Singular Integrals and Feller Semigroups

Real Analysis Methods for Markov Processes

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Preface

This book is an easy-to-read reference providing a link with functional analysis, real analysis, partial differential equations and probability. Most mathematicians working in partial differential equations are only vaguely familiar with the powerful ideas of stochastic analysis. On the other hand, the additional intuition which this book conveys might provide better insight and be helpful to their work. In addition, the book provides a compendium for a large variety of facts from functional analysis, real analysis, singular integral operators and Markov processes - for looking up quickly a theorem. This book gives better coverage of important examples and applications, and is amply illustrated and all figures and tables are provided with appropriate captions.

The purpose of this book is a self-contained account of the functional analytic approach to the problem of construction of Markov processes with Ventcel' (Wentzell) boundary conditions in probability. More precisely, we prove existence theorems for Feller semigroups with Dirichlet boundary condition, oblique derivative boundary condition and first-order Ventcel' boundary condition for second-order, uniformly elliptic differential operators with *discontinuous* coefficients. Our approach here is distinguished by the extensive use of the ideas and techniques characteristic of the recent developments in the Calderón–Zygmund theory of singular integral operators with non-smooth kernels.

It should be emphasized that singular integral operators with non-smooth kernels provide a powerful tool to deal with smoothness of solutions of partial differential equations, with minimal assumptions of regularity on the coefficients. The Calderón–Zygmund theory of singular integrals continues to be one of the most influential works in modern history of analysis, and is a very refined mathematical tool whose full power is yet to be exploited.

This book is addressed to advanced undergraduates or beginning-graduate students and also mathematicians with interest in real analysis, functional analysis and partial differential equations. For the former, it may serve as an effective introduction to these four interrelated fields of analysis. For the latter, it provides a method for the study of elliptic boundary value problems with *discontinuous* coefficients, a powerful method clearly capable of extensive further development. Bibliographical references are discussed primarily in Notes and Comments at the end of each chapter. These notes are intended to supplement the text and place it in better perspective. This book will lead to a better insight into the study of singular integrals and elliptic boundary value problems for graduate students about to enter the subject, and mathematicians in the field looking for a coherent overview.

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