natural injection

$$\lambda_{YX}^{IJ} : F_{IJ}(Y)^{*}\otimes F_{IJ}(X)\to\mathrm{Hom}_{\mathcal{TC}}(X,Y)$$

which, by naturality in X, is to be seen as a map

$$\lambda_{Y}^{IJ} : F_{IJ}(Y)^{*}\to\mathrm{Hom}_{\mathcal{RTC}}\left(F_{IJ},Y^{\#}\right)$$

This paper, consisting of five sections, aims to give a graphical description of λ_{YX}^{IJ} and therefore λ_{Y}^{IJ}, to which composition is easily described (Proposition 2.4), allowing of identification of the primitive idempotents in $\mathrm{End}_{\mathcal{TC}}(X)$ (Corollary 5.7) corresponding to the irreducible summands of $X^{\#}$ in $\mathcal{RTC}$ and being able to be thought of as categorical analogues of *Ocneanu projections*. A synopsis of the paper goes as follows. §2 records some basic results arising from the Yoneda Lemma, Lemma 2.3 implying that λ_{Y}^{IJ} is characterized as being the unique opposite of the canonical Yoneda map

$$\mu_{Y}^{IJ} : F_{IJ}(Y)\to\mathrm{Hom}_{\mathcal{RTC}}\left(Y^{\#},F_{IJ}\right)$$

The graphical calculus of MTC's is built in §3, culminating in Lemma 3.11 and Proposition 3.14. §4 provides an introduction to the tube category, while §5 gives a graphical candidate for λ_{Y}^{IJ}, establishing that it is opposite to μ_{Y}^{IJ}.

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