Part V

Oblique Derivative Problems for Elliptic Differential Equations with Discontinuous Coefficients

16
Oblique Derivative Problems in Sobolev Spaces

This chapter is devoted to the study of the *regular* oblique derivative problem for a second-order, uniformly elliptic differential operator with discontinuous coefficients in the framework of L^p Sobolev spaces. More precisely, we consider a second-order, uniformly elliptic differential operator with VMO coefficients and an oblique derivative boundary operator that is *nowhere* tangential to the boundary. We state global regularizing property of the oblique derivative problem in the framework of L^p Sobolev spaces (Theorem 16.1). Furthermore, we state an exsitence and uniqueness theorem for the oblique derivative problem in the framework of L^p Sobolev spaces (Theorem 16.2).

16.1 Formulation of the Oblique Derivative Problem

Let Ω be a bounded domain of $\mathbf{R}^n$, $n \geq 3$, with boundary $\partial\Omega$ of class $C^{1,1}$. In the interior Ω, we consider a second-order, elliptic differential operator $\mathcal{L}$ with real *discontinuous* coefficients of the form

$$\mathcal{L}u := \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} \quad \text{for } x \in \Omega.$$

More precisely, we assume that the coefficients $a^{ij}(x)$ satisfy the following three conditions (1), (2) and (3):

(1) $a^{ij}(x) \in \mathrm{VMO} \cap L^\infty(\Omega)$ for $1 \leq i, j \leq n$.
(2) $a^{ij}(x) = a^{ji}(x)$ for almost all $x \in \Omega$ and $1 \leq i, j \leq n$.
(3) There exists a positive constant λ such that

$$\frac{1}{\lambda}|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2 \tag{16.1}$$

for almost all $x \in \Omega$ and for all $\xi \in \mathbf{R}^n$.

If $\eta^{ij}(r)$ is the VMO modulus of $a^{ij}(x)$, then we let

$$\eta(r) := \left(\sum_{i,j=1}^{n} \eta^{ij}(r)^2 \right)^{1/2}.$$

On the boundary $\partial\Omega$, we consider a first-order boundary operator $\mathcal{B}$ with real continuous coefficients of the from

$$\mathcal{B}u := \frac{\partial u}{\partial \ell} + \sigma(x')u = \sum_{i=1}^{n} \ell^i(x') \frac{\partial u}{\partial x_i} + \sigma(x')u \quad \text{for } x' \in \partial\Omega. \tag{16.2}$$

Concerning the boundary operator $\mathcal{B}$, we assume that the following three conditions (16.3a), (16.3b) and (16.3c) are satisfied:

$$\ell^i(x') \text{ and } \sigma(x') \text{ are Lipschitz continuous functions on } \partial\Omega. \tag{16.3a}$$

$$\langle \ell(x'), \mathbf{n}(x') \rangle := \sum_{i=1}^{n} \ell^i(x')\, n_i(x') > 0 \quad \text{on } \partial\Omega. \tag{16.3b}$$

$$\sigma(x') < 0 \quad \text{on } \partial\Omega. \tag{16.3c}$$

Here $\mathbf{n}(x') = (n_1(x'), n_2(x'), \ldots, n_n(x'))$ is the unit *interior* normal to $\partial\Omega$ and $\ell(x') = (\ell^1(x'), \ell^2(x'), \ldots, \ell^n(x'))$ is a vector field on $\partial\Omega$. It should be emphasized that the boundary operator $\mathcal{B}$ is given by a directional derivative with respect to the vector field $\ell(x')$ on $\partial\Omega$. The simple geometric meaning of conditions (16.3b) is that the vector field $\ell(x')$ is *nowhere* tangential to the boundary $\partial\Omega$ (see Figure 16.1).

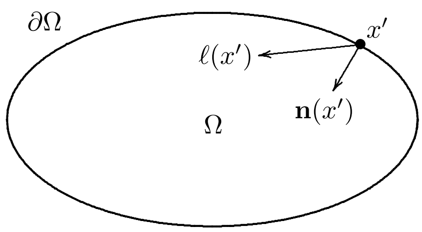

Fig. 16.1. the unit interior normal $\mathbf{n}(x')$ and the oblique vector field $\ell(x')$ at x'

To interpret the boundary condition (16.2) in the sense of traces on $\partial\Omega$, we recall some definitions and useful notations. If $1 < p < \infty$ and if $k = 1$ or $k = 2$, we define the L^p Sobolev space

$W^{k,p}(\Omega) =$ the space of (equivalence classes of) functions

$u \in L^p(\Omega)$ whose derivatives $D^\alpha u$, $|\alpha| \leq k$, in the sense of distributions are in $L^p(\Omega)$,

and define the boundary space of traces of functions

$$B^{k-1/p,p}(\partial\Omega) = \text{ the space of the traces } \gamma_0 u \text{ of functions } u \in W^{k,p}(\Omega).$$

In the space $B^{k-1/p,p}(\partial\Omega)$, we introduce a norm

$$|\varphi|_{B^{k-1/p,p}(\partial\Omega)} = \inf\left\{\|u\|_{W^{k,p}(\Omega)} : u \in W^{k,p}(\Omega),\ \gamma_0 u = \varphi \text{ on } \partial\Omega\right\}.$$

The space $B^{k-1/p,p}(\partial\Omega)$ is a Banach space with respect to the norm $|\cdot|_{B^{k-1/p,p}(\partial\Omega)}$. We recall that the space $B^{k-1/p,p}(\partial\Omega)$ is a Besov space (see the trace theorem (Theorem 7.4)).

The purpose of this chapter is to study global regularity and solvability in the framework of Sobolev spaces of the the following non-homogenous oblique derivative problem:

$$\begin{cases} \mathcal{L}u(x) = f(x) & \text{for almost all } x \in \Omega, \\ \mathcal{B}u(x') = \varphi(x') & \text{in the sense of traces on } \partial\Omega. \end{cases} \tag{16.4}$$

It should be emphasized that the boundary value problem (16.4) is a *regular* oblique derivative problem, since the vector field $\ell(x')$ is nowhere tangential to the boundary $\partial\Omega$.

The interest in the study of oblique derivative problems for elliptic operators with VMO coefficients increased significantly in the last twenty years. This is mainly due to the fact that VMO contains as a proper subspace $C(\overline{\Omega})$ which ensures the extension of the L^p Schauder theory of operators with continuous coefficients to *discontinuous* coefficients (see [33], [40]). On the other hand, the Sobolev spaces $W^{1,n}(\Omega)$ and $W^{\theta,n/\theta}(\Omega)$, $0 < \theta < 1$, are also contained in VMO; hence the VMO discontinuity of the $a^{ij}(x)$ becomes more general than those studied before (see [49], [50], [88]).

16.2 Statement of Main Results (Theorems 16.1 and 16.2)

The first main result of this chapter is stated as follows (see [47, Chapter 2, Theorem 2.2.1]):

Theorem 16.1 (the regularity theorem). *Let* $1 < p < \infty$, *and assume that conditions* (16.1) *and* (16.3) *are satisfied. If a function* $u \in W^{2,q}(\Omega)$, $1 < q < p < \infty$, *is a solution of problem* (16.4) *with* $f \in L^p(\Omega)$ *and*

$\varphi \in B^{1-1/p,p}(\partial\Omega)$, *then it follows that* $u \in W^{2,p}(\Omega)$. *Moreover, we have the global* a priori *estimate*

$$\|u\|_{W^{2,p}(\Omega)} \leq C_1 \left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \|\varphi\|_{B^{1-1/p,p}(\partial\Omega)} \right), \qquad (16.5)$$

with a positive constant $C_1 = C_1(n, p, \lambda, \eta, \ell, \sigma, \partial\Omega)$.

The proof of Theorem 16.1 can be visualized in the following diagram:

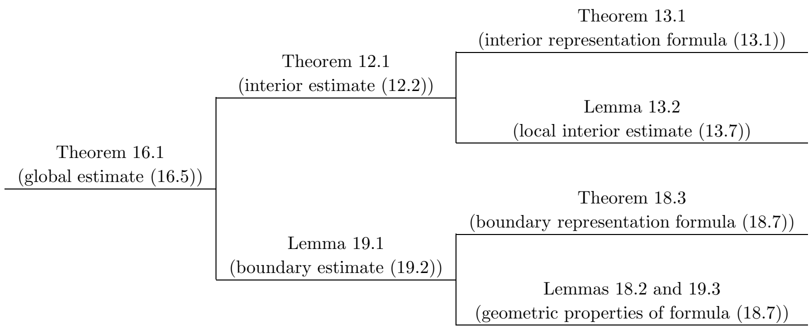

Table 16.1. *A flowchart for the proof of Theorem 16.1*

The regularizing property of the couple $(\mathcal{L}, \mathcal{B})$ implies the well-posedness of problem (16.4) in the framewok of L^p Sobolev spaces. More precisely, the second main result of this chapter is stated as follows (see [47, Chapter 2, Theorem 2.2.2], [84, Theorem 1.1], [86, Theorem 1.2]):

Theorem 16.2 (the existence and uniqueness theorem). *Let* $1 < p < \infty$, *and assume that conditions* (16.1) *and* (16.3) *are satisfied. Then, for any* $f \in L^p(\Omega)$ *and any* $\varphi \in B^{1-1/p,p}(\partial\Omega)$ *there exists a unique solution of problem* (16.4). *Moreover, we have the global* a priori *estimate*

$$\|u\|_{W^{2,p}(\Omega)} \leq C_2 \left(\|f\|_{L^p(\Omega)} + \|\varphi\|_{B^{1-1/p,p}(\partial\Omega)} \right), \qquad (16.6)$$

with a positive constant $C_2 = C_2(n, p, \lambda, \eta, \ell, \sigma, \partial\Omega)$.

Remark 16.1. The results presented here can be applied to the study of Sobolev regularity of the solutions of problem (16.4) for a general second-order elliptic operator

$$Au = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial u}{\partial x_i} + c(x)u,$$

where the lower order coefficients $b^i(x)$ and $c(x)$ satisfy suitable Lebesgue integrability conditions (see [46]) such as study the case where

$$b^i(x),\ c(x) \in L^\infty(\Omega),$$

and

$$c(x) \leq 0 \quad \text{almost everywhere in } \Omega.$$

Remark 16.2. The condition that

$$\sigma(x') < 0 \quad \text{on } \partial\Omega \tag{16.7}$$

is not necessary for the regularity result of problem (16.4). In fact, problem (16.4) for a sign-changing function $\sigma(x')$ may be reduced to the case considered here if we take a function $F(x) \in C^{1,1}(\overline{\Omega})$ which satisfies the condition

$$\frac{\partial F}{\partial \ell} = -\sigma(x') - 1 \quad \text{on } \partial\Omega,$$

and let

$$u(x) = v(x)e^{F(x)}.$$

On the other hand, in the proof of Theorem 16.2, condition (16.7) is essential for the uniqueness result of problem (16.4).

The crucial point of our investigations is the local boundary Sobolev regularity of the solutions of problem (16.4). Our approach is based on explicit integral representation formulas (18.8) for the second derivatives of solutions of problem (16.4) with constant coefficients operators and homogeneous boundary conditions (near the boundary), in terms of singular integral operators with Calderón–Zygmund kernels and their commutators and operators with positive kernels (Theorem 18.3). This method has been already used in the study of the Dirichlet problem in Part III of this book. In order to deal with non-homogeneous oblique derivative boundary conditions with variable coefficients, we introduce a special auxiliary function which, roughly speaking, absorbs the right-hand side of the boundary condition (16.4) (Lemma 17.1). Moreover, we make use of special non-dimensional norms (17.1) and (17.6) to estimate effectively the Sobolev norms of the second derivatives of solutions of problem (16.4).

Finally, it should be emphasized that VMO functions are invariant under $C^{1,1}$-diffeomorphisms (see [1, Proposition 1.3]).

16.3 Notes and Comments

This chapter is adapted from Di Fazio–Palagachev [22] and Maugeri–Palagachev–Softova [47].

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Oblique Derivative Boundary Conditions

In this chapter, for a given boundary function, we construct an auxiliary function that satisfies an oblique derivative boundary condition. More precisely, for a given boundary function $\varphi(x') \in B^{1-1/p,p}(\partial\Omega)$, we construct a special extension $\phi(x) \in W^{2,p}(\Omega)$ which satisfies the oblique derivative boundary condition (see Figure 16.1)

$$\frac{\partial \phi}{\partial \ell} = \sum_{i=1}^{n} \ell^i(x') \frac{\partial \phi}{\partial x_i} = \varphi \quad \text{on } \partial\Omega.$$

This result (Lemma 17.1) will allow us to represent, locally near the boundary, the solution of the non-homogeneous oblique derivative problem (16.4) in Chapter 19 (see formula (19.10)). In this way, we are reduced to the study of the *homogeneous* oblique derivative problem:

$$\begin{cases} \mathcal{L}u(x) = f(x) & \text{for almost all } x \in \Omega, \\ \mathcal{B}u(x') = 0 & \text{in the sense of traces on } \partial\Omega. \end{cases}$$

17.1 Construction of Auxiliary Functions

Let $\widetilde{\Gamma}$ be a bounded portion of the hyperplane

$$\begin{aligned} &\{x_n = 0\} \\ &= \left\{x = (x', x_n) \in \mathbf{R}^n : x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbf{R}^{n-1},\ x_n = 0\right\}. \end{aligned}$$

Let $\widetilde{\varphi}(x')$ be a function defined on $\widetilde{\Gamma}$ which belongs to $B^{1-1/p,p}(\widetilde{\Gamma})$. The space $B^{1-1/p,p}(\widetilde{\Gamma})$, $1 < p < \infty$, is a Banach space equipped with the norm (cf. the norm (7.11) with $m := 0$ and $\theta := 1 - 1/p$)

$$\|\widetilde{\varphi}\|^*_{B^{1-1/p,p}(\widetilde{\Gamma})} = \left(\int_{\widetilde{\Gamma}} |\widetilde{\varphi}(x')|^p \, dx'\right)^{1/p} \tag{17.1}$$

$$+ d^{1/2} \left(\int_{\widetilde{\Gamma}} \int_{\widetilde{\Gamma}} \frac{|\widetilde{\varphi}(x') - \widetilde{\varphi}(y')|^p}{|x' - y'|^{p+n-2}} \, dx' \, dy' \right)^{1/p},$$

where $d = \operatorname{diam} \widetilde{\Gamma}/2$.

First, following the proof of [33, Theorem 6.26] we take a bell-shaped function $\zeta(x')$ on $\mathbf{R}^{n-1}$ which satisfies the following four conditions:

$$\zeta(x') \in C_0^2(\mathbf{R}^{n-1}). \tag{17.2a}$$

$$\zeta(x') \geq 0 \quad \text{on } \mathbf{R}^{n-1}. \tag{17.2b}$$

$$\operatorname{supp} \zeta \subset \left\{ x' \in \mathbf{R}^{n-1} : |x'| \leq 1 \right\}. \tag{17.2c}$$

$$\int_{\mathbf{R}^{n-1}} \zeta(x') \, dx' = 1. \tag{17.2d}$$

We take an arbitrary point $x_0 = (x_0', 0)$ of the hyperplane $\{x_n = 0\}$ and $R > 0$, and let (see Figure 17.1)

$$B_R^+ := B_R(x_0) \cap \{x_n > 0\},$$

and

$$\Gamma_R := B_R(x_0) \cap \{x_n = 0\}.$$

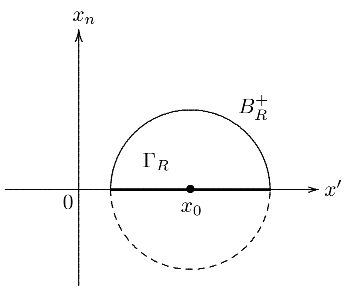

Fig. 17.1. The semi-ball B_R^+ in $\mathbf{R}_+^n$ and the ball $\Gamma_R = B_R \cap \{x_n = 0\}$ in $\mathbf{R}^{n-1}$

Without loss of generality, we may take Γ_R instead of $\widetilde{\Gamma}$ in the above definition of the norm $\|\widetilde{\varphi}\|^*_{B^{1-1/p,p}(\widetilde{\Gamma})}$, and take $d = R$. For a given function $\widetilde{\varphi} \in B^{1-1/p,p}(\Gamma_R)$, we may assume that $\widetilde{\varphi}(x')$ can be extended to the whole hyperplane $\{x_n = 0\}$ as a function with compact support, preserving its $B^{1-1/p,p}$-norm.

Assuming that the boundary $\partial\Omega$ is locally flattened out near the point x_0 such that $\Omega \subset \{x_n > 0\}$ (see Figure 17.2 below), we remark that the *regular* oblique derivative boundary condition (16.3b) implies that

$$\ell^n(x_0) > 0. \tag{17.3}$$

Consider now the function

$$\phi(x) := \frac{x_n}{\ell^n(x_0)} \int_{\mathbf{R}^{n-1}} \widetilde{\varphi}(x' - x_n y')\zeta(y')\, dy' \quad \text{for } x = (x', x_n) \in \mathbf{R}^n_+. \tag{17.4}$$

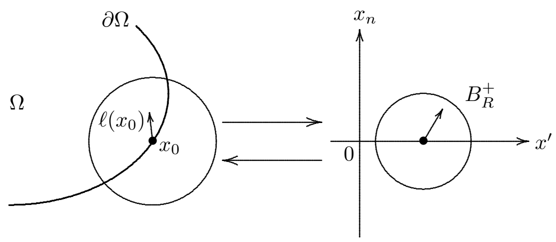

Fig. 17.2. The semi-ball B_R^+ in $\mathbf{R}^n_+$ and the oblique vector field $\ell(x_0)$ at x_0

The next lemma is an essential step in our further considerations:

Lemma 17.1. *Let* $1 < p < \infty$. *The function* $\phi(x)$, *defined by formula* (17.4), *belongs to the space* $W^{2,p}(B_R^+)$ *and satisfies the conditions*

$$\phi(x', 0) = 0 \quad \text{on } \Gamma_R, \tag{17.5a}$$

$$\frac{\partial \phi}{\partial x_n}(x', 0) = \frac{\widetilde{\varphi}(x')}{\ell^n(x_0)} \quad \text{on } \Gamma_R. \tag{17.5b}$$

If we introduce a norm

$$\|\phi\|^*_{W^{2,p}(B_R^+)} := \|\phi\|_{L^p(B_R^+)} + R \left\|\nabla^2 \phi\right\|_{L^p(B_R^+)}, \tag{17.6}$$

then we have the estimate

$$\|\phi\|^*_{W^{2,p}(B_R^+)} \leq C R^{1/2} \|\widetilde{\varphi}\|^*_{B^{1-1/p,p}(\Gamma_R)}, \tag{17.7}$$

with a positive constant $C = C(n, p, \lambda, \ell, \zeta)$.

Proof. The proof of Lemma 17.1 is divided into three steps.

Step 1: First, we prove that

$$\|\phi\|_{L^p(B_R^+)} \leq C_1 R^{1+1/p} \|\widetilde{\varphi}\|^*_{B^{1-1/p,p}(\Gamma_R)} \tag{17.8}$$

for some constant $C_1 = C_1(n, p, \ell, \zeta) > 0$, Without loss of generality, we may assume (see Figure 17.3) that

$$x_0 = 0,$$
$$B_R^+ = B_R(0) \cap \{x_n > 0\} = \{x = (x', x_n) \in \mathbf{R}^n : |x| < R,\ x_n > 0\},$$

and

$$\Gamma_R = B_R(0) \cap \{x_n = 0\} = \{x = (x', 0) \in \mathbf{R}^n : |x'| < R\}.$$

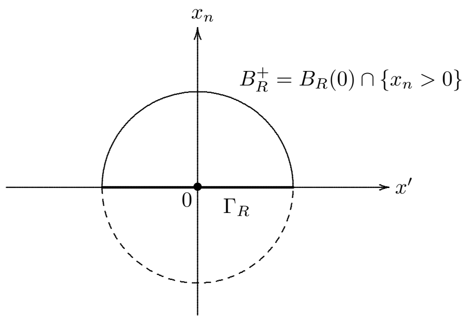

Fig. 17.3. The semi-ball B_R^+ in $\mathbf{R}_+^n$ and the ball $\Gamma_R = B_R \cap \{x_n = 0\}$ in $\mathbf{R}^{n-1}$

By using Minkowski's inequality for integrals (Theorem 3.18) and Fubini's theorem (Theorem 3.10), we obtain that

$$\begin{aligned}
&\int_{B_R^+} |\phi(x)|^p \, dx \qquad (17.9) \\
&= \frac{1}{|\ell^n(x_0)|^p} \int_{B_R^+} \left| x_n \int_{\mathbf{R}^{n-1}} \widetilde{\varphi}(x' - x_n y') \zeta(y') \, dy' \right|^p dx \\
&\le C(n, p, \ell, \zeta) \int_{\mathbf{R}^{n-1}} |\zeta(y')|^p \left(\int_{B_R^+} x_n^p |\widetilde{\varphi}(x' - x_n y')|^p \, dx \right) dy' \\
&= C(n, p, \ell, \zeta) \int_{\mathbf{R}^{n-1}} |\zeta(y')|^p I_{B_R^+}(y') \, dy',
\end{aligned}$$

where

$$I_{B_R^+}(y') := \int_{B_R^+} x_n^p |\widetilde{\varphi}(x' - x_n y')|^p \, dx.$$

However, by letting (see Figure 17.4)

$$Q_R$$

$$:= \Big\{ x = (x_1, x_2, \dots, x_{n-1}, x_n) \in \mathbf{R}^n : |x_1| \leq R,\ |x_2| \leq R, \dots,$$
$$|x_{n-1}| \leq R,\ 0 \leq x_n \leq R \Big\},$$

we obtain that

$$I_{B_R^+}(y') \leq I_{Q_R}(y') = \int_{Q_R} x_n^p |\widetilde{\varphi}(x' - x_n y')|^p \, dx' \, dx_n$$
$$\leq \int_0^R x_n^p \left(\int_{Q'_R(x_n)} |\widetilde{\varphi}(z')|^p \, dz' \right) dx_n,$$

where

$$z' = (z_1, z_2, \dots, z_{n-1}) = x' - x_n \, y'$$
$$= (x_1 - x_n y_1,\, x_2 - x_n y_2, \dots, x_{n-1} - x_n y_{n-1}),$$

and

$$Q'_R(x_n) = \Big\{ z' = (z_1, z_2, \dots, z_{n-1}) \in \mathbf{R}^{n-1} : |z_1 + x_n y_1| \leq R,$$
$$|z_2 + x_n y_2| \leq R, \dots, |z_n + x_n y_{n-1}| \leq R \Big\}.$$

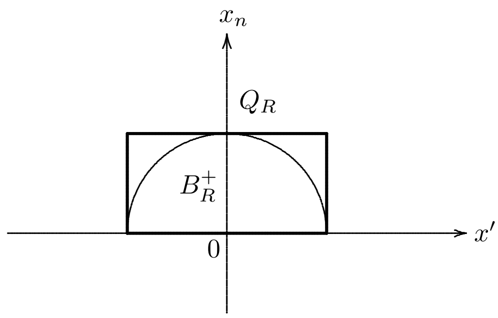

Fig. 17.4. The semi-ball B_R^+ and the cube Q_R in $\mathbf{R}^n_+$

Since we have, for some positive constant c,

$$\int_{Q'_R(x_n)} |\widetilde{\varphi}(z')|^p \, dz' \leq c \left(\|\widetilde{\varphi}\|^*_{B^{1-1/p,p}(\Gamma_R)} \right)^p \quad \text{for } 0 \leq x_n \leq R,$$

it follows that

$$\begin{aligned} I_{B_R^+}(y') &= \int_{B_R^+} x_n^p |\widetilde{\varphi}(x' - x_n y')|^p \, dx \\ &\le c \left(\|\widetilde{\varphi}\|^*_{B^{1-1/p,p}(\Gamma_R)} \right)^p \int_0^R x_n^p \, dx_n \\ &= \frac{cR^{p+1}}{p+1} \left(\|\widetilde{\varphi}\|^*_{B^{1-1/p,p}(\Gamma_R)} \right)^p \quad \text{for } y' \in \mathbf{R}^{n-1}. \end{aligned} \tag{17.10}$$

Therefore, by combining estimates (17.9) and (17.10) we obtain that

$$\begin{aligned} &\int_{B_R^+} |\phi(x)|^p \, dx \\ &\le C(n,p,\ell,\zeta) \, \frac{cR^{p+1}}{p+1} \int_{\mathbf{R}^{n-1}} |\zeta(y')|^p \, dy' \left(\|\widetilde{\varphi}\|^*_{B^{1-1/p,p}(\Gamma_R)} \right)^p. \end{aligned}$$

This proves the desired estimate (17.8), with

$$C_1 := \left(\frac{c\, C(n,p,\ell,\zeta)}{p+1} \right)^{1/p} \|\zeta\|_{L^p(\mathbf{R}^{n-1})}.$$

Step 2: Secondly, we prove that

$$R \left\| \nabla^2 \phi \right\|_{L^p(B_R^+)} \le C_2 R^{1/2} \|\widetilde{\varphi}\|^*_{B^{1-1/p,p}(\Gamma_R)}, \tag{17.11}$$

for some a positive constant $C_2 = C_2(n,p,\ell,\zeta)$.

To do this, we calculate now the first and second derivatives of the function $\phi(x)$ defined by formula (17.4).

We use the shorthand

$$D_i = \frac{\partial}{\partial x_i}, \quad D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$$

for derivatives on $\mathbf{R}^n$.

By letting $z' = x' - x_n y'$ in formula (17.4), it follows that

$$\phi(x', x_n) = \frac{x_n^{2-n}}{\ell^n(x_0)} \int_{\mathbf{R}^{n-1}} \widetilde{\varphi}(z') \zeta \left(\frac{x' - z'}{x_n} \right) dz',$$

so that

$$\begin{aligned} D_i \phi(x', x_n) &= \frac{x_n^{1-n}}{\ell^n(x_0)} \int_{\mathbf{R}^{n-1}} \widetilde{\varphi}(z') \frac{\partial \zeta}{\partial x_i} \left(\frac{x' - z'}{x_n} \right) dz' \\ &\qquad \text{for } 1 \le i \le n-1; \\ D_n \phi(x', x_n) &= \frac{(2-n) x_n^{1-n}}{\ell^n(x_0)} \int_{\mathbf{R}^{n-1}} \widetilde{\varphi}(z') \zeta \left(\frac{x' - z'}{x_n} \right) dz' \end{aligned}$$

$$- \frac{x_n^{-n}}{\ell^n(x_0)} \int_{\mathbf{R}^{n-1}} \widetilde{\varphi}(z') \nabla' \zeta \left(\frac{x' - z'}{x_n} \right) \cdot (x' - z') \, dz,$$

where

$$\nabla' = \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots, \frac{\partial}{\partial y_{n-1}} \right)$$

and

$$x' \cdot y' = \sum_{j=1}^{n-1} x_j y_j.$$

Hence we have, for the gradient $\nabla \phi = (\partial \phi / \partial x_1, \partial \phi / \partial x_2, \dots, \partial \phi / \partial x_n)$,

$$D_i \phi(x', x_n) = \frac{1}{\ell^n(x_0)} \int_{\mathbf{R}^{n-1}} \widetilde{\varphi}(x' - x_n y') D_i \zeta(y') \, dy' \qquad (17.12a)$$

$$\text{for } 1 \leq i \leq n-1;$$

$$D_n \phi(x', x_n) = \frac{2-n}{\ell^n(x_0)} \int_{\mathbf{R}^{n-1}} \widetilde{\varphi}(x' - x_n y') \zeta(y') \, dy' \qquad (17.12b)$$

$$- \frac{1}{\ell^n(x_0)} \int_{\mathbf{R}^{n-1}} \widetilde{\varphi}(x' - x_n y') \nabla' \zeta(y') \cdot y' \, dy'.$$

However, by conditions (17.2) it follows from an application of the divergence theorem (Theorem 5.2) that

$$\int_{\mathbf{R}^{n-1}} D_i \zeta(y') \, dy' = \int_{\mathbf{R}^{n-1}} D_{ij} \zeta(y') \, dy' \qquad (17.13a)$$

$$= \int_{\mathbf{R}^{n-1}} \nabla'(D_i \zeta)(y') \cdot y' \, dy' = 0 \qquad \text{for } 1 \leq i, j \leq n-1;$$

$$\int_{\mathbf{R}^{n-1}} \zeta(y') \, dy' = 1, \quad \int_{\mathbf{R}^{n-1}} \nabla' \zeta(y') \cdot y' \, dy' = 1 - n, \qquad (17.13b)$$

and

$$\int_{\mathbf{R}^{n-1}} \nabla' \left(\nabla' \zeta(y') \cdot y' \right) \cdot y' \, dy' = (1-n)^2. \qquad (17.13c)$$

Therefore, the desired formulas (17.5) follow from three formulas (17.4), (17.12b) and (17.13) if we take $x_n = 0$.

Furthermore, since $\zeta(y') \in C_0^2(\mathbf{R}^{n-1})$ we can differentiate formulas (17.12) once again to obtain that

$$D_{ij} \phi(x', x_n) = \frac{1}{\ell^n(x_0)} \frac{1}{x_n} \int_{\mathbf{R}^{n-1}} \widetilde{\varphi}(x' - x_n y') D_{ij} \zeta(y') \, dy'$$

$$\text{for } 1 \leq i, j \leq n-1;$$

$$\begin{aligned}D_{in}\phi(x', x_n) &= \frac{1-n}{\ell^n(x_0)} \frac{1}{x_n} \int_{\mathbf{R}^{n-1}} \widetilde{\varphi}(x' - x_n y') D_i \zeta(y') \, dy' \\ &\quad - \frac{1}{\ell^n(x_0)} \frac{1}{x_n} \int_{\mathbf{R}^{n-1}} \widetilde{\varphi}(x' - x_n y') \nabla' (D_i \zeta)(y') \cdot y' \, dy' \\ &\qquad \text{for } 1 \leq i \leq n-1; \\ D_{nn}\phi(x', x_n) &= \frac{(2-n)(1-n)}{\ell^n(x_0)} \frac{1}{x_n} \int_{\mathbf{R}^{n-1}} \widetilde{\varphi}(x' - x_n y') \zeta(y') \, dy' \\ &\quad + \frac{2n-3}{\ell^n(x_0)} \frac{1}{x_n} \int_{\mathbf{R}^{n-1}} \widetilde{\varphi}(x' - x_n y') \nabla' \zeta(y') \cdot y' \, dy' \\ &\quad + \frac{1}{\ell^n(x_0)} \frac{1}{x_n} \int_{\mathbf{R}^{n-1}} \widetilde{\varphi}(x' - x_n y') \nabla' \left(\nabla' \zeta(y') \cdot y' \right) \cdot y' \, dy'.\end{aligned}$$

Hence, by combining these three formulas and formulas (17.13) we obtain that

$$\begin{aligned}&D_{ij}\phi(x', x_n) \\ &= \frac{1}{\ell^n(x_0)} \frac{1}{x_n} \int_{\mathbf{R}^{n-1}} [\widetilde{\varphi}(x' - x_n y') - \widetilde{\varphi}(x')] \, D_{ij} \zeta(y') \, dy' \\ &\qquad \text{for } 1 \leq i, j \leq n-1;\end{aligned} \tag{17.14a}$$

$$\begin{aligned}&D_{in}\phi(x', x_n) \\ &= \frac{1-n}{\ell^n(x_0)} \frac{1}{x_n} \int_{\mathbf{R}^{n-1}} [\widetilde{\varphi}(x' - x_n y') - \widetilde{\varphi}(x')] \, D_i \zeta(y') \, dy' \\ &\quad - \frac{1}{\ell^n(x_0)} \frac{1}{x_n} \int_{\mathbf{R}^{n-1}} [\widetilde{\varphi}(x' - x_n y') - \widetilde{\varphi}(x')] \, \nabla' (D_i \zeta)(y') \cdot y' \, dy' \\ &\qquad \text{for } 1 \leq i \leq n-1;\end{aligned} \tag{17.14b}$$

and

$$\begin{aligned}&D_{nn}\phi(x', x_n) \\ &= \frac{(2-n)(1-n)}{\ell^n(x_0)} \frac{1}{x_n} \int_{\mathbf{R}^{n-1}} [\widetilde{\varphi}(x' - x_n y') - \widetilde{\varphi}(x')] \, \zeta(y') \, dy' \\ &\quad + \frac{2n-3}{\ell^n(x_0)} \frac{1}{x_n} \int_{\mathbf{R}^{n-1}} [\widetilde{\varphi}(x' - x_n y') - \widetilde{\varphi}(x')] \, \nabla' \zeta(y') \cdot y' \, dy' \\ &\quad + \frac{1}{\ell^n(x_0)} \frac{1}{x_n} \int_{\mathbf{R}^{n-1}} [\widetilde{\varphi}(x' - x_n y') - \widetilde{\varphi}(x')] \, \nabla' \left(\nabla' \zeta(y') \cdot y' \right) \cdot y' \, dy'.\end{aligned} \tag{17.14c}$$

Here it should be noticed that the integrals in formulas (17.14) are all of the type

$$\psi(x) = \psi(x', x_n) = \frac{1}{x_n} \int_{\mathbf{R}^{n-1}} [\widetilde{\varphi}(x' - x_n y') - \widetilde{\varphi}(x')] \, \mu(y') \, dy', \tag{17.15}$$

where $\mu(y')$ is (modulo a constant multiplier) one of the functions $\zeta(y')$, $D_i\zeta(y')$, $y' \cdot \nabla'\zeta(y')$, $D_{ij}\zeta(y')$ and $\nabla'(\nabla'\zeta \cdot y') \cdot y'$.

However, we can find a positive constant C_3 such that

$$\int_{B_R^+} |\psi(x)|^p \, dx \leq C_3 \int_{\Gamma_R} \int_{\Gamma_R} \frac{|\widetilde{\varphi}(x') - \widetilde{\varphi}(y')|^p}{|x' - y'|^{p+n-2}} \, d\sigma_{x'} \, d\sigma_{y'}. \tag{17.16}$$

The proof of estimate (17.16) will be given in the next Section 17.2, due to its length.

In view of definition (17.1), we obtain from formulas (17.14) and (17.15) that, for some positive constants C_4 and C_5,

$$R \left\| \nabla^2 \phi \right\|_{L^p(B_R^+)} \leq C_4 R \|\psi\|_{L^p(B_R^+)} \leq C_5 R^{1/2} \|\widetilde{\varphi}\|^*_{B^{1-1/p,p}(\Gamma_R)}.$$

This proves the desired estimate (17.11).

Step 3: Finally, our main estimate (17.6) follows by combining estimates (17.8) and (17.11).

The proof of Lemma 17.1 is now complete, apart from the proof of estimate (17.16). □

17.2 Proof of Estimate (17.16)

The proof is divided into five steps.

Step (I): First, we can rewrite formula (17.15) in the form

$$\begin{aligned}\psi(x', x_n) &= \frac{1}{x_n} \int_{\mathbf{R}^{n-1}} [\widetilde{\varphi}(x' - x_n y') - \widetilde{\varphi}(x')] \, \mu(y') \, dy' \\ &= \frac{1}{x_n^n} \int_{\mathbf{R}^{n-1}} [\widetilde{\varphi}(x' - z') - \widetilde{\varphi}(x')] \, \mu\left(\frac{z'}{x_n}\right) dz' \\ &= \frac{1}{x_n^n} \int_{|z'| \leq x_n} [\widetilde{\varphi}(x' - z') - \widetilde{\varphi}(x')] \, \mu\left(\frac{z'}{x_n}\right) dz',\end{aligned}$$

since we have the assertion

$$\operatorname{supp} \mu \subset \left\{ x' \in \mathbf{R}^{n-1} : |x'| \leq 1 \right\}.$$

Therefore, we obtain that

$$\begin{aligned}|\psi(x', x_n)| &\leq \|\mu\|_{L^\infty(\mathbf{R}^{n-1})} \, \frac{1}{x_n^n} \\ &= \frac{1}{x_n^n} \int_{|z'| \leq x_n} [\widetilde{\varphi}(x' - z') - \widetilde{\varphi}(x')] \, dz'.\end{aligned} \tag{17.17}$$

Step (II): Now, if we let

$$w(y') := \|\widetilde{\varphi}(\cdot - y) - \widetilde{\varphi}(\cdot)\|_{L^p(\mathbf{R}^{n-1})} \quad \text{for } y' \in \mathbf{R}^{n-1},$$

then, by applying Minkowski's inequality for integrals (Theorem 3.18) to inequality (17.17) we obtain that

$$\|\psi(\cdot, x_n)\|_{L^p(\mathbf{R}^{n-1})} \leq \|\mu\|_{L^\infty(\mathbf{R}^{n-1})} \, \frac{1}{x_n^n} \int_{|z'| \leq x_n} w(z')\, dz'. \tag{17.18}$$

Therefore, we obtain from inequality (17.18) that

$$\|\psi(\cdot, \cdot)\|_{L^p(\mathbf{R}^n_+)}^p \leq C_1^p \left\{ \int_0^\infty \frac{1}{\rho^{pn}} \left(\int_{|y'| \leq \rho} w(y')\, dy \right)^p d\rho \right\}, \tag{17.19}$$

where

$$C_1 = \|\mu\|_{L^\infty(\mathbf{R}^{n-1})}.$$

Step (III): We estimate the last integral of inequality (17.19). To do this, we rewrite the integral in the form

$$\begin{aligned}
&\int_0^\infty \frac{1}{\rho^{pn}} \left(\int_{|y'| \leq \rho} w(y')\, dy' \right)^p d\rho \\
&= \int_0^\infty \frac{1}{\rho^{pn}} \left\{ \int_0^\rho \left(\int_{\Sigma_{n-1}} w(r\sigma)\, d\sigma \right) r^{n-2}\, dr \right\}^p d\rho \\
&= \int_0^\infty \left(\rho^{-n+1} \int_0^\rho r^{n-2} \, \frac{h(r)}{\rho^\alpha}\, dr \right)^p d\rho,
\end{aligned}$$

where Σ_{n-1} is the unit sphere in $\mathbf{R}^{n-1}$ and

$$h(r) := \int_{\Sigma_{n-1}} w(r\sigma)\, d\sigma.$$

Since we have, for $0 < r \leq \rho$,

$$\frac{1}{\rho} \leq \frac{1}{r},$$

it follows that

$$\begin{aligned}
&\int_0^\infty \frac{1}{\rho^{pn}} \left(\int_{|y'| \leq \rho} w(y')\, dy' \right)^p d\rho \qquad (17.20)\\
&= \int_0^\infty \left(\rho^{-n+1} \int_0^\rho r^{n-2} \, \frac{h(r)}{\rho}\, dr \right)^p d\rho \\
&\leq \int_0^\infty \left(\rho^{-n+1+1/p} \int_0^\rho r^{n-3} \, h(r)\, dr \right)^p \frac{d\rho}{\rho}.
\end{aligned}$$

By applying Hardy's inequality (Theorem 3.20) with

$$\gamma := -n + 1 + \frac{1}{p},$$
$$f(r) := r^{n-3}\, h(r),$$

we can estimate the last integral of inequality (17.20) as follows:

$$\begin{aligned}&\int_0^\infty \left(\rho^{-n+1+1/p} \int_0^\rho r^{n-3}\, h(r)\, dr\right)^p \frac{d\rho}{\rho} \\ &\le \frac{1}{|-n+1+1/p|^p} \left(\int_0^\infty \frac{h(r)^p}{r^p}\, dr\right).\end{aligned} \tag{17.21}$$

Step (IV): Moreover, we estimate the last integral of inequality (17.21). Since we have, by Hölder's inequality (Theorem 3.14),

$$\begin{aligned}h(r) &= \int_{\Sigma_{n-1}} w(r\sigma)\, d\sigma \\ &\le \left(\int_{\Sigma_{n-1}} w(r\sigma)^p\, d\sigma\right)^{1/p} \left(\int_{\Sigma_{n-1}} 1^q\, d\sigma\right)^{1/q} \\ &= \omega_{n-1}^{1/q} \left(\int_{\Sigma_{n-1}} w(r\sigma)^p\, d\sigma\right)^{1/p},\end{aligned}$$

it follows that

$$\begin{aligned}\int_0^\infty \frac{h(r)^p}{r^p}\, dr &\le \omega_{n-1}^{p/q} \int_0^\infty \frac{1}{r^p} \left(\int_{\Sigma_{n-1}} w(r\sigma)^p\, d\sigma\right) dr \\ &= \omega_{n-1}^{p/q} \int_0^\infty \int_{\Sigma_{n-1}} \left(\frac{w(r\sigma)^p}{r^{n+p-2}}\right) r^{n-2}\, dr\, d\sigma \\ &= \omega_{n-1}^{p/q} \int_{\mathbf{R}^{n-1}} \frac{w(y')^p}{|y'|^{n+p-2}}\, dy',\end{aligned} \tag{17.22}$$

where ω_{n-1} is the surface area of the unit sphere Σ_{n-1}

$$\omega_{n-1} := |\Sigma_{n-1}| = \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)}.$$

Step (V): By combining inequalities (17.19) through (17.22), we obtain that

$$\begin{aligned}\|\psi\|^p_{L^p(\mathbf{R}^n_+)} &\le C_1^p \int_0^\infty \frac{1}{\rho^{pn}} \left(\int_{|y'|\le\rho} w(y')\, dy'\right)^p d\rho \\ &\le C_1^p \int_0^\infty \left(\rho^{-n+1+1/p} \int_0^\rho r^{n-3}\, h(r)\, dr\right)^p \frac{d\rho}{\rho}\end{aligned}$$

$$\leq \frac{C_1^p}{|-n+1+1/p|^p} \int_0^\infty \frac{h(r)^p}{r^p}\,dr$$
$$\leq \frac{C_1^p \omega_{n-1}^{p/q}}{|-n+1+1/p|^p} \int_{\mathbf{R}^{n-1}} \frac{w(y')^p}{|y'|^{n+p-2}}\,dy'.$$

However, we have, by Fubini's theorem (Theorem 3.10),

$$\begin{aligned}\int_{\mathbf{R}^{n-1}} \frac{w(y')^p}{|y'|^{n+p-2}}\,dy' &= \int_{\mathbf{R}^{n-1}} \frac{\|\widetilde{\varphi}(\cdot - y') - \widetilde{\varphi}(\cdot)\|^p_{L^p(\mathbf{R}^{n-1})}}{|y'|^{n+p-2}}\,dy' \\ &= \int_{\mathbf{R}^{n-1}} \int_{\mathbf{R}^{n-1}} \frac{|\widetilde{\varphi}(x'-y') - \widetilde{\varphi}(x')|^p}{|y'|^{n+p-2}}\,dx'\,dy'.\end{aligned}$$

Summing up, we have proved that

$$\|\psi\|_{L^p(\mathbf{R}^n_+)} \leq C_2 \left(\int_{\mathbf{R}^{n-1}} \int_{\mathbf{R}^{n-1}} \frac{|\widetilde{\varphi}(x'-y') - \widetilde{\varphi}(x')|^p}{|y'|^{n+p-2}}\,dx'\,dy' \right)^{1/p}, \tag{17.23}$$

where

$$C_2 := \frac{C_1\,\omega_{n-1}^{1/q}}{n-1-1/p}.$$

The desired estimate (17.16) follows from inequality (17.23).

The proof of estimate (17.16) (and hence that of Lemma 17.1) is now complete. □

17.3 Notes and Comments

The results of this chapter are adapted from Di Fazio–Palagachev [22] and Maugeri–Palagachev–Softova [47].

18
Boundary Representation Formula for Solutions

In this chapter we prove boundary representation formulas for solutions of problem (16.4), by using the half space Green function for the uniformly elliptic differential operator

$$\mathcal{L} = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}.$$

The first step is to derive a boundary representation formula (18.4) for the solution of problem (16.4) with the constant coefficients differential operator $\mathcal{L}_0$ and the constant coefficients boundary operator $\mathcal{B}_0$ (Lemma 18.1). The second step is to derive integral representation formulas (18.8) for the second derivatives of solutions of the oblique derivative problem for the variable coefficients differential operator $\mathcal{L}$ and the constant coefficients boundary operator $\mathcal{B}_0$ (Theorem 18.3). The third step for the general couple $(\mathcal{L}, \mathcal{B})$ will be carried out in the next Chapter 19 (Lemma 19.1).

18.1 Integral representation formulas for the oblique derivative problem

Now we take an arbitrary point $x_0 = (x_0', 0)$ of the hyperplane $\{x_n = 0\}$ and $r > 0$, and we let (see Figure 18.1 below)

$$\begin{aligned} B_r &:= B_r(x_0) = \{x \in \mathbf{R}^n : |x - x_0| < r\}, \\ B_r^+ &:= B_r(x_0) \cap \{x_n > 0\} \\ &= \{x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : |x - x_0| < r,\ x_n > 0\}, \end{aligned}$$

and

$$C_r := B_r(x_0) \cap \{x_n = 0\}$$

$$= \{x = (x_1, x_2, \ldots, x_{n-1}, 0) \in \mathbf{R}^n : |x' - x_0'| < r\}.$$

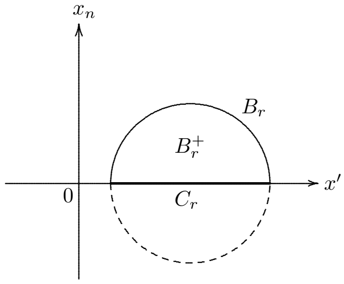

Fig. 18.1. The semi-ball B_r^+ in $\mathbf{R}_+^n$ and the ball $C_r = B_r \cap \{x_n = 0\}$ in $\mathbf{R}^{n-1}$

We consider the second-order, elliptic differential operator

$$\mathcal{L}_0 := \sum_{i,j=1}^{n} a^{ij}(x_0) \frac{\partial^2}{\partial x_i \partial x_j}$$

with constant coefficients, and the first-order boundary operator

$$\mathcal{B}_0 := \frac{\partial}{\partial \ell(x_0')} + \sigma(x_0') = \sum_{i=1}^{n} \ell^i(x_0') \frac{\partial}{\partial x_i} + \sigma(x_0')$$

prescribed by a directional derivative with respect to the constant vector field

$$\ell(x_0') = (\ell^1(x_0'), \ell^2(x_0'), \ldots, \ell^n(x_0')).$$

Here we assume that

$$\sigma(x_0') < 0, \quad \ell^n(x_0') > 0. \tag{18.1}$$

In the following we shall denote the matrix $(a^{ij}(x_0))$ by $\boldsymbol{a}(x_0)$ and its inverse matrix $(A^{ij}(x_0))$ by $\boldsymbol{A}(x_0)$, respectively. Then the fundamental solution $\Gamma(x_0, \xi)$ of $\mathcal{L}_0$ is given by the formula

$$\Gamma(x_0, \xi) = \frac{1}{(2-n)\omega_n \sqrt{\det \boldsymbol{a}(x_0)}} \left(\sum_{i,j=1}^{n} A^{ij}(x_0) \xi_i \xi_j \right)^{(2-n)/2},$$

where

$$\omega_n := |\Sigma_n| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

is the surface area of the unit sphere Σ_n in $\mathbf{R}^n$. By arguing just as in [33, Section 6.7], we can verify that the half space Green function $G(x_0, x, y)$ for the constant coefficients elliptic differential operator $\mathcal{L}_0$ is given by the formula (cf. [33, Section 6.7, formula (6.62)])

$$\begin{aligned} &G(x_0, x, y) \\ &:= \Gamma(x_0, x-y) - \Gamma(x_0, T(x; x_0) - y) + \theta(x_0, T(x; x_0) - y), \end{aligned} \tag{18.2}$$

where

$$T(x; x_0) := x - \frac{2x_n}{a^{nn}(x_0)} \boldsymbol{a}^n(x_0) = \begin{pmatrix} x_1 - 2x_n \frac{a^{1n}(x_0)}{a^{nn}(x_0)} \\ x_2 - 2x_n \frac{a^{2n}(x_0)}{a^{nn}(x_0)} \\ \vdots \\ -x_n \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

and

$$\begin{aligned} &\theta(x_0, \xi) \\ &= \frac{2}{\omega_n \sqrt{\det \boldsymbol{a}(x_0)}} \frac{\ell^n(x_0')}{a^{nn}(x_0)} \\ &\times \int_0^\infty \frac{e^{\sigma(x_0')s}(\xi_n + s\, T_n(\ell(x_0')))}{\left(\sum_{i,j=1}^n A^{ij}(x_0)(\xi_i + s\, T_i(\ell(x_0')))(\xi_j + s\, T_j(\ell(x_0')))\right)^{n/2}}\, ds, \end{aligned} \tag{18.3}$$

with

$$\begin{aligned} \xi + s\, T(\ell(x_0')) &= \begin{pmatrix} \xi_1 + s\, T_1(\ell(x_0')) \\ \xi_2 + s\, T_2(\ell(x_0')) \\ \vdots \\ \xi_n + s\, T_n(\ell(x_0')) \end{pmatrix} \\ &= \begin{pmatrix} \xi_1 + s\, \ell^1(x_0') - 2s\, \ell^n(x_0') \frac{a^{1n}(\ell(x_0'))}{a^{nn}(\ell(x_0'))} \\ \xi_2 + s\, \ell^2(x_0') - 2s\, \ell^n(x_0') \frac{a^{2n}(\ell(x_0'))}{a^{nn}(\ell(x_0'))} \\ \vdots \\ \xi_n - s\, \ell^n(x_0') \end{pmatrix}. \end{aligned}$$

More precisely, we can prove the following lemma (see [22, Lemma 3.1], [47, Chapter 2, Lemma 2.2.5]):

Lemma 18.1. *Assume that condition* (18.1) *is satisfied. If a function* $u \in C_0^\infty(B_{2r})$, *with* $\operatorname{supp} u \subset B_r$, *is a solution of the oblique derivative problem (see Figure 18.2 below)*

$$\mathcal{L}_0 u = f \quad \text{in } B_{2r}^+,$$

$$\mathcal{B}_0 u = 0 \quad \textit{on } C_{2r},$$

then we have, for all $x \in B_r^+$,

$$\begin{aligned} & u(x) \\ &= \int_{B_{2r}^+} G(x_0, x, y) f(y)\, dy \\ &= \int_{B_{2r}^+} (\Gamma(x_0, x - y) - \Gamma(x_0, T(x; x_0) - y) + \theta(x_0, T(x; x_0) - y)) f(y) dy. \end{aligned} \tag{18.4}$$

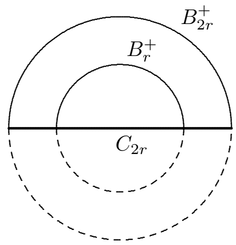

Fig. 18.2. The semi-ball B_{2r}^+ in $\mathbf{R}_+^n$ and the ball $C_{2r} = B_{2r} \cap \{x_n = 0\}$ in $\mathbf{R}^{n-1}$

Let B_r be an open ball of radius r, and we assume that the functions $a^{ij}(x) \in \mathrm{VMO} \cap L^\infty(\Omega)$ satisfy the following two conditions (i) and (ii):

(i) $a^{ij}(x) = a^{ji}(x)$ for almost all $x \in B_r$ and $1 \leq i, j \leq n$.
(ii) There exists a positive constant λ such that

$$\frac{1}{\lambda}|\xi|^2 \leq \sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2$$
$$\text{for almost all } x \in B_r \text{ and all } \xi \in \mathbf{R}^n.$$

If $\widetilde{B}_r$ is the subset of B_r where conditions (i) and (ii) hold true, then we let

$$\Gamma(x, \xi) := \frac{1}{(2-n)\omega_n} \frac{1}{\sqrt{\det \boldsymbol{a}(x)}} \left(\sum_{i,j=1}^n A_{ij}(x)\xi_i\xi_j \right)^{(2-n)/2},$$
$$\text{for all } x \in \widetilde{B}_r \text{ and all } \xi \in \mathbf{R}^n \setminus \{0\},$$

and

$$\theta(x, \xi)$$

$$= \frac{2}{\omega_n \sqrt{\det \boldsymbol{a}(x)}} \frac{\ell^n(x')}{a^{nn}(x)}$$
$$\times \int_0^\infty \frac{e^{\sigma(x')s}(\xi_n + sT_n(\ell(x')))}{\left(\sum_{i,j=1}^n A^{ij}(x)(\xi_i + sT_i(\ell(x')))(\xi_j + sT_j(\ell(x')))\right)^{n/2}} \, ds.$$

Here:

$(A_{ij}(x)) =$ the inverse matrix of $\mathbf{a}(x) = (a^{ij}(x))$,

$\omega_n := |\Sigma_n| = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$ (the surface area of the unit sphere Σ_n in $\mathbf{R}^n$),

and

$$\xi + sT(\ell(x')) = \begin{pmatrix} \xi_1 + sT_1(\ell(x')) \\ \xi_2 + sT_2(\ell(x')) \\ \vdots \\ \xi_n + sT_n(\ell(x')) \end{pmatrix}$$
$$= \begin{pmatrix} \xi_1 + s\ell^1(x') - 2s\ell^n(x') \frac{a^{1n}(\ell(x'))}{a^{nn}(\ell(x'))} \\ \xi_2 + s\ell^2(x_0') - 2s\ell^n(x') \frac{a^{2n}(\ell(x'))}{a^{nn}(\ell(x'))} \\ \vdots \\ \xi_n - s\ell^n(x') \end{pmatrix}.$$

The next lemma states an important property of the function $\theta(x, \xi)$ (see [22, Remark 3.1]):

Lemma 18.2. *The function* $\theta\left(x, T(x; x_0) - y\right)$ *satisfies the estimate*

$$\left| D_\xi^\alpha \theta\left(x, T(x; x_0) - y\right)\right| \leq \frac{C_\alpha}{|T(x; x_0) - y|^{n-2+|\alpha|}}, \tag{18.5}$$

with a positive constant $C_\alpha = C(n, |\alpha|, \ell, \boldsymbol{a})$.

Proof. The proof is divided into three steps.

Step 1: First, we let

$$\psi(x; \tau, \eta)$$
$$:= \frac{2}{\omega_n \sqrt{\det \boldsymbol{a}(x)}} \frac{\ell^n(x')}{a^{nn}(x)}$$
$$\times \int_0^\infty \frac{e^{\sigma(x')\tau t}(\eta_n + tT_n(\ell(x')))}{\left(\sum_{i,j=1}^n A^{ij}(x)(\eta_i + tT_i(\ell(x')))(\eta_j + tT_j(\ell(x')))\right)^{n/2}} \, dt,$$

where

$$\eta_i := \frac{\xi_i}{|T(x;x_0) - y|} \quad \text{for } 1 \leq i \leq n.$$

Then it is easy to see that

$$\begin{aligned}
&\int_0^\infty \frac{e^{\sigma(x')s}(\xi_n + sT_n(\ell(x')))}{\left(\sum_{i,j=1}^n A^{ij}(x)(\xi_i + sT_i(\ell(x')))(\xi_j + sT_j(\ell(x')))\right)^{n/2}}\, ds \\
&= \frac{1}{|T(x;x_0) - y|^{n-2}} \\
&\quad \times \int_0^\infty \frac{e^{\sigma(x')|T(x;x_0)-y|t}(\eta_n + tT_n(\ell(x')))}{\left(\sum_{i,j=1}^n A^{ij}(x)(\eta_i + tT_i(\ell(x')))(\eta_j + tT_j(\ell(x')))\right)^{n/2}}\, dt.
\end{aligned}$$

This proves that the function $\theta(x, \xi)$ is expressed in the form

$$\theta(x,\xi) = \frac{1}{|T(x;x_0) - y|^{n-2}} \psi\left(x; |T(x;x_0) - y|, \eta\right), \tag{18.6}$$

with

$$\eta = \frac{\xi}{|T(x;x_0) - y|}. \tag{18.7}$$

Step 2: Secondly, by condition (18.1) it follows that the angle between the vectors η and $T(\ell(x'))$ is less than π for all $\eta \in \mathbf{R}^n_- = \{x_n < 0\}$ (see Figure 18.3 below). Hence we have, for some constant $0 < \delta_0 < 1$,

$$\langle \eta, T(\ell(x')) \rangle \geq -|\eta|\ |T(\ell(x'))|\delta_0,$$

and so

$$\begin{aligned}
|\eta + tT(\ell(x'))|^2 &= |\eta|^2 + 2t \langle \eta, T(\ell(x')) \rangle + t^2 |T(\ell(x'))|^2 \\
&\geq |\eta|^2 - 2t|\eta|\ |T(\ell(x'))|\delta_0 + t^2|T(\ell(x'))|^2 \\
&= |T(\ell(x'))|^2 \left(t - \frac{\delta_0}{|T(\ell(x'))|}|\eta|\right)^2 + (1 - \delta_0^2)|\eta|^2 \\
&\geq (1 - \delta_0^2)|\eta|^2.
\end{aligned}$$

Moreover, we have, for all $t \geq 4|\eta|\delta_0/|T(\ell(x'))|$,

$$\begin{aligned}
&|\eta + tT(\ell(x'))|^2 \\
&= |\eta|^2 + 2t \langle \eta, T(\ell(x')) \rangle + t^2|T(\ell(x'))|^2 \\
&\geq |\eta|^2 - 2t|\eta||T(\ell(x'))|\delta_0 + t^2|T(\ell(x'))|^2
\end{aligned}$$

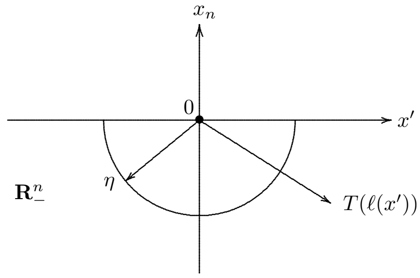

Fig. 18.3. The angle between the vectors $T(\ell(x'))$ and $\eta \in \mathbf{R}^n_- = \{x_n < 0\}$

$$
\begin{aligned}
&= |\eta|^2 + \frac{t}{2}\left(t|T(\ell(x'))|^2 - 4|\eta|\delta_0|T(\ell(x'))|\right) + \frac{t^2}{2}|T(\ell(x'))|^2 \\
&\geq |\eta|^2 + \frac{t^2}{2}|T(\ell(x'))|^2.
\end{aligned}
$$

By the positivity of $\boldsymbol{A}$, we obtain that, for some positive constant c_0,

$$
\begin{aligned}
&\sum_{i,j=1}^{n} A^{ij}(x)(\eta_i + tT_i(\ell(x')))(\eta_j + tT_j(\ell(x'))) \\
&\geq c_0|\eta + tT(\ell(x'))|^2 \\
&\geq \begin{cases} c_0\left(1-\delta_0^2\right)|\eta|^2 & \text{for all } t \geq 0, \\ c_0\left(|\eta|^2 + \frac{t^2}{2}|T(\ell(x'))|^2\right) & \text{for all } t \geq \frac{4|\eta|\delta_0}{|T(\ell(x'))|}. \end{cases}
\end{aligned}
$$

Therefore, we find that $\psi(x;\tau,\eta)$ is a smooth function of (τ,η) for $\tau > 0$ and $\eta \in \mathbf{R}^n_-$, since the denominator of the integrand is bounded away from zero and since $n \geq 3$.

Step 3: Finally, it is easy to see that, for some positive constant $C'_\alpha = C'(n,|\alpha|,\ell,\boldsymbol{a})$,

$$
\left| D^\alpha_\eta \psi\left(x; |T(x;x_0)-y|, \frac{T(x;x_0)-y}{|T(x;x_0)-y|}\right)\right| \leq C'_\alpha.
$$

This proves the desired estimate (18.5), since we have, by formulas (18.6) and (18.7),

$$
D^\alpha_\xi \theta(x,\xi) = \frac{1}{|T(x;x_0)-y|^{|\alpha|+n-2}} D^\alpha_\eta \psi\left(x; |T(x;x_0)-y|, \eta\right).
$$

The proof of Lemma 18.2 is complete. □

In the following we shall use the notation

$$\Gamma_i(x,\xi) = \frac{\partial}{\partial \xi_i}\Gamma(x,\xi), \quad 1 \le i \le n,$$
$$\Gamma_{ij}(x,\xi) = \frac{\partial^2}{\partial \xi_i \partial \xi_j}\Gamma(x,\xi), \quad 1 \le i,j \le n,$$

and

$$\theta_i(x,\xi) = \frac{\partial}{\partial \xi_i}\theta(x,\xi), \quad 1 \le i \le n,$$
$$\theta_{ij}(x,\xi) = \frac{\partial^2}{\partial \xi_i \partial \xi_j}\theta(x,\xi), \quad 1 \le i,j \le n.$$

The next theorem gives integral representation formulas for the second derivatives of solutions of the oblique derivative problem for the variable coefficients elliptic differential operator

$$\mathcal{L} = \sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}$$

and the constant coefficients boundary operator

$$\mathcal{B}_0 = \frac{\partial}{\partial \ell(x_0')} + \sigma(x_0') = \sum_{i=1}^{n} \ell^i(x_0')\frac{\partial}{\partial x_i} + \sigma(x_0').$$

Theorem 18.3. *Let $\widetilde{B}_r$ be the subset of B_r where conditions (i) and (ii) hold true, and let $f \in L^p(B_r^+)$ for $1 < p < \infty$. If a function $u \in W^{2,p}(B_r^+)$ is a solution of the oblique derivative problem*

$$\mathcal{L}u = f \quad \text{in } B_r^+,$$
$$\mathcal{B}_0 u = 0 \quad \text{on } C_r,$$

then we have, for all $x \in \widetilde{B}_r^+ := \widetilde{B}_r \cap \{x_n > 0\}$,

$$\begin{aligned} &\frac{\partial^2 u}{\partial x_i \partial x_j}(x) \\ &= \text{v. p.} \int_{B_r^+} \Gamma_{ij}(x_0, x-y) \\ &\times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) \right] dy \\ &+ c_{ij}(x_0)f(x) - I_{ij}(x;x_0) + J_{ij}(x;x_0) \quad \textit{for } 1 \le i,j \le n. \end{aligned} \tag{18.8}$$

Here:

$$c_{ij}(x_0) := \int_{|\xi|=1} \Gamma_i(x_0,t) t_j \, d\sigma_t; \tag{18.9}$$

and the terms $I_{ij}(x;x_0)$ *are defined respectively as follows:*

- $I_{ij}(x;x_0) \qquad (1 \le i,j \le n-1)$ (18.10a)

$$:= \int_{B_r^+} \Gamma_{ij}(x_0, T(x;x_0)-y) \times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) \right] dy.$$

- $I_{in}(x;x_0) = I_{ni}(x;x_0) \qquad (1 \le i \le n-1)$ (18.10b)

$$:= \int_{B_r^+} \sum_{\ell=1}^{n} \Gamma_{i\ell}(x_0, T(x;x_0)-y) B_\ell(x_0) \times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) \right] dy.$$

- $I_{nn}(x;x_0)$ (18.10c)

$$:= \int_{B_r^+} \sum_{\ell,m=1}^{n} \Gamma_{\ell m}(x_0, T(x;x_0)-y) B_\ell(x_0) B_m(x_0) \times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) \right] dy.$$

The terms $J_{ij}(x;x_0)$ *are defined respectively as follows:*

- $J_{ij}(x;x_0) \qquad (1 \le i,j \le n-1)$ (18.11a)

$$:= \int_{B_r^+} \theta_{ij}(x_0, T(x;x_0)-y) \times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) \right] dy.$$

- $J_{in}(x;x_0) = J_{ni}(x;x_0) \qquad (1 \le i \le n-1)$ (18.11b)

$$:= \int_{B_r^+} \sum_{j=1}^{n} \theta_{ij}(x_0, T(x;x_0)-y) B_j(x_0)$$

$$\times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) \right] dy.$$

$$\begin{aligned} & J_{nn}(x;x_0) \qquad\qquad (18.11c) \\ & := \int_{B_r^+} \sum_{i,j=1}^{n} \theta_{ij}(x_0, T(x;x_0) - y) B_i(x_0) B_j(x_0) \\ & \quad \times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) \right] dy. \end{aligned}$$

Moreover, the map $T(x;x_0)$ is defined by the formula

$$T(x;x_0) := \begin{pmatrix} x_1 - 2x_n \frac{a^{1n}(x_0)}{a^{nn}(x_0)} \\ x_2 - 2x_n \frac{a^{2n}(x_0)}{a^{nn}(x_0)} \\ \vdots \\ -x_n \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

and the vector $B(x_0)$ is defined by the formula

$$B(x_0) = \begin{pmatrix} B_1(x_0) \\ B_2(x_0) \\ \vdots \\ B_n(x_0) \end{pmatrix} := \frac{\partial T}{\partial x_n}(x;x_0) = \begin{pmatrix} -2\frac{a^{1n}(x_0)}{a^{nn}(x_0)} \\ -2\frac{a^{2n}(x_0)}{a^{nn}(x_0)} \\ \vdots \\ -1 \end{pmatrix}.$$

Proof. By a density argument, it suffices to prove formula (18.8) for all $u \in C^\infty(B_r^+)$. Indeed, the general case can be proved by using Theorems 14.2 and 14.5 (and Remark 14.2). More precisely, we obtain the following three assertions (I), (II) and (III):

(I) If we let

$$\widetilde{K}_{ij} f(x) := \int_{\mathbf{R}_+^n} \Gamma_{ij}(x, T(x;x_0) - y) f(y)\, dy,$$

then there exists a positive constant $C_1 = C_1(n, p, \lambda, M)$ such that (Theorem 14.2)

$$\|\widetilde{K}_{ij} f\|_{L^p(\mathbf{R}_+^n)} \leq C_1 \|f\|_{L^p(\mathbf{R}_+^n)} \quad \text{for all } f \in L^p(\mathbf{R}_+^n).$$

(II) Let $a \in L^\infty(\mathbf{R}^n)$. If we let

$$\widetilde{C}[a, K_{ij}] f(x) := \int_{\mathbf{R}_+^n} \Gamma_{ij}(x, T(x;x_0) - y) |a(x) - a(y)| f(y)\, dy,$$

then there exists a constant $C_2 = C_2(n, p, \lambda, M) > 0$ such that (Theorem 14.5)

$$\|\widetilde{C}[a, K_{ij}]f\|_{L^p(\mathbf{R}^n_+)} \leq C_2 \|a\|_* \|f\|_{L^p(\mathbf{R}^n_+)} \quad \text{for all } f \in L^p(\mathbf{R}^n_+).$$

(III) Let $a \in L^\infty(\mathbf{R}^n)$. If we let

$$\widetilde{C}[a, \Theta_{ij}]f(x) := \int_{\mathbf{R}^n_+} \theta_{ij}(x, T(x; x_0) - y)[a(x) - a(y)]f(y)\, dy,$$

then there exists a constant $C_3 = C_3(n, p, \lambda, M) > 0$ such that

$$\|\widetilde{C}[a, \Theta_{ij}]f\|_{L^p(\mathbf{R}^n_+)} \leq C_3 \|a\|_* \|f\|_{L^p(\mathbf{R}^n_+)} \quad \text{for all } f \in L^p(\mathbf{R}^n_+).$$

Indeed, it suffices to note that we have, by Lemma 18.2 with $|\alpha| := 2$ and Lemma 19.2 in the next Chapter 19,

$$|\theta_{ij}(x_0, T(x; x_0) - y)| \leq \frac{C}{|T(x; x_0) - y|^n} \leq \frac{C'}{|\tilde{x} - y|^n},$$

with some positive constants C and C'.

Let x_0 be an arbitrary point of $\widetilde{B}_r$. Since we have the formula

$$\mathcal{L}u(x) = \sum_{i,j=1}^n a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x),$$

it follows that

$$\begin{aligned}\mathcal{L}_0 u(x) &= \sum_{i,j=1}^n a^{ij}(x_0) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \\ &= \sum_{i,j=1}^n \left[a^{ij}(x_0) - a^{ij}(x)\right] \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + f(x) \quad \text{in } B_r^+,\end{aligned}$$

and further that

$$\mathcal{B}_0 u(x') = 0 \quad \text{on } C_r.$$

However, we remark that the half space Green function $G(x_0, x, y)$ for the operator $\mathcal{L}_0$ is given by formula (18.2)

$$\begin{aligned}G(x_0, x, y) := \Gamma(x_0, x - y) &- \Gamma(x_0, T(x; x_0) - y) \\ &+ \theta(x_0, T(x; x_0) - y),\end{aligned} \tag{18.2}$$

Hence we have the formula

$$u(x) \tag{18.12}$$

$$= \int_{B_r^+} G(x_0, x, y) \left(\sum_{i,j=1}^{n} \left[a^{ij}(x_0) - a^{ij}(y) \right] \frac{\partial^2 u}{\partial x_i \partial x_j}(y) + f(y) \right) dy$$

$$= \int_{B_r^+} \Gamma(x_0, x - y) \left(\sum_{i,j=1}^{n} \left[a^{ij}(x_0) - a^{ij}(y) \right] \frac{\partial^2 u}{\partial x_i \partial x_j}(y) + f(y) \right) dy$$

$$- \int_{B_r^+} \Gamma(x_0, T(x; x_0) - y)$$

$$- \int_{B_r^+} \times \left(\sum_{i,j=1}^{n} \left[a^{ij}(x_0) - a^{ij}(y) \right] \frac{\partial^2 u}{\partial x_i \partial x_j}(y) + f(y) \right) dy$$

$$+ \int_{B_r^+} \theta(x_0, T(x; x_0) - y)$$

$$+ \int_{B_r^+} \times \left(\sum_{i,j=1}^{n} \left[a^{ij}(x_0) - a^{ij}(y) \right] \frac{\partial^2 u}{\partial x_i \partial x_j}(y) + f(y) \right) dy$$

$$:= H(x; x_0) - I(x; x_0) + J(x; x_0).$$

We can differentiate the first term

$$H(x; x_0) = \int_{B_r^+} \Gamma(x_0, x - y) \left(\sum_{i,j=1}^{n} \left[a^{ij}(x_0) - a^{ij}(y) \right] \frac{\partial^2 u}{\partial x_i \partial x_j}(y) + f(y) \right) dy$$

in formula (18.12) twice to obtain that

$$\begin{aligned} \frac{\partial^2 H}{\partial x_i \partial x_j}(x; x_0) &= \text{v. p.} \int_{B_r^+} \Gamma_{ij}(x_0, x - y) \\ &\quad \times \left(\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y) \right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) \right) dy \\ &\quad + f(x) \left(\int_{|t|=1} \Gamma_i(x_0, t) t_j \, d\sigma_t \right) \quad \text{for } 1 \leq i, j \leq n. \end{aligned} \tag{18.13}$$

As for the two terms

$$I(x; x_0) = \int_{B_r^+} \left(\sum_{i,j=1}^{n} \left[a^{ij}(x_0) - a^{ij}(y) \right] \frac{\partial^2 u}{\partial x_i \partial x_j}(y) + f(y) \right) dy,$$

$$J(x;x_0) = \int_{B_r^+} \left(\sum_{i,j=1}^{n} \left[a^{ij}(x_0) - a^{ij}(y)\right] \frac{\partial^2 u}{\partial x_i \partial x_j}(y) + f(y) \right) dy$$

in formula (18.12), we can differentiate them under the integral sign to obtain that

$$\bullet \ \frac{\partial^2 I}{\partial x_i \partial x_j}(x;x_0) \tag{18.14a}$$
$$= \int_{B_r^+} \Gamma_{ij}(x_0, T(x;x_0) - y)$$
$$\times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) \right] dy$$
$$= I_{ij}(x;x_0) \quad \text{for } 1 \le i,j \le n-1;$$

$$\bullet \ \frac{\partial^2 I}{\partial x_i \partial x_n}(x;x_0) \tag{18.14b}$$
$$= \int_{B_r^+} \sum_{\ell=1}^{n} \Gamma_{i\ell}(x_0, T(x;x_0) - y) \frac{\partial T_\ell}{\partial x_n}(x;x_0)$$
$$\times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) \right] dy$$
$$= I_{in}(x;x_0) \quad \text{for } 1 \le i \le n-1;$$

and

$$\bullet \ \frac{\partial^2 I}{\partial x_n \partial x_n}(x;x_0) \tag{18.14c}$$
$$= \int_{B_r^+} \sum_{\ell,m=1}^{n} \Gamma_{\ell m}(x_0, T(x;x_0) - y) \frac{\partial T_\ell}{\partial x_n}(x;x_0) \frac{\partial T_m}{\partial x_n}(x;x_0)$$
$$\times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) \right] dy$$
$$= I_{nn}(x;x_0).$$

and

$$\bullet \ \frac{\partial^2 J}{\partial x_i \partial x_j}(x;x_0) \tag{18.15a}$$
$$= \int_{B_r^+} \theta_{ij}(x_0, T(x;x_0) - y)$$

$$\times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y) \right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) \right] dy$$

$$= J_{ij}(x; x_0) \quad \text{for } 1 \leq i, j \leq n-1;$$

$$\bullet \; \frac{\partial^2 J}{\partial x_i \partial x_n}(x; x_0) \tag{18.15b}$$

$$= \int_{B_r^+} \sum_{\ell=1}^{n} \theta_{i\ell}(x_0, T(x; x_0) - y) \frac{\partial T_\ell}{\partial x_n}(x; x_0)$$

$$\times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y) \right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) \right] dy$$

$$= J_{in}(x; x_0) \quad \text{for } 1 \leq i \leq n-1;$$

and

$$\bullet \; \frac{\partial^2 J}{\partial x_n \partial x_n}(x; x_0) \tag{18.15c}$$

$$= \int_{B_r^+} \sum_{\ell,m=1}^{n} \theta_{\ell m}(x_0, T(x; x_0) - y) \frac{\partial T_\ell}{\partial x_n}(x; x_0) \frac{\partial T_m}{\partial x_n}(x; x_0)$$

$$\times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y) \right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) \right] dy$$

$$= J_{nn}(x; x_0).$$

Therefore, the desired formula (18.8) follows from formulas (18.13), (18.14) and (18.15).

The proof of Theorem 18.3 is complete. □

18.2 Notes and Comments

The results of this chapter are adapted from Gilbarg–Trudinger [33, Section 6.7] and Di Fazio–Palagachev [22].

19

Boundary Regularity of Solutions

The purpose of this chapter is to prove boundary Sobolev regularity of the solutions of problem (16.4) towards the proof of Theorem 16.1 (Lemma 19.1). A combination of this regularity result with the interior regularity (Theorem 12.1) will prove the main result in the next Chapter 20.

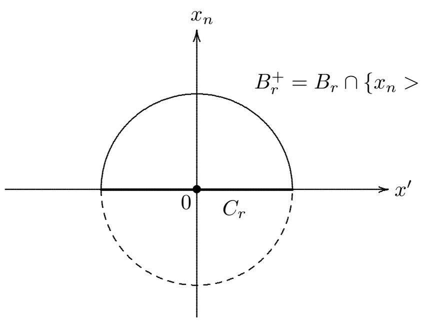

Fig. 19.1. The semi-ball B_r^+ in $\mathbf{R}^n_+$ and the ball $C_r = B_r \cap \{x_n = 0\}$ in $\mathbf{R}^{n-1}$

19.1 Boundary regularity of the solutions of problem (16.4)

Without loss of generality, we may assume (see Figure 19.1) that

$$x_0 = 0,$$

and we let

$$B_r := B_r(0) = \{x \in \mathbf{R}^n : |x| < r\},$$
$$B_r^+ := B_r \cap \{x_n > 0\} = \{x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : |x| < r,\ x_n > 0\},$$

and

$$C_r := B_r \cap \{x_n = 0\} = \{x = (x_1, x_2, \ldots, x_{n-1}, 0) \in \mathbf{R}^n : |x'| < r\}.$$

As in the previous chapter, we assume that the boundary $\partial\Omega$ is locally flattened near a point $x_0 \in \partial\Omega$ such that

$$\Omega \subset \mathbf{R}^n_+ = \{x = (x', x_n) \in \mathbf{R}^n : x_n > 0\}$$

(see Figure 17.2). The following result implies the boundary regularizing property of the couple $(\mathcal{L}, \mathcal{B})$ in the framework of Sobolev spaces:

Lemma 19.1. *Let $1 < q < p < \infty$, and assume that conditions* (16.1) *and* (16.3) *are satisfied. If $r > 0$ and $u \in W^{2,q}(B_r^+)$ is a solution of the non-homogeneous oblique derivative problem*

$$\mathcal{L}u = f \quad \text{in } B_r^+, \tag{19.1a}$$
$$\mathcal{B}u = \varphi \quad \text{on } C_r, \tag{19.1b}$$

with $f \in L^p(B_r^+)$ and $\varphi \in B^{1-1/p,p}(C_r)$, then there exists a constant $0 < R < r$ sufficiently small so that $u \in W^{2,p}(B_R^+)$ (see Figure 19.2). Moreover, we have the boundary a priori *estimate*

$$\left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^p(B_R^+)} \tag{19.2}$$
$$\leq C \left(\|u\|_{L^p(B_R^+)} + \|f\|_{L^p(B_R^+)} + \|\varphi\|_{B^{1-1/p,p}(C_R)} \right) \quad \text{for } 1 \leq i, j \leq n,$$

with a positive constant $C = C(n, p, \lambda, \eta, \ell, \sigma, \partial\Omega)$.

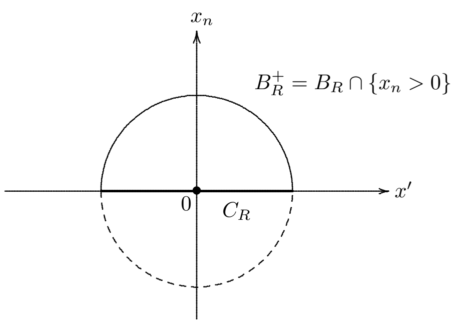

Fig. 19.2. The semi-ball B_R^+ in $\mathbf{R}^n_+$ and the ball $C_R = B_R \cap \{x_n = 0\}$ in $\mathbf{R}^{n-1}$

Proof. We make use of the explicit representation formula (18.7) of the second derivatives $\nabla^2 u$. The proof is divided into four steps.

Step 1: Without loss of generality, we may assume that the ball B_r is centered at the origin. Let $x_0 = (x_0', x_{0n})$ be an arbitray point of B_r^+, with $x_0' \in \mathbf{R}^{n-1}$. Then we have the formulas

$$\begin{aligned} \mathcal{L}_0 u(x) &= \sum_{i,j=1}^{n} a^{ij}(x_0) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \\ &= \sum_{i,j=1}^{n} \left[a^{ij}(x_0) - a^{ij}(x) \right] \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + f(x) \quad \text{in } B_r^+, \end{aligned} \tag{19.3}$$

and

$$\begin{aligned} \mathcal{B}_0 u(x') &= \frac{\partial u}{\partial \ell(x_0')}(x') + \sigma(x_0') u(x') \\ &= \sum_{i=1}^{n} \ell^i(x_0') \frac{\partial u}{\partial x_i}(x') + \sigma(x_0') u(x') \\ &= \sum_{i=1}^{n} \left[\ell^i(x_0') - \ell^i(x') \right] \frac{\partial u}{\partial x_i}(x') + \left[\sigma(x_0') - \sigma(x') \right] u(x') \\ &\quad + \varphi(x') \quad \text{on } C_r. \end{aligned} \tag{19.4}$$

Now we let

$$\begin{aligned} \widetilde{\varphi}(x') := \widetilde{\varphi}(x', u) = \sum_{i=1}^{n} \left[\ell^i(x_0') - \ell^i(x') \right] \frac{\partial u}{\partial x_i}(x') \\ + \left[\sigma(x_0') - \sigma(x') \right] u(x') + \varphi(x'), \end{aligned} \tag{19.5}$$

and define a function $\phi(x) = \phi(x, u)$ by formula (17.4), that is,

$$\begin{aligned} &\phi(x) = \phi(x, u) \\ &:= \frac{x_n}{\ell^n(x_0)} \int_{\mathbf{R}^{n-1}} \widetilde{\varphi}(x' - x_n y') \zeta(y') \, dy' \\ &= \frac{x_n}{\ell^n(x_0)} \sum_{i=1}^{n} \int_{\mathbf{R}^{n-1}} \Bigg(\left[\ell^i(x_0') - \ell^i(x' - x_n y') \right] \frac{\partial u}{\partial x_i}(x')(x' - x_n y') \\ &\quad + \left[\sigma(x_0') - \sigma(x' - x_n y') \right] u(x' - x_n y') + \varphi(x') \Bigg) \zeta(y') \, dy', \end{aligned} \tag{19.6}$$

where $\zeta(x')$ is a bell-shaped function on $\mathbf{R}^{n-1}$ which satisfies the four conditions (17.2).

Then, by formulas (17.5) it follows that

$$\begin{aligned} \mathcal{B}_0\phi(x') &= \frac{\partial \phi}{\partial \ell(x_0')}(x') + \sigma(x_0')\phi(x') \\ &= \sum_{i=1}^{n} \ell^i(x_0')\frac{\partial \phi}{\partial x_i}(x') + \sigma(x_0')\phi(x') \\ &= \widetilde{\varphi}(x') \quad \text{on } C_r. \end{aligned} \tag{19.7}$$

Hence, by combining formulas (19.3), (19.4) and (19.7) we find that the function $u(x) - \phi(x)$ satisfies the conditions

$$\begin{aligned} \mathcal{L}_0(u-\phi)(x) &= \sum_{i,j=1}^{n} a^{ij}(x_0)\frac{\partial^2 (u-\phi)}{\partial x_i \partial x_j}(x) \\ &= \sum_{i,j=1}^{n} \left[a^{ij}(x_0) - a^{ij}(x)\right] \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + f(x) \\ &\quad - \sum_{i,j=1}^{n} a^{ij}(x_0)\frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) \quad \text{in } B_r^+, \end{aligned} \tag{19.8}$$

and

$$\begin{aligned} \mathcal{B}_0(u-\phi)(x') &= \frac{\partial (u-\phi)}{\partial \ell(x_0')}(x') + \sigma(x_0')(u(x') - \phi(x')) \\ &= 0 \quad \text{on } C_r. \end{aligned} \tag{19.9}$$

Therefore, by applying Lemma 18.1 to the function $u(x) - \phi(x)$ we obtain from formulas (19.8) and (19.9) that the solution of the non-homogeneous oblique derivative problem (19.1) can be expressed as follows:

$$\begin{aligned} &u(x) \\ &= \phi(x,u) + \int_{B_r^+} G(x_0,x,y)\Bigg\{ \left[\sum_{i,j=1}^{n} a^{ij}(x_0) - a^{ij}(y)\right] \frac{\partial^2 u}{\partial x_i \partial x_j}(y) \\ &\quad + f(y) - \sum_{i,j=1}^{n} a^{ij}(x_0)\frac{\partial^2 \phi}{\partial x_i \partial x_j}(y,u)\Bigg\} dy, \end{aligned} \tag{19.10}$$

where $G(x_0, x, y)$ is the half space Green function for the elliptic operator $\mathcal{L}_0$ given by the formula (18.2)

$$G(x_0,x,y) = \Gamma(x_0, x-y) - \Gamma(x_0, T(x;x_0) - y) + \theta(x_0, T(x;x_0) - y).$$

Here it should be emphasized that the function $\phi(x,u)$ defined by formula (19.6) depends *affinely* on u.

On the other hand, by applying Theorem 18.3 the function $u(x)-\phi(x)$ we obtain from formulas (19.8) and (19.9) that

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x_i \partial x_j}(x) \qquad (19.11)\\
&= \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x,u) + \text{v. p.} \int_{B_r^+} \Gamma_{ij}(x_0, x-y) \\
&\quad \times \left\{ \sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) - \mathcal{L}_0 \phi(y,u) \right\} dy \\
&\quad + c_{ij}(x_0)\left(f(x) - \mathcal{L}\phi(x,u)\right) - \widetilde{I}_{ij}(x;x_0) + \widetilde{J}_{ij}(x;x_0) \\
&\qquad \text{for almost all } x \in B_r^+,
\end{aligned}$$

where the function $c_{ij}(x_0)$ is defined by the formula (18.8)

$$c_{ij}(x_0) = \int_{|\xi|=1} \Gamma_i(x_0,t) t_j \, d\sigma_t,$$

and the terms $\widetilde{I}_{ij}(x;x_0)$ are defined respectively as follows (see formulas (18.9)):

- $$\begin{aligned}
&\widetilde{I}_{ij}(x;x_0) \qquad (1 \leq i,j \leq n-1) \qquad (19.12a)\\
&:= \int_{B_r^+} \Gamma_{ij}(x_0, T(x;x_0) - y) \\
&\quad \times \left\{ \sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) - \mathcal{L}_0 \phi(y,u) \right\} dy.
\end{aligned}$$

- $$\begin{aligned}
&\widetilde{I}_{in}(x;x_0) = \widetilde{I}_{ni}(x;x_0) \qquad (1 \leq i \leq n-1) \qquad (19.12b)\\
&:= \int_{B_r^+} \sum_{\ell=1}^{n} \Gamma_{i\ell}(x_0, T(x;x_0) - y) B_\ell(x_0) \\
&\quad \times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) - \mathcal{L}_0 \phi(y,u) \right] dy.
\end{aligned}$$

- $$\begin{aligned}
&\widetilde{I}_{nn}(x;x_0) \qquad (19.12c)\\
&:= \int_{B_r^+} \sum_{\ell,m=1}^{n} \Gamma_{\ell m}(x_0, T(x;x_0) - y) B_\ell(x_0) B_m(x_0)
\end{aligned}$$

$$\times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) - \mathcal{L}_0 \phi(y,u) \right] dy.$$

The terms $\widetilde{J}_{ij}(x;x_0)$ are defined respectively as follows (see formulas (18.10)):

- $\widetilde{J}_{ij}(x;x_0) \qquad (1 \leq i,j \leq n-1)$ (19.13a)

$$:= \int_{B_r^+} \theta_{ij}(x_0, T(x;x_0) - y) \times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) - \mathcal{L}_0 \phi(y,u) \right] dy.$$

- $\widetilde{J}_{in}(x;x_0) = \widetilde{J}_{ni}(x;x_0) \qquad (1 \leq i \leq n-1)$ (19.13b)

$$:= \int_{B_r^+} \sum_{j=1}^{n} \theta_{ij}(x_0, T(x;x_0) - y) B_j(x_0) \times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) - \mathcal{L}_0 \phi(y,u) \right] dy.$$

- $\widetilde{J}_{nn}(x;x_0)$ (19.13c)

$$:= \int_{B_r^+} \sum_{i,j=1}^{n} \theta_{ij}(x_0, T(x;x_0) - y) B_i(x_0) B_j(x_0) \times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + f(y) - \mathcal{L}_0 \phi(y,u) \right] dy.$$

Here recall that the vector $B(x_0)$ is defined by the formula

$$B(x_0) = \begin{pmatrix} B_1(x_0) \\ B_2(x_0) \\ \vdots \\ B_n(x_0) \end{pmatrix} := \frac{\partial T}{\partial x_n}(x;x_0) = \begin{pmatrix} -2\frac{a^{1n}(x_0)}{a^{nn}(x_0)} \\ -2\frac{a^{2n}(x_0)}{a^{nn}(x_0)} \\ \vdots \\ -1 \end{pmatrix}.$$

Step 2: Now we assume that $q < p$, and let $q \leq s \leq p$. If w is a function in $W^{2,s}(B_r^+)$, then we define a mapping $\mathcal{S}$ by the formula (cf. formula (19.10))

$$\mathcal{S}w = \phi(x,w) \tag{19.14}$$

$$+ \int_{B_r^+} G(x_0, x, y) \Bigg\{ \sum_{i,j=1}^{n} \left[a^{ij}(x_0) - a^{ij}(y) \right] \frac{\partial^2 w}{\partial x_i \partial x_j}(y)$$

$$+ f(y) - \sum_{i,j=1}^{n} a^{ij}(x_0) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(y, w) \Bigg\} \, dy.$$

Here the function $\phi(x, w)$ is defined by formula (19.6), that is,

$$\phi(x, w) := \frac{x_n}{\ell^n(x_0)} \int_{\mathbf{R}^{n-1}} \widetilde{\varphi}(x' - x_n y', w) \zeta(y') \, dy', \tag{19.15}$$

where

$$\widetilde{\varphi}(x', w) := \sum_{i=1}^{n} \left[\ell^i(x_0') - \ell^i(x') \right] \frac{\partial w}{\partial x_i}(x') \\ + \left[\sigma(x_0') - \sigma(x') \right] w(x'). \tag{19.16}$$

We show that

$$\mathcal{S} \colon W^{2,s}(B_R^+) \longrightarrow W^{2,s}(B_R^+)$$

is a *contraction mapping* for each $s \in [q, p]$ provided $0 < R < r$ is sufficiently small.

In the following we shall use the shorthand

$$D_i = \frac{\partial}{\partial x_i}, \quad 1 \le i \le n,$$

$$D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}, \quad 1 \le i, j \le n,$$

for derivatives on $\mathbf{R}^n$.

Step 2-1: Let w_1 and w_2 be two arbitrary functions in the Sobolev space $W^{2,s}(B_r^+)$. By letting

$$w := w_1 - w_2,$$

we obtain from formula (19.14) that

$$\begin{aligned} &\mathcal{S}w_1 - \mathcal{S}w_2 \\ &= \phi(x, w) + \int_{B_r^+} G(x_0, x, y) \\ &\quad \times \left[\sum_{i,j=1}^{n} \left[a^{ij}(x_0) - a^{ij}(y) \right] D_{ij} w(y) - \sum_{i,j=1}^{n} a^{ij}(x_0) D_{ij} \phi(y, w) \right] dy. \end{aligned} \tag{19.17}$$

Then we have, for some positive constant $C(n, \lambda)$,

$$\| \mathcal{S}w_1 - \mathcal{S}w_2 \|_{L^s(B_r^+)}$$

$$\leq \|\phi(\cdot, w)\|_{L^s(B_r^+)}$$
$$+ \left\| \int_{B_r^+} G(x_0, \cdot, y) \sum_{i,j=1}^{n} \left[a^{ij}(x_0) - a^{ij}(y)\right] D_{ij}w(y)\, dy \right\|_{L^s(B_r^+)}$$
$$+ C(n, \lambda) \left\| \int_{B_r^+} G(x_0, \cdot, y) \sum_{i,j=1}^{n} D_{ij}\phi(y, w)\, dy \right\|_{L^s(B_r^+)}.$$

However, by Lemma 18.2 it follows that

$$G(x_0, x, y) = O(|x - y|^{2-n}) \quad \text{as } |x - y| \to 0.$$

Hence we find that the two integrals

$$\int_{B_r^+} G(x_0, x, y) \sum_{i,j=1}^{n} \left[a^{ij}(x_0) - a^{ij}(y)\right] D_{ij}w(y)\, dy,$$
$$\int_{B_r^+} G(x_0, x, y) \sum_{i,j=1}^{n} D_{ij}\phi(y, w)\, dy$$

are Riesz potentials.

Now, we need potential estimates in the classical potential theory (see Section 3.10).

Since $a^{ij}(x) \in L^\infty(\Omega)$, it follows from an application of Theorem 3.32 with

$$\Omega := B_r^+, \quad \mu := \frac{2}{n}, \quad p = q := s, \quad \delta := 0$$

that we have, for some positive constants $C_1(n, s, \lambda)$ and $C_2(n, s, \lambda)$,

$$\left\| \int_{B_r^+} G(x_0, \cdot, y) \sum_{i,j=1}^{n} \left[a^{ij}(x_0) - a^{ij}(y)\right] D_{ij}w(y)\, dy \right\|_{L^s(B_r^+)}$$
$$\leq C_1(n, s, \lambda) r^2 \|\nabla^2 w\|_{L^s(B_r^+)},$$

and

$$\left\| \int_{B_r^+} G(x_0, \cdot, y) \sum_{i,j=1}^{n} D_{ij}\phi(\cdot, w)\, dy \right\|_{L^s(B_r^+)}$$
$$\leq C_2(n, s, \lambda) r^2 \|\nabla^2 \phi(\cdot, w)\|_{L^s(B_r^+)}.$$

Therefore, we have proved that

$$\|\mathcal{S}w_1 - \mathcal{S}w_2\|_{L^s(B_r^+)} \tag{19.18}$$
$$\leq \|\phi(\cdot, w)\|_{L^s(B_r^+)} + C(n, s, \lambda) r^2 \left(\|\nabla^2 \phi(\cdot, w)\|_{L^s(B_r^+)} + \|\nabla^2 w\|_{L^s(B_r^+)} \right)$$

for all w_1, $w_2 \in W^{2,s}(B_r^+)$.

Step 2-2: Furthermore, we have, for almost all $x \in B_r^+$,

$$\begin{aligned} & D_{ij}(\mathcal{S}w_1 - \mathcal{S}w_2)(x) \\ &= D_{ij}\phi(x,w) \\ &\quad + \text{v. p.} \int_{B_r^+} \Gamma_{ij}(x_0, x-y) \\ &\quad \times \sum_{k,\ell=1}^{n} \left\{ \left[a^{k\ell}(x_0) - a^{k\ell}(y)\right] D_{k\ell}w(y) - \mathcal{L}_0\phi(y,w) \right\} dy \\ &\quad - c_{ij}(x_0)\mathcal{L}\phi(x,w) + \widetilde{I}_{ij}(x;x_0,w) + \widetilde{J}_{ij}(x;x_0,w) \quad \text{for } 1 \le i,j \le n, \end{aligned}$$

where the terms $\widetilde{I}_{ij}(x;x_0,w)$ and $\widetilde{J}_{ij}(x;x_0,w)$ are defined respectively as in formulas (19.12) and (19.13) with u replaced by w and with $f(y) \equiv 0$.

Since the $\Gamma_{ij}(x_0,\xi)$ are Calderón–Zygmund kernels in the variable ξ, it follows from an application of Corollary 14.9 and Remark 14.1 that

$$\begin{aligned} & \left\| \text{v. p.} \int_{B_r^+} \Gamma_{ij}(x_0, \cdot - y) \sum_{k,\ell=1}^{n} \left[a^{k\ell}(x_0) - a^{k\ell}(y)\right] D_{k\ell}w(y)\, dy \right\|_{L^s(B_r^+)} \qquad (19.19a) \\ & \le C_1(n,s,\lambda,\eta,M,\partial\Omega)\, \eta(r) \left\| \nabla^2 w \right\|_{L^s(B_r^+)}, \end{aligned}$$

and that

$$\begin{aligned} & \left\| \text{v. p.} \int_{B_r^+} \Gamma_{ij}(x_0, \cdot - y) \mathcal{L}_0\phi(y,w)\, dy \right\|_{L^s(B_r^+)} \qquad (19.19b) \\ & \le C_2(n,s,\lambda,\eta,M,\partial\Omega) \left\| \mathcal{L}_0\phi(\cdot,w) \right\|_{L^s(B_r^+)} \\ & \le C_3(n,s,\lambda,\eta,M,\partial\Omega) \left\| \nabla^2\phi(\cdot,w) \right\|_{L^s(B_r^+)}. \end{aligned}$$

Here we recall that

$$M := \max_{1\le i,j\le n} \max_{|\alpha|\le 2n} \left\| \frac{\partial^\alpha \Gamma_{ij}}{\partial \xi^\alpha}(\cdot,\cdot) \right\|_{L^\infty(\Omega\times\Sigma_n)},$$

$$\eta(r) := \left(\sum_{k,\ell=1}^{n} \eta^{k\ell}(r)^2 \right)^{1/2},$$

where $\eta^{k\ell}(r)$ is the VMO modulus of $a^{k\ell}$.

Step 2-3: Before the proof of Theorem 16.1, we need the following geometric properties of the map $T(x;x_0)$ analogous to Lemma 14.11:

Lemma 19.2. *For the map $T(x;x_0)$, there exist positive constants c_1, c_2 such that, for all $y \in \mathbf{R}^n_+$ and all $x \in \mathbf{R}^n_+$ for which $T(x;x_0)$ is defined, we have the inequalities*

$$c_1|\tilde{x} - y| \leq |T(x;x_0) - y| \leq c_2|\tilde{x} - y|, \tag{19.20}$$

where

$$\tilde{x} = (x', -x_n) \quad \textit{for } x = (x', x_n) \in \mathbf{R}^n_+.$$

Proof. The proof of Lemma 19.2 is divided into two steps.

(1) First, we have, for all $y = (y', y_n) \in \mathbf{R}^n_+$,

$$\begin{aligned} &T(x;x_0) - y \\ &= (T_1(x;x_0) - y_1, T_2(x;x_0) - y_2, \ldots, T_{n-1}(x;x_0) - y_{n-1}, -x_n - y_n) \\ &:= (T(x;x_0)' - y', -x_n - y_n), \end{aligned}$$

and so

$$|T(x;x_0) - y| \geq x_n + y_n \geq x_n.$$

Hence it follows that, for all $y \in \mathbf{R}^n_+$ and all $x \in \mathbf{R}^n_+$ for which $T(x)$ is defined,

$$\begin{aligned} \frac{|T(x;x_0) - \tilde{x}|}{|T(x;x_0) - y|} &\leq \frac{|T(x;x_0) - \tilde{x}|}{x_n} \\ &= \frac{1}{x_n}\,|(x', 0) + x_n A(x_0) - (x', -x_n)| \\ &= |(0', 1) + A(x_0)|\,, \end{aligned} \tag{19.21}$$

where the vector $A(x_0)$ is defined by the formula

$$A(x_0) = \left(-2\frac{a^{1n}(x_0)}{a^{nn}(x_0)}, -2\frac{a^{2n}(x_0)}{a^{nn}(x_0)}, \ldots, -1 \right)$$

However, we can find a positive constant $C_1(n,\mu)$ such that

$$|(0', 1) + A(x_0)| \leq C_1(n,\mu) \quad \text{for all } x_0 \in \mathbf{R}^n_+.$$

Hence we have, for all $y \in \mathbf{R}^n_+$ and all $x \in \mathbf{R}^n_+$ for which $T(x)$ is defined,

$$\frac{|T(x;x_0) - \tilde{x}|}{|T(x;x_0) - y|} \leq C_1(n,\mu). \tag{19.22}$$

Therefore, we obtain from inequalities (19.21) and (19.22) that, for all $y \in \mathbf{R}^n_+$ and for all $x \in \mathbf{R}^n_+$ for which $T(x;x_0)$ is defined,

$$|\tilde{x} - y| \leq |T(x;x_0) - \tilde{x}| + |T(x;x_0) - y|$$

$$\leq |T(x;x_0) - y| \left(1 + \frac{|T(x;x_0) - \tilde{x}|}{|T(x;x_0) - y|}\right)$$
$$\leq (1 + C_1(n,\mu))|T(x;x_0) - y|.$$

This proves the desired inequality (19.20) with

$$c_1 := \frac{1}{1 + C_1(n,\mu)}.$$

(2) On the other hand, it follows that we have, for all $y = (y', y_n) \in \mathbf{R}^n_+$ and for all $x = (x', x_n) \in \mathbf{R}^n_+$,

$$|\tilde{x} - y| = \sqrt{|x' - y'|^2 + (x_n + y_n)^2},$$
$$|T(x;x_0) - y| = \sqrt{|T(x;x_0)' - y'|^2 + (x_n + y_n)^2},$$

where

$$T(x;x_0)' - y'$$
$$= x' - y' - 2x_n \left(\frac{a^{1n}(x_0)}{a^{nn}(x_0)}, \frac{a^{2n}(x_0)}{a^{nn}(x_0)}, \ldots, \frac{a^{n-1n}(x_0)}{a^{nn}(x_0)}\right).$$

We remark that, for some constant $C_2(n,\mu) > 0$,

$$|T(x;x_0)' - y'| \leq |x' - y'| + 2C_2(n,\mu)x_n$$
$$\leq |x' - y'| + 2C_2(n,\mu)(x_n + y_n).$$

Hence we have, for all $y \in \mathbf{R}^n_+$ and for all $x \in \mathbf{R}^n_+$ for which $T(x;x_0)$ is defined,

$$|T(x;x_0) - y| \leq |T(x;x_0)' - y'| + (x_n + y_n)$$
$$\leq |x' - y'| + (1 + 2C_2(n,\mu))(x_n + y_n)$$
$$\leq (2C_2(n,\mu) + 1)\left(|x' - y| + (x_n + y_n)\right)$$
$$\leq \sqrt{2}\left(2C_2(n,\mu) + 1\right)\sqrt{|x' - y|^2 + (x_n + y_n)^2}$$
$$= \sqrt{2}\left(2C_2(n,\mu) + 1\right)|\tilde{x} - y|.$$

This proves the desired inequality (19.20) with

$$c_2 := \sqrt{2}\left(2C_2(n,\mu) + 1\right).$$

The proof of Lemma 19.2 is complete. □

Step 2-4: Therefore, by applying Theorems 14.2 and 14.5 (and Remark 14.2) to our situation we obtain that

$$\left\|\widetilde{I}_{ij}(\cdot;x_0,w)\right\|_{L^s(B_r^+)}, \quad \left\|\widetilde{J}_{ij}(\cdot;x_0,w)\right\|_{L^s(B_r^+)} \tag{19.23}$$

$$\leq C_1(n,s,\lambda,\eta,M,\partial\Omega)\left\|\sum_{h,k=1}^{n}\widetilde{C}(a^{hk},D_{hk}w)+\widetilde{K}(\mathcal{L}\phi(\cdot,w))\right\|_{L^s(B_r^+)}$$
$$\leq C_2(n,s,\lambda,\eta,M,\partial\Omega)\left(\eta(r)\left\|\nabla^2 w\right\|_{L^s(B_r^+)}+\|\mathcal{L}\phi(\cdot,w)\|_{L^s(B_r^+)}\right)$$
$$\leq C_3(n,s,\lambda,\eta,M,\partial\Omega)\left(\eta(r)\left\|\nabla^2 w\right\|_{L^s(B_r^+)}+\left\|\nabla^2\phi(\cdot,w)\right\|_{L^s(B_r^+)}\right).$$

Indeed, it suffices to note that we have, by Lemma 18.2 with $|\alpha| := 2$ and Lemma 19.2,

$$|\theta_{ij}(x_0,T(x;x_0)-y)|\leq\frac{C}{|T(x;x_0)-y|^n}\leq\frac{C'}{|\tilde{x}-y|^n},$$

with some positive constants C and C'.

Finally, it follows that

$$\|c_{ij}(x_0)\mathcal{L}\phi(\cdot,w)\|_{L^s(B_r^+)}\leq C_3\|\nabla^2\phi(\cdot,w)\|_{L^s(B_r^+)}. \tag{19.24}$$

By combining inequalities (19.19), (19.23) and (19.24), we have proved that

$$\begin{aligned}&\|\nabla^2(\mathcal{S}w_1-\mathcal{S}w_2)\|_{L^s(B_r^+)} \qquad (19.25)\\
&\leq\|\nabla^2\phi(\cdot,w)\|_{L^s(B_r^+)}+C_1\left(\eta(r)\|\nabla^2 w\|_{L^s(B_r^+)}+\|\nabla^2\phi(\cdot,w)\|_{L^s(B_r^+)}\right)\\
&\quad+C_2\left(\eta(r)\|\nabla^2 w\|_{L^s(B_r^+)}+\|\nabla^2\phi(\cdot,w)\|_{L^s(B_r^+)}\right)\\
&\quad+C_3\|\nabla^2\phi(\cdot,w)\|_{L^s(B_r^+)}.\end{aligned}$$

Step 2-5: Therefore, we obtain from inequalities (19.18) and (19.25) that

$$\begin{aligned}&\|\mathcal{S}w_1-\mathcal{S}w_2\|^*_{W^{2,s}(B_r^+)} \qquad (19.26)\\
&=\|\mathcal{S}w_1-\mathcal{S}w_2\|_{L^s(B_r^+)}+r\|\nabla^2(\mathcal{S}w_1-\mathcal{S}w_2)\|_{L^s(B_r^+)}\\
&\leq C_4\Big(r\eta(r)\|\nabla^2(w_1-w_2)\|_{L^s(B_r^+)}+r^2\|\nabla^2(w_1-w_2)\|_{L^s(B_r^+)}\\
&\qquad+r\|\nabla^2\phi(\cdot,w)\|_{L^s(B_r^+)}+\|\phi(\cdot,w)\|_{L^s(B_r^+)}\Big)\\
&=C_4\Big((\eta(r)+r)\,r\|\nabla^2(w_1-w_2)\|_{L^s(B_r^+)}\\
&\qquad+r\|\nabla^2\phi(\cdot,w)\|_{L^s(B_r^+)}+\|\phi(\cdot,w)\|_{L^s(B_r^+)}\Big)\\
&\leq C_4\Big((\eta(r)+r)\|w_1-w_2\|^*_{W^{2,s}(B_r^+)}\end{aligned}$$

$$+ r\|\nabla^2\phi(\cdot,w)\|_{L^s(B_r^+)} + \|\phi(\cdot,w)\|_{L^s(B_r^+)}\Bigg)$$

for all $w_1,\, w_2 \in W^{2,s}(B_r^+)$,

with a positive constant $C_4 = C_4(n, s, \lambda, \eta, M, \partial\Omega)$.

Step 2-6: Now we estimate the last two terms of inequality (19.26) in terms of the norm

$$\|w\|^*_{W^{2,s}(B_r^+)} = \|w_1 - w_2\|^*_{W^{2,s}(B_r^+)}.$$

To do this, by applying Lemma 17.1 with $p := s$ and $R := r$ we obtain that

$$\begin{aligned} \|\phi(\cdot,w)\|^*_{W^{2,s}(B_r^+)} &:= \|\phi(\cdot,w)\|_{L^s(B_r^+)} + r\|\nabla^2\phi(\cdot,w)\|_{L^s(B_r^+)} \\ &\le C r^{1/2}\, \|\widetilde{\varphi}(\cdot,w)\|^*_{B^{1-1/s,s}(C_r)}\,. \end{aligned} \tag{19.27}$$

On the other hand, we find from formula (19.16) that

$$\begin{aligned} & r^{1/2}\, \|\widetilde{\varphi}(\cdot,w)\|^*_{B^{1-1/s,s}(C_r)} \\ & \le r^{1/2} C_5 \Bigg(\sum_{i=1}^n \|(\ell^i(x_0') - \ell^i(\cdot))D_i w\|^*_{B^{1-1/s,s}(C_r)} \\ & \quad + \|(\sigma(x_0') - \sigma(\cdot))w\|^*_{B^{1-1/s,s}(C_r)} \Bigg). \end{aligned} \tag{19.28}$$

Here we recall that the norm $\|\cdot\|^*_{B^{1-1/s,s}(C_r)}$ is given by the formula (see the norm (17.1) and the norm (7.11) with $m := 0$ and $p := s$)

$$\begin{aligned} \|\widetilde{\psi}\|^*_{B^{1-1/s,s}(C_r)} &= \left(\int_{C_r} |\widetilde{\psi}(x')|^s\, dx' \right)^{1/s} \\ &\quad + r^{1/2} \left(\int_{C_r} \int_{C_r} \frac{|\widetilde{\psi}(x') - \widetilde{\psi}(y')|^s}{|x'-y'|^{s+n-2}}\, dx'\, dy' \right)^{1/s}. \end{aligned}$$

In order to estimate the first and second terms on inequality (19.29), we make use of Rademacher's theorem. Since the functions $\ell^i(x')$ are Lipschitz continuous, it follows from an application of Rademacher's theorem (Theorem 5.1) and the trace theorem (Theorem 7.4) that the first term on inequality (19.28) is estimated as follows:

$$\begin{aligned} & r^{1/2} \sum_{i=1}^n \left\|(\ell^i(x_0') - \ell^i(\cdot))D_i w\right\|^*_{B^{1-1/s,s}(C_r)} \\ & \le C_6 \Bigg(r^{3/2}\, \|D_i w\|_{L^s(C_r)} \end{aligned} \tag{19.29}$$

$$+ r^{1+1/s} \left(\int_{C_r} \int_{C_r} \frac{|D_i w(x') - D_i w(y')|^s}{|x' - y'|^{s+n-2}} \, dx' \, dy' \right)^{1/s} \Bigg)$$
$$\leq C_6 \left(r^{3/2} + r^{1+1/s} \right) \|\nabla w\|_{W^{1,s}(B_r^+)} .$$

Moreover, by applying interpolation inequality (13.17) with $\varepsilon := r^{1/2}$ (Theorem 13.4) we obtain that

$$\|\nabla w\|_{L^s(B_r^+)} \leq r^{1/2} \left\|\nabla^2 w\right\|_{L^s(B_r^+)} + \frac{C}{r^{1/2}} \|w\|_{L^s(B_r^+)} .$$

Hence we have the inequality

$$\begin{aligned} & r^{3/2} \|\nabla w\|_{W^{1,s}(B_r^+)} && (19.30) \\ & \leq r^{3/2} C_7 \left(\left\|\nabla^2 w\right\|_{L^s(B_r^+)} + \|\nabla w\|_{L^s(B_r^+)} \right) \\ & \leq C_7 \left(r^{3/2} \left\|\nabla^2 w\right\|_{L^s(B_r^+)} + r^2 \left\|\nabla^2 w\right\|_{L^s(B_r^+)} + r \|w\|_{L^s(B_r^+)} \right) \\ & \leq C_7 r^{1/2} \|w\|^*_{W^{2,s}(B_r^+)} \quad \text{for all } w \in W^{2,s}(B_r^+). \end{aligned}$$

Similarly, we have, by interpolation inequality (13.17) with $\varepsilon := r^{1/s}$,

$$\begin{aligned} & r^{1+1/s} \|\nabla w\|_{W^{1,s}(B_r^+)} && (19.31) \\ & \leq r^{1+1/s} C_8 \left(\|\nabla^2 w\|_{L^s(B_r^+)} + \|\nabla w\|_{L^s(B_r^+)} \right) \\ & \leq C_8 \left(r^{1+1/s} \|\nabla^2 w\|_{L^s(B_r^+)} + r^{1+2/s} \|\nabla^2 w\|_{L^s(B_r^+)} + r \|w\|_{L^s(B_r^+)} \right) \\ & \leq C_8 r^{1/s} \|w\|^*_{W^{2,s}(B_r^+)} \quad \text{for all } w \in W^{2,s}(B_r^+).. \end{aligned}$$

By combining inequalities (19.29), (19.30) and (19.31), we obtain that

$$\begin{aligned} & r^{1/2} \sum_{i=1}^{n} \|(\ell^i(x_0') - \ell^i(\cdot)) D_i w\|^*_{B^{1-1/s,s}(C_r)} && (19.32) \\ & \leq C' \left(r^{1/2} + r^{1/s} \right) \|w\|^*_{W^{2,s}(B_r^+)} \\ & = C' \left(r^{1/2} + r^{1/s} \right) \|w_1 - w_2\|^*_{W^{2,s}(B_r^+)} \quad \text{for all } w_1,\, w_2 \in W^{2,s}(B_r^+). \end{aligned}$$

On the other hand, since the function $\sigma(x')$ is Lipschitz continuous, it follows from an application of Rademacher's theorem (Theorem 5.1) and the trace theorem (Theorem 7.4) that the second term on inequality (19.28) is estimated as follows:

$$\begin{aligned} & r^{1/2} \|(\sigma(x_0') - \sigma(\cdot)) w\|^*_{B^{1-1/s,s}(C_r)} && (19.33) \\ & \leq C'' \left(r^{2/s+1} \|\nabla^2 w\|_{L^s(B_r^+)} + r \|w\|_{L^s(B_r^+)} \right) \end{aligned}$$

$$\leq C'' r^{1/s} \left(r \|\nabla^2 w\|_{L^s(B_r^+)} + r^{1-1/s} \|w\|_{L^s(B_r^+)} \right)$$
$$\leq C'' r^{1/s} \|w\|^*_{W^{2,s}(B_r^+)}$$
$$= C'' r^{1/s} \|w_1 - w_2\|^*_{W^{2,s}(B_r^+)} \quad \text{for all } w_1,\, w_2 \in W^{2,s}(B_r^+).$$

Therefore, by combining inequalities (19.27), (19.28), (19.32) and (19.33) we can estimate the last two terms of inequality (19.26) as follows:

$$\begin{aligned} &\|\phi(\cdot, w)\|^*_{W^{2,s}(B_r^+)} \qquad (19.34)\\ &:= \|\phi(\cdot, w)\|_{L^s(B_r^+)} + r \|\nabla^2 \phi(\cdot, w)\|_{L^s(B_r^+)}\\ &\leq C r^{1/2} \|\widetilde{\varphi}(\cdot, w)\|^*_{B^{1-1/s,s}(C_r)}\\ &\leq C C_5 \left(C' r^{1/s} + C'' \left(r^{1/2} + r^{1/s} \right) \right) \|w\|^*_{W^{2,s}(B_r^+)}\\ &= C C_5 \left(C' r^{1/s} + C'' \left(r^{1/2} + r^{1/s} \right) \right) \|w_1 - w_2\|^*_{W^{2,s}(B_r^+)}\\ &\quad \text{for all } w_1,\, w_2 \in W^{2,s}(B_r^+). \end{aligned}$$

Step 2-7: By combining estimates (19.26) and (19.34), we obtain that

$$\|\mathcal{S} w_1 - \mathcal{S} w_2\|^*_{W^{2,s}(B_r^+)} \leq C(r) \|w_1 - w_2\|^*_{W^{2,s}(B_r^+)} \quad \text{for all } w_1,\, w_2 \in W^{2,s}(B_r^+),$$

with a positive constant

$$C(r) := C_4 \left(\eta(r) + r \right) + C C_5 \left(C' r^{1/s} + C'' \left(r^{1/2} + r^{1/s} \right) \right) = o(1) \qquad \text{as } r \downarrow 0.$$

Hence we can take $r := R$ so small that

$$C(R) < 1.$$

This proves that $\mathcal{S}$ is a contraction mapping of $W^{2,s}(B_R^+)$, equipped with the norm $\| \cdot \|^*_{W^{2,s}(B_R^+)}$, into itself, for each $s \in [q, p]$.

Step 3: Now we assume that a function $u \in W^{2,q}(B_R^+)$ with $q < p$ is a solution of problem (19.1). Then we obtain from formulas (19.10) and (19.14) that $u \in W^{2,q}(B_R^+)$ is a fixed point of $\mathcal{S}$, that is,

$$u = \mathcal{S} u,$$

and that

$$W^{2,p}(B_R^+) \subset W^{2,q}(B_R^+).$$

Hence, by the uniqueness of fixed points of $\mathcal{S}$ it follows that

$$u \in W^{2,p}(B_R^+).$$

Step 4: To prove estimate (19.2), it suffices to take the L^p-norm of the both sides of formula (19.10), just as in the proof of estimate (19.26). More precisely, taking $w_1 := u$ and $w_2 := 0$ we obtain that

$$\begin{aligned} \|\nabla^2 u\|_{L^p(B_R^+)} &= \|\nabla^2(\mathcal{S}u)\|_{L^p(B_R^+)} \\ &\leq \|\nabla^2(\mathcal{S}u - \mathcal{S}0)\|_{L^p(B_R^+)} + \|\nabla^2(\mathcal{S}0)\|_{L^p(B_R^+)} \\ &\quad \text{for all } u \in W^{2,p}(B_R^+). \end{aligned} \tag{19.35}$$

Step 4-1: In order to estimate the first term on inequality (19.35), we remark that the term

$$\begin{aligned} &\mathcal{S}u - \mathcal{S}0 \\ &= \phi(x,u) + \int_{B_r^+} G(x_0,x,y) \\ &\quad \times \left[\sum_{i,j=1}^n \left[a^{ij}(x_0) - a^{ij}(y)\right] D_{ij}u(y) - \sum_{i,j=1}^n a^{ij}(x_0) D_{ij}\phi(y,u) \right] dy \end{aligned}$$

is estimated exactly as in estimate (19.25) with $w = w_1 := u$ and $w_2 := 0$ as follows:

$$\begin{aligned} \|\nabla^2(\mathcal{S}u - \mathcal{S}0)\|_{L^p(B_R^+)} \leq C_1 \Big(& R\eta(R)\|\nabla^2 u\|_{L^p(B_R^+)} + R^2\|\nabla^2 u\|_{L^p(B_R^+)} \\ & + R\|\nabla^2\phi(\cdot,u)\|_{L^p(B_R^+)} + \|\phi(\cdot,u)\|_{L^p(B_R^+)} \Big) \\ & \text{for all } u \in W^{2,p}(B_R^+), \end{aligned}$$

with some positive constant $C_1 = C_1(n,s,\lambda,\eta,M,\partial\Omega)$.

Therefore, by applying inequality (17.6) we obtain that

$$\begin{aligned} \|\nabla^2(\mathcal{S}u - \mathcal{S}0)\|_{L^p(B_R^+)} &\leq C_1\left(R\eta(R) + R^2\right)\|\nabla^2 u\|_{L^p(B_R^+)} \\ &\quad + C_2 R^{1/2}\|\widetilde{\varphi}(\cdot,u)\|^*_{B^{1-1/p,p}(C_R)}. \end{aligned} \tag{19.36}$$

However, by formula (19.16) with $w := u$ it follows that

$$\widetilde{\varphi}(x',u) = \mathcal{B}_0 u(x') - \mathcal{B}u(x') = \mathcal{B}_0 u(x') - \varphi(x'),$$

so that, by the trace theorem (Theorem 7.4),

$$\begin{aligned} &\|\widetilde{\varphi}(\cdot,u)\|^*_{B^{1-1/p,p}(C_R)} \\ &\leq \|\mathcal{B}_0 u\|^*_{B^{1-1/p,p}(C_R)} + \|\varphi\|^*_{B^{1-1/p,p}(C_R)} \end{aligned}$$

$$\leq C_3 \left(\|u\|_{W^{2,p}(B_R^+)} + \|\varphi\|_{B^{1-1/p,p}(C_R)} \right)$$
$$\leq C_3 \left(\|u\|_{L^p(B_R^+)} + \|\nabla u\|_{L^p(B_R^+)} + \|\nabla^2 u\|_{L^p(B_R^+)} \right)$$
$$+ C_3 \|\varphi\|_{B^{1-1/p,p}(C_R)}.$$

Hence, we have, by interpolation inequality (13.17) with $\varepsilon := 1/2$,

$$\begin{aligned} &\|\widetilde{\varphi}(\cdot, u)\|^*_{B^{1-1/p,p}(C_R)} \\ &\leq C_3 \left(\|u\|_{L^p(B_R^+)} + \|\nabla u\|_{L^p(B_R^+)} + \|\nabla^2 u\|_{L^p(B_R^+)} \right) \\ &\quad + C_3 \|\varphi\|_{B^{1-1/p,p}(C_R)} \\ &\leq C_4 \left(\|u\|_{L^p(B_R^+)} + \|\nabla^2 u\|_{L^p(B_R^+)} + \|\varphi\|_{B^{1-1/p,p}(C_R)} \right). \end{aligned} \tag{19.37}$$

By combining inequalities (19.36) and (19.37), we have proved that

$$\begin{aligned} &\|\nabla^2 (\mathcal{S}u - \mathcal{S}0)\|_{L^p(B_R^+)} \\ &\leq C_1 \left(R\eta(R) + R^2 \right) \|\nabla^2 u\|_{L^p(B_R^+)} \\ &\quad + C_2 C_4 R^{1/2} \left(\|u\|_{L^p(B_R^+)} + \|\nabla^2 u\|_{L^p(B_R^+)} + \|\varphi\|_{B^{1-1/p,p}(C_R)} \right) \\ &\leq \left(C_1 \left(R\eta(R) + R^2 \right) + C_2 C_4 R^{1/2} \right) \|\nabla^2 u\|_{L^p(B_R^+)} \\ &\quad + C_5 \left(\|u\|_{L^p(B_R^+)} + \|\varphi\|_{B^{1-1/p,p}(C_R)} \right) \end{aligned} \tag{19.38}$$

for all $u \in W^{2,p}(B_R^+)$.

Step 4-2: In order to estimate the second term on inequality (19.35), we remark that the term

$$\mathcal{S}0 = \int_{B_r^+} G(x_0, x, y) f(y)\, dy$$

is estimated as follows:

$$\|\nabla^2 (\mathcal{S}0)\|_{L^p(B_R^+)} \leq C_2 \|f\|_{L^p(B_R^+)}, \tag{19.39}$$

with some positive constant $C_2 = C_2(n, s, \lambda, \eta, M, \partial\Omega)$.

Step 4-3: Summing up, we obtain from inequalities (19.35), (19.38) and (19.39) that

$$\begin{aligned} \|\nabla^2 u\|_{L^p(B_R^+)} &= \|\nabla^2 (\mathcal{S}u)\|_{L^p(B_R^+)} \\ &\leq \|\nabla^2 (\mathcal{S}u - \mathcal{S}0)\|_{L^p(B_R^+)} + \|\nabla^2 (\mathcal{S}0)\|_{L^p(B_R^+)} \\ &\leq C(R) \|\nabla^2 u\|_{L^p(B_R^+)} \\ &\quad + C_5 \left(\|u\|_{L^p(B_R^+)} + \|\varphi\|_{B^{1-1/p,p}(C_R)} + \|f\|_{L^p(B_R^+)} \right) \end{aligned}$$

$$\text{for all } u \in W^{2,p}(B_R^+),$$

with a positive constant

$$C(R) := C_1 \left(R\eta(R) + R^2\right) + C_2 C_4 R^{1/2} = o(1) \quad \text{as } R \downarrow 0.$$

Therefore, if we take R so small that

$$C(R) < \frac{1}{2},$$

then we have the desired boundary *a priori* estimate (19.2)

$$\left\|\nabla^2 u\right\|_{L^p(B_R^+)} \leq C \left(\|u\|_{L^p(B_R^+)} + \|f\|_{L^p(B_R^+)} + \|\varphi\|_{B^{1-1/p,p}(C_R)}\right)$$
$$\text{for } u \in W^{2,p}(B_R^+).$$

Now the proof of Lemma 19.1 is complete. □

19.2 Notes and Comments

This chapter is adapted from Maugeri–Palagachev–Softova [47].

20

Proof of Theorems 16.1 and 16.2

This chapter is devoted to the study of the following *non-homogeneous* oblique derivative problem

$$\begin{cases} \mathcal{L}u(x) := \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x) & \text{for almost all } x \in \Omega, \\ \mathcal{B}u(x') := \frac{\partial u}{\partial \ell} + \sigma(x')u = \varphi(x') & \text{on } \partial\Omega. \end{cases} \tag{16.4}$$

Concerning the operator $\mathcal{L}$, we assume that the following three conditions (1), (2) and (3) are satisfied:

(1) $a^{ij}(x) \in \mathrm{VMO} \cap L^\infty(\Omega)$ for $1 \leq i, j \leq n$.
(2) $a^{ij}(x) = a^{ji}(x)$ for almost all $x \in \Omega$ and $1 \leq i, j \leq n$.
(3) There exists a positive constant λ such that

$$\frac{1}{\lambda}|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2 \tag{16.1}$$

for almost all $x \in \Omega$ and for all $\xi \in \mathbf{R}^n$.

Concerning the boundary operator $\mathcal{B}$, we assume that the following three conditions (16.3a), (16.3b) and (16.3c) are satisfied:

$$\ell^i(x') \text{ and } \sigma(x') \text{ are Lipschitz continuous functions on } \partial\Omega. \tag{16.3a}$$

$$\langle \ell(x'), \mathbf{n}(x') \rangle := \sum_{i=1}^{n} \ell^i(x') n_i(x') > 0 \quad \text{on } \partial\Omega. \tag{16.3b}$$

$$\sigma(x') < 0 \quad \text{on } \partial\Omega. \tag{16.3c}$$

We prove the regularity, existence and uniqueness theorems (Theorems 16.1 and 16.2) for problem (16.4). By Lemma 17.1, for any given function $\varphi \in B^{1-1/p,p}(\partial\Omega)$ we can construct a function

$$v \in W^{2,p}(\Omega)$$

such that

$$\mathcal{B}v = \varphi \quad \text{on } \partial\Omega$$

and further that

$$\|v\|_{W^{2,p}(\Omega)} \leq C\|\varphi\|_{B^{1-1/p,p}(\partial\Omega)} \quad \text{for some positive constant } C. \tag{20.1}$$

Hence, by letting

$$w = u - v,$$

we are reduced to the study of the following *homogeneous* oblique derivative problem:

$$\mathcal{L}w = \mathcal{L}u - \mathcal{L}v = f - \mathcal{L}v \quad \text{in } \Omega, \tag{20.2a}$$

$$\mathcal{B}w = \mathcal{B}u - \mathcal{B}v = 0 \quad \text{on } \partial\Omega. \tag{20.2b}$$

Our proof is based on some interior and boundary *a priori* estimates for the solutions of the homogeneous oblique derivative problem (20.2) (Theorem 12.1 and Lemma 19.1). Both the interior and boundary *a priori* estimates are consequences of explicit representation formulas (19.10) and (19.11) for the solutions of problem (19.2) and also of the L^p-boundedness of Calderón–Zygmund singular integral operators and boundary commutators appearing in those representation formulas (Theorems 14.2 and 14.5). It should be emphasized that the VMO assumption on the coefficients a^{ij} is of the greatest relevance in the study of singular commutators.

20.1 Proof of Theorem 16.1

The purpose of this section is to prove Theorem 16.1. The proof of Theorem 16.1 is divided into two steps (see [18], [19], [47]).

Step 1: First, in view of Lemma 19.1, we obtain that if a function $w \in W^{2,q}(\Omega)$, $1 < q < p < \infty$, is a solution of problem (20.2) with

$$f - \mathcal{L}v \in L^p(\Omega),$$

then it follows that

$$w = u - v \in W^{2,p}(\Omega).$$

This proves that if a function $u \in W^{2,q}(\Omega)$, $1 < q < p < \infty$, is a solution of problem (16.4) with $f \in L^p(\Omega)$ and $\varphi \in B^{1-1/p,p}(\partial\Omega)$, then it follows that

$$u = v + w \in W^{2,p}(\Omega).$$

Step 2: Secondly, we prove the global *a priori* estimate (16.5). However, by estimate (20.1) we may assume that $u \in W^{2,p}(\Omega)$ is a solution of the homogeneous oblique derivative problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{20.3}$$

Therefore, it suffices to prove the following global *a priori* estimate:

$$\|u\|_{W^{2,p}(\Omega)} \leq c_1 \left(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \right) \quad \text{for all } u \in \mathfrak{B}. \tag{20.4}$$

Here

$$\mathfrak{B} := \left\{ u \in W^{2,p}(\Omega) : \mathcal{B}u = 0 \text{ on } \partial\Omega \right\},$$

and $c_1 > 0$ is a constant depending on the coefficients a^{ij} only through the ellipticity constant λ, the bound on the norms $\|a^{ij}\|_{L^p(\Omega)}$ and the VMO moduli of the a^{ij}. The proof of estimate (20.4) is carried out in a standard way from the interior *a priori* estimate (Theorem 12.1) and the boundary *a priori* estimate (Lemma 19.1) by a covering argument, by flattening the boundary $\partial\Omega$ and by using interpolation inequalities (Theorem 13.4), just as in the proof of Theorem 12.2.

Step 2-1: Now we choose a finite covering $\{U_j\}_{j=1}^N$ of the boundary $\partial\Omega$ by open subsets of $\mathbf{R}^n$ and $C^{1,1}$-diffeomorphisms G_j of $U_j \cap \Omega$ onto B_r^+ in each of which the boundary *a priori* estimate (19.2) holds true (see Figure 20.1 below). Furthermore, we choose an open subset U_0 of Ω, bounded away from $\partial\Omega$, such that (see Figure 20.2 below)

$$\Omega \subset U_0 \cup \left(\cup_{j=1}^N U_j \right).$$

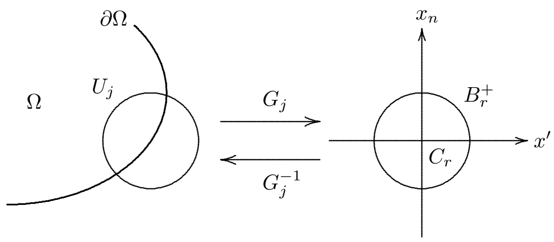

Fig. 20.1. The covering $\{U_j\}$ of $\partial\Omega$ and the $C^{1,1}$-diffeomorphism G_j of $U_j \cap \Omega$ onto B_r^+

Step 2-2: We take a partition of unity $\{\alpha_k\}_{k=0}^N$ subordinate to the

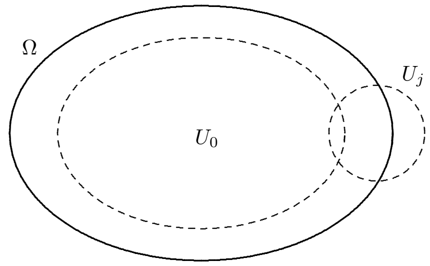

Fig. 20.2. The open covering $\{U_j\}$ of $\partial\Omega$ and the open set U_0 bounded away from $\partial\Omega$

open covering $\{U_k\}_{k=0}^N$ of Ω. Then, by applying the interior *a priori* estimate (12.2) to the function $\alpha_0 u$ we obtain that

$$\begin{aligned}
&\|u\|_{W^{2,p}(\Omega)} \qquad (20.5)\\
&\leq \sum_{k=0}^{N} \|\alpha_k u\|_{W^{2,p}(\Omega)}\\
&\leq \|\alpha_0 u\|_{W^{2,p}(\Omega)}\\
&\quad + \sum_{k=1}^{N} \left(\|\nabla^2(\alpha_k u)\|_{L^p(\Omega)} + \|\nabla(\alpha_k u)\|_{L^p(\Omega)} + \|\alpha_k u\|_{L^p(\Omega)}\right)\\
&\leq C_0 \left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\right)\\
&\quad + \sum_{k=1}^{N} \left(\|\nabla^2(\alpha_k u)\|_{L^p(U_k\cap\Omega)} + \|\nabla(\alpha_k u)\|_{L^p(U_k\cap\Omega)} + \|u\|_{L^p(\Omega)}\right).
\end{aligned}$$

(I) In order to estimate the terms $\nabla(\alpha_k u)$ for $1 \leq k \leq N$, we recall the interpolation inequality (Theorem 13.4)

$$\|\nabla v\|_{L^p(\Omega)} \leq \varepsilon \, \|\nabla^2 v\|_{L^p(\Omega)} + \frac{C}{\varepsilon} \, \|v\|_{L^p(\Omega)} \quad \text{for all } v \in W^{2,p}(\Omega). \quad (13.17)$$

Since we have the formula

$$\nabla(\alpha_k u) = \alpha_k \, \nabla u + u(\nabla \alpha_k) \quad \text{for } 1 \leq k \leq N,$$

by applying inequality (13.17) to the function $u \in W^{2,p}(\Omega)$, we obtain that, for some positive constants C_1 and C_2,

$$\begin{aligned}
&\|\nabla(\alpha_k u)\|_{L^p(U_k\cap\Omega)} \qquad (20.6)\\
&\leq \|\alpha_k(\nabla u)\|_{L^p(U_k\cap\Omega)} + \|u(\nabla\alpha_k)\|_{L^p(U_k\cap\Omega)}\\
&\leq \|\nabla u\|_{L^p(\Omega)} + C_1 \|u\|_{L^p(\Omega)}
\end{aligned}$$

$$\leq \varepsilon \, \|\nabla^2 u\|_{L^p(\Omega)} + \frac{C_2}{\varepsilon}\|u\|_{L^p(\Omega)} + C_1\|u\|_{L^p(\Omega)} \quad \text{for } 1 \leq k \leq N.$$

(II) Moreover, by applying the boundary *a priori* estimate (19.2) to the functions $\alpha_k u \in W^{2,p}(U_k \cap \Omega)$ for $1 \leq k \leq N$, we obtain that

$$\begin{aligned} &\|\nabla^2(\alpha_k u)\|_{L^p(U_k\cap\Omega)} \qquad (20.7)\\ &\leq C\big(\|\alpha_k u\|_{L^p(U_k\cap\Omega)} + \|\mathcal{L}(\alpha_k u)\|_{L^p(U_k\cap\Omega)} + \|\mathcal{B}(\alpha_k u)\|_{B^{1-1/p,p}(U_k\cap\partial\Omega)}\big). \end{aligned}$$

However, since u is a solution of the homogeneous oblique derivative problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (20.3)$$

it follows that

$$\begin{aligned} \mathcal{L}(\alpha_k u) &= \alpha_k(\mathcal{L}u) + u\,\mathcal{L}(\alpha_k) + 2\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial \alpha_k}{\partial x_i}\frac{\partial u}{\partial x_j} \qquad (20.8\text{a})\\ &= \alpha_k f + u\,\mathcal{L}(\alpha_k) + 2\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial \alpha_k}{\partial x_i}\frac{\partial u}{\partial x_j} \quad \text{in } \Omega, \end{aligned}$$

and further that

$$\mathcal{B}(\alpha_k u) = \alpha_k(\mathcal{B}u) + \frac{\partial \alpha_k}{\partial \ell}u = \frac{\partial \alpha_k}{\partial \ell}u \quad \text{on } \partial\Omega. \qquad (20.8\text{b})$$

Hence, we have, by inequality (20.7) and formulas (20.8),

$$\begin{aligned} &\|\nabla^2(\alpha_k u)\|_{L^p(U_k\cap\Omega)} \qquad (20.9)\\ &\leq C\left(\|\alpha_k u\|_{L^p(U_k\cap\Omega)} + \|\mathcal{L}(\alpha_k u)\|_{L^p(U_k\cap\Omega)} + \|\mathcal{B}(\alpha_k u)\|_{B^{1-1/p,p}(U_k\cap\partial\Omega)}\right)\\ &\leq C\Bigg(\|\alpha_k f\|_{L^p(U_k\cap\Omega)} + \|u\,\mathcal{L}(\alpha_k)\|_{L^p(U_k\cap\Omega)}\\ &\quad + 2\sum_{i,j=1}^{n}\left\|a^{ij}\frac{\partial \alpha_k}{\partial x_i}\frac{\partial u}{\partial x_j}\right\|_{L^p(U_k\cap\Omega)}\Bigg) + C_3\left\|u\frac{\partial \alpha_k}{\partial \ell}\right\|_{B^{1-1/p,p}(U_k\cap\partial\Omega)}\\ &\leq C_4\left(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}\right) + C_5\|u\|_{B^{1-1/p,p}(\partial\Omega)}. \end{aligned}$$

On the other hand, by applying the trace theorem (Theorem 7.4) we obtain that

$$\begin{aligned} \|u\|_{B^{1-1/p,p}(\partial\Omega)} &\leq C_6\|u\|_{W^{1,p}(\Omega)} \qquad (20.10)\\ &\leq C_7\left(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}\right). \end{aligned}$$

Therefore, it follows from inequalities (20.9), (20.10) and (13.17) that

$$\begin{aligned} &\|\nabla^2(\alpha_k u)\|_{L^p(U_k\cap\Omega)} \\ &\le C_4 \left(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}\right) \\ &\quad + C_5C_7 \left(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}\right) \\ &\le \varepsilon \, \|\nabla^2 u\|_{L^p(\Omega)} + \frac{C_8}{\varepsilon}\|u\|_{L^p(\Omega)} + C_9 \left(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right) \\ &\qquad \text{for } 1 \le k \le N. \end{aligned} \tag{20.11}$$

(III) By combining inequalities (20.5), (20.6) and (20.11), we obtain that

$$\begin{aligned} \|u\|_{W^{2,p}(\Omega)} &\le 2N\varepsilon \, \|\nabla^2 u\|_{L^p(\Omega)} + (C_0 + NC_9)\left(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right) \\ &\quad + \frac{N(C_2 + C_8)}{\varepsilon}\|u\|_{L^p(\Omega)} + (NC_1 + 1)\,\|u\|_{L^p(\Omega)}. \end{aligned}$$

This proves the desired global *a priori* estimate (20.4), if we take

$$\varepsilon = \frac{1}{4N}.$$

The proof of Theorem 16.1 is now complete.

20.2 Proof of Theorem 16.2

This section is devoted to the proof of Theorem 16.2. More precisely, we prove the following existence and uniqueness theorem for the homogeneous oblique derivative problem (20.3) due to Di Fazio–Palagachev [22, Theorem 1.2]:

Theorem 20.1. *Let $n < p < \infty$ and assume that conditions* (16.1) *and* (16.3) *are satisfied. For any given function $f \in L^p(\Omega)$, there exists a unique solution $u \in W^{2,p}(\Omega)$ of problem* (20.3). *Moreover, we have the* a priori *estimate*

$$\|u\|_{W^{2,p}(\Omega)} \le C_3\|f\|_{L^p(\Omega)}, \tag{20.12}$$

with a positive constant $C_3 = C_3(n, p, \lambda, \eta, \ell, \sigma, \partial\Omega)$.

Proof. The proof of Theorem 20.1 is divided into three steps.

Step 1: First, the *uniqueness result* of problem (20.3) follows from an application of the Bakel'man and Aleksandrov maximum principle (see [43, Corollary 2.4]):

Theorem 20.2 (Bakel'man–Aleksandrov)**.** *Assume that a function* $u \in W^{2,p}(\Omega)$*,* $n < p < \infty$*, satisfies the conditions*

$$\begin{cases} \mathcal{L}u \leq 0 & \textit{almost everywhere in } \Omega, \\ \mathcal{B}u \leq 0 & \textit{on } \partial\Omega. \end{cases}$$

Then it follows that $u(x) \geq 0$ *in* Ω*.*

Proof. First, it should be noticed that we have, by Sobolev's imbedding theorem (Theorem 7.4),

$$W^{2,p}(\Omega) \subset C^1(\overline{\Omega}),$$

since $2 - n/p > 1$ for $n < p < \infty$.

(1) If $u(x)$ is a constant function in Ω, then it follows that

$$\sigma(x')u = \mathcal{B}u \leq 0 \quad \text{on } \partial\Omega.$$

This proves that $u(x) \geq 0$ in Ω, since $\sigma(x') < 0$ on $\partial\Omega$.

(2) Now we consider the case where $u(x)$ is not a constant function in Ω. We assume, to the contrary, that $u(x)$ takes a negative minimum at a point $x_0 \in \overline{\Omega}$. Then, by applying the weak maximum principle (Theorem 8.5) to the function $-u(x)$ we obtain that

$$0 < -u(x_0) = \max_{\overline{\Omega}}(-u) \leq \max_{\partial\Omega}(-u)^+.$$

Hence we have, for some point $x_0' \in \partial\Omega$,

$$-u(x_0') = -u(x_0) = \max_{\overline{\Omega}}(-u) > 0.$$

Moreover, it follows from an application of Hopf's boundary point lemma (Lemma 8.7) that

$$-\frac{\partial u}{\partial \mathbf{n}}(x_0') = \frac{\partial(-u)}{\partial \mathbf{n}}(x_0') < 0,$$

so that

$$\frac{\partial u}{\partial \mathbf{n}}(x_0') > 0.$$

However, by conditions (16.3b) and (16.3c) this implies that

$$0 \geq \mathcal{B}u(x_0') = \frac{\partial u}{\partial \ell}(x_0') + \sigma(x_0')u(x_0') > 0.$$

This contradiction proves that $u(x) \geq 0$ in Ω.

The proof of Theorem 20.2 is complete. $\square$

By applying Theorem 20.2 to the functions $\pm u(x)$, we obtain from condition (16.3c) that

$$\begin{cases} \mathcal{L}u = 0 & \text{almost everywhere in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega \end{cases}$$
$$\Longrightarrow \quad u = 0 \quad \text{in } \Omega.$$

Namely, the mapping

$$\mathcal{A} = (\mathcal{L}, \mathcal{B}) : W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \oplus B^{1-1/p,p}(\partial\Omega)$$

is *injective* for $n < p < \infty$. This proves the uniqueness of solutions of the homogeneous oblique derivative problem (20.3).

Step 2: In order to prove the *existence result* of problem (20.3), we make use of the method of continuity (Theorem 2.14). We shall apply Theorem 2.14 with

$$\begin{aligned} \mathfrak{B} &:= \left\{ u \in W^{2,p}(\Omega) : \mathcal{B}u = 0 \right\}, \\ \mathfrak{V} &:= L^p(\Omega), \\ \mathcal{L}_1 &:= \mathcal{L} = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \\ \mathcal{L}_0 &:= \Delta. \end{aligned}$$

Here it should be noticed that the space $\mathfrak{B}$ is a closed subspace of $W^{2,p}(\Omega)$, since the trace theorem (Theorem 7.4) asserts that the boundary operator

$$\mathcal{B} = \frac{\partial}{\partial \ell} + \sigma(x') : W^{2,p}(\Omega) \longrightarrow B^{1-1/p,p}(\partial\Omega)$$

is continuous.

Step 2-1: The essential step in our proof is how to show inequality (2.12) for a family of elliptic differential operators

$$\begin{aligned} \mathcal{L}_t &:= t\mathcal{L} + (1-t)\Delta \\ &= \sum_{i,j=1}^{n} \left(t\, a^{ij}(x) + (1-t)\delta^{ij} \right) \frac{\partial^2}{\partial x_i \partial x_j} \quad \text{for } 0 \le t \le 1. \end{aligned}$$

More precisely, we consider, instead of the original oblique derivative problem (20.3)

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases}$$

a family of oblique derivative problems

$$\begin{cases} \mathcal{L}_t u = (t\mathcal{L} + (1-t)\Delta)\, u = f & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega. \end{cases}$$

By the uniqueness result of problem (20.3), we can get rid of the term $\|u\|_{L^p(\Omega)}$ on the right-hand side of estimate (20.4). Namely, we shall prove the *a priori* estimate (corresponding to inequality (2.12))

$$\|u\|_{W^{2,p}(\Omega)} \leq c_2 \|\mathcal{L}_t u\|_{L^p(\Omega)} \quad \text{for all } u \in \mathfrak{B}. \tag{20.13}$$

Here $c_2 > 0$ is a constant depending on the coefficients a^{ij} only through the ellipticity constant λ, the bound on the norms $\|a^{ij}\|_{L^p(\Omega)}$ and the VMO moduli of the a^{ij}. First, it should be noticed that the coefficients

$$a^{ij}_{(t)}(x) := t\, a^{ij}(x) + (1-t)\delta^{ij}, \quad 0 \leq t \leq 1,$$

satisfy the following three conditions (i), (ii) and (iii): ,

(i) $a^{ij}_{(t)}(x) \in \mathrm{VMO} \cap L^\infty(\mathbf{R}^n)$. Indeed, we have, for all $0 \leq t \leq 1$,

$$\|a^{ij}_{(t)}\|_{L^\infty(\mathbf{R}^n)} \leq \|a^{ij}\|_{L^\infty(\mathbf{R}^n)} + 1,$$

and

$$\|a^{ij}_{(t)}\|_* \leq t\|a^{ij}\|_* \leq \|a^{ij}\|_*.$$

(ii) $a^{ij}_{(t)}(x) = a^{ji}_{(t)}(x)$ for almost all $x \in \Omega$ and $1 \leq i, j \leq n$.

(iii) We have, for almost all $x \in \Omega$ and all $\xi \in \mathbf{R}^n$,

$$\frac{1}{\lambda+1}|\xi|^2 \leq \sum_{i,j=1}^n a^{ij}_{(t)}(x)\xi_i\xi_j \leq (\lambda+1)|\xi|^2,$$

where λ is the same constant as in condition (16.1).

To prove estimate (20.13), we assume, to the contrary, that estimate (20.13) does not hold true. Then we can find a sequence of elliptic differential operators

$$\mathcal{L}^{(m)} = \sum_{i,j=1}^n a^{ij}_{(m)}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \quad m = 1, 2, \ldots,$$

and a sequence of functions

$$u^{(m)} \in \mathfrak{B} = \left\{ u \in W^{2,p}(\Omega) : \mathcal{B}u = 0 \right\}, \quad m = 1, 2, \ldots,$$

such that the coefficients $a^{ij}_{(m)}$ and the functions $u^{(m)}$ satisfy the following four conditions (i), (ii), (iii) and (iv):

(i) $a^{ij}_{(m)}(x) \in \mathrm{VMO} \cap L^\infty(\mathbf{R}^n)$ for $1 \le i, j \le n$ and

$$\|a^{ij}_{(m)}\|_{L^\infty(\mathbf{R}^n)} \le \|a^{ij}\|_{L^\infty(\mathbf{R}^n)} + 1, \tag{20.14a}$$

$$\eta^{ij}_{(m)}(r) \le \eta^{ij}(r). \tag{20.14b}$$

Here we recall that

$$\eta^{ij}_{(m)}(r) = \sup_{\rho \le r} \frac{1}{|B|} \int_B \left| a^{ij}_{(m)}(y) - \left(a^{ij}_{(m)}\right)_B \right| dy,$$

$$\eta^{ij}(r) = \sup_{\rho \le r} \frac{1}{|B|} \int_B \left| a^{ij}(y) - \left(a^{ij}\right)_B \right| dy,$$

where the supremum is taken over all balls B with radius $\rho \le r$.

(ii) $a^{ij}_{(m)}(x) = a^{ji}_{(m)}(x)$ for almost all $x \in \Omega$ and $1 \le i, j \le n$.

(iii) We have, for almost all $x \in \Omega$ and all $\xi \in \mathbf{R}^n$,

$$\frac{1}{\lambda + 1}|\xi|^2 \le \sum_{i,j=1}^n a^{ij}_{(m)}(x)\xi_i\xi_j \le (\lambda + 1)|\xi|^2. \tag{20.15}$$

(iv) $u^{(m)} \in W^{2,p}(\Omega)$ and

$$Bu^{(m)} = 0 \quad \text{on } \partial\Omega, \tag{20.16a}$$

$$\|u^{(m)}\|_{W^{2,p}(\Omega)} = 1, \tag{20.16b}$$

$$\|\mathcal{L}^{(m)}u^{(m)}\|_{L^p(\Omega)} \longrightarrow 0. \tag{20.16c}$$

Step 2-1a: First, it follows that the sequence $\{(a^{ij}_{(m)})_B\}$ is bounded for every $1 \le i, j \le n$. Indeed, we have, by inequality (15.9a),

$$\left|(a^{ij}_{(m)})_B\right| \le \|a^{ij}_{(m)}\|_{L^\infty(\mathbf{R}^n)} \le \|a^{ij}\|_{L^\infty(\mathbf{R}^n)} + 1.$$

Moreover, we have the following lemma:

Lemma 20.3. *For any ball B in $\mathbf{R}^n$, every sequence*

$$\left\{a^{ij}_{(m)}(x) - \left(a^{ij}_{(m)}\right)_B\right\}, \quad 1 \le i, j \le n,$$

is compact *in the space $L^1(B)$.*

Proof. First, we take a bell-shaped function $\varphi(x)$ on $\mathbf{R}^n$ which satisfies the following four conditions:

$$\varphi(x) \in C_0^\infty(\mathbf{R}^n).$$
$$\varphi(x) \ge 0 \quad \text{on } \mathbf{R}^n.$$
$$\operatorname{supp} \varphi \subset B(0, 1) := \{x \in \mathbf{R}^n : |x| \le 1\}.$$

$$\int_{\mathbf{R}^n} \varphi(x)\, dx = 1.$$

For each $\varepsilon > 0$, we let

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n} \varphi\left(\frac{x}{\varepsilon}\right).$$

Then it is easy to verify that

$$\begin{aligned}
&\varphi_\varepsilon(x) \in C_0^\infty(\mathbf{R}^n). \\
&\varphi_\varepsilon(x) \geq 0 \quad \text{on } \mathbf{R}^n. \\
&\operatorname{supp} \varphi_\varepsilon \subset B(0,\varepsilon) := \{x \in \mathbf{R}^n : |x| \leq \varepsilon\}, \\
&\int_{\mathbf{R}^n} \varphi_\varepsilon(x)\, dx = 1.
\end{aligned}$$

The functions $\{\varphi_\varepsilon\}$ are called *Friedrichs' mollifiers.*

(1) For simplicity, we write

$$a_{(m)}(x) := a^{ij}_{(m)}(x),$$

and let

$$f_{(m)}(x) := a_{(m)}(x) - \left(a_{(m)}\right)_B, \tag{20.17a}$$

$$f^\varepsilon_{(m)}(x) := a_{(m)} * \varphi_\varepsilon(x) - \left(a_{(m)} * \varphi_\varepsilon\right)_B, \tag{20.17b}$$

where

$$a_{(m)} * \varphi_\varepsilon(x) = \int_{\mathbf{R}^n} a_{(m)}(x-y)\varphi_\varepsilon(y)\, dy$$

is a mollification of $a_{(m)}$.

Then we can prove the following two claims 20.1 and 20.2 (see Claims 15.1 and 15.2):

Claim 20.1. The sequence $\{f^\varepsilon_{(m)}\}$ is *uniformly bounded* and *equicontinuous* in B, for each $\varepsilon > 0$.

Proof. (a) The *uniform boundedness* of $\{f^\varepsilon_{(m)}\}$: First, we have, for all $x \in B$,

$$\begin{aligned}
|a_{(m)} * \varphi_\varepsilon(x)| &\leq \int_{\mathbf{R}^n} |a_{(m)}(x-y)|\, \varphi_\varepsilon(y)\, dy \leq \|a_{(m)}\|_{L^\infty(\mathbf{R}^n)} \\
&\leq \|a\|_{L^\infty(\mathbf{R}^n)} + 1,
\end{aligned}$$

and

$$\left|\left(a_{(m)} * \varphi_\varepsilon\right)_B\right| \leq \frac{1}{|B|} \int_{\mathbf{R}^n} |a_{(m)}(x-y)|\, \varphi_\varepsilon(y)\, dy \leq \|a_{(m)}\|_{L^\infty(\mathbf{R}^n)}$$

$$\leq \|a\|_{L^\infty(\mathbf{R}^n)} + 1.$$

Hence it follows that

$$\begin{aligned}|f^\varepsilon_{(m)}(x)| &\leq |a_{(m)} * \varphi_\varepsilon(x)| + |\left(a_{(m)} * \varphi_\varepsilon\right)_B| \\ &\leq 2\left(\|a\|_{L^\infty(\mathbf{R}^n)} + 1\right) \quad \text{for all } x \in B.\end{aligned}$$

(b) The *equicontinuity* of $\{f^\varepsilon_{(m)}\}$: It suffices to show that $\{\nabla f^\varepsilon_{(m)}\}$ is uniformly bounded in B, for each $\varepsilon > 0$.

We have, for all $x \in B$,

$$\begin{aligned}&|\nabla f^\varepsilon_{(m)}(x)| \qquad (20.18)\\ &\leq \frac{1}{\varepsilon^{n+1}} \int_{B(x,\varepsilon)} \left|a_{(m)}(y) - \left(a_{(m)}\right)_{B(x,\varepsilon)}\right| \left|\nabla\varphi\left(\frac{x-y}{\varepsilon}\right)\right| dy \\ &\leq \frac{1}{\varepsilon^{n+1}} \left(\sup_{\mathbf{R^n}} |\nabla\varphi|\right) \int_{B(x,\varepsilon)} \left|a_{(m)}(y) - \left(a_{(m)}\right)_{B(x,\varepsilon)}\right| dy \\ &\leq \frac{1}{\varepsilon^{n+1}} \left(\sup_{\mathbf{R^n}} |\nabla\varphi|\right) |B(x,\varepsilon)|\, \eta_{(m)}(\varepsilon) \\ &= \frac{1}{\varepsilon^{n+1}} \left(\sup_{\mathbf{R^n}} |\nabla\varphi|\right) (\varepsilon^n \omega_n)\, \eta_{(m)}(\varepsilon) \\ &\leq \frac{\omega_n}{\varepsilon} \left(\sup_{\mathbf{R^n}} |\nabla\varphi|\right) \eta(\varepsilon).\end{aligned}$$

Here we recall that

$$\begin{aligned}\omega_n &= \frac{2\pi^{n/2}}{\Gamma(n/2)}, \\ \eta_{(m)}(r) = \eta^{ij}_{(m)}(r) &:= \sup_{\rho \leq r} \frac{1}{|B|} \int_B \left|a^{ij}_{(m)}(y) - \left(a^{ij}_{(m)}\right)_B\right| dy, \\ \eta(r) &:= \left(\sum_{i,j=1}^n \eta^{ij}_{(m)}(r)^2\right)^{1/2}.\end{aligned}$$

Thus, we obtain from inequality (20.14) that $\{\nabla f^\varepsilon_{(m)}\}$ is uniformly bounded in B, for each $\varepsilon > 0$. □

Claim 20.2. There exists a positive constant C, independent of ε, such that

$$\int_B \left|f^\varepsilon_{(m)}(x) - f_{(m)}(x)\right| dx \leq C|B|\eta(\varepsilon). \tag{20.19}$$

Proof. First, by Fubini's theorem (Theorem 3.10) it follows that

$$\left(a_{(m)} * \varphi_\varepsilon\right)_B = \frac{1}{|B|} \int_B a_{(m)} * \varphi_\varepsilon(x)\, dx \tag{20.20}$$

$$
\begin{aligned}
&= \frac{1}{|B|}\int_B \left(\int_{B(0,\varepsilon)} a_{(m)}(x-y)\varphi_\varepsilon(y)\,dy\right) dx \\
&= \int_{B(0,\varepsilon)} \left(\frac{1}{|B|}\int_B a_{(m)}(x-y)\,dx\right)\varphi_\varepsilon(y)\,dy \\
&= \int_{B(0,\varepsilon)} \left(\frac{1}{|B-y|}\int_{B-y} a_{(m)}(z)\,dz\right)\varphi_\varepsilon(y)\,dy \\
&= \int_{B(0,\varepsilon)} \big(a_{(m)}\big)_{B-y}\,\varphi_\varepsilon(y)\,dy,
\end{aligned}
$$

where $B-y$ is the translation of the ball B by y-units

$$B - y = \{x - y : x \in B\}.$$

Hence we have, by formulas (20.17) and (20.20),

$$
\begin{aligned}
&\int_B \left| f^\varepsilon_{(m)}(x) - f_{(m)}(x)\right| dx \qquad (20.21)\\
&= \int_B \left|a_{(m)} * \varphi_\varepsilon(x) - \big(a_{(m)} * \varphi_\varepsilon\big)_B - \big(a_{(m)}(x) - \big(a_{(m)}\big)_B\big)\right| dx \\
&= \int_B \left|\int_{B(0,\varepsilon)} \big(a_{(m)}(x-y) - a_{(m)}(x)\big)\right. \\
&\qquad \left. - \Big(\big(a_{(m)}\big)_{B-y} - \big(a_{(m)}\big)_B\Big)\varphi_\varepsilon(y)\,dy\right| dx \\
&\le \int_B \int_{B(0,\varepsilon)} \left|\big(a_{(m)}(x-y) - a_{(m)}(x)\big)\right. \\
&\qquad \left. - \Big(\big(a_{(m)}\big)_{B-y} - \big(a_{(m)}\big)_B\Big)\right|\varphi_\varepsilon(y)\,dy\,dx \\
&\le \int_{B(0,\varepsilon)} \varphi_\varepsilon(y)\left(\int_B \left|\big(a_{(m)}(x-y) - a_{(m)}(x)\big)\right.\right. \\
&\qquad \left.\left. - \Big(\big(a_{(m)}\big)_{B-y} - \big(a_{(m)}\big)_B\Big)\right| dx\right) dy.
\end{aligned}
$$

However, we have the formula

$$
\begin{aligned}
\big(a_{(m)}\big)_{B-y} &= \frac{1}{|B-y|}\int_{B-y} a_{(m)}(w)\,dw = \frac{1}{|B|}\int_B a_{(m)}(z-y)\,dz \\
&= a_{(m)}(\cdot - y)_B.
\end{aligned}
$$

Hence it follows from an application of inequality (4.9) with $f := a_{(m)}$ that

$$\frac{1}{|B|}\int_B \left|\big(a_{(m)}(x-y) - a_{(m)}(x)\big) - \Big(\big(a_{(m)}\big)_{B-y} - \big(a_{(m)}\big)_B\Big)\right| dx \quad (20.22)$$

$$= \frac{1}{|B|} \int_B \left| \left(a_{(m)}(x-y) - a_{(m)}(x) \right) - \left(a_{(m)}(\cdot - y) - a_{(m)} \right)_B \right| dx$$
$$\leq \| a_{(m)}(\cdot - y) - a_{(m)}(\cdot) \|_*$$
$$\leq C\eta_{(m)}(\varepsilon), \quad |y| < \varepsilon.$$

Therefore, by combining inequalities (20.21) and (20.22) we obtain that

$$\begin{aligned} \int_B \left| f^{\varepsilon}_{(m)}(x) - f_{(m)}(x) \right| dx &\leq C \left(\int_{\{|y| \leq \varepsilon\}} \varphi_\varepsilon(y) \eta_{(m)}(\varepsilon) \, dy \right) |B| \\ &= C\, \eta_{(m)}(\varepsilon)\, |B| \\ &\leq C|B|\eta(\varepsilon) \quad \text{for all } \varepsilon > 0. \end{aligned}$$

This proves the desired inequality (20.19). □

(2) By combining Claims 20.1 and 20.2 and the Ascoli–Arzelá theorem, we find that the sequence $\{f_{(m)}\}$ is *totally bounded* in $L^1(B)$, so that it is compact in $L^1(B)$.

The proof of Lemma 20.3 is complete. □

Step 2-1b: By using Lemma 20.3 and the Bolzano–Weierstrass theorem, we can choose a subsequence of the sequence

$$a^{ij}_{(m)}(x) = \left(a^{ij}_{(m)}(x) - (a^{ij}_{(m)})_B \right) + (a^{ij}_{(m)})_B, \quad m = 1, 2, \ldots,$$

which converges almost everywhere in B. Therefore, by considering an exhaustive sequence of balls of $\mathbf{R}^n$ we can choose a subsequence of $a^{ij}_{(m)}$, denoted again by $a^{ij}_{(m)}$, which converges to a function α^{ij} almost everywhere in $\mathbf{R}^n$, as $m \to \infty$:

$$a^{ij}_{(m)}(x) \longrightarrow \alpha^{ij}(x) \quad \text{almost everywhere in } \mathbf{R}^n, \text{ as } m \to \infty. \tag{20.23}$$

Then it is easy to verify the following three assertions (i), (ii) and (iii):

(i) $\alpha^{ij}(x) \in \mathrm{VMO} \cap L^\infty(\mathbf{R}^n)$ for $1 \leq i, j \leq n$. Indeed, since we have the inequality

$$\|a^{ij}_{(m)}\|_{L^\infty(\mathbf{R}^n)} \leq \|a^{ij}\|_{L^\infty(\mathbf{R}^n)} + 1,$$

it follows that

$$\|\alpha^{ij}\|_{L^\infty(\mathbf{R}^n)} \leq \|a^{ij}\|_{L^\infty(\mathbf{R}^n)} + 1.$$

Moreover, since we have, for all balls B with radius $\rho \leq r$,

$$\frac{1}{|B|} \int_B |a^{ij}_{(m)}(x) - (a^{ij}_{(m)})_B| \, dx \leq \eta^{ij}_{(m)}(r) \leq \eta^{ij}(r),$$

and since we have, by the Lebesgue dominated convergence theorem (Theorem 3.9),

$$\begin{aligned}(a^{ij}_{(m)})_B &= \frac{1}{|B|}\int_B a^{ij}_{(m)}(y)\,dy \longrightarrow \frac{1}{|B|}\int_B \alpha^{ij}(y)\,dy \\ &= (\alpha^{ij})_B \quad \text{as } m \to \infty,\end{aligned}$$

it follows that, for all balls B with radius $\rho \le r$,

$$\begin{aligned}\frac{1}{|B|}\int_B |\alpha^{ij}(x) - (\alpha^{ij})_B|\,dx &= \lim_{m\to\infty} \frac{1}{|B|}\int_B |a^{ij}_{(m)}(x) - (a^{ij}_{(m)})_B|\,dx \\ &\le \eta^{ij}(r).\end{aligned}$$

This proves that the VMO modulus of α^{ij} is dominated by $\eta^{ij}(r)$:

$$\sup_{\rho\le r} \frac{1}{|B|}\int_B |\alpha^{ij}(x) - (\alpha^{ij})_B|\,dx \le \eta^{ij}(r).$$

(ii) $\alpha^{ij}(x) = \alpha^{ji}(x)$ for almost all $x \in \Omega$ and $1 \le i, j \le n$. Indeed, we have, for almost all $x \in \Omega$,

$$\alpha^{ij}(x) = \lim_{m\to\infty} a^{ij}_{(m)}(x) = \lim_{m\to\infty} a^{ji}_{(m)}(x) = \alpha^{ji}(x).$$

(iii) We have, for almost all $x \in \Omega$ and all $\xi \in \mathbf{R}^n$,

$$\frac{1}{\lambda+1}|\xi|^2 \le \sum_{i,j=1}^{n} \alpha^{ij}(x)\xi_i\xi_j \le (\lambda+1)|\xi|^2.$$

These inequalities can be obtained by passing to the limit in inequalities (20.15).

We introduce a new second-order, uniformly elliptic differential operator A by the formula

$$\mathcal{A} := \sum_{i,j=1}^{n} \alpha^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}.$$

Step 2-1c: By using the Eberlein–Shmulyan theorem (Theorem 2.5) and the Rellich–Kondrachov theorem (Theorem 7.6), we can obtain the following two assertions (A) and (B):

(A) Let X be a reflexive Banach space, and let $\{x_n\}$ be any sequence which is norm bounded. Then there exists a subsequence $\{x_{n'}\}$ which converges *weakly* to an element of X.

(B) The injection $W^{2,p}(\Omega) \to W^{1,p}(\Omega)$ is *compact*.

Therefore, we can find a subsequence of $\{u^{(m)}\}$, denoted again by $\{u^{(m)}\}$, which converges weakly to a function

$$v \in W^{2,p}(\Omega)$$

in the space $W^{2,p}(\Omega)$ and also converges strongly to v in the space $L^p(\Omega)$:

$$\begin{aligned} u^{(m)} &\longrightarrow v \quad \text{weakly in } W^{2,p}(\Omega) \text{ as } m \to \infty, \\ u^{(m)} &\longrightarrow v \quad \text{in } L^p(\Omega) \text{ as } m \to \infty. \end{aligned}$$

We shall prove that $v = 0$, that is,

$$u^{(m)} \longrightarrow 0 \quad \text{weakly in } W^{2,p}(\Omega) \text{ as } m \to \infty, \tag{20.24a}$$

$$u^{(m)} \longrightarrow 0 \quad \text{in } L^p(\Omega) \text{ as } m \to \infty. \tag{20.24b}$$

(I) First, we prove that

$$Bv = 0 \quad \text{on } \partial\Omega. \tag{20.25}$$

To do this, it should be noticed that the set

$$\mathfrak{B} = \left\{u \in W^{2,p}(\Omega) : \mathcal{B}u = 0 \text{ on } \partial\Omega\right\}$$

is balanced, convex and *strongly closed* in $W^{2,p}(\Omega)$. Indeed, it suffices to note that if $\{u_j\}$ is any sequence in $\mathfrak{B}$ which converges strongly to a function u in $W^{2,p}(\Omega)$, then we have, by the continuity of $\mathcal{B} : W^{2,p}(\Omega) \to B^{1-1/p,p}(\partial\Omega)$ (Theorem 7.4),

$$\mathcal{B}u = \lim_{j\to\infty} \mathcal{B}u_j = 0 \quad \text{in } B^{1-1/p,p}(\partial\Omega).$$

Hence, it follows that the set $\mathfrak{B}$ is *weakly closed* in $W^{2,p}(\Omega)$, by applying Mazur's theorem (Theorem 2.25) with

$$X := W^{2,p}(\Omega), \quad M := \mathfrak{B}.$$

Therefore, we obtain assertion (20.25), that is,

$$v \in \mathfrak{B}.$$

Indeed, it suffices to note by assertion (20.16a) that $u^{(m)} \in \mathfrak{B}$ and further that $\{u^{(m)}\}$ converges weakly to $v \in W^{2,p}(\Omega)$.

(II) Next, let φ be an arbitrary function in $L^q(\Omega)$ with $q = p/(p-1)$. Then we have, by Hölder's inequality (Theorem 3.14) and condition (20.16b),

$$\left| \int_\Omega \left(\mathcal{L}^{(m)} u^{(m)} - \mathcal{A}v \right) \varphi \, dx \right| \tag{20.26}$$

$$
\begin{aligned}
&= \left| \sum_{i,j=1}^{n} \int_\Omega \left(a^{ij}_{(m)}(x) \frac{\partial^2 u^{(m)}}{\partial x_i \partial x_j} - \alpha^{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} \right) \varphi \, dx \right| \\
&\leq \sum_{i,j=1}^{n} \left| \int_\Omega \left(a^{ij}_{(m)}(x) \frac{\partial^2 u^{(m)}}{\partial x_i \partial x_j} - \alpha^{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} \right) \varphi \, dx \right| \\
&\leq \sum_{i,j=1}^{n} \left| \int_\Omega \left(a^{ij}_{(m)}(x) - \alpha^{ij}(x) \right) \frac{\partial^2 u^{(m)}}{\partial x_i \partial x_j} \, \varphi \, dx \right| \\
&\quad + \sum_{i,j=1}^{n} \left| \int_\Omega \alpha^{ij}(x) \left(\frac{\partial^2 u^{(m)}}{\partial x_i \partial x_j} - \frac{\partial^2 v}{\partial x_i \partial x_j} \right) \varphi \, dx \right| \\
&\leq \sum_{i,j=1}^{n} \left\| \nabla^2 u^{(m)} \right\|_{L^p(\Omega)} \left\| \left(a^{ij}_{(m)} - \alpha^{ij} \right) \varphi \right\|_{L^q(\Omega)} \\
&\quad + \sum_{i,j=1}^{n} \left| \int_\Omega \alpha^{ij}(x) \varphi(x) \cdot \left(\frac{\partial^2 u^{(m)}}{\partial x_i \partial x_j} - \frac{\partial^2 v}{\partial x_i \partial x_j} \right) dx \right| \\
&\leq \sum_{i,j=1}^{n} \left\| \left(a^{ij}_{(m)} - \alpha^{ij} \right) \varphi \right\|_{L^q(\Omega)} \\
&\quad + \sum_{i,j=1}^{n} \left| \int_\Omega \alpha^{ij}(x) \varphi(x) \cdot \left(\frac{\partial^2 u^{(m)}}{\partial x_i \partial x_j} - \frac{\partial^2 v}{\partial x_i \partial x_j} \right) dx \right| .
\end{aligned}
$$

However, by assertions (20.14a) and (20.23) it follows from an application of the Lebesgue dominated convergence theorem (Theorem 3.9) that the first term on the last inequality (20.26) tends to zero as $m \to \infty$:

$$
\sum_{i,j=1}^{n} \left\| \left(a^{ij}_{(m)} - \alpha^{ij} \right) \varphi \right\|_{L^q(\Omega)} \longrightarrow 0.
$$

Moreover, we recall that $\{u^{(m)}\}$ converges weakly to a function $v \in W^{2,p}(\Omega)$. Since $\alpha^{ij}(x)\varphi(x) \in L^q(\Omega)$, we find that the second term on the last inequality (20.26) tends to zero as $m \to \infty$:

$$
\sum_{i,j=1}^{n} \left| \int_\Omega \alpha^{ij}(x) \varphi(x) \cdot \left(\frac{\partial^2 u^{(m)}}{\partial x_i \partial x_j} - \frac{\partial^2 v}{\partial x_i \partial x_j} \right) dx \right| \longrightarrow 0.
$$

Hence we have, by inequality (20.26),

$$
\int_\Omega \mathcal{L}^{(m)} u^{(m)} \cdot \varphi \, dx \longrightarrow \int_\Omega \mathcal{A} v \cdot \varphi \, dx \quad \text{as } m \to \infty.
$$

On the other hand, by Hölder's inequality (Theorem 3.14) and assertion

(20.16c) it follows that

$$\left| \int_\Omega \mathcal{L}^{(m)} u^{(m)} \cdot \varphi \, dx \right| \leq \|\mathcal{L}^{(m)} u^{(m)}\|_{L^p(\Omega)} \cdot \|\varphi\|_{L^q(\Omega)} \longrightarrow 0 \quad \text{as } m \to \infty.$$

This proves that

$$\int_\Omega \mathcal{A}v \cdot \varphi \, dx = \lim_{n\to\infty} \int_\Omega \mathcal{L}^{(m)} u^{(m)} \cdot \varphi \, dx = 0 \quad \text{for all } \varphi \in L^q(\Omega).$$

Summing up, we have proved that

$$\begin{aligned} &v \in W^{2,p}(\Omega), \\ &\mathcal{A}v = 0 \quad \text{in } \Omega, \\ &\mathcal{B}v = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

By applying the uniqueness result of problem (20.3) to the operator $\mathcal{A}$ (Step 1), we obtain that

$$v = 0 \quad \text{in } \Omega.$$

This proves the desired assertions (20.24).

Step 2-1d: By combining assertions (20.16c) and (20.24b), we have proved that

$$\begin{aligned} &\mathcal{L}^{(m)} u^{(m)} \longrightarrow 0 \quad \text{in } L^p(\Omega), \\ &u^{(m)} \longrightarrow 0 \quad \text{in } L^p(\Omega). \end{aligned}$$

Therefore, by applying estimate (20.4) to the operators $\{L^{(m)}\}$ we obtain that

$$\|u^{(m)}\|_{W^{2,p}(\Omega)} \leq c_1 \left(\|\mathcal{L}^{(m)} u^{(m)}\|_{L^p(\Omega)} + \|u^{(m)}\|_{L^p(\Omega)} \right),$$

so that

$$u^{(m)} \longrightarrow 0 \quad \text{in } W^{2,p}(\Omega).$$

However, this assertion contradicts condition (20.16b):

$$\|u^{(m)}\|_{W^{2,p}(\Omega)} = 1.$$

This contradiction proves the desired *a priori* estimate (20.13).

Step 2-2: By virtue of estimate (20.13), we can apply Theorem 2.14 to obtain that the oblique derivative problem is uniquely solvable for the operator $\mathcal{L}_0$ if and only if it is uniquely solvable for the operator $\mathcal{L}_1$. However, it is known (see [93, Theorem 3.29]) that the oblique derivative problem is uniquely solvable for the Laplace operator $\mathcal{L}_0 = \Delta$: More precisely, for any $f \in L^p(\Omega)$ there exists a unique solution $u \in W^{2,p}(\Omega)$ of the oblique derivative problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega. \end{cases}$$

This proves that $\mathcal{L}_0 = \Delta$ maps $\mathfrak{B} := \left\{u \in W^{2,p}(\Omega) : \mathcal{B}u = 0\right\}$ onto $L^p(\Omega)$.

Therefore, it follows from an application of Theorem 2.14 (the method of continuity) that $\mathcal{L}_1 := \mathcal{L}$ maps $\mathfrak{B}$ *onto* $L^p(\Omega)$. Namely, for any $f \in L^p(\Omega)$ there exists a unique solution $u \in W^{2,p}(\Omega)$ of problem (20.3).

Step 3: Finally, the *a priori* estimate (20.12) follows from the *a priori* estimate (20.13) with $t := 1$.

Now the proof of Theorem 20.1 (and hence that of Theorem 16.2) is complete. □

20.3 Notes and Comments

Section 20.1: The proof of Theorem 16.1 is adapted from Chiarenza–Frasca–Longo [18], [19] and Maugeri–Palagachev–Softova [47]).

Section 20.2: The proof of Theorem 16.2 (Theorem 20.1) is essentially due to Di Fazio–Palagachev [22, Theorem 1.2].

Part VI

Construction of Feller Semigroups with Discontinuous Coefficients

21

Markov Processes and Feller Semigroups

This chapter is devoted to the functional analytic approach to the study of Markov processes.

In Section 21.1, we summarize the basic definitions and results about Markov processes, and formulate Markov processes in terms of transition functions. From the viewpoint of functional analysis, the transition function is something more convenient than the Markov process itself (Theorem 21.1). Indeed, we can associate with each transition function in a natural way a family of bounded linear operators acting on the space of continuous functions on the state space, and the so-called Markov property implies that this family forms a semigroup. The semigroup approach to Markov processes can be traced back to the pioneering work of Feller [26] and [27] in early 1950s.

Transition functions and their associated semigroups are studied in Section 21.2 (Theorem 21.8). These semigroups are called Feller semigroups.

In Section 21.3, by using the Hille–Yosida theory of semigroups we characterize Feller semigroups in terms of their infinitesimal generators. In particular, we prove a version of the Hille–Yosida theorem adapted to the present context (Theorem 21.9), which forms a functional analytic background for the proof of Theorem 1.1 (Theorem 21.1). The construction of Feller semigroups will be carried out in Chapter 25.

21.1 Markov Processes and Transition Functions

In this section, we summarize the basic definitions and results about Markov processes, and formulate Markov processes in terms of transition functions. From the viewpoint of functional analysis, the transition function is something more convenient than the Markov process itself,

21.1.1 Definition of a Markov Process

Let K be a locally compact, separable metric space and $\mathcal{B}$ the σ-algebra of all Borel sets in K, that is, the smallest σ-algebra containing all open sets in K. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function X defined on Ω taking values in K is called a *random variable* if it satisfies the condition

$$X^{-1}(E) = \{X \in E\} := \{\omega \in \Omega : X(\omega) \in E\} \in \mathcal{F} \quad \text{for all } E \in \mathcal{B}.$$

We express this by saying that X is $\mathcal{F}/\mathcal{B}$-measurable. A family $\{x_t\}_{t \geq 0}$ of random variables is called a *stochastic process*, and it may be thought of as the motion in time of a physical particle. The space K is called the *state space* and Ω the *sample space*. For a fixed $\omega \in \Omega$, the function $x_t(\omega)$ for $t \geq 0$ defines in the state space K a *trajectory* or *path* of the process corresponding to the sample point ω.

Now we introduce a class of *Markov processes* which we will deal with in this book.

Definition 21.1. Assume that we are given the following:

(1) A locally compact, separable metric space K and the σ-algebra $\mathcal{B}$ of all Borel sets in K. A point ∂ is adjoined to K as the *point at infinity* if K is not compact, and as an isolated point if K is compact. We let

$$\begin{aligned} K_\partial &= K \cup \{\partial\}, \\ \mathcal{B}_\partial &= \text{the } \sigma\text{-algebra in } K_\partial \text{ generated by } \mathcal{B}. \end{aligned}$$

(2) The space Ω of all mappings $\omega \colon [0, \infty] \to K_\partial$ such that $\omega(\infty) = \partial$ and that if $\omega(t) = \partial$ then $\omega(s) = \partial$ for all $s \geq t$. Let ω_∂ be the constant map $\omega_\partial(t) = \partial$ for all $t \in [0, \infty]$.

(3) For each $t \in [0, \infty]$, the coordinate map x_t defined by $x_t(\omega) = \omega(t)$, $\omega \in \Omega$.

(4) For each $t \in [0, \infty]$, a pathwise shift mapping $\theta_t : \Omega \to \Omega$ defined by the formula $\theta_t \omega(s) = \omega(t + s)$ for $\omega \in \Omega$. We remark that $\theta_\infty \omega = \omega_\partial$ and that $x_t \circ \theta_s = x_{t+s}$ for all t, $s \in [0, \infty]$.

(5) A σ-algebra $\mathcal{F}$ in Ω and an increasing family $\{\mathcal{F}_t\}_{0 \leq t \leq \infty}$ of sub-σ-algebras of $\mathcal{F}$.

(6) For each $x \in K_\partial$, a probability measure P_x on $(\Omega, \mathcal{F})$.

We say that these elements define a (temporally homogeneous) *Markov process* $\mathcal{X} = (x_t, \mathcal{F}, \mathcal{F}_t, P_x)$ if the following four conditions are satisfied:

(i) For each $0 \leq t < \infty$, the function x_t is $\mathcal{F}_t/\mathcal{B}_\partial$- measurable, that is,

$$\{x_t \in E\} \in \mathcal{F}_t \quad \text{for all } E \in \mathcal{B}_\partial.$$

(ii) For each $0 \leq t < \infty$ and $E \in \mathcal{B}$, the function

$$p_t(x, E) = P_x \{x_t \in E\}$$

is a Borel measurable function of $x \in K$.

(iii) $P_x\{\omega \in \Omega : x_0(\omega) = x\} = 1$ for each $x \in K_\partial$.

(iv) For all t, $h \in [0, \infty]$, $x \in K_\partial$ and $E \in \mathcal{B}_\partial$, we have the formula

$$P_x \{x_{t+h} \in E \mid \mathcal{F}_t\} = p_h(x_t, E) \quad \text{a. e.},$$

or equivalently

$$P_x(A \cap \{x_{t+h} \in E\}) = \int_A p_h(x_t(\omega), E)\, dP_x(\omega) \quad \text{for all } A \in \mathcal{F}_t.$$

Here is an intuitive way of thinking about the above definition of a Markov process. The sub-σ-algebra $\mathcal{F}_t$ may be interpreted as the collection of events which are observed during the time interval $[0, t]$. The value $P_x(A)$, $A \in \mathcal{F}$, may be interpreted as the probability of the event A under the condition that a particle starts at position x; hence the value $p_t(x, E)$ expresses the transition probability that a particle starting at position x will be found in the set E at time t. The function $p_t(x, \cdot)$ is called the *transition function* of the process $\mathcal{X}$. The transition function $p_t(x, \cdot)$ specifies the probability structure of the process. The intuitive meaning of the crucial condition (iv) is that the future behavior of a particle, knowing its history up to time t, is the same as the behavior of a particle starting at $x_t(\omega)$, that is, a particle starts afresh.

A Markovian particle moves in the space K until it "dies" or "disappear" at the time when it reaches the point ∂; hence the point ∂ is called the *terminal point* or *cemetery.* With this interpretation in mind, we let

$$\zeta(\omega) = \inf\{t \in [0, \infty] : x_t(\omega) = \partial\}.$$

The random variable ζ is called the *lifetime* of the process $\mathcal{X}$. The process $\mathcal{X}$ is said to be *conservative* if it satisfies the condition

$$P_x\{\zeta = \infty\} = 1 \quad \text{for all } x \in K.$$

21.1.2 Transition Functions

Our first job is to give the precise definition of a transition function adapted to the Hille–Yosida theory of semigroups:

Definition 21.2. Let (K, ρ) be a locally compact, separable metric space and $\mathcal{B}$ the σ-algebra of all Borel sets in K. A function $p_t(x, E)$, defined for all $t \geq 0$, $x \in K$ and $E \in \mathcal{B}$, is called a (temporally homogeneous) *Markov transition function* on K if it satisfies the following four conditions (a) through (d):

(a) $p_t(x, \cdot)$ is a non-negative measure on $\mathcal{B}$ and $p_t(x, K) \leq 1$ for each $t \geq 0$ and each $x \in K$.

(b) $p_t(\cdot, E)$ is a Borel measurable function for each $t \geq 0$ and each $E \in \mathcal{B}$.

(c) $p_0(x, \{x\}) = 1$ for each $x \in K$.

(d) (The Chapman–Kolmogorov equation) For any t, $s \geq 0$, any $x \in K$ and any $E \in \mathcal{B}$, we have the formula

$$p_{t+s}(x, E) = \int_K p_t(x, dy) p_s(y, E). \tag{21.1}$$

It is just condition (d) which reflects the Markov property that a particle starts afresh. Here is an intuitive way of thinking about the above definition of a Markov transition function. The value $p_t(x, E)$ expresses the transition probability that a physical particle starting at position x will be found in the set E at time t. Equation (21.1) expresses the idea that a transition from the position x to the set E in time $t + s$ is composed of a transition from x to some position y in time t, followed by a transition from y to the set E in the remaining time s; the latter transition has probability $p_s(y, E)$ which depends only on y. Thus a physical particle "starts afresh"; this property is called the *Markov property.*

A Markov transition function $p_t(x, \cdot)$ is said to be *normal* if it satisfies the condition

$$p_{+0}(x, K) = \lim_{t \downarrow 0} p_t(x, K) = 1 \quad \text{for all } x \in K.$$

The next theorem, due to Dynkin [24, Chapter 4, Section 2], justfies the definition of a transition function, and hence it will be fundamental for our further study of Markov processes:

Theorem 21.1. *For every Markov process, the function p_t, defined by the formula*

$$p_t(x, E) = P_x \{x_t \in E\} \quad \text{for } x \in K,\ E \in \mathcal{B} \text{ and } t \geq 0,$$

is a Markov transition function. Conversely, every normal Markov transition function corresponds to some Markov process.

21.1.3 Feller Transition Functions

Let (K, ρ) be a locally compact, separable metric space. Let $C(K)$ be the space of real-valued, bounded continuous functions on K; $C(K)$ is a Banach space with the supremum norm

$$\|f\|_\infty = \sup_{x \in K} |f(x)|.$$

We say that a function $f \in C(K)$ converges to 0 as $x \to \partial$ if, for each $\varepsilon > 0$, there exists a compact subset E of K such that

$$|f(x)| < \varepsilon \quad \text{for all } x \in K \setminus E,$$

and write

$$\lim_{x \to \partial} f(x) = 0.$$

Let $C_0(K)$ be the subspace of $C(K)$ which consists of all functions satisfying $\lim_{x \to \partial} f(x) = 0$; $C_0(K)$ is a closed subspace of $C(K)$. We remark that $C_0(K)$ may be identified with $C(K)$ if K is compact.

Now we introduce some condition on the measures $p_t(x, \cdot)$ related to continuity in $x \in K$ for every fixed $t \geq 0$.

Definition 21.3. A transition function p_t is called a *Feller function* if the function

$$T_t f(x) = \int_K p_t(x, dy) f(y)$$

is a continuous function of $x \in K$ whenever f is bounded and continuous on K, that is, $C(K)$ is an invariant space for the operators T_t. We say that p_t is a C_0-*function* if $C_0(K)$ is an invariant subspace of $C(K)$ for the operators T_t.

Remark 21.1. The Feller property is equivalent to saying that the measures $p_t(x, \cdot)$ depend continuously on $x \in K$ in the usual weak topology, for every fixed $t \geq 0$.

21.1.4 Path Functions of Markov Processes

A Markov process $\mathcal{X} = (x_t, \mathcal{F}, \mathcal{F}_t, P_x)$ is said to be *right-continuous* provided that, for each $x \in K$,

$$P_x\{\omega \in \Omega : \text{the mapping } t \mapsto x_t(\omega) \text{ is a right-continuous function from } [0,\infty) \text{ into } K_\partial\} = 1.$$

Furthermore, we say that $\mathcal{X}$ is *continuous* provided that, for each $x \in K$,

$$P_x\{\omega \in \Omega : \text{the mapping } t \mapsto x_t(\omega) \text{ is a continuous function from } [0,\zeta) \text{ into } K_\partial\} = 1.$$

Here ζ is the lifetime of the process $\mathcal{X}$.

Now we give some useful criteria for path-continuity in terms of transition functions (see Dynkin [24, Chapter 6], [25, Chapter 3, Section 2]):

Theorem 21.2. *Let p_t be a normal transition function on K.*

(i) Assume that the following two conditions (L) and (M) are satisfied:

(L) For each $s > 0$ and each compact $E \subset K$, we have the assertion

$$\lim_{x \to \partial} \sup_{0 \leq t \leq s} p_t(x, E) = 0.$$

(M) For each $\varepsilon > 0$ and each compact $E \subset K$, we have the assertion

$$\lim_{t \downarrow 0} \sup_{x \in E} p_t(x, K \setminus U_\varepsilon(x)) = 0,$$

where $U_\varepsilon(x) = \{y \in K : \rho(y,x) < \varepsilon\}$ is an ε-neighborhood of x.

Then there exists a Markov process $\mathcal{X}$ with transition function p_t whose paths are right-continuous on $[0,\infty)$ and have left-hand limits on $[0,\zeta)$ almost surely.

(ii) Assume that condition (L) and the following condition (N) (replacing condition (M)) are satisfied:

(N) For each $\varepsilon > 0$ and each compact $E \subset K$, we have the assertion

$$\lim_{t \downarrow 0} \frac{1}{t} \sup_{x \in E} p_t(x, K \setminus U_\varepsilon(x)) = 0.$$

Then there exists a Markov process $\mathcal{X}$ with transition function p_t whose paths are almost surely continuous on $[0, \zeta)$.

Remark 21.2. Some remarks are in order:

(1) Condition (L) is trivially satisfied if the state space K is compact.
(2) It is known (see Dynkin [24, Lemma 6.2]) that if the paths of a Markov process are right-continuous, then the transition function p_t satisfies the condition

$$\lim_{t \downarrow 0} p_t(x, U_\varepsilon(x)) = 1 \quad \text{for every } x \in K.$$

21.1.5 Strong Markov Processes

A Markov process is called a *strong Markov process* if the "starting afresh" property holds not only for every fixed moment but also for suitable random times.

We formulate precisely this "strong" Markov property. Let

$$\mathcal{X} = (x_t, \mathcal{F}, \mathcal{F}_t, P_x)$$

be a Markov process. A mapping $\tau : \Omega \to [0, \infty]$ is called a *stopping time* or *Markov time* with respect to $\{\mathcal{F}_t\}$ if it satisfies the condition

$$\{\tau \leq t\} \in \mathcal{F}_t \quad \text{for all } t \in [0, \infty).$$

Intuitively, this means that the events $\{\tau \leq t\}$ depend on the process only up to time t, but not on the "future" after time t. It should be noticed that any non-negative constant mapping is a stopping time.

If τ is a stopping time with respect to $\{\mathcal{F}_t\}$, we let

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \quad \text{for all } t \in [0, \infty)\}.$$

Intuitively, we may think of $\mathcal{F}_\tau$ as the "past" up to the random time τ. It is easy to verify that $\mathcal{F}_\tau$ is a σ-algebra. If $\tau \equiv t_0$ for some constant $t_0 \geq 0$, then $\mathcal{F}_\tau$ reduces to $\mathcal{F}_{t_0}$.

For each $t \in [0, \infty]$, we define a mapping

$$\Phi_t \colon [0, t] \times \Omega \longrightarrow K_\partial$$

by the formula

$$\Phi_t(s, \omega) = x_s(\omega).$$

A Markov process $\mathcal{X} = (x_t, \mathcal{F}, \mathcal{F}_t, P_x)$ is said to be *progressively measurable* with respect to $\{\mathcal{F}_t\}$ if the mapping Φ_t is $\mathcal{B}_{[0,t]} \times \mathcal{F}_t/\mathcal{B}_\partial$-measurable for each $t \in [0, \infty]$, that is, if we have the condition

$$\Phi_t^{-1}(E) = \{\Phi_t \in E\} \in \mathcal{B}_{[0,t]} \times \mathcal{F}_t \quad \text{for all } E \in \mathcal{B}_\partial.$$

Here $\mathcal{B}_{[0,t]}$ is the σ-algebra of all Borel sets in the interval $[0, t]$ and $\mathcal{B}_\partial$ is the σ-algebra in K_∂ generated by $\mathcal{B}$. It should be noticed that if $\mathcal{X}$ is progressively measurable and if τ is a stopping time, then the mapping $x_\tau \colon \omega \mapsto x_{\tau(\omega)}(\omega)$ is $\mathcal{F}_\tau/\mathcal{B}_\partial$- measurable.

Definition 21.4. We say that a progressively measurable Markov process $\mathcal{X} = (x_t, \mathcal{F}, \mathcal{F}_t, P_x)$ has the *strong Markov property* with respect to $\{\mathcal{F}_t\}$ if the following condition is satisfied:

For all $h \geq 0$, $x \in K_\partial$, $E \in \mathcal{B}_\partial$ and all stopping times τ, we have the formula

$$P_x\{x_{\tau+h} \in E \mid \mathcal{F}_\tau\} = p_h(x_\tau, E),$$

or equivalently,

$$P_x(A \cap \{x_{\tau+h} \in E\}) = \int_A p_h(x_{\tau(\omega)}(\omega), E)\, dP_x(\omega) \quad \text{for every } A \in \mathcal{F}_\tau.$$

This expresses the idea of "starting afresh" at random times.

The next result gives a useful criterion for the strong Markov property (see [24, Theorem 5.10]):

Theorem 21.3. *Every right-continuous Markov process has the strong Markov property if its transition function has C_0-property.*

We state a simple criterion for the strong Markov property in terms of transition functions. To do this, we introduce the following:

Definition 21.5. A transition function p_t on K is said to be *uniformly stochastically continuous* on K if it satisfies the following condition:

For each $\varepsilon > 0$ and each compact $E \subset K$, we have the assertion

$$\lim_{t \downarrow 0} \sup_{x \in E} [1 - p_t(x, U_\varepsilon(x))] = 0, \tag{21.2}$$

where $U_\varepsilon(x) = \{y \in K : \rho(y, x) < \varepsilon\}$ is an ε-neighborhood of x.

It should be emphasized that every uniformly stochastically continuous transition function is normal and satisfies condition (M) in Theorem 21.2. By combining part (i) of Theorem 21.2 and Theorem 21.3, we obtain the following (see [24, Theorem 6.3]):

Theorem 21.4. *If a uniformly stochastically continuous, C_0-transition function satisfies condition (L), then it is the transition function of some strong Markov process whose paths are right-continuous and have no discontinuities other than jumps.*

We remark that Theorem 21.4 can be visualized as follows:

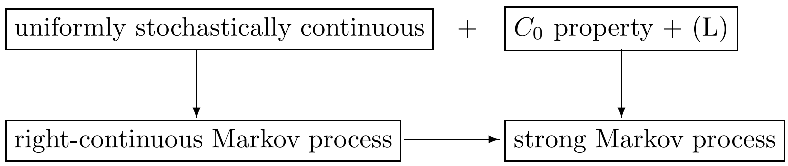

Fig. 21.1. A functional analytic approach to strong Markov processes in Theorem 21.4

A continuous strong Markov process is called a *diffusion process.*

The next result states a sufficient condition for the existence of a diffusion process with a prescribed transition function:

Theorem 21.5. *If a uniformly stochastically continuous, C_0-transition function satisfies conditions (L) and (N), then it is the transition function of some diffusion process.*

This is an immediate consequence of part (ii) of Theorem 21.2 and Theorem 21.4.

21.2 Transition Functions and Feller Semigroups

In this section we study the semigroups associated with Feller transition functions.

Let (K, ρ) be a locally compact, separable metric space and let $C(K)$ be the Banach space of real-valued, bounded continuous functions on K.

Then we have the following:

Theorem 21.6. *If p_t is a Feller transition function on K, then the associated operators $\{T_t\}_{t \geq 0}$, defined by the formula*

$$T_t f(x) = \int_K p_t(x, dy) f(y) \quad \text{for } f \in C(K), \tag{21.3}$$

form a non-negative and contraction semigroup on $C(K)$:

(i) $T_{t+s} = T_t \cdot T_s$, t, $s \geq 0$ (the semigroup property); $T_0 = I$.

(ii) $f \in C(K)$, $0 \leq f(x) \leq 1$ *on* $K \implies 0 \leq T_t f(x) \leq 1$ *on* K.

Conversely, if $\{T_t\}_{t \geq 0}$ *is a non-negative and contraction semigroup on the space* $C_0(K)$*, then there exists a unique* C_0*-transition function* p_t *on* K *such that formula* (21.3) *holds true.*

It should be emphasized that the C_0-property deals with continuity of a transition function $p_t(x, E)$ in x, and does not, by itself, have no concern with continuity in t.

Now we give a necessary and sufficient condition on $p_t(x, E)$ in order that its associated semigroup $\{T_t\}_{t \geq 0}$ is strongly continuous in t on the space $C_0(K)$ (cf. [25, Lemma 2.6]):

$$\lim_{s \downarrow 0} \|T_{t+s} f - T_t f\|_\infty = 0 \quad \text{for every } f \in C_0(K). \tag{21.4}$$

Theorem 21.7. *Let* p_t *be a* C_0*-transition function on* K*. Then the associated semigroup* $\{T_t\}_{t \geq 0}$*, defined by formula* (21.3)*, is strongly continuous in* t *on* $C_0(K)$ *if and only if* p_t *is uniformly stochastically continuous on* K *and satisfies condition (L) of Theorem 21.2.*

Remark 21.3. Since the semigroup $\{T_t\}_{t \geq 0}$ is a contraction semigroup, we find that the strong continuity (21.4) of $\{T_t\}$ in t for $t \geq 0$ is equivalent to the strong continuity at $t = 0$:

$$\lim_{t \downarrow 0} \|T_t f - f\|_\infty = 0 \quad \text{for every } f \in C_0(K). \tag{21.4$'$}$$

Definition 21.6. A family $\{T_t\}_{t \geq 0}$ of bounded linear operators acting on $C_0(K)$ is called a *Feller semigroup* on K if it satisfies the following three conditions (i), (ii) and (iii):

(i) $T_{t+s} = T_t \cdot T_s$, $t, s \geq 0$ (the semigroup property); $T_0 = I$.

(ii) $\{T_t\}$ is strongly continuous in t for $t \geq 0$:

$$\lim_{s \downarrow 0} \|T_{t+s} f - T_t f\|_\infty = 0, \quad f \in C_0(K).$$

(iii) $\{T_t\}$ is non-negative and contractive on $C_0(K)$:

$$f \in C_0(K),\ 0 \leq f(x) \leq 1 \quad \text{on } K \implies 0 \leq T_t f(x) \leq 1 \quad \text{on } K.$$

By combining Theorems 21.6 and 21.7, we obtain the following:

Theorem 21.8 (Dynkin)**.** *If* p_t *is a uniformly stochastically continuous,* C_0*-transition function on* K *which satisfies condition (L) of Theorem 21.2, then its associated operators* $\{T_t\}_{t \geq 0}$*, defined by formula* (21.3)*,* *form a Feller semigroup on K. Conversely, if $\{T_t\}_{t\geq 0}$ is a Feller semigroup on K, then there exists a uniformly stochastically continuous, C_0-transition p_t on K, satisfying condition (L), such that formula* (21.3) *holds.*

21.3 Feller Semigroups and their Infinitesimal Generators

Let K be a locally compact, separable metric space. If $\{T_t\}_{t\geq 0}$ is a Feller semigroup on K, we define its infinitesimal generator $\mathfrak{A}$ by the formula

$$\mathfrak{A}u = \lim_{t\downarrow 0} \frac{T_t u - u}{t}, \tag{21.5}$$

provided that the limit (21.5) exists in $C_0(K)$. More precisely, the generator $\mathfrak{A}$ is a linear operator from $C_0(K)$ into itself defined as follows.

(1) The domain $D(\mathfrak{A})$ of $\mathfrak{A}$ is the set

$$D(\mathfrak{A}) = \{u \in C_0(K) : \text{the limit (21.5) exists}\}.$$

(2) $\mathfrak{A}u = \lim_{t\downarrow 0} \dfrac{T_t u - u}{t}$ for every $u \in D(A)$.

The next theorem is a version of the Hille–Yosida theorem adapted to the present context:

Theorem 21.9 (Hille–Yosida)**.** *(i) Let $\{T_t\}_{t\geq 0}$ be a Feller semigroup on K and let $\mathfrak{A}$ be its infinitesimal generator. Then we have the following four assertions (a), (b), (c) and (d):*

(a) The domain $D(\mathfrak{A})$ is dense *in the space $C_0(K)$.*

(b) For each $\alpha > 0$, the equation $(\alpha I - \mathfrak{A})u = f$ has a unique solution u in $D(\mathfrak{A})$ for any $f \in C_0(K)$. Hence, for each $\alpha > 0$, the Green operator *$(\alpha I - \mathfrak{A})^{-1} : C_0(K) \to C_0(K)$ can be defined by the formula*

$$u = (\alpha I - \mathfrak{A})^{-1} f, \quad f \in C_0(K).$$

(c) For each $\alpha > 0$, the operator $(\alpha I - \mathfrak{A})^{-1}$ is non-negative on $C_0(K)$:

$$f \in C_0(K),\ f(x) \geq 0 \quad \text{on } K \implies (\alpha I - A)^{-1} f(x) \geq 0 \quad \text{on } K.$$

(d) For each $\alpha > 0$, the operator $(\alpha I - \mathfrak{A})^{-1}$ is bounded on $C_0(K)$ with norm

$$\|(\alpha I - \mathfrak{A})^{-1}\| \leq \frac{1}{\alpha}.$$

(ii) Conversely, if $\mathfrak{A}$ *is a linear operator from* $C_0(K)$ *into itself satisfying condition (a) and if there is a constant* $\alpha_0 \geq 0$ *such that, for all* $\alpha > \alpha_0$*, conditions (b) through (d) are satisfied, then* $\mathfrak{A}$ *is the infinitesimal generator of some Feller semigroup* $\{T_t\}_{t \geq 0}$ *on* K.

Now, let K be a *compact* metric space. We remark that the space $C_0(K)$ may be identified with $C(K)$. Then we have the following:

Corollary 21.10. *Let* K *be a* compact *metric space and let* $\mathfrak{A}$ *be the infinitesimal generator of a Feller semigroup on* K. *Assume that the constant function* 1 *belongs to the domain* $D(\mathfrak{A})$ *of* $\mathfrak{A}$ *and that we have, for some constant* c,

$$\mathfrak{A}1 \leq -c \quad \text{on } K. \tag{21.6}$$

Then the operator $\mathfrak{A}' = \mathfrak{A} + cI$ *is the infinitesimal generator of some Feller semigroup* $\{T_t\}_{t \geq 0}$ *on* K.

Although Theorem 21.9 tells us precisely when a linear operator $\mathfrak{A}$ is the infinitesimal generator of some Feller semigroup, it is usually difficult to verify conditions (b) through (d). So we give useful criteria in terms of the *maximum principle* which have evolved from the ideas of Bony–Courrège–Priouret[11], Dynkin [25] and Sato–Ueno [62] (cf. [57]):

Theorem 21.11 (Hille–Yosida–Ray)**.** *Let* K *be a* compact *metric space. Then we have the following two assertions (i) and (ii):*

(i) Let B *be a linear operator from* $C(K) = C_0(K)$ *into itself, and assume that the following two conditions are satisfied:*

- (α) *The domain* $D(B)$ *of* B *is* dense *in* $C(K)$.
- (β) *There exists an open and dense subset* K_0 *of* K *such that if* $u \in D(B)$ *takes a* positive *maximum at a point* x_0 *of* K_0*, then we have the inequality*

$$Bu(x_0) \leq 0.$$

Then the operator B *is* closable *in* $C(K)$.

(ii) Let B *be as in part (i), and further assume that the following two conditions are satisfied:*

- (β') *If* $u \in D(B)$ *takes a* positive *maximum at a point* x' *of* K*, then we have the inequality*

$$Bu(x') \leq 0.$$

(γ) *For some* $\alpha_0 \geq 0$*, the range* $R(\alpha_0 I - B)$ *of* $\alpha_0 I - B$ *is* dense *in* $C(K)$.

Then the minimal closed extension $\overline{B}$ *of* B *is the infinitesimal generator of some Feller semigroup on* K.

Corollary 21.12. *Let* M *be a bounded linear operator on* $C(K)$ *into itself. If* $\mathfrak{A}$ *generates a Feller semigroup* $\{T_t\}_{t \geq 0}$ *on* K *and if either* M *or* $\mathfrak{A}' = \mathfrak{A} + M$ *satisfies condition* (β') *of Theorem 21.11, then the operator* $\mathfrak{A}'$ *is the infinitesimal generator of some Feller semigroup on* K.

21.4 Notes and Comments

For more leisurely treatments of Markov processes and Feller semigroups, the reader is referred to Blumenthal–Getoor [8], Dynkin [24], [25], Lamperti [41], Revuz–Yor [59] and also Taira [79].

22

Feller Semigroups with Dirichlet Condition

In this chapter we consider the Dirichlet problem for the diffusion operator with VMO coefficients in the framework of L^p Sobolev spaces, and prove an existence and uniqueness theorem for the Dirichlet problem (Theorem 22.2). The uniqueness result in Theorem 22.2 follows from a variant of the Bakel'man–Aleksandrov maximum principle in the framework of Sobolev spaces due to Bony [9] (Theorem 8.5). Moreover, we construct a Feller semigroup associated with absorption phenomenon at the boundary (see Theorem 1.3 and Figure 1.7).

22.1 Formulation of the Dirichlet Problem

Let Ω be a bounded domain in Euclidean space $\mathbf{R}^n$, $n \geq 3$, with boundary $\partial\Omega$ of class $C^{1,1}$. If $1 < p < \infty$ and if $k = 1$ or $k = 2$, we define the Sobolev space

$$\begin{aligned} W^{k,p}(\Omega) = &\text{ the space of (equivalence classes of) functions} \\ &\ u \in L^p(\Omega) \text{ whose derivatives } D^\alpha u,\ |\alpha| \leq k, \text{ in the} \\ &\ \text{sense of distributions are in } L^p(\Omega), \end{aligned}$$

and the boundary space

$$B^{k-1/p,p}(\partial\Omega) = \text{the space of the traces } \gamma_0 u \text{ of functions } u \in W^{k,p}(\Omega).$$

In the space $B^{k-1/p,p}(\partial\Omega)$, we introduce a norm

$$|\varphi|_{B^{k-1/p,p}(\partial\Omega)} = \inf\left\{ \|u\|_{W^{k,p}(\Omega)} : u \in W^{k,p}(\Omega),\ \gamma_0 u = \varphi \text{ on } \partial\Omega \right\}.$$

We recall that the space $B^{k-1/p,p}(\partial\Omega)$ is a Besov space (see the trace theorem (Theorem 7.4)).

Let A be a second-order, elliptic differential operator with real *discontinuous* coefficients of the form

$$Au := \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial u}{\partial x_i} + c(x)u.$$

Here the functions $a^{ij}(x)$, $b^i(x)$ and $c(x)$ satisfy the following three conditions (1), (2) and (3):

(1) $a^{ij}(x) \in \mathrm{VMO} \cap L^\infty(\Omega)$, $a^{ij}(x) = a^{ji}(x)$ for almost all $x \in \Omega$ and $1 \leq i, j \leq n$, and there exist a constant $\lambda > 0$ such that

$$\frac{1}{\lambda}|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i \xi_j \leq \lambda |\xi|^2$$

for almost all $x \in \Omega$ and all $\xi \in \mathbf{R}^n$.

(2) $b^i(x) \in L^\infty(\Omega)$ for $1 \leq i \leq n$.

(3) $c(x) \in L^\infty(\Omega)$ and $c(x) \leq 0$ for almost all $x \in \Omega$.

In this section we consider the following non-homogeneous Dirichlet boundary value problem: Given functions $f(x)$ and $\varphi(x')$ defined in Ω and on $\partial\Omega$, respectively, find a function $u(x)$ in Ω such that

$$\begin{cases} Au = f & \text{in } \Omega, \\ \gamma_0 u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{22.1}$$

The first main result of this chapter is stated as follows (cf. Vitanza [95, Theorem 2.2]):

Theorem 22.1 (the regularity theorem)**.** *Let* $1 < p < \infty$*. Assume that conditions* (16.1) *and* (16.3) *are satisfied. If a function* $u \in W^{2,q}(\Omega)$*,* $1 < q < p < \infty$*, is a solution of the Dirichlet problem* (22.1) *with* $f \in L^p(\Omega)$ *and* $\varphi \in B^{2-1/p,p}(\partial\Omega)$*, then it follows that* $u \in W^{2,p}(\Omega)$*. Moreover, we have the global* a priori *estimate*

$$\|u\|_{W^{2,p}(\Omega)} \leq C_1 \left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \|\varphi\|_{B^{2-1/p,p}(\partial\Omega)} \right), \tag{22.2}$$

with a positive constant $C_1 = C_1(n, p, \lambda, \eta, \ell, \sigma, \partial\Omega)$*.*

The second main result of this chapter is a generalization of Bony [9, Théorème 3] to the VMO case (cf. [95, Theorems 2.3 and 2.4]):

Theorem 22.2 (the existence and uniqueness theorem)**.** *Let* $1 < p < \infty$*. Assume that*

$$c(x) \leq 0 \quad \textit{for almost all } x \in \Omega.$$

Then, for any $f \in L^p(\Omega)$ *and any* $\varphi \in B^{2-1/p,p}(\partial\Omega)$ *there exists a unique solution* $u \in W^{2,p}(\Omega)$ *of the Dirichlet problem* (22.1).

Remark 22.1. Theorem 22.2 plays an essential role in the study of the existence of positive solutions of semilinear Dirichlet eigenvalue problems for diffusive logistic equations with discontinuous coefficients which model population dynamics in environments with spatial heterogeneity (see [76]).

If we associate with problem (22.1) a linear operator

$$\mathcal{A}_p = (A, \gamma_0) \colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \oplus B^{2-1/p,p}(\partial\Omega),$$

then we obtain from the trace theorem (Theorem 7.4) and Theorem 22.2 that the operator $\mathcal{A}_p$ is continuous and *bijective* for all $1 < p < \infty$.

22.2 Proof of Theorem 22.1

The proof of the regularity theorem for the Dirichlet problem (Theorem 22.1) is divided into three steps.

Step 1: For any $\varphi \in B^{2-1/p,p}(\partial\Omega)$, we can find a function $v \in W^{2,p}(\Omega)$ such that $\gamma_0 v = \varphi$ on $\partial\Omega$, and further that the mapping

$$B^{2-1/p,p}(\partial\Omega) \ni \varphi \longmapsto v \in W^{2,p}(\Omega) \tag{22.3}$$

is continuous (see Stein [67, Theorem]). On the other hand, it should be noticed (see [2, Theorem 5.37]) that the closure $W_0^{1,p}(\Omega)$ of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$ may be characterized as follows:

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega) : \gamma_0 u = 0 \text{ on } \partial\Omega \right\} \quad \text{for all } 1 < p < \infty.$$

Therefore, we have only to prove Theorem 22.1 in the case where $\varphi := 0$ and $1 < q < p < \infty$:

$$\begin{cases} u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), \\ Au = f \in L^p(\Omega) \end{cases} \Longrightarrow u \in W^{2,p}(\Omega). \tag{22.4}$$

Step 2: In order to prove assertion (22.4), we need the Sobolev imbedding theorems (Theorem 7.4):

$$W^{2,p}(\Omega) \subset \begin{cases} L^{nq/(n-2q)}(\Omega) & \text{if } 1 < q < n/2, \\ L^r(\Omega) \quad \text{for all } n/2 \leq r < \infty & \text{if } q = n/2, \\ L^\infty(\Omega) & \text{if } n/2 < q < \infty. \end{cases} \tag{22.5}$$

$$W^{1,q}(\Omega) \subset \begin{cases} L^{nq/(n-q)}(\Omega) & \text{if } 1 < q < n, \\ L^r(\Omega) \quad \text{for all } n \le r < \infty & \text{if } q = n, \\ L^\infty(\Omega) & \text{if } n < q < \infty. \end{cases} \tag{22.6}$$

We consider the following three cases (A), (B) and (C).

(A) The case where $1 < q < n/2$: By assertions (22.5) and (22.6), it follows that

$$\begin{cases} u \in W^{2,q}(\Omega) \subset L^{nq/(n-2q)}(\Omega), \\ \frac{\partial u}{\partial x_i} \in W^{1,q}(\Omega) \subset L^{nq/(n-q)}(\Omega) \quad \text{for all } 1 \le i \le n. \end{cases}$$

Hence we have the assertion

$$\begin{aligned} \mathcal{L}u &= \sum_{i,j=1}^n a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} \\ &= f - \sum_{i=1}^n b^i(x) \frac{\partial u}{\partial x_i} + c(x) u \in L^{nq/(n-q)}(\Omega). \end{aligned}$$

By applying the global regularity theorem (Theorem 12.2), we obtain that

$$u \in L^{p_1}(\Omega),$$

where

$$p_1 = \min\left\{p, \frac{nq}{n-q}\right\}.$$

It is easy to see that

$$p_1 > q,$$

and further that

$$p_1 - q > \frac{1}{n-1}.$$

Therefore, by making use of a standard "bootstrap argument" we can conclude that

$$u \in L^p(\Omega).$$

(B) The case where $n/2 \le q < n$: By assertions (22.5) and (22.6), it follows that

$$\begin{cases} u \in W^{2,q}(\Omega) \subset L^p(\Omega), \\ \frac{\partial u}{\partial x_i} \in W^{1,q}(\Omega) \subset L^{nq/(n-q)}(\Omega) \quad \text{for all } 1 \le i \le n. \end{cases}$$

Hence we have the assertion

$$\mathcal{L}u = f - \sum_{i=1}^{n} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u \in L^{nq/(n-q)}(\Omega).$$

Therefore, by making use of a standard "bootstrap argument" we can conclude that

$$u \in L^p(\Omega).$$

(C) The case where $n \leq q < \infty$: By assertions (22.5) and (22.6), it follows that

$$\begin{cases} u \in W^{2,q}(\Omega) \subset L^\infty(\Omega), \\ \frac{\partial u}{\partial x_i} \in W^{1,q}(\Omega) \subset L^p(\Omega) \quad \text{for all } 1 \leq i \leq n. \end{cases}$$

Hence we have the assertion

$$\mathcal{L}u = f - \sum_{i=1}^{n} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u \in L^p(\Omega).$$

By applying the global regularity theorem (Theorem 12.2), we obtain that

$$u \in L^p(\Omega).$$

Summing up, we have proved the desired assertion (22.4).

Step 3: Finally, it remains to prove the global *a priori* estimate (22.2).

First, by using the global *a priori* estimate (12.3) we obtain that

$$\|u\|_{W^{2,p}(\Omega)} \leq C_1 \left(\|u\|_{L^p(\Omega)} + \|\mathcal{L}u\|_{L^p(\Omega)} + \|\varphi\|_{B^{2-1/p,p}(\partial\Omega)} \right), \quad (22.7)$$

with a positive constant C_1. Indeed, it suffices to note that the mapping (22.3) is continuous.

On the other hand, we have the inequality

$$\begin{aligned} &\|\mathcal{L}u\|_{L^p(\Omega)} \qquad (22.8) \\ &\leq \left\| f - \sum_{i=1}^{n} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u \right\|_{L^p(\Omega)} \\ &\leq \|f\|_{L^p(\Omega)} + \sum_{i=1}^{n} \|b^i\|_{L^\infty(\Omega)} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)} + \|c\|_{L^\infty(\Omega)} \|u\|_{L^p(\Omega)}. \end{aligned}$$

Hence, by combining two inequalities (22.7) and (22.8) we obtain that

$$\begin{aligned} &\|u\|_{W^{2,p}(\Omega)} \qquad (22.9) \\ &\leq C_2 \left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \|\varphi\|_{B^{2-1/p,p}(\partial\Omega)} + \|\nabla u\|_{L^p(\Omega)} \right), \end{aligned}$$

with a positive constant C_2.

However, recall the *interpolation inequality* (13.17)

$$\|\nabla u\|_{L^p(\Omega)} \leq \varepsilon \, \|\nabla^2 u\|_{L^p(\Omega)} + \frac{C_3}{\varepsilon} \, \|u\|_{L^p(\Omega)} \quad \text{for all } \varepsilon > 0. \tag{22.10}$$

Therefore, the desired estimate (22.2) follows from inequalities (22.9) and (22.10) if we take

$$\varepsilon = \frac{1}{2C_2}.$$

Now the proof of Theorem 22.1 is complete. □

Let $\mathcal{A}_p$ be a continuous linear operator defined by the formula

$$\mathcal{A}_p = (A, \gamma_0) \colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \oplus B^{2-1/p,p}(\partial\Omega) \quad \text{for } 1 < p < \infty.$$

Then we have the following:

Corollary 22.3. *The null space*

$$\begin{aligned} N\left(\mathcal{A}_p\right) &= \left\{u \in W^{2,p}(\Omega) : Au = 0 \text{ in } \Omega,\, \gamma_0 u = 0\right\} \\ &= \left\{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : Au = 0 \text{ in } \Omega\right\} \end{aligned}$$

of $\mathcal{A}_p$ is independent *of p for $1 < p < \infty$.*

Proof. If $1 < p_1 < p_2 < \infty$, then it follows that

$$N\left(\mathcal{A}_{p_2}\right) \subset N\left(\mathcal{A}_{p_1}\right).$$

Conversely, if $u \in N\left(\mathcal{A}_{p_1}\right)$, then we have, by assertion (22.4),

$$\begin{cases} u \in W^{2,p_1}(\Omega) \cap W_0^{1,p_1}(\Omega), \\ Au = 0 \in L^{p_2}(\Omega) \end{cases} \implies u \in W^{2,p_2}(\Omega).$$

This proves that

$$N\left(\mathcal{A}_{p_1}\right) \subset N\left(\mathcal{A}_{p_2}\right).$$

The proof of Corollary 22.3 is complete. □

22.3 Proof of Theorem 22.2

The proof of the unique solvability theorem for the Dirichlet problem (Theorem 22.2) is divided into four steps.

Step 1: Our proof is based on the following existence and uniqueness theorem for the *homogeneous* Dirichlet problem (see Theorem 15.1):

Theorem 22.4. *Let $1 < p < \infty$ and*

$$\mathcal{L}u = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Then, for any $f \in L^p(\Omega)$ there exists a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ of the Dirichlet problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ \gamma_0 u = 0 & \text{on } \partial\Omega. \end{cases} \tag{22.11}$$

Moreover, we have the a priori *estimate*

$$\|u\|_{W^{2,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)}, \tag{22.12}$$

with a positive constant $C = C(N, p, \lambda, \eta, M, \partial\Omega)$.

Now, for any $\varphi \in B^{2-1/p,p}(\partial\Omega)$ we can find a function $v \in W^{2,p}(\Omega)$ such that $\gamma_0 v = \varphi$ on $\partial\Omega$ (see Theorem 7.4). Hence we have the following existence and uniqueness theorem for the non-homogeneous Dirichlet problem:

Corollary 22.5. *Let $1 < p < \infty$. For any $f \in L^p(\Omega)$ and any $\varphi \in B^{2-1/p,p}(\partial\Omega)$, there exists a unique solution $u \in W^{2,p}(\Omega)$ of the Dirichlet problem*

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ \gamma_0 u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{22.13}$$

We recall (see Section 2.7) that a linear operator T from a Banach space X into a Banach space Y is called a *Fredholm operator* if it satisfies the following five conditions (i), (ii), (iii), (iv) and (v):

- (i) The domain $D(T)$ of T is dense in X.
- (ii) T is a closed operator.
- (iii) The null space $N(T) = \{x \in D(T) : Tx = 0\}$ of T has finite dimension in Y; $\dim N(T) < \infty$.
- (iv) The range $R(T) = \{Tx : x \in D(T)\}$ of T is closed in Y.
- (v) The range $R(T)$ of T has finite codimension in Y; $\operatorname{codim} R(T) = \dim Y/R(T) < \infty$.

Then the *index* of T is defined by the formula

$$\operatorname{ind} T := \dim N(T) - \operatorname{codim} R(T).$$

If we associate with problem (22.13) a continuous linear operator

$$\mathcal{A}_0 = (\mathcal{L}, \gamma_0) \colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \oplus B^{2-1/p,p}(\partial\Omega),$$

then Corollary 22.5 asserts that the mapping $\mathcal{A}_0$ is an algebraic and topological *isomorphism* for all $1 < p < \infty$. In particular, we have the assertion

$$\operatorname{ind} \mathcal{A}_0 = 0. \tag{22.14}$$

Step 2: If we let

$$Bu := \sum_{i=1}^{n} b^i(x) \frac{\partial u}{\partial x_i} + c(x)u,$$

then it is clear that the operator

$$B \colon W^{2,p}(\Omega) \longrightarrow W^{1,p}(\Omega)$$

is continuous for all $1 < p < \infty$. Moreover, it follows from an application of the Rellich–Kondrachov theorem (Theorem 7.6) that the injection $W^{1,p}(\Omega) \to L^p(\Omega)$ is compact. Hence we find that the mapping

$$B \colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega)$$

is *compact* for all $1 < p < \infty$.

Therefore, we obtain that the mapping

$$\mathcal{A}_p = \mathcal{A}_0 + (B, 0) \colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \oplus B^{2-1/p,p}(\partial\Omega)$$

is a Fredholm operator with index *zero* for all $1 < p < \infty$, since we have, by Theorem 2.55 and assertion (22.14),

$$\operatorname{ind} \mathcal{A}_p = \operatorname{ind} \mathcal{A}_0 = 0 \quad \text{for all } 1 < p < \infty. \tag{22.15}$$

Step 3: In order to prove that

$$\dim N(\mathcal{A}_p) = 0 \quad \text{for all } n < p < \infty, \tag{22.16}$$

we need the weak maximum principle (Theorem 8.5) due to Bony [9], By applying Theorem 8.5 to the functions $\pm u(x)$, we obtain that

$$\begin{cases} u \in W^{2,p}(\Omega) & \text{for } n < p < \infty, \\ Au = 0 & \text{in } \Omega, \\ \gamma_0 u = 0 & \text{on } \partial\Omega \end{cases} \implies u = 0 \quad \text{in } \Omega.$$

This proves the desired assertion (22.16).

In view of Corollary 22.3, we have proved that

$$\dim N(\mathcal{A}_p) = 0 \quad \text{for all } 1 < p < \infty,$$

that is, the mapping $\mathcal{A}_p$ is *injective* for all $1 < p < \infty$.

Therefore, it is also *surjective* for $1 < p < \infty$, since we have, by assertion (22.15),

$$\operatorname{ind} \mathcal{A}_p = \dim N(\mathcal{A}) - \operatorname{codim} R(\mathcal{A}) = 0.$$

Step 4: Summing up, we have proved that the mapping

$$\mathcal{A}_p = (A, \gamma_0) \colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \oplus B^{2-1/p,p}(\partial\Omega)$$

is an algebraic and topological *isomorphism* for all $1 < p < \infty$. Indeed, the continuity of the inverse of $\mathcal{A}_p$ follows immediately from an application of Banach's open mapping theorem (Theorem 2.39).

Now the proof of Theorem 22.2 is complete. □

22.4 Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. To do so, we shall apply a version of the Hille–Yosida theorem (Theorem 21.9).

22.4.1 The Space $C_0(\overline{\Omega})$

First, we consider a one-point compactification $K_\partial = K \cup \{\partial\}$ of the locally compact space $K = \Omega$. We say that two points x and y of $\overline{\Omega}$ are equivalent modulo $\partial\Omega$ if $x = y$ or x, $y \in \partial\Omega$. We denote by $\overline{\Omega}/\partial\Omega$ the totality of equivalence classes modulo $\partial\Omega$. On the set $\overline{\Omega}/\partial\Omega$ we define the quotient topology induced by the projection

$$q \colon \overline{\Omega} \longrightarrow \overline{\Omega}/\partial\Omega.$$

Then it is easy to see that the topological space $\overline{\Omega}/\partial\Omega$ is a *one-point compactification* K_∂ of the space Ω and that the *point at infinity* ∂ corresponds to the boundary $\partial\Omega$ (see Figure 22.1):

$$\begin{aligned} K_\partial &= \overline{\Omega}/\partial\Omega, \\ \partial &= \partial\Omega. \end{aligned}$$

Furthermore, we have the following two assertions (i) and (ii):

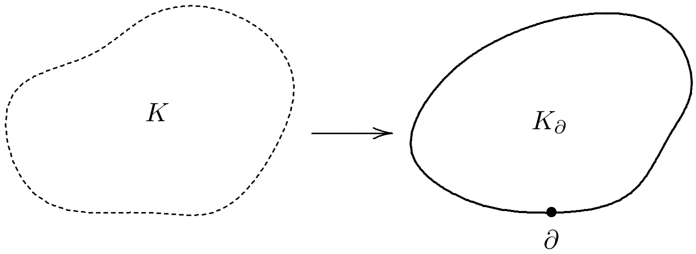

Fig. 22.1. The one-point compactification $\overline{\Omega}/\partial\Omega$ and the point at infinity ∂

(i) If $\widetilde{u}$ is a continuous function defined on $\overline{\Omega}/\partial\Omega$, then the function $\widetilde{u} \circ q$ is continuous on $\overline{\Omega}$ and constant on $\partial\Omega$.
(ii) Conversely, if u is a continuous function defined on $\overline{\Omega}$ and constant on $\partial\Omega$, then it defines a continuous function $\widetilde{u}$ on $\overline{\Omega}/\partial\Omega$.

In other words, we have the isomorphism

$$C(K_\partial) \cong \left\{u \in C(\overline{\Omega}) : \, u \text{ is constant on } \partial\Omega\right\}. \tag{22.17}$$

Now we introduce a closed subspace of $C(K_\partial)$ as follows:

$$C_0(K) = \{u \in C(K_\partial) : u(\partial) = 0\}.$$

Then we have, by assertion (22.17),

$$C_0(K) \cong C_0(\overline{\Omega}) = \left\{u \in C(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\right\}.$$

22.4.2 End of Proof of Theorem 1.3

Now it remains to verify all conditions (a) through (d) in Theorem 21.9 with

$$K := \Omega,$$
$$C_0(K) := C_0(\overline{\Omega}),$$
$$\mathfrak{A} := \mathfrak{A}_D.$$

Recall that $\mathfrak{A}_D \colon C_0(\overline{\Omega}) \to C_0(\overline{\Omega})$ is a linear operator defined as follows:

(a) The domain $D(\mathfrak{A}_D)$ is the set

$$D(\mathfrak{A}_D) = \left\{u \in W^{2,p}(\Omega) \cap C_0(\overline{\Omega}) : Au \in C_0(\overline{\Omega})\right\} \text{ for } n < p < \infty. \tag{1.6}$$

(b) $\mathfrak{A}_D u = Au$ for every $u \in D(\mathfrak{A}_D)$.

The proof is divided into four steps.

Step 1: (b) For each $\alpha > 0$, the equation

$$(\alpha I - \mathfrak{A}_D)u = f$$

has a unique solution $u \in D(\mathfrak{A}_D)$ for any $f \in C_0(\overline{\Omega})$.

Since we have the inequality

$$c(x) - \alpha \leq -\alpha \quad \text{for almost all } x \in \Omega,$$

by applying Theorem 22.2 to the operator $A - \alpha$ we obtain that the Dirichlet problem

$$\begin{cases} (\alpha - A)u = f & \text{almost everywhere in } \Omega, \\ \gamma_0 u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in W^{2,p}(\Omega)$ for any $f \in L^p(\Omega)$ with $n < p < \infty$. In particular, for any $f \in C_0(\overline{\Omega})$ there exists a unique function $u \in W^{2,p}(\Omega) \cap C_0(\overline{\Omega})$ such that

$$(\alpha - A)u = f \quad \text{in } \Omega.$$

Hence we have the assertion

$$Au = \alpha u - f \in C_0(\overline{\Omega}).$$

By formula (1.6), this proves that

$$\begin{cases} u \in D(\mathfrak{A}_D), \\ (\alpha I - \mathfrak{A}_D)u = f. \end{cases}$$

Step 2: (c) For each $\alpha > 0$, the *Green operator* $G_\alpha^0 = (\alpha I - \mathfrak{A}_D)^{-1}$ is non-negative on the space $C_0(\overline{\Omega})$:

$$f \in C_0(\overline{\Omega}),\ f(x) \geq 0 \quad \text{in } \Omega \implies u(x) = G_\alpha^0 f(x) \geq 0 \quad \text{in } \Omega.$$

Indeed, if we let

$$v(x) = -u(x) = -G_\alpha^0 f(x),$$

then it follows that

$$\begin{cases} (A - \alpha)v = f \geq 0 & \text{in } \Omega, \\ \gamma_0 v = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, by applying Theorem 8.5 to our situation we obtain that

$$v(x) \leq 0 \quad \text{in } \Omega,$$

so that

$$u(x) = -v(x) \geq 0 \quad \text{in } \Omega.$$

Step 3: (d) For each $\alpha > 0$, the Green operator $G_\alpha^0 = (\alpha I - \mathfrak{A}_D)^{-1}$ is bounded on the space $C_0(\overline{\Omega})$ with norm $1/\alpha$: $\|G_\alpha^0\| \leq 1/\alpha$.

Let $f(x)$ be an arbitrary function in $C_0(\overline{\Omega})$. If we let

$$u_\pm(x) = \pm\alpha G_\alpha^0 f(x) - \|f\|_{C(\overline{\Omega})} \in W^{2,p}(\Omega),$$

it suffices to show that

$$u_\pm(x) \leq 0 \quad \text{in } \Omega. \tag{22.18}$$

Indeed, it follows that

$$\begin{aligned}(A-\alpha)u_\pm(x) &= \mp\alpha f(x) + (\alpha - c(x))\|f\|_{C(\overline{\Omega})} \\ &= \alpha(\|f\|_{C(\overline{\Omega})} \mp f(x)) + (-c(x))\|f\|_{C(\overline{\Omega})} \\ &\geq 0 \quad \text{in } \Omega.\end{aligned}$$

Thus, by applying Theorem 8.5 to the operator $A - \alpha$ we obtain that the function $u_\pm(x)$ may take its positive maximum only on $\partial\Omega$. This proves assertion (22.18), since we have the inequality

$$\gamma_0(u_\pm) = -\|f\|_{C(\overline{\Omega})} < 0 \quad \text{on } \partial\Omega.$$

Step 4: (a) The domain $D(\mathfrak{A}_D)$ is dense in $C_0(\overline{\Omega})$. More precisely, we prove that we have the assertion

$$\lim_{\alpha\to+\infty} \left\|\alpha G_\alpha^0 u - u\right\|_{C(\overline{\Omega})} = 0 \quad \text{for each } u \in C_0(\overline{\Omega}). \tag{22.19}$$

To do this, we introduce an extension $\widetilde{G_\alpha^0}$ of the Green operator G_α^0 to the space $L^\infty(\Omega)$: By Theorem 22.2, we find that the Dirichlet problem

$$\begin{cases}(\alpha - A)u = f & \text{almost everywhere in } \Omega, \\ \gamma_0 u = 0 & \text{on } \partial\Omega\end{cases}$$

has a unique solution $u \in W^{2,p}(\Omega) \cap C_0(\overline{\Omega})$ $(n < p < \infty)$ for any $f \in L^\infty(\Omega)$. If we let

$$u := \widetilde{G_\alpha^0} f,$$

then it is easy to verify that the operator $\widetilde{G_\alpha^0}$ is an extension of G_α^0 to $L^\infty(\Omega)$. Moreover, just as in Steps 2 and 3, we can prove the following two assertions (A) and (B):

(A) The operator $\widetilde{G_\alpha^0}\colon L^\infty(\Omega) \to C_0(\overline{\Omega})$ is non-negative.

(B) The operator $\widetilde{G_\alpha^0}\colon L^\infty(\Omega) \to C_0(\overline{\Omega})$ is bounded with norm $1/\alpha$: $\|\widetilde{G_\alpha^0}\| \leq 1/\alpha$.

The operators G_α^0 and $\widetilde{G_\alpha^0}$ can be visualized as follows:

$$\begin{array}{ccc} D(\widetilde{G_\alpha^0}) = L^\infty(\Omega) & \xrightarrow{\widetilde{G_\alpha^0}} & C_0(\overline{\Omega}) \\ \uparrow & & \downarrow \\ D(G_\alpha^0) = C_0(\overline{\Omega}) & \xrightarrow{G_\alpha^0} & D(\mathfrak{A}_D) \end{array}$$

Fig. 22.2. The operators G_α^0 and $\widetilde{G_\alpha^0}$

Since the space $C_0^2(\overline{\Omega}) := C^2(\overline{\Omega}) \cap C_0(\overline{\Omega})$ is dense in $C_0(\overline{\Omega})$, it suffices to prove assertion (22.19) for any $u \in C_0^2(\overline{\Omega})$.

First, since $a^{ij}(x)$, $b^i(x)$, $c(x) \in L^\infty(\Omega)$, it follows that

$$Au = \sum_{i,j=1}^n a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b^i(x) \frac{\partial u}{\partial x_i} + c(x)u \in L^\infty(\Omega) \tag{22.20}$$

for all $u \in C_0^2(\overline{\Omega})$.

Now, if we let

$$w = \alpha G_\alpha^0 u - \widetilde{G_\alpha^0}(Au),$$

then we have the assertions

$$\begin{cases} w \in W^{2,p}(\Omega) \cap C_0(\overline{\Omega}), \\ (A-\alpha)w = -\alpha u + Au = (A-\alpha)u \quad \text{in } \Omega, \end{cases}$$

and so

$$\begin{cases} w - u \in W^{2,p}(\Omega) \cap C_0(\overline{\Omega}), \\ (A-\alpha)(w-u) = 0 \quad \text{in } \Omega. \end{cases}$$

By Theorem 22.2, this implies that $w - u = 0$ in Ω, that is,

$$u = w = \alpha G_\alpha^0 u - \widetilde{G_\alpha^0}(Au).$$

Therefore, assertion (22.19) for any $u \in C_0^2(\overline{\Omega})$ follows from an application of assertion (B) and assertion (22.20), since we have, for all $\alpha > 0$,

$$\|u - \alpha G_\alpha^0 u\|_{C(\overline{\Omega})} = \|\widetilde{G_\alpha^0}(Au)\|_{C(\overline{\Omega})} \leq \frac{1}{\alpha}\|Au\|_{L^\infty(\Omega)}.$$

Now the proof of Theorem 1.3 is complete. □

22.5 Proof of Remark 1.3

Finally, we prove that the domain

$$D(\mathfrak{A}_D) = \left\{ u \in C_0(\overline{\Omega}) \cap W^{2,p}(\Omega) : Au \in C_0(\overline{\Omega}) \right\}$$

is *independent* of p, for $n < p < \infty$.

We let

$$\mathcal{D}_p := \left\{ u \in W^{2,p}(\Omega) \cap C_0(\overline{\Omega}) : Au \in C_0(\overline{\Omega}) \right\}.$$

In order to prove Remark 1.3, it suffices to show that

$$\mathcal{D}_{p_1} = \mathcal{D}_{p_2} \quad \text{for } n < p_1 < p_2 < \infty.$$

First, it follows that

$$\mathcal{D}_{p_2} \subset \mathcal{D}_{p_1},$$

since we have $L^{p_2}(\Omega) \subset L^{p_1}(\Omega)$ for $p_2 > p_1$.

Conversely, let v be an arbitrary element of $\mathcal{D}_{p_1}$:

$$v \in W^{2,p_1}(\Omega) \cap C_0(\overline{\Omega}), \quad Av \in C_0(\overline{\Omega}).$$

Then, since we have the assertions

$$v,\ Av \in C_0(\overline{\Omega}) \subset L^{p_2}(\Omega),$$

it follows from an application of Theorem 22.2 with $p := p_2$ that there exists a unique function $u \in W^{2,p_2}(\Omega)$ such that

$$\begin{cases} Au = Av & \text{in } \Omega, \\ \gamma_0 u = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence we have the assertions

$$\begin{cases} u - v \in W^{2,p_1}(\Omega), & \\ A(u - v) = 0 & \text{in } \Omega, \\ \gamma_0(u - v) = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, by applying again Theorem 22.2 with $p := p_1$ we obtain that $u - v = 0$, so that $v = u \in W^{2,p_2}(\Omega)$. This implies that

$$v \in \mathcal{D}_{p_2}.$$

The proof of Remark 1.3 is complete. □

22.6 Notes and Comments

Section 22.1: Theorem 22.1 is inspired by Vitanza [95, Theorem 2.2] and Theorem 22.2 is inspired by Bony [9, Théorème 3], respectively.

Section 22.4: The proof of Theorem 1.3 is adapted from [77, Theorem 1.2].

23

Feller Semigroups with Oblique Derivative Condition

In this chapter we study the oblique derivative problem in the framework of L^p Sobolev spaces, and prove an existence and uniqueness theorem for the oblique derivative problem with VMO coefficients (Theorem 23.2). The uniqueness result in Theorem 23.2 follows from a variant of the Bakel'man–Aleksandrov maximum principle in the framework of L^p Sobolev spaces due to Lieberman [43] (Theorem 23.5). Moreover, we construct a Feller semigroup associated with absorption, reflection and drift phenomena at the boundary (Theorem 1.2 and Figure 1.6).

23.1 Formulation of the Oblique Derivative Problem

Let A be a second-order, elliptic differential operator with real *discontinuous* coefficients of the form

$$Au := \sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u. \tag{1.1}$$

Here the functions $a^{ij}(x)$, $b^i(x)$ and $c(x)$ satisfy the following three conditions (1), (2) and (3):

(1) $a^{ij}(x) \in \mathrm{VMO} \cap L^\infty(\Omega)$, $a^{ij}(x) = a^{ji}(x)$ for almost all $x \in \Omega$ and $1 \leq i, j \leq n$, and there exist a constant $\lambda > 0$ such that

$$\frac{1}{\lambda}|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2$$

for almost all $x \in \Omega$ and all $\xi \in \mathbf{R}^n$.

(2) $b^i(x) \in L^\infty(\Omega)$ for $1 \leq i \leq n$.

(3) $c(x) \in L^\infty(\Omega)$ and $c(x) \leq 0$ for almost all $x \in \Omega$.

In this chapter, we consider an oblique derivative boundary operator of the form

$$\mathcal{B}u := \mu(x')\frac{\partial u}{\partial \mathbf{n}} + \beta(x') \cdot u + \gamma(x')u. \tag{23.1}$$

Here the functions $\mu(x')$, $\beta(x')$ and $\gamma(x')$ satisfy the following three conditions (4), (5) and (6):

(4) $\mu(x')$ is a Lipschitz continuous function on $\partial\Omega$ and $\mu(x') \geq 0$ on $\partial\Omega$.
(5) $\beta(x')$ is a Lipschitz continuous vector field on $\partial\Omega$.
(6) $\gamma(x')$ is a Lipschitz continuous function on $\partial\Omega$ and $\gamma(x') \leq 0$ on $\partial\Omega$.
(7) $\mathbf{n} = (n_1, n_2, \ldots, n_n)$ is the unit interior normal to the boundary $\partial\Omega$ (see Figure 23.1).

The purpose of this section is to prove an existence and uniqueness theorem for the following non-homogeneous oblique derivative problem in the framework of L^p Sobolev spaces:

$$\begin{cases} Au = f & \text{in } \Omega, \\ \mathcal{B}u = \varphi & \text{on } \partial\Omega \end{cases} \tag{23.2}$$

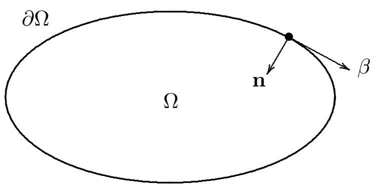

Fig. 23.1. The vector field β and the unit interior normal $\mathbf{n}$

The first main result of this chapter is the following regularity theorem for the oblique derivative problem (23.2) (cf. [47, Theorem 2.2.1]):

Theorem 23.1 (the regularity theorem). *Let $1 < p < \infty$. Assume that the functions $\mu(x')$ and $\gamma(x')$ satisfy the conditions*

$$\mu(x') > 0 \quad \textit{on } \partial\Omega, \tag{H.1}$$

and

$$\gamma(x') < 0 \quad \textit{on } \partial\Omega. \tag{H.2}$$

If a function $u \in W^{2,q}(\Omega)$, $1 < q < p < \infty$, is a solution of the oblique *derivative problem* (23.2) *with* $f \in L^p(\Omega)$ *and* $\varphi \in B^{1-1/p,p}(\partial\Omega)$*, then it follows that* $u \in W^{2,p}(\Omega)$*. Moreover, we have the global* a priori *estimate*

$$\|u\|_{W^{2,p}(\Omega)} \leq C_1 \left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \|\varphi\|_{B^{1-1/p,p}(\partial\Omega)} \right), \tag{23.3}$$

with a positive constant $C_1 = C_1(n, p, \lambda, \eta, \mu, \beta, \gamma, \partial\Omega)$.

The second main result of this chapter is the following existence and uniqueness theorem for the oblique derivative problem (23.2) (cf. [46, Theorem 4.1], [47, Theorem 2.2.2]):

Theorem 23.2 (the existence and uniqueness theorem)**.** *Let* $1 < p < \infty$*, and assume that conditions (H.1) and (H.2) are satisfied. Then, for any* $f \in L^p(\Omega)$ *and any* $\varphi \in B^{1-1/p,p}(\partial\Omega)$ *there exists a unique solution* $u \in W^{2,p}(\Omega)$ *of the oblique derivative problem* (23.2)*. Moreover, we have the global* a priori *estimate*

$$\|u\|_{W^{2,p}(\Omega)} \leq C_2 \left(\|f\|_{L^p(\Omega)} + \|\varphi\|_{B^{1-1/p,p}(\partial\Omega)} \right), \tag{23.4}$$

with a positive constant $C_2 = C_2(n, p, \lambda, \eta, \mu, \beta, \gamma, \partial\Omega)$.

If we associate with problem (23.2) a continuous linear operator

$$\mathcal{A}_p = (A, \mathcal{B})\colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \oplus B^{1-1/p,p}(\partial\Omega),$$

then we obtain from the trace theorem (Theorem 7.4) and Theorem 23.2 that the operator $\mathcal{A}_p$ is continuous and *bijective* for all $1 < p < \infty$.

23.2 Proof of Theorem 23.1

The proof of the regularity theorem for the oblique derivative problem (Theorem 23.1) is divided into three steps.

Step 1: For any $\varphi \in B^{1-1/p,p}(\partial\Omega)$, we can find a function $v \in W^{2,p}(\Omega)$ such that $\mathcal{B}v = \varphi$ on $\partial\Omega$, and further that the mapping

$$B^{1-1/p,p}(\partial\Omega) \ni \varphi \longmapsto v \in W^{2,p}(\Omega) \tag{23.5}$$

is continuous (see Lemma 17.1).

Therefore, we have only to prove Theorem 23.1 in the case where $\varphi := 0$ and $1 < q < p < \infty$:

$$\begin{cases} u \in W^{2,q}(\Omega), \\ Au = f \in L^p(\Omega), \\ \mathcal{B}u = 0 \text{ on } \partial\Omega \end{cases} \tag{23.6}$$

$$\Longrightarrow \ u \in W^{2,p}(\Omega).$$

Step 2: In order to prove assertion (23.6), we need the Sobolev imbedding theorems (Theorem 7.4):

$$W^{2,p}(\Omega) \subset \begin{cases} L^{nq/(n-2q)}(\Omega) & \text{if } 1 < q < n/2, \\ L^r(\Omega) \quad \text{for all } n/2 \le r < \infty & \text{if } q = n/2, \\ L^\infty(\Omega) & \text{if } n/2 < q < \infty. \end{cases} \tag{22.5}$$

$$W^{1,q}(\Omega) \subset \begin{cases} L^{nq/(n-q)}(\Omega) & \text{if } 1 < q < n, \\ L^r(\Omega) \quad \text{for all } n \le r < \infty & \text{if } q = n, \\ L^\infty(\Omega) & \text{if } n < q < \infty. \end{cases} \tag{22.6}$$

We consider the following three cases (A), (B) and (C).

(A) The case where $1 < q < n/2$: By assertions (22.5) and (22.6), it follows that

$$\begin{cases} u \in W^{2,q}(\Omega) \subset L^{nq/(n-2q)}(\Omega), \\ \frac{\partial u}{\partial x_i} \in W^{1,q}(\Omega) \subset L^{nq/(n-q)}(\Omega) \quad \text{for all } 1 \le i \le n. \end{cases}$$

Hence we have the assertion

$$\begin{aligned} \mathcal{L}u &= \sum_{i,j=1}^n a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} \\ &= f - \sum_{i=1}^n b^i(x) \frac{\partial u}{\partial x_i} + c(x) u \in L^{nq/(n-q)}(\Omega). \end{aligned}$$

By applying the global regularity theorem (Theorem 16.1), we obtain that

$$u \in L^{p_1}(\Omega),$$

where

$$p_1 = \min \left\{ p, \frac{nq}{n-q} \right\}.$$

It is easy to see that

$$p_1 > q,$$

and further that

$$p_1 - q > \frac{1}{n-1}.$$

Therefore, by making use of a standard bootstrap argument we can conclude that

$$u \in L^p(\Omega).$$

(B) The case where $n/2 \leq q < n$: By assertions (22.5) and (22.6), it follows that

$$\begin{cases} u \in W^{2,q}(\Omega) \subset L^p(\Omega), \\ \frac{\partial u}{\partial x_i} \in W^{1,q}(\Omega) \subset L^{nq/(n-q)}(\Omega) \quad \text{for all } 1 \leq i \leq n. \end{cases}$$

Hence we have the assertion

$$\mathcal{L}u = f - \sum_{i=1}^{n} b^i(x) \frac{\partial u}{\partial x_i} + c(x)u \in L^{nq/(n-q)}(\Omega).$$

Therefore, by making use of a standard bootstrap argument we can conclude that

$$u \in L^p(\Omega).$$

(C) The case where $n \leq q < \infty$: By assertions (22.5) and (22.6), it follows that

$$\begin{cases} u \in W^{2,q}(\Omega) \subset L^\infty(\Omega), \\ \frac{\partial u}{\partial x_i} \in W^{1,q}(\Omega) \subset L^p(\Omega) \quad \text{for all } 1 \leq i \leq n. \end{cases}$$

Hence we have the assertion

$$\mathcal{L}u = f - \sum_{i=1}^{n} b^i(x) \frac{\partial u}{\partial x_i} + c(x)u \in L^p(\Omega).$$

By applying the global regularity theorem (Theorem 16.1), we obtain that

$$u \in L^p(\Omega).$$

Summing up, we have proved the desired assertion (23.6).

Step 3: Finally, it remains to prove the global *a priori* estimate (23.3).

First, by using the global *a priori* estimate (16.5) we obtain that

$$\|u\|_{W^{2,p}(\Omega)} \leq C_1 \left(\|u\|_{L^p(\Omega)} + \|\mathcal{L}u\|_{L^p(\Omega)} + \|\varphi\|_{B^{1-1/p,p}(\partial\Omega)} \right), \quad (23.7)$$

with a positive constant C_1. Indeed, it suffices to note that the mapping (23.5) is continuous.

On the other hand, we have the inequality

$$\|\mathcal{L}u\|_{L^p(\Omega)} \quad (23.8)$$

$$\leq \left\| f - \sum_{i=1}^{n} b^i(x) \frac{\partial u}{\partial x_i} + c(x) u \right\|_{L^p(\Omega)}$$
$$\leq \|f\|_{L^p(\Omega)} + \sum_{i=1}^{n} \|b^i\|_{L^\infty(\Omega)} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)} + \|c\|_{L^\infty(\Omega)} \|u\|_{L^p(\Omega)}.$$

Hence, by combining two inequalities (23.7) and (23.8) we obtain that

$$\|u\|_{W^{2,p}(\Omega)} \leq C_2 \left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \|\varphi\|_{B^{1-1/p,p}(\partial\Omega)} + \|\nabla u\|_{L^p(\Omega)} \right), \tag{23.9}$$

with a positive constant C_2.

However, we recall the *interpolation inequality* (13.17)

$$\|\nabla u\|_{L^p(\Omega)} \leq \varepsilon \, \|\nabla^2 u\|_{L^p(\Omega)} + \frac{C_3}{\varepsilon} \, \|u\|_{L^p(\Omega)} \quad \text{for all } \varepsilon > 0. \tag{22.10}$$

Therefore, the desired estimate (23.3) follows from inequalities (23.9) and (22.10) if we take

$$\varepsilon = \frac{1}{2C_2}.$$

Now the proof of Theorem 23.1 is complete. □

Let $\mathcal{A}_p$ be a continuous linear operator defined by the formula

$$\mathcal{A}_p = (A, \mathcal{B}) \colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \oplus B^{1-1/p,p}(\partial\Omega) \quad \text{for } 1 < p < \infty.$$

Then we have the following:

Corollary 23.3. *The null space*

$$N(\mathcal{A}_p) = \left\{ u \in W^{2,p}(\Omega) : Au = 0 \text{ in } \Omega, \, \mathcal{B}u = 0 \text{ on } \partial\Omega \right\}$$

of $\mathcal{A}_p$ *is* independent *of* p *for* $1 < p < \infty$.

Proof. If $1 < p_1 < p_2 < \infty$, then it follows that

$$N(\mathcal{A}_{p_2}) \subset N(\mathcal{A}_{p_1}).$$

Conversely, if $u \in N(\mathcal{A}_{p_1})$, then we have, by assertion (23.6),

$$\begin{cases} u \in W^{2,p_1}(\Omega), \\ Au = 0 \in L^{p_2}(\Omega), \\ \mathcal{B}u = 0 \text{ on } \partial\Omega \end{cases}$$
$$\Longrightarrow \; u \in W^{2,p_2}(\Omega).$$

This proves that

$$N(\mathcal{A}_{p_1}) \subset N(\mathcal{A}_{p_2}).$$

The proof of Corollary 23.3 is complete. □

23.3 Proof of Theorem 23.2

The proof of Theorem 23.2 is divided into four steps. First, since **n** is the unit inward normal to the boundary $\partial\Omega$, it follows that (see Figure 23.1)

$$\langle \mu(x')\mathbf{n} + \beta(x'), \mathbf{n}\rangle = \mu(x')\,\langle \mathbf{n}, \mathbf{n}\rangle = \mu(x'), \quad x' \in \partial\Omega.$$

Therefore, we find that condition (H.1) is equivalent to condition (16.3b) for $\mathcal{B}$, and further that condition (H.2) is equivalent to condition (16.3c) for $\mathcal{B}$.

Step 1: Our proof is based on the following existence and uniqueness theorem for the *non-homogeneous* oblique derivative problem (Theorem 16.2):

Theorem 23.4. *Let $1 < p < \infty$ and*

$$\mathcal{L}u = \sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Assume that conditions (H.1) and (H.2) are satisfied. Then, for any $f \in L^p(\Omega)$ and any $\varphi \in B^{1-1/p,p}(\partial\Omega)$ there exists a unique solution $u \in W^{2,p}(\Omega)$ of the oblique derivative problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ \mathcal{B}u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{23.10}$$

Moreover, we have the global a priori *estimate*

$$\|u\|_{W^{2,p}(\Omega)} \leq C_2 \left(\|f\|_{L^p(\Omega)} + \|\varphi\|_{B^{1-1/p,p}(\partial\Omega)} \right), \tag{23.11}$$

with a positive constant $C_2 = C_2(n, p, \lambda, \eta, \mu, \beta, \gamma, \partial\Omega)$.

If we associate with problem (23.10) a continuous linear operator

$$\mathcal{A}_0 = (\mathcal{L}, \mathcal{B})\colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \oplus B^{1-1/p,p}(\partial\Omega),$$

then we obtain from the trace theorem (Theorem 7.4) and Theorem 23.4 that the mapping $\mathcal{A}_0$ is an algebraic and topological *isomorphism*, for all $1 < p < \infty$. In particular, we have the assertion

$$\operatorname{ind} \mathcal{A}_0 = \dim N(\mathcal{A}_0) - \operatorname{codim} R(\mathcal{A}_0) = 0 \quad \text{for all } 1 < p < \infty. \tag{23.12}$$

Step 2: If we let

$$Bu := \sum_{i=1}^{n} b^i(x) \frac{\partial u}{\partial x_i} + c(x)u,$$

then it follows that the operator

$$B \colon W^{2,p}(\Omega) \longrightarrow W^{1,p}(\Omega)$$

is continuous for all $1 < p < \infty$. Moreover, it follows from an application of the Rellich–Kondrachov theorem (Theorem 7.6) that the injection $W^{1,p}(\Omega) \to L^p(\Omega)$ is compact for all $1 < p < \infty$. Hence we find that the mapping

$$B \colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega)$$

is *compact* for all $1 < p < \infty$. It should be noticed that

$$\mathcal{A}_p = (A, \mathcal{B}) = (\mathcal{L}, \mathcal{B}) + (B, 0) = \mathcal{A}_0 + (B, 0).$$

Therefore, we obtain that the mapping

$$\mathcal{A}_p = \mathcal{A}_0 + (B, 0) \colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \oplus B^{1-1/p,p}(\partial\Omega)$$

is a Fredholm operator with index *zero* for all $1 < p < \infty$, since we have, by Theorem 2.55 and assertion (23.12),

$$\operatorname{ind} \mathcal{A}_p = \operatorname{ind} \mathcal{A}_0 = 0. \tag{23.13}$$

Step 3: In order to prove that

$$\dim N(\mathcal{A}_p) = 0 \quad \text{for all } n < p < \infty, \tag{23.14}$$

we need the following Bakel'man and Aleksandrov maximum principle (see [43, Corollary 2.4], [94]):

Theorem 23.5 (Bakel'man–Aleksandrov). *Let $n < p < \infty$. Assume that conditions (H.1) and (H.2) are satisfied. If a function $u \in W^{2,p}(\Omega)$ satisfies the conditions*

$$\begin{cases} Au \leq 0 & \text{almost everywhere in } \Omega, \\ \mathcal{B}u \leq 0 & \text{on } \partial\Omega, \end{cases}$$

then it follows that either $u(x)$ is a non-negative constant function or $u(x) > 0$ on $\overline{\Omega}$.

Proof. First, it should be noticed that we have, by Sobolev's imbedding theorem (see [2, Theorem 4.12, Part II]; [80, Chapter 4]),

$$W^{2,p}(\Omega) \subset C^1(\overline{\Omega}),$$

since $2 - n/p > 1$ for $n < p < \infty$.

We have only to consider the case where $u(x)$ is not a constant function in Ω. We assume, to the contrary, that $u(x)$ takes a non-positive minimum at a point $x_0 \in \overline{\Omega}$. If we let

$$v(x) = -u(x),$$

then we have the assertions

$$\begin{cases} v \in W^{2,p}(\Omega) & \text{for } 1 < p < \infty, \\ Av = -(Au) \geq 0 & \text{in } \Omega, \\ \mathcal{B}v \geq 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, by applying the strong maximum principle (Theorem 8.9) to the function $v(x)$ we obtain that, for some boundary point $x_0' \in \partial\Omega$,

$$v(x_0') = v(x_0) = \max_{\overline{\Omega}}(-u) \geq 0.$$

Moreover, it follows from an application of Hopf's boundary point lemma (Lemma 8.7) that

$$\frac{\partial v}{\partial \mathbf{n}}(x_0') < 0.$$

By conditions (H.1) and (H.2), this implies that

$$0 \leq \mathcal{B}v(x_0') = \mu(x_0')\frac{\partial u}{\partial \mathbf{n}}(x_0') + \gamma(x_0')u(x_0') \leq \mu(x_0')\frac{\partial u}{\partial \mathbf{n}}(x_0') < 0.$$

This contradiction proves that $u(x) > 0$ on $\overline{\Omega}$.

The proof of Theorem 23.5 is complete. □

By applying Theorem 23.5 to the functions $\pm u(x)$, it follows that

$$\begin{cases} u \in W^{2,p}(\Omega) & \text{for } n < p < \infty, \\ Au = 0 & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega \end{cases}$$
$$\Longrightarrow \quad u = 0 \quad \text{in } \Omega.$$

This proves the desired assertion (23.14).

In view of Corollary 23.3, we have proved that

$$\dim N(\mathcal{A}_p) = 0 \quad \text{for all } 1 < p < \infty,$$

that is, the mapping $\mathcal{A}_p$ is *injective* for all $1 < p < \infty$.

Therefore, it is also *surjective* for $1 < p < \infty$, since we have, by assertion (23.13),

$$\operatorname{ind} \mathcal{A}_p = \dim N(\mathcal{A}_p) - \operatorname{codim} R(\mathcal{A}_p) = 0.$$

Step 4: Summing up, we have proved that the mapping

$$\mathcal{A}_p = (A, \mathcal{B}) \colon W^{2,p}(\Omega) \longrightarrow L^p(\Omega) \oplus B^{1-1/p,p}(\partial\Omega)$$

is an algebraic and topological *isomorphism* for all $1 < p < \infty$. Indeed, the continuity of the inverse of $\mathcal{A}_p$ follows immediately from an application of Banach's open mapping theorem (Theorem 2.39).

The proof of Theorem 23.2 is complete. □

23.4 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. To do so, we have only to verify all conditions (a) through (d) in Theorem 21.9 with

$$\begin{aligned} &K := \overline{\Omega}, \\ &C_0(K) := C(\overline{\Omega}), \\ &\mathfrak{A} := \mathfrak{A}_\nu. \end{aligned}$$

The proof is divided into four steps.

Step 1: First, we prove that, for each $\alpha \geq 0$, the equation $(\alpha I - A)u = f$ has a unique solution $u \in D(\mathfrak{A}_\nu)$ for any $f \in C(\overline{\Omega})$.

By applying Theorem 23.2, we obtain that the oblique derivative problem

$$\begin{cases} (\alpha - A)u = f & \text{almost everywhere in } \Omega, \\ L_\nu u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution

$$u \in W^{2,p}(\Omega)$$

for any $f \in L^p(\Omega)$ with $n < p < \infty$. In particular, for any $f \in C(\overline{\Omega})$ there exists a function $u \in W^{2,p}(\Omega)$ such that

$$(\alpha - A)u = f \quad \text{in } \Omega.$$

Hence we have the assertion

$$Au = \alpha u - f \in C(\overline{\Omega}).$$

By formula (1.5), this proves that

$$\begin{cases} u \in D(\mathfrak{A}_\nu), \\ (\alpha I - \mathfrak{A}_\nu)u = f. \end{cases}$$

Step 2: Secondly, we prove that, for each $\alpha \geq 0$, the *Green operator* $G_\alpha^\nu = (\alpha I - \mathfrak{A}_\nu)^{-1}$ is non-negative on the space $C(\overline{\Omega})$:

$$f \in C(\overline{\Omega}),\ f(x) \geq 0 \quad \text{in } \Omega \implies u(x) = G_\alpha^\nu f(x) \geq 0 \quad \text{in } \Omega.$$

More precisely, we prove the following assertion:

$$\begin{aligned} &f \in C(\overline{\Omega}),\ f(x) \geq 0,\ f(x) \not\equiv 0 \quad \text{in } \Omega \\ &\implies u(x) = G_\alpha^\nu f(x) > 0 \quad \text{on } \overline{\Omega}. \end{aligned} \tag{23.15}$$

Since we have the assertions

$$\begin{cases} u \in W^{2,p}(\Omega) & \text{for } n < p < \infty, \\ (A - \alpha)u = -f \leq 0 & \text{almost everywhere in } \Omega, \\ L_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

by applying Theorem 23.5 we obtain that either $u(x)$ is a non-negative constant function or $u(x) > 0$ on $\overline{\Omega}$. However, if $u(x) \equiv 0$ in Ω, then it follows that

$$f(x) = (\alpha - A)u(x) \equiv 0 \quad \text{in } \Omega.$$

This contradiction proves that either $u(x)$ is a positive constant function or $u(x) > 0$ on $\overline{\Omega}$; that is,

$$G_\alpha^\nu f(x) > 0 \quad \text{on } \overline{\Omega}.$$

Step 3: Thirdly, we prove that, for each $\alpha > 0$, the Green operator $G_\alpha^\nu = (\alpha I - \mathfrak{A}_\nu)^{-1}$ is bounded on the space $C(\overline{\Omega})$ with norm $1/\alpha$: $\|G_\alpha^\nu\| \leq 1/\alpha$.

By assertion (23.15), it suffices to show that

$$\alpha G_\alpha^\nu 1(x) \leq 1 \quad \text{on } \overline{\Omega}.$$

If we let

$$v(x) := \alpha G_\alpha^\nu 1(x) - 1,$$

then we have the assertions

$$\begin{cases} u \in W^{2,p}(\Omega) & \text{for } n < p < \infty, \\ (A - \alpha)v = 0 & \text{in } \Omega, \\ L_\nu v = 0 & \text{on } \partial\Omega. \end{cases}$$

By applying Theorem 23.5 to the function $-v(x)$, we arrive at a contradiction that

$$\max_{\overline{\Omega}} v > 0 \implies v(x) \equiv 0 \quad \text{in } \Omega.$$

This proves that

$$\max_{\overline{\Omega}} v \leq 0,$$

that is,

$$\alpha G_\alpha^\nu 1(x) \leq 1 \quad \text{on } \overline{\Omega}.$$

Step 4: The *closedness* of $\mathfrak{A}_\nu$ is an immediate consequence of the boundedness of $G_\alpha^\nu = (\alpha I - \mathfrak{A}_\nu)^{-1}$. Indeed, it suffices to note the formula

$$\mathfrak{A}_\nu = \alpha I - \left(G_\alpha^\nu\right)^{-1}.$$

Step 5: Finally, we prove that the domain $D(\mathfrak{A}_\nu)$ is *dense* in $C(\overline{\Omega})$. More precisely, we prove that

$$\lim_{\alpha \to +\infty} \|\alpha G_\alpha^\nu u - u\|_{C(\overline{\Omega})} = 0 \quad \text{for any } u \in C(\overline{\Omega}). \tag{23.16}$$

Step 5-1: It suffices to prove assertion (23.16) for any $v \in C^2(\overline{\Omega})$ such that $L_\nu v = 0$ on $\partial\Omega$. In fact, we have the following density theorem (see [4, Lemma 3.2]):

Lemma 23.6. *Let $u \in C(\overline{\Omega})$. For any given $\varepsilon > 0$, we can find a function $v \in C^2(\overline{\Omega})$ such that*

$$\begin{cases} \|u - v\|_{C(\overline{\Omega})} < \varepsilon, \\ L_\nu v = 0 \quad \textit{on } \partial\Omega. \end{cases} \tag{23.17}$$

Proof. First, it follows from an application of the Weierstrass approximation theorem that there exists a polynomial $g(x)$ such that

$$\|u - g\|_{C(\overline{\Omega})} < \frac{\varepsilon}{2}.$$

Secondly, we can construct a function $h(x) \in C^2(\overline{\Omega})$ such that (see Lemma 17.1)

$$h = 0 \quad \text{on } \partial\Omega,$$

$$\frac{\partial h}{\partial \mathbf{n}} = \frac{1}{\mu(x')} L_\nu g \quad \text{on } \partial\Omega,$$
$$\|h\|_{C(\overline{\Omega})} < \frac{\varepsilon}{2}.$$

This implies that

$$L_\nu h = \mu(x')\frac{\partial h}{\partial \mathbf{n}} + \beta(x') \cdot h + \gamma(x')h = \mu(x')\frac{\partial h}{\partial \mathbf{n}} = L_\nu g \quad \text{on } \partial\Omega.$$

Therefore, it is easy to verify that the function $v(x) = g(x) - h(x)$ satisfies the desired conditions (23.17).

The proof of Lemma 23.6 is complete. □

Step 5-2: To prove assertion (23.16) for any $v \in C^2(\overline{\Omega})$ such that $L_\nu v = 0$, we introduce an extension $\widetilde{G_\alpha^\nu}$ of the Green operator G_α^ν to the space $L^p(\Omega)$ for $N < p < \infty$. By applying Theorem 23.2, we obtain that the oblique derivative problem

$$\begin{cases} (\alpha - A)u = f & \text{almost everywhere in } \Omega, \\ L_\nu u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in W^{2,p}(\Omega)$ for any $f \in L^p(\Omega)$. If we let

$$u := \widetilde{G_\alpha^\nu} f,$$

then it is easy to verify that the operator $\widetilde{G_\alpha^\nu}$ is an extension of G_α^ν to $L^p(\Omega)$. Moreover, just as in Steps 2 and 3 we can prove the following two assertions (A) and (B):

(A) The operator $\widetilde{G_\alpha^\nu}\colon L^p(\Omega) \to C(\overline{\Omega})$ is non-negative.
(B) The operator $\widetilde{G_\alpha^\nu}\colon L^\infty(\Omega) \to C(\overline{\Omega})$ is bounded with norm $1/\alpha$: $\|\widetilde{G_\alpha^\nu}\| \leq 1/\alpha$.

The operators G_α^ν and $\widetilde{G_\alpha^\nu}$ can be visualized as follows:
First, since $a^{ij}(x)$, $b^i(x)$, $c(x) \in L^\infty(\Omega)$ and $v \in C^2(\overline{\Omega})$, it follows that

$$Av = \sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x)\frac{\partial v}{\partial x_i} + c(x)v \in L^\infty(\Omega). \tag{23.18}$$

Thus, if we let

$$w := \alpha G_\alpha^\nu v + \widetilde{G_\alpha^\nu}(Av),$$

$$\begin{array}{ccc}
D(\widetilde{G_\alpha^\nu}) = L^p(\Omega) & \xrightarrow{\widetilde{G_\alpha^\nu}} & C(\overline{\Omega}) \\
\uparrow & & \| \\
D(\widetilde{G_\alpha^\nu}) = L^\infty(\Omega) & \xrightarrow{\widetilde{G_\alpha^\nu}} & C(\overline{\Omega}) \\
\uparrow & & \uparrow \\
D(G_\alpha^\nu) = C(\overline{\Omega}) & \xrightarrow{G_\alpha^\nu} & D(\mathfrak{A}_\nu)
\end{array}$$

Fig. 23.2. The operators G_α^ν and $\widetilde{G_\alpha^\nu}$

then we have the assertions

$$\begin{cases} w \in W^{2,p}(\Omega) \cap C^1(\overline{\Omega}) & \text{for } n < p < \infty, \\ (A-\alpha)w = (A-\alpha)v & \text{almost everywhere in } \Omega, \\ L_\nu w = 0 & \text{on } \partial\Omega, \end{cases}$$

and so

$$\begin{cases} w - v \in W^{2,p}(\Omega) \cap C(\overline{\Omega}) & \text{for } n < p < \infty, \\ (A-\alpha)(w-v) = 0 & \text{almost everywhere in } \Omega, \\ L_\nu (w-v) = 0 & \text{on } \partial\Omega. \end{cases}$$

By applying Theorem 23.2 to the function $w(x) - v(x)$, we obtain that $w - v = 0$ in Ω. This implies that

$$v = w = \alpha G_\alpha^\nu v + \widetilde{G_\alpha^\nu}(Av).$$

Therefore, the desired assertion (23.16) for any $v \in C^2(\overline{\Omega})$ such that $L_\nu v = 0$ follows from an application of assertion (B) and assertion (23.18), since we have, for all $\alpha > 0$,

$$\|v - \alpha G_\alpha^\nu v\|_{C(\overline{\Omega})} = \|\widetilde{G_\alpha^\nu}(Av)\|_{C(\overline{\Omega})} \leq \frac{1}{\alpha}\|Av\|_{L^\infty(\Omega)}.$$

Now the proof of Theorem 1.2 is complete. □

23.5 Proof of Remark 1.2

Finally, we prove that the domain

$$D(\mathfrak{A}_\nu) = \left\{u \in W^{2,p}(\Omega) : Au \in C(\overline{\Omega}),\ L_\nu u = 0 \text{ on } \partial\Omega\right\}$$

is *independent* of p, for $n < p < \infty$.

We let

$$\mathcal{D}_p := \left\{ u \in W^{2,p}(\Omega) : Au \in C(\overline{\Omega}),\ L_\nu u = 0 \text{ on } \partial\Omega \right\}.$$

In order to prove Remark 1.2, it suffices to show that

$$\mathcal{D}_{p_1} = \mathcal{D}_{p_2} \quad \text{for } n < p_1 < p_2 < \infty.$$

First, it follows that

$$\mathcal{D}_{p_2} \subset \mathcal{D}_{p_1},$$

since we have the assertion

$$L^{p_2}(\Omega) \subset L^{p_1}(\Omega) \quad \text{for } p_2 > p_1.$$

Conversely, let v be an arbitrary element of $\mathcal{D}_{p_1}$:

$$v \in W^{2,p_1}(\Omega), \quad Av \in C(\overline{\Omega}),\ L_\nu u = 0 \text{ on } \partial\Omega.$$

Then, since we have the assertions

$$v,\ Av \in C(\overline{\Omega}) \subset L^{p_2}(\Omega),$$

it follows from an application of Theorem 23.2 with $p := p_2$ that there exists a unique function $u \in W^{2,p_2}(\Omega)$ such that

$$\begin{cases} (A-\alpha)u = (A-\alpha)v & \text{in } \Omega, \\ L_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence we have the assertions

$$\begin{cases} u - v \in W^{2,p_1}(\Omega), \\ (A-\alpha)(u-v) = 0 & \text{in } \Omega, \\ L_\nu(u-v) = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, by applying again Theorem 8.5 with $p := p_1$ we obtain that $u - v = 0$, so that $v = u \in W^{2,p_2}(\Omega)$. This implies that

$$v \in \mathcal{D}_{p_2}.$$

The proof of Remark 1.2 is complete. □

23.6 Notes and Comments

Section 23.2: Theorem 23.1 is inspired by Maugeri–Palagachev–Softova [47, Theorem 2.2.1].

Section 23.3: Theorem 23.2 is adapted from Maugeri–Palagachev [46, Theorem 4.1] and Maugeri–Palagachev–Softova [47, Theorem 2.2.2].

Section 23.3: The proof of Theorem 1.2 is adapted from Taira [78, Subsection 4.3].

24

Feller Semigroups and Boundary Value Problems

The purpose of this chapter is to prove a general existence theorem for Feller semigroups in terms of boundary value problems (Theorem 24.9), following the main idea of Taira [73, Section 9.6] and [79, Chapter 10] (cf. Bony–Courrège–Priouret [11], Sato–Ueno [62]). Intuitively, Theorem 24.9 asserts that we can "piece together" a Markov process on the boundary $\partial\Omega$ with A-diffusion in the interior Ω to construct a Markov process on the closure $\overline{\Omega} = \Omega \cup \partial\Omega$ (see Remark 24.5).

Let Ω be a bounded domain in Euclidean space $\mathbf{R}^n$, $n \geq 3$, with boundary $\partial\Omega$ of class $C^{1,1}$. Let A be a second-order, elliptic differential operator with real *discontinuous* coefficients of the form

$$Au := \sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u. \qquad (1.1)$$

We assume that the coefficients $a^{ij}(x)$, $b^i(x)$ and $c(x)$ of the differential operator A satisfy the following three conditions (1), (2) and (3):

(1) $a^{ij}(x) \in \mathrm{VMO} \cap L^\infty(\Omega)$, $a^{ij}(x) = a^{ji}(x)$ for almost all $x \in \Omega$ and $1 \leq i, j \leq n$, and there exist a constant $\lambda > 0$ such that

$$\frac{1}{\lambda}|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2$$

for almost all $x \in \Omega$ and all $\xi \in \mathbf{R}^n$.

(2) $b^i(x) \in L^\infty(\Omega)$ for $1 \leq i \leq n$.
(3) $c(x) \in L^\infty(\Omega)$ and $c(x) \leq 0$ for almost all $x \in \Omega$.

Let L be a first-order, *Ventcel' boundary condition* of the form

$$Lu := \mu(x')\frac{\partial u}{\partial \mathbf{n}} + \beta(x') \cdot u + \gamma(x')u - \delta(x')(Au|_{\partial\Omega}) \qquad (1.3)$$

$$:= L_\nu u - \delta(x')(Au|_{\partial\Omega}) \quad \text{on } \partial\Omega.$$

We assume that the coefficients $\mu(x')$, $\beta(x')$, $\gamma(x')$ and $\delta(x')$ of the boundary operator L satisfy the following four conditions (4), (5), (6) and (7):

(4) $\mu(x')$ is a Lipschitz continuous function on $\partial\Omega$ and $\mu(x') \geq 0$ on $\partial\Omega$.
(5) $\beta(x')$ is a Lipschitz continuous vector field on $\partial\Omega$.
(6) $\gamma(x')$ is a Lipschitz continuous function on $\partial\Omega$ and $\gamma(x') \leq 0$ on $\partial\Omega$.
(7) $\delta(x')$ is a Lipschitz continuous function on $\partial\Omega$ and $\delta(x') \geq 0$ on $\partial\Omega$.
(8) $\mathbf{n} = (n_1, n_2, \ldots, n_n)$ is the unit interior normal to the boundary $\partial\Omega$ (see Figure 1.2).

Now we are interested in the following functional analytic problem of construction of Markov processes with boundary conditions in probability:

Problem. Given a differential operator A and a Ventcel' boundary condition L, can we construct a Feller semigroup $\{T_t\}_{t\geq 0}$ on $\overline{\Omega}$ whose infinitesimal generator $\mathfrak{A}$ is characterized by the data (A, L) ?

24.1 Green Operators and Harmonic Operators

Let $n < p < \infty$ and $\alpha > 0$. Since we have the inequality

$$c(x) - \alpha \leq -\alpha \quad \text{for almost all } x \in \Omega,$$

by applying Theorem 23.1 to the operator $A - \alpha$ we obtain that the Dirichlet problem

$$\begin{cases} (\alpha - A)u = f & \text{almost everywhere in } \Omega, \\ u = \varphi & \text{on } \partial\Omega \end{cases} \tag{24.1}$$

has a unique solution

$$u \in W^{2,p}(\Omega)$$

for any $f \in C(\overline{\Omega})$ and any $\varphi \in C^2(\partial\Omega)$, since $C(\overline{\Omega}) \subset L^p(\Omega)$ and $C^2(\partial\Omega) \subset B^{2-1/p,p}(\partial\Omega)$. Therefore, we can introduce two linear operators

$$G_\alpha^0 \colon C(\overline{\Omega}) \longrightarrow C(\overline{\Omega}),$$

and

$$H_\alpha : C^2(\partial\Omega) \longrightarrow C(\overline{\Omega})$$

as follows:

(a) For any $f \in C(\overline{\Omega})$, the function $G_\alpha^0 f \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ is the unique solution of the problem

$$\begin{cases} (\alpha - A) G_\alpha^0 f = f & \text{in } \Omega, \\ G_\alpha^0 f = 0 & \text{on } \partial\Omega. \end{cases} \tag{24.2}$$

(b) For any $\varphi \in C^2(\partial\Omega)$, the function $H_\alpha \varphi \in W^{2,p}(\Omega)$ is the unique solution of the problem

$$\begin{cases} (\alpha - A) H_\alpha \varphi = 0 & \text{in } \Omega, \\ H_\alpha \varphi = \varphi & \text{on } \partial\Omega. \end{cases} \tag{24.3}$$

The operators G_α^0 and H_α can be visualized as follows: Here it

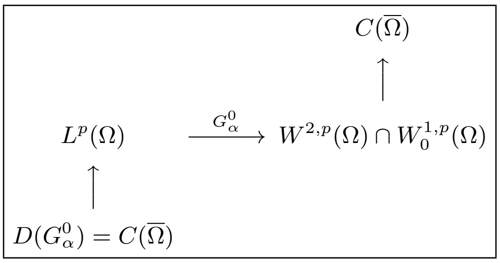

Fig. 24.1. The operator G_α^0

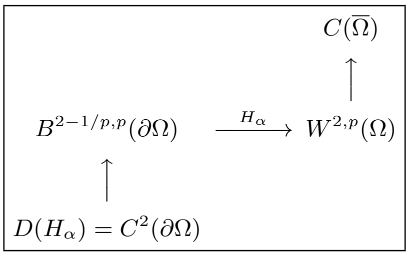

Fig. 24.2. The operator H_α

should be noticed that we have, by Sobolev's imbedding theorem (see [2, Theorem 4.12, Part II]; [80, Chapter 4]),

$$W^{2,p}(\Omega) \subset C^1(\overline{\Omega}) \quad \text{for } n < p < \infty,$$

and, by an imbedding theorem for Besov spaces (see [2, Theorem 7.34]),

$$C^2(\partial\Omega) \subset B^{2-1/p,p}(\partial\Omega) \subset C^1(\partial\Omega) \quad \text{for } n < p < \infty,$$

since $2 - n/p > 1$ and $(1 - 1/p)p = p - 1 > n - 1$ for $n < p < \infty$.

The operator G_α^0 is called the *Green operator* and the operator H_α is called the *harmonic operator*, respectively.

Then we have the following fundamental results for the operators G_α^0 and H_α:

Theorem 24.1. *Let $n < p < \infty$ and $\alpha > 0$. Then we have the following two assertions:*

(i) (a) The Green operators G_α^0 are non-negative and bounded with norm

$$\|G_\alpha^0\| = \|G_\alpha^0 1\|_{C(\overline{\Omega})} \leq \frac{1}{\alpha}. \tag{24.4}$$

(b) For any $f \in C(\overline{\Omega})$, we have the assertion

$$G_\alpha^0 f = 0 \quad \text{on } \partial\Omega.$$

(c) For all α, $\beta > 0$, the resolvent equation *holds true:*

$$G_\alpha^0 f - G_\beta^0 f + (\alpha - \beta) G_\alpha^0 (G_\beta^0 f) = 0 \quad \text{for } f \in C(\overline{\Omega}). \tag{24.5}$$

(d) For any $f \in C(\overline{\Omega})$, we have the assertion

$$\lim_{\alpha \to +\infty} \alpha G_\alpha^0 f(x_0) = f(x_0) \quad \text{for every point } x_0 \in \Omega. \tag{24.6}$$

Furthermore, if $f|_{\partial\Omega} = 0$, that is, if $f \in C_0(\overline{\Omega})$, then this convergence is uniform in $x \in \overline{\Omega}$. In other words, we have the assertion

$$\lim_{\alpha \to +\infty} \alpha G_\alpha^0 f = f \quad \text{in } C_0(\overline{\Omega}). \tag{24.7}$$

(ii) (e) The harmonic operators H_α, $\alpha > 0$, can be uniquely extended to non-negative, bounded linear operators on $C(\partial\Omega)$ into $C(\overline{\Omega})$, denoted again by H_α, with norm $\|H_\alpha\| = 1$.

(f) For any $\varphi \in C(\partial\Omega)$, we have the assertion

$$H_\alpha \varphi = \varphi \quad \text{on } \partial\Omega.$$

(g) For all α, $\beta > 0$, we have the equation

$$H_\alpha \varphi - H_\beta \varphi + (\alpha - \beta) G^0_\alpha (H_\beta \varphi) = 0 \quad \text{for } \varphi \in C(\partial\Omega). \tag{24.8}$$

Proof. (i) *Assertion (a)*: First, we show that the operators G^0_α are non-negative for all $\alpha > 0$:

$$f \in C(\overline{\Omega}),\ f(x) \geq 0 \quad \text{in } \Omega \implies G^0_\alpha f(x) \geq 0 \quad \text{in } \Omega.$$

If we let

$$v(x) := -G^0_\alpha f(x),$$

then it follows that

$$\begin{cases} (A - \alpha)v = f \geq 0 & \text{in } \Omega, \\ \gamma_0 v = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, by applying the weak maximum principle (Theorem 8.5) with $A := A - \alpha$ we obtain that

$$v(x) \leq 0 \quad \text{in } \Omega,$$

so that

$$G^0_\alpha f(x) = -v(x) \geq 0 \quad \text{in } \Omega.$$

Secondly, we show that the operators G^0_α are bounded with norm $1/\alpha$, for all $\alpha > 0$. To do this, it suffices to show that

$$G^0_\alpha 1(x) \leq \frac{1}{\alpha} \quad \text{on } \overline{\Omega}. \tag{24.4$'$}$$

since G^0_α are non-negative on $C(\overline{\Omega})$.

If we let

$$u(x) := \alpha G^0_\alpha 1(x) - 1 \in W^{2,p}(\Omega),$$

then it follows that

$$(A - \alpha)u(x) = -\alpha + (\alpha - c(x)) = -c(x) \geq 0 \quad \text{in } \Omega,$$

and that

$$u = -1 \quad \text{on } \partial\Omega.$$

Thus, by applying Theorem 8.5 with $A := A - \alpha$ we obtain that

$$\alpha G^0_\alpha 1(x) - 1 = u(x) \leq 0 \quad \text{on } \overline{\Omega}.$$

This proves the desired assertion (24.4′).

Assertion (b): It suffices to note that the function

$$G_\alpha^0 f \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$$

is the unique solution of the Dirichlet problem (24.2).

Assertion (c): This is an immediate consequence of the uniqueness theorem for problem (24.1) (Theorem 23.1). Indeed, it follows that the function

$$u := G_\alpha^0 f - G_\beta^0 f + (\alpha - \beta) G_\alpha^0 (G_\beta^0 f) \in W^{2,p}(\Omega)$$

satisfies the equation

$$\begin{aligned}
(\alpha - A)u &= f - (\alpha - A) G_\beta^0 f + (\alpha - \beta) G_\beta^0 f \\
&= f - (\beta - A + \alpha - \beta)\, G_\beta^0 f + (\alpha - \beta) G_\beta^0 f \\
&= f - f - (\alpha - \beta) G_\beta^0 f + (\alpha - \beta) G_\beta^0 f \\
&= 0 \quad \text{in } \Omega,
\end{aligned}$$

and the boundary condition

$$u = 0 \quad \text{on } \partial\Omega.$$

By applying Theorem 23.1 to the operator $A - \alpha$, we obtain that

$$u = 0 \quad \text{in } \Omega.$$

This proves the resolvent equation (24.5) for $f \in C(\overline{\Omega})$.

Assertion (d): First, let $f(x)$ be an arbitrary function in $C(\overline{\Omega})$ satisfying $f = 0$ on $\partial\Omega$. Then it follows from the uniqueness theorem for problem (24.1) that we have, for all α, $\beta > 0$,

$$f - \alpha G_\alpha^0 f = G_\alpha^0 \left((\beta - A) f \right) - \beta G_\alpha^0 f.$$

Thus we have, by estimate (24.4),

$$\left\| f - \alpha G_\alpha^0 f \right\|_{C(\overline{\Omega})} \leq \frac{1}{\alpha} \|(\beta - A) f\|_{C(\overline{\Omega})} + \frac{\beta}{\alpha} \|f\|_{C(\overline{\Omega})},$$

so that

$$\lim_{\alpha \to +\infty} \left\| f - \alpha G_\alpha^0 f \right\|_{C(\overline{\Omega})} = 0.$$

To prove assertion (24.6), let $f(x)$ be an arbitrary function in $C(\overline{\Omega})$ and let x_0 be an *arbitrary point* of Ω. Take a function $\psi(x) \in C(\overline{\Omega})$ such that

$$\begin{cases} 0 \le \psi(x) \le 1 & \text{on } \overline{\Omega}, \\ \psi(x) = 0 & \text{in a neighborhood of } x_0, \\ \psi(x) = 1 & \text{near the boundary } \partial\Omega. \end{cases}$$

Then it follows from the non-negativity of G_α^0 and estimate (24.4) that

$$0 \le \alpha G_\alpha^0 \psi(x_0) + \alpha G_\alpha^0 (1-\psi)(x_0) = \alpha G_\alpha^0 1(x_0) \le 1. \tag{24.9}$$

However, by applying assertion (24.7) to the function $1-\psi(x)$ we obtain that

$$\lim_{\alpha\to+\infty} \alpha G_\alpha^0 (1-\psi)(x_0) = (1-\psi)(x_0) = 1.$$

In view of inequalities (24.9), this implies that

$$\lim_{\alpha\to+\infty} \alpha G_\alpha^0 \psi(x_0) = 0.$$

Thus, since we have the inequality

$$-\|f\|_{C(\overline{\Omega})}\psi \le f\psi \le \|f\|_{C(\overline{\Omega})}\psi \quad \text{on } \overline{\Omega},$$

it follows that

$$|\alpha G_\alpha^0 (f\psi)(x_0)| \le \|f\|_{C(\overline{\Omega})}\ \alpha G_\alpha^0 \psi(x_0) \to 0 \quad \text{as } \alpha \to +\infty.$$

Therefore, by applying assertion (24.7) to the function $(1-\psi(x))f(x)$ we obtain that

$$\begin{aligned} f(x_0) &= \left((1-\psi)f\right)(x_0) \\ &= \lim_{\alpha\to+\infty} \alpha G_\alpha^0 \left((1-\psi)f\right)(x_0) \\ &= \lim_{\alpha\to+\infty} \alpha G_\alpha^0 f(x_0) \quad \text{for every point } x_0 \in \Omega. \end{aligned}$$

This proves the desired assertion (24.6).

(ii) *Assertion (e)*: First, let $\varphi(x')$ be an arbitrary function in $C^2(\partial\Omega)$ such that $\varphi \ge 0$ on $\partial\Omega$. Then we have the assertions

$$\begin{cases} (A-\alpha)(-H_\alpha\varphi) = 0 & \text{in } \Omega, \\ -H_\alpha\varphi = -\varphi \le 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, by applying Theorem 8.5 with $A := A - \alpha$ to the function $u := -H_\alpha\varphi$ we obtain that

$$H_\alpha\varphi \ge 0 \quad \text{on } \overline{\Omega}.$$

This proves the non-negativity of H_α.

In order to prove the boundedness of H_α

$$\|H_\alpha\| = 1,$$

it suffices to show that

$$H_\alpha 1(x) \leq 1 \quad \text{in } \Omega,$$

since H_α is non-negative.

To do this, we remark that the function $H_\alpha 1 - 1$ satisfies the conditions

$$\begin{cases} (A-\alpha)(H_\alpha 1 - 1) = -c(x) + \alpha \geq 0 & \text{in } \Omega, \\ H_\alpha 1 - 1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, by applying Theorem 8.5 with $A := A - \alpha$ and $u := H_\alpha 1 - 1$ we obtain that

$$H_\alpha 1(x) - 1 \leq 0 \quad \text{in } \Omega.$$

Since the space $C^2(\partial\Omega)$ is dense in $C(\partial\Omega)$, it follows that the operator

$$H_\alpha \colon C^2(\partial\Omega) \longrightarrow C(\overline{\Omega})$$

can be uniquely extended to a non-negative, bounded linear operator, denoted again by H_α,

$$H_\alpha \colon C(\partial\Omega) \longrightarrow C(\overline{\Omega}).$$

Assertion (f): This assertion follows from formula (24.3), since the space $C^2(\partial\Omega)$ is dense in $C(\partial\Omega)$ and since the operator $H_\alpha \colon C(\partial\Omega) \to C(\overline{\Omega})$ is bounded.

Assertion (g): We find from the uniqueness theorem for problem (24.3) (Theorem 23.1) that the desired equation (24.8) holds true for all $\varphi \in C^2(\partial\Omega)$. Hence it holds true for all $\varphi \in C(\partial\Omega)$, since the space $C^2(\partial\Omega)$ is dense in $C(\partial\Omega)$ and since the operators G_α^0 and H_α are bounded.

The proof of Theorem 24.1 is now complete. □

Summing up, we have the following diagrams for the Green operators

$$G_\alpha^0 \colon C(\overline{\Omega}) \longrightarrow C(\overline{\Omega})$$

and the harmonic operators

$$H_\alpha \colon C(\partial\Omega) \longrightarrow C(\overline{\Omega}).$$

The operators G_α^0 and H_α can be visualized as follows:

$$
\boxed{\begin{array}{ccc}
 & & C(\overline{\Omega}) \\
 & & \uparrow \\
C(\overline{\Omega}) & \xrightarrow{G_\alpha^0} & W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)
\end{array}}
$$

Fig. 24.3. The operator G_α^0

$$
\boxed{\begin{array}{ccc}
C(\partial\Omega) & \xrightarrow{H_\alpha} & C(\overline{\Omega}) \\
\uparrow & & \uparrow \\
B^{2-1/p,p}(\partial\Omega) & \xrightarrow{H_\alpha} & W^{2,p}(\Omega)
\end{array}}
$$

Fig. 24.4. The operator H_α

24.2 General Boundary Value Problems

Now we consider the following general boundary value problem in the framework of the spaces of *continuous functions*:

$$
\begin{cases} (\alpha - A)u = f & \text{in } \Omega, \\ Lu = 0 & \text{on } \partial\Omega. \end{cases} \tag{24.10}
$$

To do this, we introduce *three* linear operators associated with problem (24.10).

Step (I): First, we introduce a linear operator

$$
\boxed{\overline{\mathcal{A}} \colon C(\overline{\Omega}) \longrightarrow C(\overline{\Omega})}
$$

as follows:

(a) The domain $D(\overline{\mathcal{A}})$ of $\overline{\mathcal{A}}$ is the space

$$
D(\overline{\mathcal{A}}) = \left\{ u \in W^{2,p}(\Omega) : Au \in C(\overline{\Omega}) \right\} \quad (n < p < \infty).
$$

(b) $\overline{\mathcal{A}}u = Au$ for every $u \in D(\overline{\mathcal{A}})$.

Then we have the following:

Lemma 24.2. *The operator $\overline{\mathcal{A}}$ is a densely defined, closed linear operator in the space $C(\overline{\Omega})$.*

Proof. First, by the definition of $\overline{\mathcal{A}}$ and $\mathfrak{A}_\nu$ it follows that (see Table 24.5)

$$\mathfrak{A}_\nu \subset \overline{\mathcal{A}}.$$

This proves the *density* of the domain $D(\overline{\mathcal{A}})$ in $C(\overline{\Omega})$, since the domain $D(\mathfrak{A}_\nu)$ is dense in $C(\overline{\Omega})$ (see assertion (23.7)).

$$\begin{array}{ccc} D(\overline{\mathcal{A}}) & \xrightarrow{\overline{\mathcal{A}}} & C(\overline{\Omega}) \\ \uparrow & & \downarrow \\ D(\mathfrak{A}_\nu) & \xrightarrow{\mathfrak{A}_\nu} & C(\overline{\Omega}) \end{array}$$

Fig. 24.5. The operators $\mathfrak{A}_\nu$ and $\overline{\mathcal{A}}$

Now, let (u, v) be an arbitrary element of the product space $C(\overline{\Omega}) \oplus C(\overline{\Omega})$ such that there exists a sequence $\{u_n\} \subset D(\overline{\mathcal{A}})$ which satisfies the conditions

$$u_n \longrightarrow u \quad \text{in } C(\overline{\Omega}),$$
$$Au_n \longrightarrow v \quad \text{in } C(\overline{\Omega}).$$

Then we have, by the boundedness of G_α^0,

$$G_\alpha^0(Au_n) = \alpha G_\alpha^0 u_n - u_n \longrightarrow \alpha G_\alpha^0 u - u \quad \text{in } C(\overline{\Omega}),$$

and also

$$G_\alpha^0(Au_n) \longrightarrow G_\alpha^0 v \quad \text{in } C(\overline{\Omega}).$$

This proves that

$$u = \alpha G_\alpha^0 u - G_\alpha^0 v \in W^{2,p}(\Omega). \tag{24.11}$$

Thus, by applying the operator $\alpha - A$ to the both hand sides of formula (24.11) we obtain that

$$(\alpha - A)u = \alpha(\alpha - A)G_\alpha^0 u - (\alpha - A)G_\alpha^0 v = \alpha u - v,$$

so that

$$Au = v \in C(\overline{\Omega}).$$

Summing up, we have proved that

$$\begin{cases} u \in D(\overline{\mathcal{A}}), \\ \overline{\mathcal{A}}u = v. \end{cases}$$

This proves the closedness of $\overline{\mathcal{A}}$.

The proof of Lemma 24.2 is complete. □

Remark 24.1. The domain $D(\overline{\mathcal{A}})$ does not depend on p, for $n < p < \infty$ (see Section 24.4).

The extended operators $G_\alpha^0 \colon C(\overline{\Omega}) \to C(\overline{\Omega})$ and $H_\alpha \colon C(\partial\Omega) \to C(\overline{\Omega})$ ($\alpha > 0$) still satisfy formulas (24.2) and (24.3) respectively in the following sense:

Lemma 24.3. *(i) For any $f \in C(\overline{\Omega})$, we have the assertions*

$$\begin{cases} G_\alpha^0 f \in D(\overline{\mathcal{A}}), \\ (\alpha I - \overline{\mathcal{A}}) G_\alpha^0 f = f. \end{cases} \tag{24.12}$$

(ii) For any $\varphi \in C(\partial\Omega)$, we have the assertions

$$\begin{cases} H_\alpha \varphi \in D(\overline{\mathcal{A}}), \\ (\alpha I - \overline{\mathcal{A}}) H_\alpha \varphi = 0. \end{cases} \tag{24.13}$$

Proof. Assertion (i): If $f \in C(\overline{\Omega})$, then it follows from the definition of G_α^0 that

$$G_\alpha^0 f \in W^{2,p}(\Omega),$$
$$A(G_\alpha^0 f) = \alpha G_\alpha^0 f - f \in C(\overline{\Omega}).$$

This proves the desired assertions (24.12).

Assertion (ii): If $\varphi \in C(\partial\Omega)$, we can find a sequence $\{\varphi_j\}$ in the space $C^2(\partial\Omega)$ such that

$$\varphi_j \longrightarrow \varphi \quad \text{in } C(\partial\Omega).$$

Hence, we have, by the boundedness of H_α,

$$H_\alpha \varphi_j \longrightarrow H_\alpha \varphi \quad \text{in } C(\overline{\Omega}).$$

However, it follows that

$$H_\alpha \varphi_j \in W^{2,p}(\Omega),$$
$$A(H_\alpha \varphi_j) = \alpha H_\alpha \varphi_j \in C(\overline{\Omega}),$$

so that

$$H_\alpha \varphi_j \in D(\overline{\mathcal{A}}).$$

Therefore, we have proved that

$$H_\alpha \varphi_j \in D(\overline{\mathcal{A}}),$$

$$H_\alpha \varphi_j \longrightarrow H_\alpha \varphi \quad \text{in } C(\overline{\Omega}),$$
$$A(H_\alpha \varphi_j) \longrightarrow \alpha\, H_\alpha \varphi \quad \text{in } C(\overline{\Omega}).$$

This proves the desired assertions

$$\begin{cases} H_\alpha \varphi \in D(\overline{\mathcal{A}}), \\ \overline{\mathcal{A}}\,(H_\alpha \varphi) = \alpha\, H_\alpha \varphi, \end{cases}$$

since the operator $\overline{\mathcal{A}}$ is closed.

The proof of Lemma 24.3 is complete. □

Corollary 24.4. *Every function u in $D(\overline{\mathcal{A}})$ can be written in the form*

$$u = G_\alpha^0 \left((\alpha I - \overline{\mathcal{A}})u \right) + H_\alpha(u|_{\partial\Omega}) \quad \text{for each } \alpha > 0. \tag{24.14}$$

Proof. We let

$$w = u - G_\alpha^0 \left((\alpha I - \overline{\mathcal{A}})u \right) - H_\alpha(u|_{\partial\Omega}).$$

Then it follows from Lemma 24.3 that the function w is in $D(\overline{\mathcal{A}})$ and satisfies the conditions

$$\begin{cases} (\alpha I - \overline{\mathcal{A}})w = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, we can apply Theorem 23.1 to the operator $A - \alpha$ to obtain that

$$0 = w = u - G_\alpha^0 \left((\alpha I - \overline{\mathcal{A}})u \right) - H_\alpha(u|_{\partial\Omega}).$$

This proves the desired formula (24.14).

The proof of Corollary 24.4 is complete. □

Step (II): Secondly, we introduce a linear operator

$$\boxed{\overline{LG_\alpha^0} \colon C(\overline{\Omega}) \longrightarrow C(\partial\Omega)}$$

as follows:

(a) The domain $D\left(\overline{LG_\alpha^0}\right)$ of $\overline{LG_\alpha^0}$ is the space $C(\overline{\Omega})$.

(b) $\overline{LG_\alpha^0} f = L(G_\alpha^0 f) = \mu(x') \dfrac{\partial}{\partial \mathbf{n}}(G_\alpha^0 f) + \delta(x')(f|_{\partial\Omega})$ for every $f \in D\left(\overline{LG_\alpha^0}\right)$.

$$\begin{array}{ccccc} L^p(\Omega) & \xrightarrow{G_\alpha^0} & W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) & \xrightarrow{L} & C(\partial\Omega) \\ \uparrow & & & & \\ D(\overline{LG_\alpha^0}) = C(\overline{\Omega}) & & & & \end{array}$$

Fig. 24.6. The operator $\overline{LG_\alpha^0}$

The operator $\overline{LG_\alpha^0}$ can be visualized as follows: Here it should be emphasized that we have, by Sobolev's imbedding theorem (see [2, Theorem 4.12, Part II]; [80, Chapter 4]),

$$G_\alpha^0 f \in W^{2,p}(\Omega) \subset C^1(\overline{\Omega}) \quad \text{for } n < p < \infty,$$

since $2 - n/p > 1$ for $n < p < \infty$.

Then we have the following:

Lemma 24.5. *The operators* $\overline{LG_\alpha^0} \colon C(\overline{\Omega}) \to C(\partial\Omega)$ *are non-negative and bounded for all* $\alpha > 0$.

Proof. Let f be an arbitrary function in $D(\overline{LG_\alpha^0}) = C(\overline{\Omega})$ such that $f(x) \geq 0$ on $\overline{\Omega}$. Then we have the assertions

$$\begin{cases} G_\alpha^0 f \in C^1(\overline{\Omega}), \\ G_\alpha^0 f \geq 0 & \text{on } \overline{\Omega}, \\ G_\alpha^0 f = 0 & \text{on } \partial\Omega, \end{cases}$$

and so

$$\begin{aligned} \overline{LG_\alpha^0} f(x') &= L(G_\alpha^0 f)(x') \\ &= \mu(x') \frac{\partial}{\partial \mathbf{n}} (G_\alpha^0 f)(x') + \delta(x') f(x') \\ &\geq 0 \quad \text{on } \partial\Omega. \end{aligned}$$

This proves that the operator $\overline{LG_\alpha^0}$ is non-negative.

By the non-negativity of $\overline{LG_\alpha^0}$, we have, for all $f \in D(\overline{LG_\alpha^0})$,

$$-\overline{LG_\alpha^0} \|f\|_{C(\overline{\Omega})} \leq \overline{LG_\alpha^0} f \leq \overline{LG_\alpha^0} \|f\|_{C(\overline{\Omega})} \quad \text{on } \partial\Omega.$$

This implies the boundedness of $\overline{LG_\alpha^0}$ with norm

$$\|\overline{LG_\alpha^0}\| = \|L(G_\alpha^0 1)\|_{C(\partial\Omega)}.$$

The proof of Lemma 24.5 is complete. □

Remark 24.2. Similarly, we can prove that the operators

$$LG_\alpha^0 \colon L^\infty(\Omega) \longrightarrow C(\partial\Omega)$$

are non-negative and bounded for all $\alpha > 0$, with norm

$$\|LG_\alpha^0\| = \|L(G_\alpha^0 1)\|_{C(\partial\Omega)}.$$

The operator LG_α^0 can be visualized as follows:

$$\begin{array}{ccccc} L^p(\Omega) & \xrightarrow{G_\alpha^0} & W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) & \xrightarrow{L} & C(\partial\Omega) \\ \uparrow & & & & \\ D(LG_\alpha^0) = L^\infty(\Omega) & & & & \end{array}$$

Fig. 24.7. The operator LG_α^0

The next lemma states a fundamental relationship between the operators $\overline{LG_\alpha^0}$ and $\overline{LG_\beta^0}$ for α, $\beta > 0$:

Lemma 24.6. *For any α, $\beta > 0$, we have the equation*

$$\overline{LG_\alpha^0} f - \overline{LG_\beta^0} f + (\alpha - \beta)\overline{LG_\alpha^0}\,(G_\beta^0 f) = 0 \quad \textit{for } f \in C(\overline{\Omega}). \tag{24.15}$$

Proof. We have, by the *resolvent equation*

$$G_\alpha^0 f - G_\beta^0 f + (\alpha - \beta) G_\alpha^0 (G_\beta^0 f) = 0. \tag{24.5}$$

Therefore, the desired formula (24.15) follows by applying the operator L to the both hand sides of equation (24.5).

The proof of Lemma 24.6 is complete. □

Remark 24.3. The equation (24.15) remains valid for $f \in L^\infty(\Omega)$:

$$LG_\alpha^0 f - LG_\beta^0 f + (\alpha - \beta)\overline{LG_\alpha^0}\,(G_\beta^0 f) = 0, \quad f \in L^\infty(\Omega). \tag{24.15$'$}$$

Indeed, it suffices to note that the function

$$u := G_\alpha^0 f - G_\beta^0 f + (\alpha - \beta) G_\alpha^0 (G_\beta^0 f) \in W^{2,p}(\Omega)$$

is a unque solution of the Dirichlet problem

$$\begin{cases} (\alpha - A)u = 0 & \text{almost everywhere in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Step (III): Finally, we introduce a linear operator

$$\boxed{LH_\alpha \colon C(\partial\Omega) \longrightarrow C(\partial\Omega)}$$

as follows:

(a) The domain $D\,(LH_\alpha)$ of LH_α is the space $B^{2-1/p,p}(\partial\Omega)$.

(b) $LH_\alpha\psi = \mu(x')\dfrac{\partial}{\partial \mathbf{n}}(H_\alpha\psi) + \beta(x') \cdot \psi + \gamma(x')\psi - \alpha\delta(x')\psi$ for every $\psi \in D\,(LH_\alpha)$.

The operator LH_α can be visualized as follows:

$$\begin{array}{ccccc} C(\partial\Omega) & \xrightarrow{H_\alpha} & W^{2,p}(\Omega) & \xrightarrow{L} & C(\partial\Omega) \\ \uparrow & & & & \\ D(LH_\alpha) = B^{2-p/1,p}(\partial\Omega) & & & & \end{array}$$

Fig. 24.8. The operator LH_α

Then we have the following:

Lemma 24.7. *For any $\alpha > 0$, the operator LH_α has its minimal closed extension $\overline{LH_\alpha}$ in the space $C(\partial\Omega)$.*

Proof. We apply part (i) of Theorem 21.11 with

$$K := \partial\Omega, \quad B := LH_\alpha.$$

To do this, it suffices to show that the operator LH_α satisfies condition (β') with $K := \partial\Omega$ (or condition (β) with $K := K_0 = \partial\Omega$) of the same theorem.

Assume that a function φ in the domain $D(LH_\alpha) = B^{2-1/p,p}(\partial\Omega)$ takes its positive maximum at some point x'_0 of $\partial\Omega$. Since the function

$$H_\alpha\varphi \in W^{2,p}(\Omega)$$

satisfies the conditions

$$\begin{cases} (A-\alpha)H_\alpha\varphi = 0 & \text{in } \Omega, \\ H_\alpha\varphi = \varphi & \text{on } \partial\Omega, \end{cases}$$

by applying the weak maximum principle (Theorem 8.5) with $A := A - \alpha$ to the function $H_\alpha\psi$, we find that the function $H_\alpha\varphi$ takes its positive maximum at a boundary point $x'_0 \in \partial\Omega$. Thus we can apply Hopf's boundary point lemma (Lemma 8.7) to obtain that

$$\frac{\partial}{\partial \mathbf{n}}(H_\alpha\varphi)(x'_0) < 0. \tag{24.16}$$

However, it should be noticed that the coefficients of the boundary condition L satisfy the conditions

$$\begin{aligned} \mu(x') &> 0 \quad \text{on } \partial\Omega, \\ \gamma(x') &< 0 \quad \text{on } \partial\Omega, \\ \delta(x') &\geq 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Hence we have the inequality

$$\begin{aligned} LH_\alpha\varphi(x'_0) &= \mu(x'_0)\frac{\partial}{\partial \mathbf{n}}(H_\alpha\varphi)(x'_0) + \beta(x'_0)\cdot\varphi(x'_0) \\ &\quad + \gamma(x'_0)\varphi(x'_0) - \alpha\delta(x'_0)\varphi(x'_0) \\ &= \mu(x'_0)\frac{\partial}{\partial \mathbf{n}}(H_\alpha\varphi)(x'_0) + \gamma(x'_0)\varphi(x'_0) - \alpha\delta(x'_0)\varphi(x'_0) \\ &< 0. \end{aligned}$$

This verifies condition (β') of Theorem 21.11.

The proof of Lemma 24.7 is complete. □

Remark 24.4. The closed operator $\overline{LH_\alpha}$ enjoys the following property:

If a function φ in the domain $D\left(\overline{LH_\alpha}\right)$ takes its *positive* maximum at some point x'_0 of $\partial\Omega$, then we have the inequality (24.17)

$$\overline{LH_\alpha}\varphi(x'_0) \leq 0.$$

The operators LH_α and $\overline{LH_\alpha}$ can be visualized as follows:

$$\begin{array}{ccc} C(\partial\Omega) & & \\ \uparrow & & \\ D(\overline{LH_\alpha}) & \xrightarrow{\overline{LH_\alpha}} & C(\partial\Omega) \\ \uparrow & & \| \\ D(LH_\alpha) = B^{2-p/1,p}(\partial\Omega) & \xrightarrow[LH_\alpha]{} & C(\partial\Omega) \end{array}$$

Fig. 24.9. The operators LH_α and $\overline{LH_\alpha}$

The next lemma states a fundamental relationship between the operators $\overline{LH_\alpha}$ and $\overline{LH_\beta}$ for α, $\beta > 0$:

Lemma 24.8. *The domain $D(\overline{LH_\alpha})$ of $\overline{LH_\alpha}$ does not depend on $\alpha > 0$; so we denote by $\mathcal{D}$ the common domain. Then we have, for all α, $\beta > 0$,*

$$\overline{LH_\alpha}\varphi - \overline{LH_\beta}\varphi + (\alpha - \beta)\overline{LG_\alpha^0}\,(H_\beta\varphi) = 0 \quad \textit{for every } \varphi \in \mathcal{D}. \quad (24.18)$$

Proof. Let φ be an arbitrary function in $D\left(\overline{LH_\beta}\right)$:

$$\varphi \in D\left(\overline{LH_\beta}\right),$$

and choose a sequence $\{\varphi_j\}$ in $D(LH_\beta) = B^{2-1/p,p}(\partial\Omega)$ such that

$$\begin{cases} \varphi_j \longrightarrow \varphi & \text{in } C(\partial\Omega), \\ LH_\beta\varphi_j \longrightarrow \overline{LH_\beta}\varphi & \text{in } C(\partial\Omega). \end{cases}$$

Then it follows from the boundedness of H_β and $\overline{LG_\alpha^0}$ that

$$\overline{LG_\alpha^0}(H_\beta\varphi_j) \longrightarrow \overline{LG_\alpha^0}(H_\beta\varphi) \quad \text{in } C(\partial\Omega).$$

Therefore, by using formula (24.8) with $\varphi := \varphi_j$ we obtain that

$$\begin{aligned} LH_\alpha\varphi_j &= LH_\beta\varphi_j - (\alpha - \beta)\overline{LG_\alpha^0}(H_\beta\varphi_j) \\ &\longrightarrow \overline{LH_\beta}\varphi - (\alpha - \beta)\overline{LG_\alpha^0}(H_\beta\varphi) \quad \text{in } C(\partial\Omega). \end{aligned}$$

Since the operator $\overline{LH_\alpha}$ is closed, it follows that

$$\begin{cases} \varphi \in D\left(\overline{LH_\alpha}\right), \\ \overline{LH_\alpha}\varphi = \overline{LH_\beta}\varphi - (\alpha - \beta)\overline{LG_\alpha^0}(H_\beta\varphi). \end{cases}$$

This proves the desired equation (24.18).

Conversely, we have, by interchanging α and β,

$$D\left(\overline{LH_\alpha}\right) \subset D\left(\overline{LH_\beta}\right),$$

and so

$$D\left(\overline{LH_\alpha}\right) = D\left(\overline{LH_\beta}\right) \quad \text{for all } \alpha,\ \beta > 0.$$

The proof of Lemma 24.8 is complete. □

24.3 General Existence Theorem for Feller Semigroups

Now we can give a general existence theorem for Feller semigroups on $\partial\Omega$ in terms of boundary value problem (24.10). The next theorem asserts that the closed operator $\overline{LH_\alpha}$ is the infinitesimal generator of some Feller semigroup on $\partial\Omega$ if and only if problem (24.10) is solvable for sufficiently *many* functions φ in the space $C(\partial\Omega)$:

Theorem 24.9. *Let $n < p < \infty$ and $\alpha > 0$. Then we have the following two assertions:*

(i) If the closed operator $\overline{LH_\alpha}$ for $\alpha > 0$ is the infinitesimal generator of a Feller semigroup on $\partial\Omega$, then, for each constant $\lambda > 0$ the boundary value problem

$$\begin{cases} (\alpha - A)u = 0 & \text{in } \Omega, \\ (\lambda - L)u = \varphi & \text{on } \partial\Omega \end{cases} \tag{24.19}$$

has a solution $u \in W^{2,p}(\Omega)$ for any φ in some dense *subset of $C(\partial\Omega)$.*

(ii) Conversely, if, for some constant $\lambda \geq 0$, problem (24.19) has a solution $u \in W^{2,p}(\Omega)$, $n < p < \infty$, for any φ in some dense *subset of $C(\partial\Omega)$, then the closed operator $\overline{LH_\alpha}$ is the infinitesimal generator of some Feller semigroup on $\partial\Omega$.*

Proof. Assertion (i): If the operator $\overline{LH_\alpha}$ generates a Feller semigroup on $\partial\Omega$, by applying part (i) of Theorem 21.11 with $K := \partial\Omega$ to the operator $\overline{LH_\alpha}$ we obtain that

$$R\left(\lambda I - \overline{LH_\alpha}\right) = C(\partial\Omega) \quad \text{for each } \lambda > 0.$$

This implies that the range $R\left(\lambda I - LH_\alpha\right)$ is a dense subset of $C(\partial\Omega)$ for each $\lambda > 0$. However, if $\varphi \in C(\partial\Omega)$ is in the range $R\left(\lambda I - LH_\alpha\right)$, and if $\varphi = \left(\lambda I - LH_\alpha\right)\psi$ with $\psi \in B^{2-1/p,p}(\partial\Omega)$, then the function

$$u = H_\alpha \psi \in W^{2,p}(\Omega)$$

is a solution of problem (24.19). This proves the desired assertion (i).

Assertion (ii): We apply part (ii) of Theorem 21.11 with $K := \partial\Omega$ to the operator LH_α. To do this, it suffices to show that the operator LH_α satisfies condition (γ) of the same theorem, since it satisfies condition (β'), as is shown in the proof of Lemma 24.7.

By the uniqueness theorem for problem (24.1) (Theorem 23.1), it follows that every function $u \in W^{2,p}(\Omega)$ which satisfies the equation

$$(\alpha - A)u = 0 \quad \text{in } \Omega$$

can be written in the form

$$u = H_\alpha\left(u|_{\partial\Omega}\right), \quad u|_{\partial\Omega} \in B^{2-1/p,p}(\partial\Omega) = D\left(LH_\alpha\right).$$

Thus we find that if there exists a solution $u \in W^{2,p}(\Omega)$ of problem (24.19) for a function $\varphi \in C(\partial\Omega)$, then we have the formula

$$(\lambda I - LH_\alpha)(u|_{\partial\Omega}) = \varphi,$$

and so

$$\varphi \in R(\lambda I - LH_\alpha).$$

Therefore, if there exists a constant $\lambda \geq 0$ such that problem (24.19) has a solution u in $W^{2,p}(\Omega)$ for any φ in some dense subset of $C(\partial\Omega)$, then the range $R(\lambda I - LH_\alpha)$ is dense in $C(\partial\Omega)$. This verifies condition (γ) of Theorem 21.11 with $\alpha_0 := \lambda$. Hence the desired assertion (ii) follows from an application of the same theorem.

The proof of Theorem 24.9 is complete. □

Remark 24.5. Intuitively, Theorem 24.9 asserts that we can "piece together" a Markov process on the boundary $\partial\Omega$ with A-diffusion in the interior Ω to construct a Markov process on the closure $\overline{\Omega} = \Omega \cup \partial\Omega$. The situation may be represented schematically by Figure 24.10.

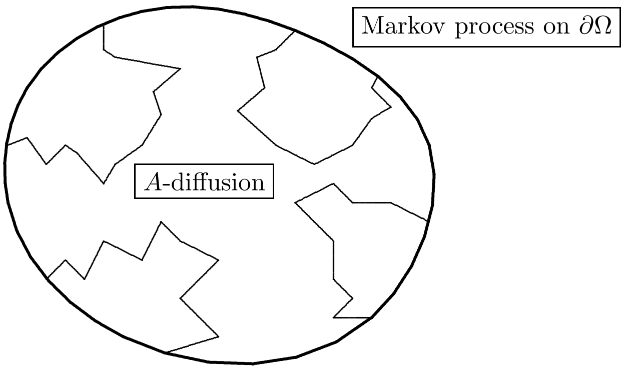

Fig. 24.10. A Markov process on $\partial\Omega$ can be "pieced together" with A-diffusion in Ω

We conclude this section by giving a precise meaning to the boundary conditions Lu for functions u in the domain $D(\overline{\mathcal{A}})$.

We let

$$\begin{aligned} D(L) &= \left\{u \in D(\overline{\mathcal{A}}) : u|_{\partial\Omega} \in \mathcal{D}\right\} \\ &= \left\{u \in W^{2,p}(\Omega) : Au \in C(\overline{\Omega}),\ u|_{\partial\Omega} \in \mathcal{D}\right\}, \end{aligned}$$

where $\mathcal{D}$ is the common domain of the operators $\overline{LH_\alpha}$ for all $\alpha > 0$ (see Lemma 24.8):

$$\mathcal{D} = D\left(\overline{LH_\alpha}\right) \quad \text{for all } \alpha > 0.$$

It should be noticed that the domain $D(L)$ contains $W^{2,p}(\Omega)$ for $n < p < \infty$, since $B^{2-1/p,p}(\partial\Omega) = D\left(LH_\alpha\right) \subset \mathcal{D}$. Moreover, Corollary 24.4 asserts that every function u in $D(L) \subset D(\overline{\mathcal{A}})$ can be written in the form

$$u = G_\alpha^0\left((\alpha I - \overline{\mathcal{A}})u\right) + H_\alpha\left(u|_{\partial\Omega}\right) \quad \text{for } \alpha > 0. \tag{24.14}$$

Then we define the boundary condition Lu by the formula

$$Lu = \overline{LG_\alpha^0}\left((\alpha I - \overline{\mathcal{A}})u\right) + \overline{LH_\alpha}\left(u|_{\partial\Omega}\right). \tag{24.20}$$

The next lemma justifies the definition (24.20) of Lu for every $u \in D(L)$:

Lemma 24.10. *The right-hand side of formula* (24.20) *depends only on* u, *not on the choice of expression* (24.14).

Proof. Assume that

$$\begin{aligned} u &= G_\alpha^0\left((\alpha I - \overline{\mathcal{A}})u\right) + H_\alpha\left(u|_{\partial\Omega}\right) \\ &= G_\beta^0\left((\beta I - \overline{\mathcal{A}})u\right) + H_\beta\left(u|_{\partial\Omega}\right), \end{aligned}$$

where $\alpha > 0$ and $\beta > 0$. Then it follows from formula (24.15) with $f := (\alpha I - \overline{\mathcal{A}})u$ and formula (24.18) with $\psi := u|_{\partial\Omega}$ that

$$\begin{aligned} &\overline{LG_\alpha^0}\left((\alpha I - \overline{\mathcal{A}})u\right) + \overline{LH_\alpha}\left(u|_{\partial\Omega}\right) \\ &= \overline{LG_\beta^0}\left((\alpha I - \overline{\mathcal{A}})u\right) - (\alpha-\beta)\overline{LG_\alpha^0}G_\beta^0\left((\alpha I - \overline{\mathcal{A}})u\right) \\ &\quad + \overline{LH_\beta}\left(u|_{\partial\Omega}\right) - (\alpha-\beta)\overline{LG_\alpha^0}H_\beta\left(u|_{\partial\Omega}\right) \\ &= \overline{LG_\beta^0}\left((\beta I - A)u\right) + \overline{LH_\beta}\left(u|_{\partial\Omega}\right) \\ &\quad + (\alpha-\beta)\left\{\overline{LG_\beta^0}u - \overline{LG_\alpha^0}G_\beta^0\left(\alpha I - \overline{\mathcal{A}}\right)u - \overline{LG_\alpha^0}H_\beta(u|_{\partial\Omega})\right\}. \end{aligned} \tag{24.21}$$

However, the last term of formula (24.21) vanishes. Indeed, it follows from formula (24.14) with $\alpha := \beta$ and formula (24.15) with $f := u$ that

$$\begin{aligned} &\overline{LG_\beta^0}u - \overline{LG_\alpha^0}\left(G_\beta^0(\alpha I - \overline{\mathcal{A}})u\right) - \overline{LG_\alpha^0}H_\beta\left(u|_{\partial\Omega}\right) \\ &= \overline{LG_\beta^0}u - \overline{LG_\alpha^0}\left(G_\beta^0(\beta I - \overline{\mathcal{A}})u + H_\beta\left(u|_{\partial\Omega}\right) + (\alpha-\beta)G_\beta^0 u\right) \\ &= \overline{LG_\beta^0}u - \overline{LG_\alpha^0}u - (\alpha-\beta)\overline{LG_\alpha^0}G_\beta^0 u \\ &= 0. \end{aligned}$$

Therefore, we obtain from formula (24.21) that

$$\overline{LG^0_\alpha}\left((\alpha I - \overline{\mathcal{A}})u\right) + \overline{LH_\alpha}\,(u|_{\partial\Omega}) = \overline{LG^0_\beta}\left((\beta I - \overline{\mathcal{A}})u\right) + \overline{LH_\beta}\,(u|_{\partial\Omega})\,.$$

This proves Lemma 24.10. □

24.4 Proof of Remark 24.1

Finally, we prove that the domain

$$D(\overline{\mathcal{A}}) = \left\{u \in W^{2,p}(\Omega) : Au \in C(\overline{\Omega})\right\}$$

is *independent* of p, for $n < p < \infty$.

We let

$$\mathcal{E}_p := \left\{u \in W^{2,p}(\Omega) : Au \in C(\overline{\Omega})\right\}.$$

In order to prove Remark 24.1, it suffices to show that

$$\mathcal{E}_{p_1} = \mathcal{E}_{p_2} \quad \text{for } n < p_1 < p_2 < \infty.$$

First, it follows that

$$\mathcal{E}_{p_2} \subset \mathcal{E}_{p_1},$$

since we have the assertion

$$L^{p_2}(\Omega) \subset L^{p_1}(\Omega) \quad \text{for } p_2 > p_1.$$

Conversely, let v be an arbitrary element of $\mathcal{E}_{p_1}$:

$$v \in W^{2,p_1}(\Omega), \quad Av \in C(\overline{\Omega}).$$

Then, since we have the assertions

$$v,\ Av \in C(\overline{\Omega}) \subset L^{p_2}(\Omega),$$

it follows from an application of Theorem 22.1 with $p := p_2$ and $\varphi := 0$ that

$$G^0_\alpha((\alpha - A)v) \in W^{2,p_2}(\Omega). \tag{24.22}$$

Moreover, we can find a sequence $\{\varphi_j\}$ in $C^2(\partial\Omega)$ such that

$$\varphi_j \longrightarrow v|_{\partial\Omega} \quad \text{in } C(\partial\Omega).$$

Then we have the assertions

$$\begin{aligned} &H_\alpha\varphi_j \in W^{2,p_2}(\Omega),\\ &H_\alpha\varphi_j \longrightarrow H_\alpha(v|_{\partial\Omega}) \quad \text{in } C(\overline{\Omega}),\\ &A(H_\alpha\varphi_j) = \alpha\, H_\alpha\varphi_j \longrightarrow \alpha\, H_\alpha(v|_{\partial\Omega}) \quad \text{in } C(\overline{\Omega}). \end{aligned}$$

However, since the operator $\overline{\mathcal{A}}\colon \mathcal{E}_{p_2} \to C(\overline{\Omega})$ is closed, it follows that

$$\begin{cases} H_\alpha(v|_{\partial\Omega}) \in \mathcal{E}_{p_2} \subset W^{2,p_2}(\Omega), \\ \overline{\mathcal{A}} H_\alpha(v|_{\partial\Omega}) = \alpha\, H_\alpha(v|_{\partial\Omega}). \end{cases} \tag{24.23}$$

Therefore, by applying Corollary 24.4 with $u := v$ we obtain from assertions (24.22) and (24.23) that

$$v = G_\alpha^0((\alpha - A)v) + H_\alpha(v|_{\partial\Omega}) \in W^{2,p_2}(\Omega).$$

This implies that

$$v \in \mathcal{E}_{p_2}.$$

The proof of Remark 24.1 is complete. □

24.5 Notes and Comments

This chapter is adapted from Bony–Courrège–Priouret [11], Sato–Ueno [62] and Taira [73, Section 9.6] and [79, Chapter 10].

25

Proof of Theorem 1.1

This Chapter 25 is devoted to the proof of Theorem 1.1. Let L be a first-order, Ventcel' boundary condition of the form

$$\begin{aligned} Lu &:= \mu(x') \frac{\partial u}{\partial \mathbf{n}} + \beta(x') \cdot u + \gamma(x') u - \delta(x') \left(Au|_{\partial\Omega}\right) \\ &:= L_\nu u - \delta(x') \left(Au|_{\partial\Omega}\right) \quad \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

Here the functions $\mu(x')$, $\beta(x')$, $\gamma(x')$ and $\delta(x')$ satisfy the following four conditions (4), (5), (6) and (7):

(i) $\mu(x')$ is a Lipschitz continuous function on $\partial\Omega$ and $\mu(x') \geq 0$ on $\partial\Omega$.
(ii) $\beta(x')$ is a Lipschitz continuous vector field on $\partial\Omega$.
(iii) $\gamma(x')$ is a Lipschitz continuous function on $\partial\Omega$ and $\gamma(x') \leq 0$ on $\partial\Omega$.
(iv) $\delta(x')$ is a Lipschitz continuous function on $\partial\Omega$ and $\delta(x') \geq 0$ on $\partial\Omega$.
(v) $\mathbf{n} = (n_1, n_2, \ldots, n_n)$ is the unit interior normal to the boundary $\partial\Omega$ (see Figure 1.2).

The crucial point in the proof is that we consider the term

$$\delta(x') \left(Au|_{\partial\Omega}\right)$$

of sticking phenomenon in the Ventcel' boundary condition

$$Lu = L_\nu u - \delta(x') \left(Au|_{\partial\Omega}\right) \quad \text{on } \partial\Omega$$

as a term of "perturbation" of the oblique derivative boundary condition

$$L_\nu u := \mu(x') \frac{\partial u}{\partial \mathbf{n}} + \beta(x') \cdot u + \gamma(x') u \quad \text{on } \partial\Omega.$$

More precisely, we make use of a generation theorem for Feller semigroups with oblique derivative boundary condition L_ν (Theorem 1.2) to verify all the conditions of a version of the Hille–Yosida theorem adapted to the present context (Theorem 21.11) for the operator $\mathfrak{A}$ defined by formula (1.4).

25.1 End of Proof of Theorem 1.1

In this section we shall prove Theorem 1.1. To do so, we apply part (ii) of Theorem 21.11 to the operator $\mathfrak{A}$ defined by formula (1.4). The proof is divided into eight steps.

Step 1: First, we prove that

The closed operator $\overline{L_\nu H_\alpha}$ is the generator of some Feller semigroup on $\partial\Omega$ for any sufficiently large $\alpha > 0$.

To do this, we apply Theorem 24.9 with $L := L_\nu$.

By applying Theorem 23.1, we obtain that the oblique derivative problem

$$\begin{cases} (A - \alpha)\, u = 0 & \text{in } \Omega, \\ L_\nu u = \varphi & \text{on } \partial\Omega \end{cases}$$

has a unique function $u \in W^{2,p}(\Omega)$ for any function $\varphi \in B^{1-1/p,p}(\partial\Omega)$, if $n < p < \infty$. Here it should be noticed that we have, by an imbedding theorem for Besov spaces (see [2, Theorem 7.34]),

$$C^1(\partial\Omega) \subset B^{1-1/p,p}(\partial\Omega) \subset C(\partial\Omega),$$

since $(1 - 1/p)p = p - 1 > n - 1$ for $n < p < \infty$.

Therefore, it follows that, for any function $\varphi \in B^{1-1/p,p}(\partial\Omega)$ there exists a unique function $\psi \in D\,(L_\nu H_\alpha) = B^{2-1/p,p}(\partial\Omega)$ such that

$$L_\nu\,(H_\alpha \psi) = \varphi.$$

This implies that the range $R\,(L_\nu H_\alpha)$ is a *dense* subset of $C(\partial\Omega)$. Hence, by applying part (ii) of Theorem 24.9 with $\lambda := 0$ we obtain that the operator $\overline{L_\nu H_\alpha}$ generates a Feller semigroup on $\partial\Omega$ for any $\alpha > 0$.

Step 2: Next we prove that

The closed operator $\overline{LH_\alpha}$ generates a Feller semigroup on $\partial\Omega$ for *any* $\alpha > 0$.

To do this, we apply Corollary 21.12 with $K := \partial\Omega$ to the operator $\overline{LH_\alpha}$ for $\alpha > 0$.

By formula (24.13), it follows that the operator $\overline{LH_\alpha}$ can be written as

$$\overline{LH_\alpha} := \overline{L_\nu H_\alpha} + M = \overline{L_\nu H_\alpha} - \alpha\delta(x'),$$

where

$$M = -\alpha\delta(x') \colon C(\partial\Omega) \longrightarrow C(\partial\Omega)$$

is a bounded linear operator. However, we find that the operator M satisfies the following condition (β') of Theorem 21.11:

(β') If $\psi \in C(\partial\Omega)$ takes a positive maximum at a point x'_0 of $\partial\Omega$, then we have the assertion

$$M\psi(x'_0) = -\alpha\delta(x'_0)\psi(x'_0) \leq 0.$$

Therefore, it follows from an application of Corollary 21.12 with

$$\mathfrak{A} := \overline{L_\nu H_\alpha}, \quad M := -\alpha\delta(x'),$$

that the closed operator $\overline{LH_\alpha}$ also generates a Feller semigroup on $\partial\Omega$.

Step 3: Now we prove that

The equation (25.1)

$$\overline{LH_\alpha}\,\psi = \varphi$$

has a unique solution ψ in $D(\overline{LH_\alpha})$ for any $\varphi \in C(\partial\Omega)$; hence the inverse $\overline{LH_\alpha}^{-1}$ of $\overline{LH_\alpha}$ can be defined on the whole space $C(\partial\Omega)$. Furthermore, the operator $-\overline{LH_\alpha}^{-1}$ is non-negative and bounded on $C(\partial\Omega)$.

Since the function $H_\alpha 1$ takes its positive maximum 1 only on the boundary $\partial\Omega$, we can apply Hopf's boundary point lemma (Lemma 8.7) to obtain that

$$\frac{\partial}{\partial \mathbf{n}}(H_\alpha 1) < 0 \quad \text{on } \partial\Omega. \tag{25.2}$$

However, it should be noticed that the coefficients of the boundary condition L satisfy the conditions

$$\begin{aligned} \mu(x') &> 0 \quad \text{on } \partial\Omega, \\ \gamma(x') &< 0 \quad \text{on } \partial\Omega, \\ \delta(x') &\geq 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Hence it follows from inequality (25.2) that

$$LH_\alpha 1(x') = \mu(x') \frac{\partial}{\partial \mathbf{n}} (H_\alpha 1)(x') + \gamma(x') - \alpha\delta(x') < 0 \quad \text{on } \partial\Omega,$$

so that

$$\ell_\alpha = - \sup_{x' \in \partial\Omega} LH_\alpha 1(x') > 0.$$

Furthermore, by using Corollary 21.12 with

$$K := \partial\Omega, \quad \mathfrak{A} := \overline{LH_\alpha}, \quad M := \ell_\alpha,$$

we obtain that the operator $\overline{LH_\alpha} + \ell_\alpha I$ is the infinitesimal generator of some Feller semigroup on $\partial\Omega$. Therefore, since $\ell_\alpha > 0$, it follows from an application of part (i) of Theorem 21.9 with $\mathfrak{A} := \overline{LH_\alpha} + \ell_\alpha I$ that the equation

$$-\overline{LH_\alpha}\,\psi = \left(\ell_\alpha I - (\overline{LH_\alpha} + \ell_\alpha I)\right)\psi = \varphi$$

has a unique solution $\psi \in D(\overline{LH_\alpha})$ for any $\varphi \in C(\partial\Omega)$, and further that the operator $-\overline{LH_\alpha}^{-1} = \left(\ell_\alpha I - (\overline{LH_\alpha} + \ell_\alpha I)\right)^{-1}$ is non-negative and bounded on the space $C(\partial\Omega)$ with norm

$$\left\| -\overline{LH_\alpha}^{-1} \right\| = \left\| \left(\ell_\alpha I - \left(\overline{LH_\alpha} + \ell_\alpha I\right)\right)^{-1} \right\| \leq \frac{1}{\ell_\alpha}.$$

Step 4: By assertion (25.1), we can define the *Green operator* G_α for $\alpha > 0$, by the formula

$$G_\alpha f = G_\alpha^0 f - H_\alpha \left(\overline{LH_\alpha}^{-1} \left(\overline{LG_\alpha^0 f} \right) \right) \quad \text{for } f \in C(\overline{\Omega}). \tag{25.3}$$

We prove that

$$G_\alpha = (\alpha I - \mathfrak{A})^{-1} \quad \text{for } \alpha > 0, \tag{25.4}$$

where $\mathfrak{A}$ is a linear operator from $C(\overline{\Omega})$ into itself defined as follows (see formula (1.4)):

(a) The domain $D(\mathfrak{A})$ is the set

$$D(\mathfrak{A}) = \left\{ u \in W^{2,p}(\Omega) : u \in D(\overline{A}),\ u|_{\partial\Omega} \in \mathcal{D},\ Lu = 0 \text{ on } \partial\Omega \right\}. \tag{25.5}$$

(b) $\mathfrak{A}u = Au$ for every $u \in D(\mathfrak{A})$.

Here $\mathcal{D}$ is the common domain of the operators $\overline{LH_\alpha}$, $\alpha > 0$ (see Lemma 24.8).

In view of Lemmas 24.3 and 24.8, it follows that we have, for any $f \in C(\overline{\Omega})$,

$$\begin{cases} G_\alpha f = G^0_\alpha f - H_\alpha \left(\overline{LH_\alpha}^{-1} \left(\overline{LG^0_\alpha f} \right) \right) \in D(\overline{\mathcal{A}}), \\ G_\alpha f|_{\partial\Omega} = -\overline{LH_\alpha}^{-1} \left(\overline{LG^0_\alpha f} \right) \in D\left(\overline{LH_\alpha}\right) = \mathcal{D}, \\ LG_\alpha f = \overline{LG^0_\alpha} f - \overline{LH_\alpha} \left(\overline{LH_\alpha}^{-1} \left(\overline{LG^0_\alpha f} \right) \right) = 0, \end{cases}$$

and also

$$\left(\alpha I - \overline{\mathcal{A}}\right) G_\alpha f = f.$$

This proves that

$$\begin{cases} G_\alpha f \in D(\mathfrak{A}), \\ (\alpha I - \mathfrak{A})\, G_\alpha f = f, \end{cases}$$

that is,

$$(\alpha I - \mathfrak{A})\, G_\alpha = I \quad \text{on } C(\overline{\Omega}).$$

Therefore, in order to prove formula (25.4) it suffices to show the injectivity of the operator $\alpha I - \mathfrak{A}$ for $\alpha > 0$.

Assume that

$$u \in D(\mathfrak{A}) \quad \text{and} \quad (\alpha I - \mathfrak{A})u = 0.$$

Then, by Corollary 24.4 it follows that the function u can be written as

$$u = H_\alpha(u|_{\partial\Omega}) \quad u|_{\partial\Omega} \in \mathcal{D} = D\left(\overline{LH_\alpha}\right).$$

Thus we have the assertion

$$\overline{LH_\alpha}(u|_{\partial\Omega}) = Lu = 0 \quad \text{on } \partial\Omega.$$

In view of assertion (25.1), this implies that

$$u|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega,$$

so that

$$u = H_\alpha\left(u|_{\partial\Omega}\right) = 0 \quad \text{in } \Omega.$$

Step 5: The *non-negativity* of G_α ($\alpha > 0$) follows immediately from formula (25.3), since the operators G^0_α, H_α, $-\overline{LH_\alpha}^{-1}$ and $\overline{LG^0_\alpha}$ are all non-negative.

Step 6: We prove that the operator G_α is *bounded* on the space $C(\overline{\Omega})$ with norm

$$\|G_\alpha\| \leq \frac{1}{\alpha} \quad \text{for all } \alpha > 0. \tag{25.6}$$

To do this, it suffices to show that

$$G_\alpha 1 \leq \frac{1}{\alpha} \quad \text{on } \overline{\Omega}, \text{ for all } \alpha > 0, \tag{25.7}$$

since G_α is non-negative on $C(\overline{\Omega})$.

First, it follows from the uniqueness property of solutions of problem (25.1) (Theorem 23.1) that

$$\alpha G_\alpha^0 1 + H_\alpha 1 = 1 + G_\alpha^0 c(x) \quad \text{on } \overline{\Omega}. \tag{25.8}$$

Indeed, it suffices to note that the both hand sides of formula (25.8) are the (unique) solution of the Dirichlet problem

$$\begin{cases} (\alpha - A)\, u = \alpha & \text{in } \Omega, \\ u|_{\partial\Omega} = 1. & \text{in } \partial\Omega \end{cases}$$

By applying the operator L to the both hand sides of formula (25.8), we obtain that

$$\begin{aligned} -LH_\alpha 1 &= -L\,(H_\alpha 1) = -L1 - L\left(G_\alpha^0 c\right) + \alpha L\left(G_\alpha^0 1\right) \\ &= -(\gamma(x') - \delta(x')c(x')) - \left(\mu(x')\frac{\partial}{\partial \mathbf{n}}(G_\alpha^0 c) + \delta(x')c(x')\right) \\ &\quad + \alpha L G_\alpha^0 1 \\ &= -\gamma(x') - \mu(x')\frac{\partial}{\partial \mathbf{n}}(G_\alpha^0 c) + \alpha L G_\alpha^0 1 \\ &\geq \alpha L G_\alpha^0 1 \quad \text{on } \partial\Omega, \end{aligned}$$

since $G_\alpha^0 c|_{\partial\Omega} = 0$ and $G_\alpha^0 c \leq 0$ on $\overline{\Omega}$. Hence we have, by the non-negativity of $-\overline{LH_\alpha}^{-1}$,

$$-\overline{LH_\alpha}^{-1}\left(LG_\alpha^0 1\right) \leq \frac{1}{\alpha} \quad \text{on } \partial\Omega. \tag{25.9}$$

By using formula (25.3) with $f := 1$, inequality (25.9) and formula (25.8), we obtain that

$$\begin{aligned} G_\alpha 1 &= G_\alpha^0 1 + H_\alpha\left(-\overline{LH_\alpha}^{-1}\left(L(G_\alpha^0 1)\right)\right) \\ &\leq G_\alpha^0 1 + \frac{1}{\alpha}\, H_\alpha 1 \\ &= \frac{1}{\alpha} + \frac{1}{\alpha}\, G_\alpha^0 c(x) \\ &\leq \frac{1}{\alpha} \quad \text{on } \overline{\Omega}, \text{ for all } \alpha > 0, \end{aligned}$$

since both the operators H_α and G_α^0 are non-negative.

Step 7: Finally, we prove that

$$\text{The domain } D(\mathfrak{A}) \text{ is } \textit{dense} \text{ in the space } C(\overline{\Omega}). \tag{25.10}$$

Step 7-1: Before the proof, we need some lemmas on the behavior of G_α^0, H_α and $-\overline{LH_\alpha}^{-1}$ as $\alpha \to +\infty$:

Lemma 25.1. *Let $\alpha > 0$. Then we have, for all $f \in C(\overline{\Omega})$,*

$$\lim_{\alpha \to +\infty} \left[\alpha G_\alpha^0 f + H_\alpha \left(f|_{\partial\Omega}\right)\right] = f \quad \textit{in } C(\overline{\Omega}). \tag{25.11}$$

Proof. Choose a constant $\beta > 0$ and let

$$g := f - H_\beta \left(f|_{\partial\Omega}\right).$$

Then, by using formula (25.8) with $\varphi := f|_{\partial\Omega}$ we obtain that

$$\alpha G_\alpha^0 g - g = \left[\alpha G_\alpha^0 f + H_\alpha \left(f|_{\partial\Omega}\right) - f\right] - \beta G_\alpha^0 H_\beta \left(f|_{\partial\Omega}\right). \tag{25.12}$$

However, we have, by estimate (25.4),

$$\lim_{\alpha \to +\infty} G_\alpha^0 H_\beta (f|_{\partial\Omega}) = 0 \quad \text{in } C(\overline{\Omega}),$$

and, by assertion (25.7),

$$\lim_{\alpha \to +\infty} \alpha G_\alpha^0 g = g \quad \text{in } C(\overline{\Omega}),$$

since $g|_{\partial\Omega} = 0$. Therefore, the desired assertion (25.11) follows by letting $\alpha \to +\infty$ in formula (25.12).

The proof of Lemma 25.1 is complete. □

Lemma 25.2. *The function*

$$\frac{\partial}{\partial \mathbf{n}} \left(H_\alpha 1\right)(x') \quad \textit{for } x' \in \partial\Omega,$$

diverges to $-\infty$ uniformly and monotonically as $\alpha \to +\infty$.

Proof. First, formula (25.8) with $\varphi := 1$ gives that

$$H_\alpha 1 = H_\beta 1 - (\alpha - \beta) G_\alpha^0 H_\beta 1.$$

Thus, in view of the non-negativity of G_α^0 and H_α it follows that

$$\alpha \geq \beta \implies H_\alpha 1 \leq H_\beta 1 \quad \text{on } \overline{\Omega}.$$

Since $H_\alpha 1|_{\partial\Omega} = H_\beta 1|_{\partial\Omega} = 1$, this implies that the functions

$$\frac{\partial}{\partial \mathbf{n}}(H_\alpha 1)(x') \quad \text{for } x' \in \partial\Omega,$$

are monotonically non-increasing in α. Furthermore, by using formula (25.6) with $f := H_\beta 1$ we find that the function

$$H_\alpha 1(x) = H_\beta 1(x) - \left(1 - \frac{\beta}{\alpha}\right) \alpha G_\alpha^0 H_\beta 1(x)$$

converges to zero monotonically as $\alpha \to +\infty$, for each interior point x of Ω.

Now, for any given constant $K > 0$ we can construct a function $u \in W^{2,p}(\Omega)$ such that

$$u = 1 \quad \text{on } \partial\Omega, \tag{25.13a}$$

$$\frac{\partial u}{\partial \mathbf{n}} \leq -K \quad \text{on } \partial\Omega. \tag{25.13b}$$

Indeed, if m is a positive integer, by applying Theorem 23.1 to our situation we obtain that the function

$$u = (H_1 1)^m$$

belongs to the space $W^{2,p}(\Omega)$ for $n < p < \infty$ and satisfies condition (25.13a), since we have the formula

$$u = (H_1 1)^m = 1 \quad \text{on } \partial\Omega,$$

and also the assertions

$$\begin{aligned}
&u = (H_1 1)^m \in L^p(\Omega), \\
&\frac{\partial u}{\partial x_i} = m\,(H_1 1)^{m-1}\,(H_1 1)_{x_i} \in L^p(\Omega) \quad \text{for } 1 \leq i \leq n, \\
&\frac{\partial^2 u}{\partial x_i \partial x_j} = m\,(H_\alpha 1)^{m-1}\,(H_1 1)_{x_i x_j} \\
&\qquad + m(m-1)\,(H_\alpha 1)^{m-2}\,(H_1 1)_{x_i}\,(H_1 1)_{x_j} \in L^p(\Omega) \quad \text{for } 1 \leq i, j \leq n.
\end{aligned}$$

Moreover, we obtain that

$$\begin{aligned}
\frac{\partial u}{\partial \mathbf{n}} &= m\,\frac{\partial}{\partial \mathbf{n}}\,(H_1 1) \\
&\leq m \sup_{x' \in \partial\Omega} \frac{\partial}{\partial \mathbf{n}}\,(H_1 1)\,(x').
\end{aligned}$$

In view of inequality (25.16) with $\varphi := 1$, this implies that the function $u = (H_1 1)^m$ satisfies condition (25.13b) for m sufficiently large.

Finally, it is easy to verify that

$$Au = A\left((H_1 1)^m\right) \in L^\infty(\Omega). \tag{25.13c}$$

Indeed, since we have the assertions

$$\begin{cases} (A-1)\, H_1 1 = 0 & \text{in } \Omega, \\ H_1 1 \in C^1(\overline{\Omega}), \end{cases}$$

it suffices to note that

$$\begin{aligned} Au &= m\,(H_1 1)^{m-1} \sum_{i,j=1}^{n} a^{ij}(x)\,(H_1 1)_{x_i x_j} \\ &\quad + m(m-1)\,(H_1 1)^{m-2} \sum_{i,j=1}^{n} a^{ij}(x)\,(H_1 1)_{x_i}\,(H_1 1)_{x_j} \\ &\quad + m\,(H_1 1)^{m-1} \sum_{i=1}^{n} b^i(x)\,(H_1 1)_{x_i} + c(x)\,(H_1 1)^m \\ &= m\,(H_1 1)^{m-1} \left(-\,(H_1 1) - c(x)\,(H_1 1)\right) \\ &\quad + m(m-1)\,(H_1 1)^{m-2} \sum_{i,j=1}^{n} a^{ij}(x)\,(H_1 1)_{x_i}\,(H_1 1)_{x_j} \\ &= -m\,(H_1 1)^m - mc(x)\,(H_1 1)^m + c(x)\,(H_1 1)^m \\ &\quad + m(m-1)\,(H_1 1)^{m-2} \sum_{i,j=1}^{n} a^{ij}(x)\,(H_1 1)_{x_i}\,(H_1 1)_{x_j} \in L^\infty(\Omega). \end{aligned}$$

We take a function $u \in W^{2,p}(\Omega)$ which satisfies conditions (25.13a), (25.13b) and (25.13c), and choose a neighborhood U of $\partial\Omega$, relative to the closure $\overline{\Omega}$, with smooth boundary ∂U such that (see Figure 25.1)

$$u \geq \frac{1}{2} \quad \text{on } U. \tag{25.14}$$

We recall that the function $H_\alpha 1$ converges to zero in Ω monotonically as $\alpha \to +\infty$. Since we have the formula

$$u|_{\partial\Omega} = H_\alpha 1|_{\partial\Omega} = 1 \quad \text{on } \partial\Omega,$$

by using Dini's theorem we can find a constant $\alpha > 1$ (depending on u and hence on K) such that

$$H_\alpha 1 \leq u \quad \text{on } \partial U \setminus \partial\Omega, \tag{25.15a}$$

$$\alpha > 2\|Au\|_{L^\infty(\Omega)}. \tag{25.15b}$$

It follows from inequalities (25.14) and (25.15b) that

$$(A-\alpha)\,(H_\alpha 1 - u) = \alpha u - Au$$

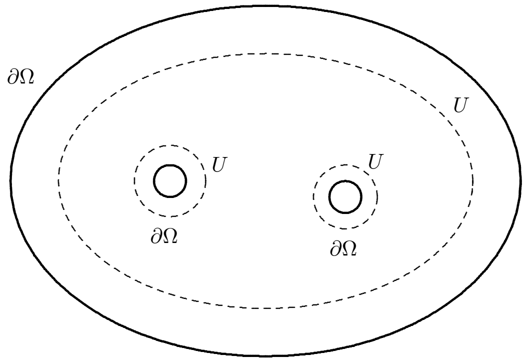

Fig. 25.1. A neighborhood U of $\partial\Omega$, relative to $\overline{\Omega}$, with smooth boundary ∂U

$$\geq \frac{\alpha}{2} - \|Au\|_{L^\infty(\Omega)}$$
$$> 0 \quad \text{in } U.$$

Thus, by applying the weak maximum principle (Theorem 8.5) with $A := A - \alpha$ to the function $H_\alpha 1 - u$ we obtain that the function $H_\alpha 1 - u$ may take its positive maximum only on the boundary ∂U. However, conditions (25.13a) and (25.15a) imply that

$$H_\alpha 1 - u \leq 0 \quad \text{on } \partial U = (\partial U \setminus \partial\Omega) \cup \partial\Omega.$$

Therefore, we have the inequality

$$H_\alpha 1 \leq u \quad \text{on } \overline{U} = U \cup \partial U,$$

and so

$$\frac{\partial}{\partial \mathbf{n}}(H_\alpha 1) \leq \frac{\partial u}{\partial \mathbf{n}} \leq -K \quad \text{on } \partial\Omega,$$

since $u|_{\partial\Omega} = H_\alpha 1|_{\partial\Omega} = 1$ on $\partial\Omega$.

The proof of Lemma 25.2 is complete. □

Corollary 25.3. *We have the assertion*

$$\lim_{\alpha \to +\infty} \left\| \overline{-LH_\alpha}^{-1} \right\| = 0.$$

Proof. Since $\mu(x') > 0$ on $\partial\Omega$, it follows from an application of Lemma 25.2 that the function

$$LH_\alpha 1(x') = \mu(x') \frac{\partial}{\partial \mathbf{n}} (H_\alpha 1)(x') + \gamma(x') - \alpha\delta(x') \quad \text{for } x' \in \partial\Omega,$$

diverges to $-\infty$ monotonically as $\alpha \to +\infty$. By Dini's theorem, this convergence is uniform in $x' \in \partial\Omega$. Hence we obtain that the function

$$\frac{1}{LH_\alpha 1(x')}$$

converges to zero uniformly in $x' \in \partial\Omega$ as $\alpha \to +\infty$. This implies that

$$\begin{aligned}\left\| -\overline{LH_\alpha}^{-1} \right\| &= \left\| -\overline{LH_\alpha}^{-1} 1 \right\|_{C(\partial\Omega)} \\ &\leq \left\| \frac{1}{LH_\alpha 1} \right\|_{C(\partial\Omega)} \longrightarrow 0 \quad \text{as } \alpha \to +\infty,\end{aligned}$$

since we have the assertion

$$1 = \frac{-LH_\alpha 1(x')}{|LH_\alpha 1(x')|} \leq \left\| \frac{1}{LH_\alpha 1} \right\|_{C(\partial\Omega)} (-LH_\alpha 1(x')) \quad \text{for all } x' \in \partial\Omega.$$

The proof of Corollary 25.3 is complete. □

Step 7-2: **Proof of Assertion** (25.10)

In view of formula (25.4) and inequality (25.6), it suffices to prove that

$$\lim_{\alpha \to +\infty} \|\alpha G_\alpha f - f\|_{C(\overline{\Omega})} = 0 \quad \text{for all } f \in C^2(\overline{\Omega}), \tag{25.16}$$

since the space $C^2(\overline{\Omega})$ is dense in $C(\overline{\Omega})$.

First, we remark that

$$\begin{aligned}\|\alpha G_\alpha f - f\|_{C(\overline{\Omega})} &= \left\| \alpha G_\alpha^0 f - \alpha H_\alpha \left(\overline{LH_\alpha}^{-1} \left(\overline{LG_\alpha^0} f \right) \right) - f \right\|_{C(\overline{\Omega})} \\ &\leq \left\| \alpha G_\alpha^0 f + H_\alpha \left(f|_{\partial\Omega} \right) - f \right\|_{C(\overline{\Omega})} \\ &\quad + \left\| -\alpha H_\alpha \left(\overline{LH_\alpha}^{-1} \left(\overline{LG_\alpha^0} f \right) \right) - H_\alpha (f|_{\partial\Omega}) \right\|_{C(\overline{\Omega})} \\ &\leq \left\| \alpha G_\alpha^0 f + H_\alpha (f|_{\partial\Omega}) - f \right\|_{C(\overline{\Omega})} \\ &\quad + \left\| -\alpha \overline{LH_\alpha}^{-1} \left(\overline{LG_\alpha^0} f \right) - f|_{\partial\Omega} \right\|_{C(\partial\Omega)}.\end{aligned}$$

Thus, in view of assertion (25.11) it suffices to show that

$$\lim_{\alpha \to +\infty} \left[-\alpha \overline{LH_\alpha}^{-1} \left(\overline{LG_\alpha^0} f \right) - f|_{\partial\Omega} \right] = 0 \quad \text{in } C(\partial\Omega). \tag{25.17}$$

Take a constant β such that $0 < \beta < \alpha$, and write

$$f = G_\beta^0 g + H_\beta \varphi,$$

where (cf. formula (25.14)):

$$\begin{cases} g = (\beta - A)\, f \in L^\infty(\Omega), \\ \varphi = f|_{\partial\Omega} \in C^2(\partial\Omega). \end{cases}$$

Then we have the assertion

$$LG^0_\beta g = L\left(G^0_\beta\, (\beta - A)\, f\right) = Lf \in C(\partial\Omega),$$

and, by Lemma 24.6 and Remark 24.3,

$$\begin{aligned} LG^0_\alpha g &= LG^0_\beta g - (\alpha - \beta)\, \overline{LG^0_\alpha}\left(G^0_\beta g\right) \\ &= Lf - (\alpha - \beta)\, \overline{LG^0_\alpha}(Lf) \in C(\partial\Omega). \end{aligned}$$

Moreover, by using the resolvent equation (25.5) with $f := g \in L^\infty(\Omega)$ and the equation (25.8) we obtain that

$$G^0_\alpha f = G^0_\alpha\left(G^0_\beta g\right) + G^0_\alpha H_\beta \varphi = \frac{1}{\alpha - \beta}\left(G^0_\beta g - G^0_\alpha g + H_\beta \varphi - H_\alpha \varphi\right).$$

Therefore, it follows that

$$\begin{aligned} &\left\| -\alpha \overline{LH_\alpha}^{-1}\left(\overline{LG^0_\alpha} f\right) - f|_{\partial\Omega} \right\|_{C(\partial\Omega)} \qquad (25.18) \\ &= \left\| \frac{\alpha}{\alpha - \beta}\left(-\overline{LH_\alpha}^{-1}\right)\left(LG^0_\beta g - LG^0_\alpha g + LH_\beta \varphi\right) + \frac{\alpha}{\alpha - \beta}\,\varphi - \varphi \right\|_{C(\partial\Omega)} \\ &= \left\| \frac{\alpha}{\alpha - \beta}\left(-\overline{LH_\alpha}^{-1}\right)\left((Lf + LH_\beta \varphi) - LG^0_\alpha g\right) + \frac{\beta}{\alpha - \beta}\,\varphi \right\|_{C(\partial\Omega)} \\ &\le \frac{\alpha}{\alpha - \beta}\left\| -\overline{LH_\alpha}^{-1} \right\| \cdot \|Lf + LH_\beta \varphi\|_{C(\partial\Omega)} \\ &\quad + \frac{\alpha}{\alpha - \beta}\left\| -\overline{LH_\alpha}^{-1} \right\| \cdot \left\| LG^0_\alpha \right\| \cdot \|g\|_{L^\infty(\Omega)} + \frac{\beta}{\alpha - \beta}\|\varphi\|_{C(\partial\Omega)}. \end{aligned}$$

By Corollary 25.3, it follows that the first term on the last inequality (25.18) converges to zero as $\alpha \to +\infty$. For the second term, by using the resolvent equation (25.5) with $f := 1$ and the non-negativity of G^0_β and LG^0_α (Remark 24.2) we find that

$$\begin{aligned} \|LG^0_\alpha\| &= \left\| L(G^0_\alpha 1) \right\|_{C(\partial\Omega)} \\ &= \left\| L(G^0_\beta 1) - (\alpha - \beta)\overline{LG^0_\alpha}(G^0_\beta 1) \right\|_{C(\partial\Omega)} \\ &\le \left\| L(G^0_\beta 1) \right\|_{C(\partial\Omega)} = \|LG^0_\beta\|. \end{aligned}$$

Hence the second term on the last inequality (25.18) also converges to zero as $\alpha \to +\infty$. It is clear that the third term on the last inequality

(25.18) converges to zero as $\alpha \to +\infty$. This completes the proof of assertion (25.17) and hence of assertion (25.16).

Step 8: Summing up, we have proved that the operator $\mathfrak{A}$, defined by formula (25.5), satisfies conditions (a) through (d) in Theorem 21.11. Hence it follows from an application of the same theorem that the operator $\mathfrak{A}$ is the infinitesimal generator of some Feller semigroup on $\overline{\Omega}$.

The proof of Theorem 1.1 is now complete. □

25.2 Notes and Comments

Section 25.1: The proof of Theorem 1.1 is adapted from [78, Section 6].

26
Concluding Remarks

This book is devoted to a careful and accessible exposition of the functional analytic approach to the problem of construction of Markov processes with Ventcel' boundary conditions in probability. More precisely, we prove existence theorems for Feller semigroups with Dirichlet boundary condition, oblique derivative boundary condition and first-order Ventcel' boundary condition for second-order, uniformly elliptic differential operators with discontinuous coefficients. Our approach here is distinguished by the extensive use of the ideas and techniques characteristic of the recent developments in the theory of Calderón and Zygmund of singular integral operators with non-smooth kernels. It should be emphasized that singular integral operators with non-smooth kernels provide a powerful tool to deal with smoothness of solutions of partial differential equations, with minimal assumptions of regularity on the coefficients.

Analytically, a Markovian particle in a domain of Euclidean space is governed by an integro-differential operator W, called Waldenfels operator, in the interior of the domain, and it obeys a boundary condition L, called Ventcel' boundary condition, on the boundary of the domain. The Waldenfels operator W takes the form of the sum of a differential operator A and an integro-differential operator S. The operator A is called a diffusion operator which describes analytically a strong Markov process with continuous paths in the interior of the domain, while the operator S is called a Lévy operator which is supposed to correspond to the jump phenomenon in the interior of the domain. Probabilistically, a Markovian particle moves both by jumps and continuously in the state space and it obeys the Ventcel' boundary condition which consists of six terms corresponding to the diffusion along the boundary, the absorption phenomenon, the reflection phenomenon, the sticking (or viscosity) phenomenon and the jump phenomenon on the boundary and the inward jump phenomenon from the boundary.

For general results on generation theorems for Feller semigroups, we give the following two overviews:

Diffusion operator A	**Lévy operator** S	proved by
Smooth coefficient case	Null	[73]
Smooth coefficient case	General case	[79]
Smooth coefficient case	Hölder continuous case	[74], [80]
VMO coefficient case	General case	[75], [77]
VMO coefficient case	Null	[75], [83]

Table 26.1. *Generation theorems for Feller semigroups via the theory of pseudo-differential operators*

Ventcel' condition L	using the theory of	proved by
Second-order case	Pseudo-differential operators	[73]
General case	Pseudo-differential operators	[79]
Degenerate Robin condition L_ρ	Pseudo-differential operators	[83], [80]
Dirichlet case	Singular integral operators	[77]
first-order case	Singular integral operators	[83]

Table 26.2. *Generation theorems for Feller semigroups via the theory of singular integral operators*

Here the boundary condition L_ρ of Robin type is given by the formula

$$L_\rho u := \mu(x')\frac{\partial u}{\partial \mathbf{n}} + \gamma(x')u \quad \text{on } \partial\Omega,$$

where the coefficients $\mu(x')$ and $\gamma(x')$ satisfy the following three conditions (i), (ii) and (iii):

(i) $\mu(x')$ is a smooth function on $\partial\Omega$ and $\mu(x') \geq 0$ on $\partial\Omega$.
(ii) $\gamma(x')$ is a smooth function on $\partial\Omega$ and $\gamma(x') \leq 0$ on $\partial\Omega$.
(iii) $\mu(x') + |\gamma(x')| > 0$ on $\partial\Omega$.

It should be emphasized that L_ρ becomes a *degenerate* boundary condition from an analytical point of view. This is due to the fact that the so-called Shapiro–Lopatinskii complementary condition is violated at the points $x' \in \partial\Omega$ where $\mu(x') = 0$ (see [3], [35], [44], [80], [98]). The intuitive meaning of hypothesis (iii) is that either the reflection phenomenon or the absorption phenomenon occurs at each point of the boundary $\partial\Omega$. More precisely, condition (iii) implies that absorption phenomenon occurs at each point of the set

$$M = \{x' \in \partial\Omega : \mu(x') = 0\},$$

while reflection phenomenon occurs at each point of the set

$$\partial\Omega \setminus M = \{x' \in \partial\Omega : \mu(x') > 0\}.$$

In other words, a Markovian particle moves continuously in the space $\overline{\Omega} \setminus M$ until it "dies" at the time when it reaches the set M (see Figure 26.1).

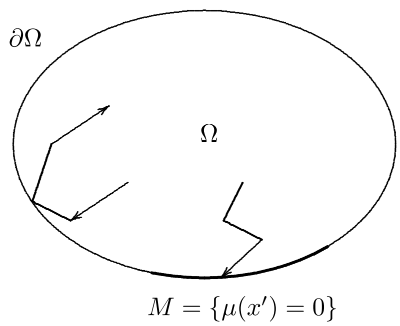

Fig. 26.1. A Markovian particle dies at the time when it reaches the set M

We give a simple example of the functions $\mu(x')$ and $\gamma(x')$ in the three conditions (i), (ii) and (iii) in Euclidean plane $\mathbf{R}^2$:

Example 26.1. Let

$$\Omega = \left\{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 < 1\right\}$$

be the *unit disk* with the boundary

$$\partial\Omega = \left\{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 = 1\right\}.$$

For a local coordinate system $x_1 = \cos\theta$, $x_2 = \sin\theta$ with $\theta \in [0, 2\pi]$ on the unit circle $\partial\Omega$, we define two functions $\mu(x_1, x_2)$ and $\gamma(x_1, x_2)$ as follows:

$$\begin{aligned} \mu(x_1, x_2) &= \mu\left(\cos\theta, \sin\theta\right) \\ &= \begin{cases} e^{\frac{2}{\pi} - \frac{1}{\theta}}\left(1 - e^{\frac{2}{\pi} + \frac{1}{\theta - \frac{\pi}{2}}}\right) & \text{for } \theta \in \left[0, \frac{\pi}{2}\right], \\ 1 & \text{for } \theta \in \left[\frac{\pi}{2}, \pi\right], \\ e^{\frac{2}{\pi} + \frac{1}{\theta - \frac{3\pi}{2}}}\left(1 - e^{\frac{2}{\pi} - \frac{1}{\theta - \pi}}\right) & \text{for } \theta \in \left[\pi, \frac{3\pi}{2}\right], \\ 0 & \text{for } \theta \in \left[\frac{3\pi}{2}, 2\pi\right], \end{cases} \end{aligned}$$

and

$$\gamma(x_1, x_2) := \mu(x_1, x_2) - 1 \quad \text{on } \partial\Omega.$$

Part VII

Appendix

Appendix 1

A Short Course to the Potential Theoretic Approach

In this appendix, follwing faithfully Gilbarg–Trudinger [33] we present a short introduction to the *potential theoretic approach* to the Dirichlet problem for Poisson's equation. The approach here can be traced back to the pioneering work of Schauder [65] and [65] on the Dirichlet problem for second-order, elliptic differential operators. This appendix is included for the sake of completeness and most of the material will be quite familiar to the reader and may be omitted.

A1.1 Hölder Continuity and Hölder Spaces

Let Ω be an open set in Euclidean space $\mathbf{R}^n$. First, we let

$$C(\Omega) = \text{ the space of continuous functions on } \Omega.$$

If k is a positive integer, we let

$$C^k(\Omega) = \text{ the space of functions of class } C^k \text{ on } \Omega.$$

Furthermore, we let

$$\begin{aligned} C(\overline{\Omega}) = &\text{the space of functions in } C(\Omega) \text{ having continuous extensions} \\ &\text{to the closure } \overline{\Omega} \text{ of } \Omega. \end{aligned}$$

If k is a positive integer, we let

$$\begin{aligned} C^k(\overline{\Omega}) = &\text{the space of functions in } C^k(\Omega) \text{ all of whose derivatives} \\ &\text{of order} \le k \text{ have continuous extensions to } \overline{\Omega}. \end{aligned}$$

Let $0 < \alpha < 1$. A function u defined on Ω is said to be *uniformly*

Hölder continuous with exponent α in Ω if the quantity

$$[u]_{\alpha;\Omega} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \tag{A.1.1}$$

is finite. We say that u is *locally Hölder continuous* with exponent α in Ω if it is uniformly Hölder continuous with exponent α on compact subsets of Ω.

If $0 < \alpha < 1$, we define the Hölder space $C^\alpha(\Omega)$ as follows:

$C^\alpha(\Omega) =$ the space of functions in $C(\Omega)$ which are locally Hölder continuous with exponent α on Ω.

If k is a positive integer and $0 < \alpha < 1$, we define the Hölder space $C^{k+\alpha}(\Omega)$ as follows:

$C^{k+\alpha}(\Omega) =$ the space of functions in $C^k(\Omega)$ all of whose k-th order derivatives are locally Hölder continuous with exponent α on Ω.

Furthermore, we let

$C^\alpha(\overline{\Omega}) =$ the space of functions in $C(\overline{\Omega})$ which are Hölder continuous with exponent α on $\overline{\Omega}$,

and

$C^{k+\alpha}(\overline{\Omega}) =$ the space of functions in $C^k(\overline{\Omega})$ all of whose k-th order derivatives are Hölder continuous with exponent α on $\overline{\Omega}$.

Let k be a non-negative integer and $0 < \alpha < 1$. We introduce various seminorms on the spaces $C^k(\Omega)$ and $C^{k+\alpha}(\Omega)$ as follows:

$$[u]_{k,0;\Omega} = \left|D^k u\right|_{0;\Omega} = \sup_{x\in\Omega} \sup_{|\beta|=k} \left|D^\beta u(x)\right|, \tag{A.1.2a}$$

$$[u]_{k,\alpha;\Omega} = \left[D^k u\right]_{\alpha;\Omega} = \sup_{|\beta|=k} \left[D^\beta u\right]_{\alpha;\Omega}. \tag{A.1.2b}$$

We can define the associated norms on the spaces $C^k(\overline{\Omega})$ and $C^{k+\alpha}(\overline{\Omega})$ as follows:

$$\|u\|_{C^k(\overline{\Omega})} = |u|_{k;\Omega} = \sum_{j=0}^{k} \left|D^j u\right|_{0;\Omega}, \tag{A.1.3a}$$

$$\|u\|_{C^{k+\alpha}(\Omega)} = |u|_{k,\alpha;\Omega} = |u|_{k;\Omega} + \left[D^k u\right]_{\alpha;\Omega}. \tag{A.1.3b}$$

Moreover, if Ω is a *bounded* domain in $\mathbf{R}^n$ with the *diameter*

$$d := \text{diam}\ \Omega = \sup_{x,y\in\Omega} |x-y|,$$

then we can inroduce non-dimensional norms $\|u\|_{C^k(\Omega)}$ and $\|u\|_{C^{k+\alpha}(\Omega)}$ equivalent respectively to the norms $\|u\|_{C^k(\Omega)}$ and $\|u\|_{C^{k+\alpha}(\Omega)}$ as follows:

$$\|u\|'_{C^k(\Omega)} = \sum_{j=0}^{k} d^j \left|D^j u\right|_{0;\Omega}, \tag{A.1.4a}$$

$$\|u\|'_{C^{k+\alpha}(\Omega)} = \sum_{j=0}^{k} d^j \left|D^j u\right|_{0;\Omega} + d^{k+\alpha} \left[D^k u\right]_{\alpha;\Omega}. \tag{A.1.4b}$$

Then we have the following claims:

Claim 1.1. If Ω is bounded, then the space $C^k(\overline{\Omega})$ is a Banach space with the norms $\|\cdot\|_{C^k(\Omega)}$ and $\|\cdot\|'_{C^k(\Omega)}$.

Claim 1.2. If Ω is bounded, then the Hölder space $C^{k+\alpha}(\overline{\Omega})$ is a Banach space with the norms $\|\cdot\|_{C^{k+\alpha}(\Omega)}$ and $\|\cdot\|'_{C^{k+\alpha}(\Omega)}$.

Claim 1.3. Let Ω be a *bounded* domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. Let k, j be non-negative integers and $0 < \alpha, \beta < 1$. Assume that

$$j + \beta < k + \alpha.$$

Then the injection

$$C^{k+\alpha}(\overline{\Omega}) \subset C^{j+\beta}(\overline{\Omega})$$

is *compact* (or *completely continuous*) ([33, Chapter 6, Lemma 6.36]).

A1.2 Interior Estimates for Harmonic Functions

First, by differntiating the Poisson integral we can obtain the following interior derivative estimates for harmonic functions ([33, Chapter 2, Theorem 2.10]):

Theorem 1.1. *Let Ω be an open set in $\mathbf{R}^n$ and let Ω' be open subset of Ω that has compact closure in Ω:*

$$\Omega' \Subset \Omega.$$

If $u(x)$ *is* harmonic *in* Ω*, then we have, for any multi-index* α*,*

$$\sup_{x\in\Omega'} |D^\alpha u(x)| \le \left(\frac{n|\alpha|}{d}\right)^{|\alpha|} \sup_{x\in\Omega} |u(x)|\,, \tag{A.1.5}$$

where

$$d = \operatorname{dist}(\Omega', \partial\Omega)\,.$$

Proof. We only prove the case where $|\alpha| = 1$. We assume that (see Figure A1.1)

$$B := B(y,R) \subset \Omega' \Subset \Omega.$$

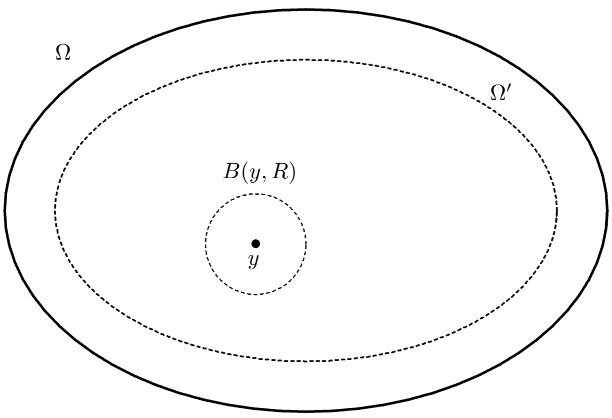

Fig. A1.1. The domains Ω and Ω'

First, we recall the mean value theorem for harmonic funtions (Theorem 5.5). Since u is harmonic in Ω, it follows that

$$\Delta(D_i u) = D_i(\Delta u) = 0 \quad \text{in } \Omega.$$

Hence, by applying the mean value theorem (Theorem 5.5) to $D_i u$ we obtain from the divergence theorem (Theorem 5.2) that

$$D_i u(y) = \frac{1}{R^n\omega_n}\int_B D_i u(x)\,dx = \frac{n}{\omega_n R^n}\int_{\partial B} u(z)\,\nu_i\,d\sigma(z).$$

Then we have the inequality

$$|D_i u(y)| \le \frac{n}{R^n\omega_n}\int_{\partial B} |u(z)\,\nu_i|\,d\sigma(z) \le \frac{n}{R^n\omega_n}\sup_{z\in\partial B}|u(z)|\cdot\int_{\partial B} d\sigma(z)$$

$$= \frac{n}{R} \sup_{z \in \partial B} |u(z)| \leq \frac{n}{R} \sup_{x \in B} |u(x)|$$
$$\leq \frac{n}{R} \sup_{x \in \Omega} |u(x)| .$$

By letting $R \uparrow d_y = \text{dist}\,(y, \partial\Omega)$, we obtain that

$$|D_i u(y)| \leq \frac{n}{d_y} \sup_{x \in \Omega} |u(x)| \quad \text{for all } y \in \Omega.$$

We remark that

$$\frac{1}{d_z} \leq \frac{1}{d} \quad \text{for all } z \in \Omega'.$$

Hence we have the inequality

$$|D_i u(z)| \leq \frac{n}{d} \sup_{x \in \Omega} |u(x)| \quad \text{for all } z \in \Omega'.$$

This proves the desired interior estimate (A.1.5) for $|\alpha| = 1$.

The proof of Theorem 1.1 is complete. □

A1.3 Hölder Regularity for the Newtonian Potential

We consider the fundamental solution $\Gamma(x-y)$ for the Laplacian in the case $n \geq 3$:

$$\Gamma(x-y) = \Gamma(|x-y|) = \frac{1}{(2-n)\omega_n} |x-y|^{2-n} . \tag{A.1.6}$$

Here

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

is the surface area of the unit ball in $\mathbf{R}^n$.

Then we have the following formulas for the fundamental solution $\Gamma(x-y)$:

$$D_i \Gamma(x-y) = \frac{\partial \Gamma}{\partial x_i}(x,y) = \frac{1}{\omega_n} (x_i - y_i) |x-y|^{-n} , \tag{A.1.7a}$$

$$D_i D_j \Gamma(x-y) = \frac{\partial^2 \Gamma}{\partial x_i \partial x_j}(x,y) \tag{A.1.7b}$$
$$= \frac{1}{\omega_n} \left\{ |x-y|^2 \delta_{ij} - n (x_i - y_i)(x_j - y_j) \right\} |x-y|^{-n-2} .$$

We remark that the fundamental solution $\Gamma(x-y)$ is *harmonic* for $x \neq y$:

$$\Delta_x \Gamma(x-y) = \sum_{i=1}^{n} \frac{\partial^2 \Gamma}{\partial x_i^2}(x-y) = 0 \quad \text{for } x \neq y.$$

By formulas (A.1.7), we have the following estimates for the fundamental solution $\Gamma(x, y)$:

$$|D_i \Gamma(x - y)| \leq \frac{1}{\omega_n} |x - y|^{1-n}, \tag{A.1.8a}$$

$$|D_i D_j \Gamma(x - y)| \leq \frac{n}{\omega_n} |x - y|^{-n}. \tag{A.1.8b}$$

Claim 1.4. Let Ω be a *smooth* domain with boundary $\partial\Omega$. If $u \in C^2(\overline{\Omega})$, then we have the *Green representation formula*

$$\begin{aligned} &u(y) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{A.1.9}) \\ &= \int_{\partial\Omega} \left(u(x) \frac{\partial \Gamma}{\partial \boldsymbol{\nu}}(x, y) - \Gamma(x, y) \frac{\partial u}{\partial \boldsymbol{\nu}}(x) \right) d\sigma(x) + \int_\Omega \Gamma(x, y) \Delta u(x) \, dx \\ &\qquad \text{for } y \in \Omega. \end{aligned}$$

Here

$$\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$$

is the unit outward normal to $\partial\Omega$.

Proof. We shall apply Green's formula (5.4b) for the fundamental solution $\Gamma(x, y)$.

Let y be an arbitrary point of Ω. We replace the domain Ω by the punctured domain $\Omega \setminus \overline{B_\rho}$ where $B_\rho = B(y, \rho)$ is a sufficiently small open ball of radius ρ about y (see Figure A1.2). By applying Green's formula

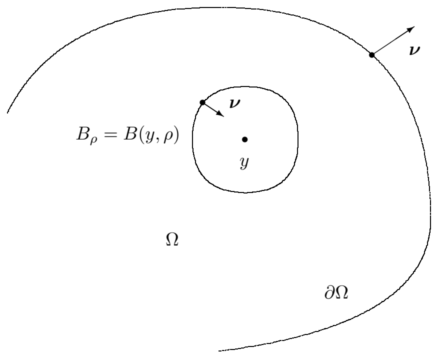

Fig. A1.2. The punctured domain $\Omega \setminus \overline{B_\rho}$

(5.4b) to our situation, we obtain that

$$\int_{\Omega \setminus B_\rho} \Gamma(x, y) \Delta u(x) \, dx \tag{A.1.10}$$

$$= \int_{\partial\Omega} \left(\Gamma(x,y) \frac{\partial u}{\partial \boldsymbol{\nu}}(x) - u(x) \frac{\partial \Gamma}{\partial \boldsymbol{\nu}}(x,y) \right) d\sigma(x)$$
$$+ \int_{\partial B_\rho} \left(\Gamma(x,y) \frac{\partial u}{\partial \boldsymbol{\nu}}(x) - u(x) \frac{\partial \Gamma}{\partial \boldsymbol{\nu}}(x,y) \right) d\sigma(x).$$

However, we have, as $\rho \downarrow 0$,

$$\begin{aligned} \int_{\partial B_\rho} \Gamma(x,y) \frac{\partial u}{\partial \boldsymbol{\nu}}(x)\, d\sigma(x) &= \Gamma(\rho) \int_{\partial B_\rho} \frac{\partial u}{\partial \boldsymbol{\nu}}(x)\, d\sigma(x) \\ &\le \frac{\rho^{2-n}}{(2-n)\omega_n}\ \omega_n \rho^{n-1} \left(\sup_{B_\rho} |Du| \right) \\ &= \frac{\rho}{2-n} \left(\sup_{B_\rho} |Du| \right) \longrightarrow 0. \end{aligned}$$

On the other hand, since we have the formula

$$\frac{\partial}{\partial \boldsymbol{\nu}} = \frac{1}{\rho} \sum_{i=1}^n (y_i - x_i) \frac{\partial}{\partial x_i} \quad \text{on the sphere } \partial B_\rho = \partial B(y,\rho),$$

it follows that

$$\begin{aligned} &\frac{\partial \Gamma}{\partial \boldsymbol{\nu}}(x,y) \\ &= \frac{1}{(2-n)\omega_n \rho} \sum_{i=1}^n (y_i - x_i) \frac{\partial}{\partial x_i} \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{1-n/2} \\ &= -\frac{1}{\omega_n \rho} \left(\sum_{i=1}^n (x_i - y_i)^2 \right) \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{-n/2} = -\frac{1}{\omega_n \rho}\, \rho^2\, \rho^{-n} \\ &= -\frac{1}{\omega_n \rho^{n-1}} \quad \text{on } \partial B_\rho. \end{aligned}$$

Hence we have, as $\rho \downarrow 0$,

$$\int_{\partial B_\rho} u(x) \frac{\partial \Gamma}{\partial \boldsymbol{\nu}}(x,y) d\,\sigma(x) = -\frac{1}{\omega_n \rho^{n-1}} \int_{\partial B_\rho} u(x)\, d\sigma(x) \longrightarrow\ -u(y).$$

Indeed, by the continuity of u it suffices to note that

$$\begin{aligned} &\left| \frac{1}{\omega_n \rho^{n-1}} \int_{\partial B_\rho} u(x)\, d\sigma(x) - u(y) \right| \\ &= \frac{1}{\omega_n \rho^{n-1}} \left| \int_{\partial B_\rho} (u(x) - u(y))\, d\sigma(x) \right| \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\omega_n \rho^{n-1}} \int_{\partial B_\rho} |u(x) - u(y)| \, d\sigma(x) \\
&\le \frac{1}{\omega_n \rho^{n-1}} \left(\sup_{|x-y|=\rho} |u(x) - u(y)| \right) \int_{\partial B_\rho} d\sigma(x) \\
&= \sup_{|x-y|=\rho} |u(x) - u(y)| \longrightarrow 0 \quad \text{as } \rho \downarrow 0.
\end{aligned}
$$

Therefore, by letting $\rho \downarrow 0$ in formula (A.1.10) we obtain the desired Green representation formula

$$
\begin{aligned}
&u(y) \\
&= \int_{\partial\Omega} \left(u(x) \frac{\partial \Gamma}{\partial \boldsymbol{\nu}}(x,y) - \Gamma(x,y) \frac{\partial u}{\partial \boldsymbol{\nu}}(x) \right) d\sigma(x) + \int_\Omega \Gamma(x,y) \Delta u(x) \, dx.
\end{aligned}
$$

The proof of Claim 1.4 is complete. □

Now we study some differentiability properties of the *Newtonian potential* of a function $f(x)$

$$
w(x) := (\Gamma * f)(x) = \int_\Omega \Gamma(x-y) \, f(y) \, dy
$$

in an open subset Ω of Euclidean space $\mathbf{R}^n$.

First, we obtain the following ([33, Chapter 4, Lemma 4.1]):

Lemma 1.2. *Let $f(x)$ be a bounded and integrable function in Ω. Then it follows that*

$$
w(x) = \int_\Omega \Gamma(x-y) \, f(y) \, dy \in C^1(\Omega),
$$

and we have, for any $x \in \Omega$,

$$
D_i w(x) = \int_\Omega D_i \Gamma(x-y) \, f(y) \, dy, \quad 1 \le i \le n. \tag{A.1.11}
$$

Secondly, we obtain the following ([33, Chapter 4, Lemma 4.2]):

Lemma 1.3. *Let $f(x)$ be a bounded and locally Hölder continuous function with exponent $0 < \alpha \le 1$ in Ω. Then it follows that*

$$
\begin{cases}
w(x) = \int_\Omega \Gamma(x-y) \, f(y) \, dy \in C^2(\Omega), \\
\Delta w = f \quad \text{in } \Omega.
\end{cases}
$$

Moreover, take an arbitrary smooth domain Ω_0 that contains Ω, and let $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ be the unit outward normal to $\partial\Omega_0$ and let $d\sigma$ be *the surface measure on $\partial\Omega_0$ (see Figure A1.3). Then we have, for any $x \in \Omega$,*

$$D_i D_j w(x) = \int_{\Omega_0} D_i D_j \Gamma(x-y) \left(f^0(y) - f(y)\right) dy \qquad \text{(A.1.12)}$$
$$- f(x) \int_{\partial\Omega_0} D_i \Gamma(x-y)\, \nu_j(y)\, d\sigma(y), \quad 1 \le i, j \le n,$$

where f^0 is the zero-extension *of f outside Ω:*

$$f^0(x) = \begin{cases} f(x) & \text{in } \Omega, \\ 0 & \text{in } \Omega_0 \setminus \Omega. \end{cases}$$

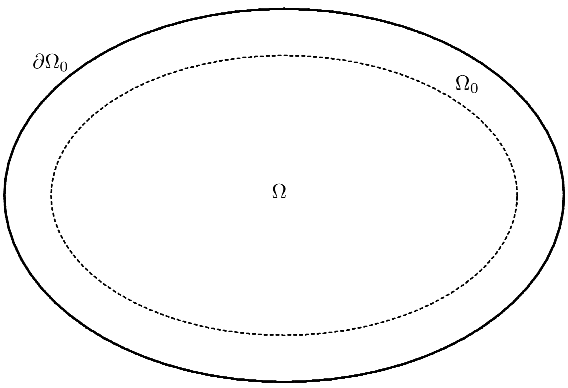

Fig. A1.3. The domains Ω and Ω_0

A1.4 Hölder Estimates for the Second Derivatives

We start with the following basic estimate ([33, Chapter 4, Lemma 4.4]):

Lemma 1.4. *Let $B_1 = B(x_0, R)$ and $B_2 = B(x_0, 2R)$ be concentric balls in $\mathbf{R}^n$ (see Figure A1.4). For a function $f \in C^\alpha(\overline{B_2})$ with $0 < \alpha < 1$, we let $w(x)$ be the Newtonian potential of f in B_2:*

$$w(x) = \int_{B_2} \Gamma(x-y)\, f(y)\, dy.$$

Then it follows that

$$w \in C^{2+\alpha}(\overline{B_1}),$$

and we have the interior estimate

$$\left|D^2 w\right|'_{0,\alpha;B_1} \leq C\,|f|'_{0,\alpha;B_2}\,, \tag{A.1.13}$$

that is,

$$\left|D^2 w\right|_{0,B_1} + (2R)^\alpha \left[D^2 w\right]_{\alpha:B_1} \leq C\left(|f|_{0,B_2} + (4R)^\alpha\,[f]_{\alpha;B_2}\right),$$

with a constant $C = C(n,\alpha) > 0$.

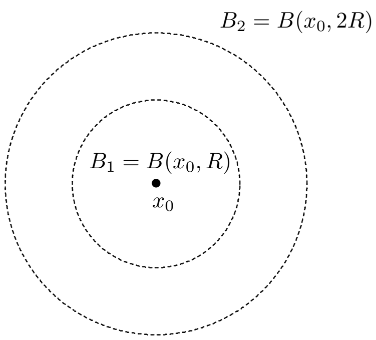

Fig. A1.4. The concentric balls B_1 and B_2

We can prove the following *interior Hölder estimate* for solutions of the Poisson equation ([33, Chapter 4, Theorem 4.6]):

Theorem 1.5. *Let* Ω *be a domain in* $\mathbf{R}^n$ *and let* $f \in C^\alpha(\Omega)$ *with* $0 < \alpha < 1$. *If a function* $u \in C^2(\Omega)$ *satisfies the Poisson equation*

$$\Delta u = f \quad \text{in } \Omega,$$

then it follows that

$$u \in C^{2+\alpha}(\Omega).$$

Moreover, for any two concentric balls (see Figure A1.5)

$$B_1 = B(x_0, R), \quad B_2 = B(x_0, 2R) \Subset \Omega,$$

we have the estimate

$$|u|'_{2,\alpha;B_1} \leq C\left(|u|_{0;B_2} + (4R)^2\,|f|'_{0,\alpha;B_2}\right), \tag{A.1.14}$$

that is,

$$|u|_{0;B_1} + (2R)\,|Du|_{0;B_1} + (2R)^2\left|D^2 u\right|_{0;B_1} + (2R)^{2+\alpha}\left[D^2 u\right]_{\alpha;B_1}$$

$$\leq C\left(|u|_{0;B_2} + (4R)^2\,|f|_{0,B_2} + (4R)^{2+\alpha}\,[f]_{\alpha;B_2}\right),$$

with a constant $C = C(n, \alpha) > 0$.

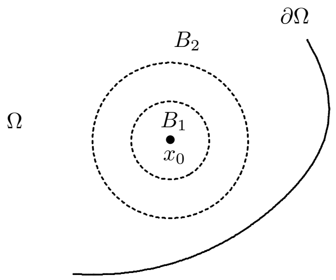

Fig. A1.5. The concentric balls B_1 and B_2 in Ω

Let Ω be an open set in $\mathbf{R}^n$. For x, $y \in \Omega$, we let

$$d_x = \operatorname{dist}(x, \partial\Omega),$$
$$d_{x,y} = \min(d_x, d_y).$$

If k is a non-negative integer and $0 < \alpha < 1$, then we introduce various interior seminorms and norms on the Hölder spaces $C^k(\Omega)$ and $C^{k+\alpha}(\Omega)$ as follows:

$$[u]^*_{k,0;\Omega} = [u]^*_{k;\Omega} = \sup_{x\in\Omega}\sup_{|\beta|=k} d_x^k\left|D^\beta u(x)\right|, \tag{A.1.15a}$$

$$|u|^*_{k;\Omega} = \sum_{j=0}^{k} [u]^*_{j;\Omega}, \tag{A.1.15b}$$

$$[u]^*_{k,\alpha;\Omega} = \sup_{x,y\in\Omega}\sup_{|\beta|=k} d_{x,y}^{k+\alpha}\frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}, \tag{A.1.15c}$$

$$|u|^*_{k,\alpha;\Omega} = |u|^*_{k;\Omega} + [u]^*_{k,\alpha;\Omega}. \tag{A.1.15d}$$

Claim 1.5. (i) If Ω is bounded with $d = \operatorname{diam}\Omega$, then we have the inequality

$$|u|^*_{k,\alpha;\Omega} \leq \max\left(1, d^{\alpha+k}\right)|u|_{k,\alpha;\Omega}. \tag{$A.1.15'$}$$

(ii) If $\Omega' \Subset \Omega$ with $\sigma = \operatorname{dist}(\Omega', \partial\Omega)$, then we have the inequality

$$\min\left(1, \sigma^{\alpha+k}\right)|u|_{k,\alpha;\Omega'} \leq |u|^*_{k,\alpha;\Omega}. \tag{$A.1.15''$}$$

Moreover, we introduce a seminorm on the Hölder space $C^{\alpha}(\Omega)$ as follows:

$$|f|^{(k)}_{0,\alpha;\Omega} = \sup_{x\in\Omega} d_x^k\,|f(x)| + \sup_{x,y\in\Omega} d_{x,y}^{k+\alpha}\,\frac{|f(x)-f(y)|}{|x-y|^{\alpha}}. \tag{A.1.16}$$

Then, by using Theorem 1.5 we can obtain a *Schauder interior estimate* for a general domain Ω ([33, Chapter 4, Theorem 4.8]):

Theorem 1.6. *Assume that a function $u \in C^2(\Omega)$ satisfies the equation*

$$\Delta u = f \quad \textit{in } \Omega$$

for a function $f \in C^{\alpha}(\Omega)$. Then we have the interior estimate

$$|u|^{*}_{2,\alpha;\Omega} \leq C\left(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}\right) \tag{A.1.17}$$

with a constant $C = C(n,\alpha) > 0$.

Proof. We have only to consider the case where

$$|u|_{0;\Omega} < \infty, \quad |f|^{(2)}_{0,\alpha;\Omega} < \infty.$$

The proof is divided into two steps.

Step 1: For each point x of Ω, we let (see Figure A1.6)

$$R := \frac{1}{3}\,d_x = \frac{1}{3}\,\text{dist}\,(x,\partial\Omega)\,,$$
$$B_1 := B(x,R),$$
$$B_2 := B(x,2R).$$

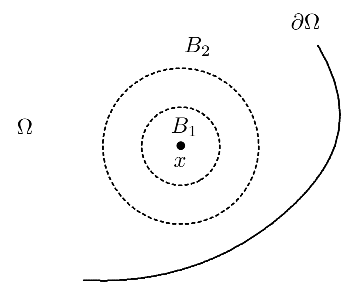

Fig. A1.6. The concentric balls B_1 and B_2 in Ω

Then we have, by estimate (A.1.14),

$$d_x\,|Du(x)| + d_x^2\,\left|D^2u(x)\right| \leq 3R\,|Du|_{0;B_1} + (3R)^2\,\left|D^2u\right|_{0,B_1} \tag{A.1.18}$$

$$\leq C\left(|u|_{0;B_2} + R^2\,|f|'_{0,\alpha;B_2}\right)$$
$$\leq C\left(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;B_2}\right).$$

However, we have the estimate

$$R^2\,|f|'_{0,\alpha;B_2} = R^2\left(|f|_{0;B_2} + (4R)^\alpha\,[f]_{\alpha;B_2}\right) \leq C\,|f|^{(2)}_{0,\alpha;\Omega}. \tag{A.1.19}$$

Indeed, since we have the inequality

$$R = \frac{1}{3}d_x \leq d_z \quad \text{for all } z \in B_2,$$

it follows that

$$R^2 \sup_{z\in B_2} |f(z)| \leq \sup_{z\in B_2} d_z^2\,|f(z)| \leq \sup_{x\in\Omega} d_x^2\,|f(x)|.$$

Similarly, since we have the inequality

$$R = \frac{1}{3}d_x \leq d_{z,w} = \min(d_z, d_w) \quad \text{for all } z,\, w \in B_2,$$

it follows that

$$R^{2+\alpha}\frac{|f(z)-f(w)|}{|z-w|^\alpha} \leq \sup_{z,w\in B_2} d_{z,w}^{2+\alpha}\frac{|f(z)-f(w)|}{|z-w|^\alpha}$$
$$\leq \sup_{x,y\in\Omega} d_{x,y}^{2+\alpha}\frac{|f(z)-f(w)|}{|z-w|^\alpha}.$$

Hence we have the inequality

$$R^2\,|f|'_{0,\alpha;B_2} \leq C\,|f|^{(2)}_{0,\alpha;\Omega}.$$

Therefore, by combining inequalities (A.1.18) and (A.1.19) we obtain that

$$|u|^*_{2;\Omega} = |u|_{0;\Omega} + \sup_{x\in\Omega} d_x\,|Du(x)| + \sup_{x\in\Omega} d_x^2\,\left|D^2u(x)\right| \tag{A.1.20}$$
$$\leq C\left(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}\right).$$

Step 2: We assume that $d_x \leq d_y$ for $x,\, y \in \Omega$, so that

$$d_x = d_{x,y} = 3R \quad \text{for } x,\, y \in \Omega.$$

Then it follows that

$$\frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \leq \begin{cases} \left[D^2u\right]_{\alpha;B_1} & \text{if } y \in B_1, \\ \frac{1}{R^\alpha}\left(\left|D^2u(x)\right| + \left|D^2u(y)\right|\right) & \text{if } y \in \Omega\setminus B_1. \end{cases}$$

Hence we have, for $x,\, y \in \Omega$,

$$d_{x,y}^{2+\alpha} \frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \tag{A.1.21}$$
$$\leq (3R)^{2+\alpha} \left[D^2u\right]_{\alpha;B_1} + 3^\alpha\,(3R)^2 \left(\left|D^2u(x)\right| + \left|D^2u(y)\right|\right).$$

However, by using inequalities (A.1.14) and (A.1.19) we can estimate the first term on the right-hand side of inequality (A.1.21) as follows:

$$\begin{aligned}(3R)^{2+\alpha} \left[D^2u\right]_{\alpha;B_1} &\leq C\left(|u|_{0;B_2} + R^2\,|f|'_{0,\alpha;B_2}\right)\\ &\leq C\left(|u|_{0,\Omega} + |f|^{(2)}_{0,\alpha;\Omega}\right).\end{aligned}$$

On the other hand, by using inequality (A.1.20) we can estimate the second term on the right-hand side of inequality (A.1.21) as follows:

$$\begin{aligned}3^\alpha\,(3R)^2 \left(\left|D^2u(x)\right| + \left|D^2u(y)\right|\right) &\leq 6\sup_{x\in\Omega} d_x^2\left|D^2u(x)\right| \leq 6\,|u|^*_{2;\Omega}\\ &\leq C\left(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}\right).\end{aligned}$$

Summing up, we obtain that

$$\sup_{x,y\in\Omega} d_{x,y}^{2+\alpha} \frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \leq C\left(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}\right). \tag{A.1.22}$$

The desired interior estimate (A.1.17) follows by combining inequalities (A.1.20) and (A.1.22).

The proof of Theorem 1.6 is complete. □

Corollary 1.7. *Let Ω be an open set in $\mathbf{R}^n$ and let Ω' be open subset of Ω that has compact closure in Ω:*

$$\Omega' \Subset \Omega.$$

Assume that a function $u \in C^2(\Omega)$ satisfies the equation

$$\Delta u = f \quad \text{in } \Omega$$

for a function $f \in C^\alpha(\Omega)$. Then we have, for any $0 < d \leq \operatorname{dist}(\Omega', \partial\Omega)$,

$$d\,|Du|_{0;\Omega'} + d^2\left|D^2u\right|_{0;\Omega'} + d^{2+\alpha}\left[D^2u\right]_{\alpha;\Omega'} \tag{A.1.23}$$
$$\leq C\left(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}\right),$$

with a constant $C = C(n,\alpha) > 0$.

Proof. For all x, $y \in \Omega'$, we have the asertions

$$d_x \geq d, \quad d_y \geq d,$$
$$d_{x,y} \geq d.$$

Hence, by using estimate (A.1.17) we obtain that

$$\begin{aligned}
&d\,|Du|_{0;\Omega'} + d^2 \left|D^2 u\right|_{0;\Omega'} + d^{2+\alpha} \left[D^2 u\right]_{\alpha;\Omega'} \\
&\leq \sup_{x\in\Omega'} d_x \,|Du(x)| + \sup_{x\in\Omega'} d_x^2 \left|D^2 u(x)\right| + \sup_{x,y\in\Omega'} d_{x,y}^{2+\alpha} \frac{|D^2 u(x) - D^2 u(y)|}{|x-y|^\alpha} \\
&\leq |u|^*_{2,\alpha;\Omega'} \leq |u|^*_{2,\alpha;\Omega} \\
&\leq C \left(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega} \right).
\end{aligned}$$

This proves the desired estimate (A.1.23).

The proof of Corollary 1.7 is complete. □

This corollary ([33, Chapter 6, Corollary 6.3]) provides a bound on the seminorms $|Du|_{0;\Omega'}$, $|D^2u|_{0;\Omega'}$ and $[D^2u]_{\alpha;\Omega'}$ in any subset Ω' of Ω for which $\operatorname{dist}(\Omega', \partial\Omega) \geq d$ (see Figure A1.7).

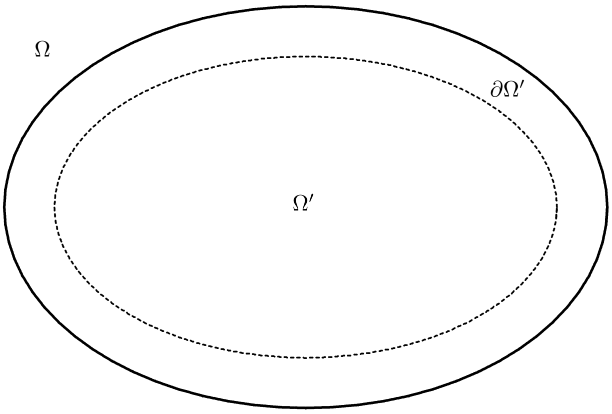

Fig. A1.7. The domains Ω and Ω′

A1.5 Hölder Estimates at the Boundary

The purpose of this section is to generalize Theorem 1.10 for a general domain Ω.

First, we introduce some notation (see Figure A1.8):

$$\mathbf{R}^n_+ = \{(x', x_n) \in \mathbf{R}^n : x_n > 0\},$$
$$T = \{(x', x_n) \in \mathbf{R}^n : x_n = 0\},$$
$$B_1 = B(x_0, R) = \{x \in \mathbf{R}^n : |x - x_0| < R\}, \quad x_0 \in \overline{\mathbf{R}^n_+},$$
$$B_2 = B(x_0, 2R) = \{x \in \mathbf{R}^n : |x - x_0| < 2R\}, \quad x_0 \in \overline{\mathbf{R}^n_+},$$
$$B_1^+ = B_1 \cap \mathbf{R}^n_+ = B(x_0, R) \cap \mathbf{R}^n_+,$$
$$B_2^+ = B_2 \cap \mathbf{R}^n_+ = B(x_0, 2R) \cap \mathbf{R}^n_+.$$

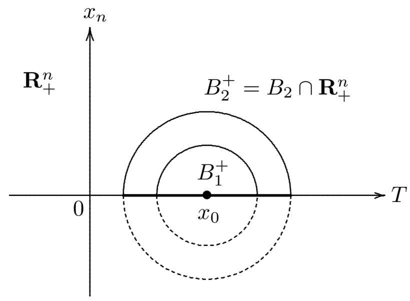

Fig. A1.8. The domains B_1^+, B_2^+ and T in $\overline{\mathbf{R}^n_+}$

The next lemma is a generalization of Lemma 1.4 to the half space $\mathbf{R}^n_+$ ([33, Chapter 4, Lemma 4.10]):

Lemma 1.8. *If $f \in C^\alpha(\overline{B_2^+})$, we let w be the Newtonian potential of f in B_2^+:*

$$w(x) = \int_{B_2^+} \Gamma(x - y)\, f(y)\, dy.$$

Then it follows that

$$w \in C^{2+\alpha}(\overline{B_1^+}),$$

and we have the boundary a priori *estimate*

$$\left|D^2 w\right|'_{0,\alpha;B_1^+} \leq C\, |f|'_{0,\alpha;B_2^+}\,, \tag{A.1.24}$$

that is,

$$\left|D^2 w\right|_{0;B_1^+} + R^\alpha \left[D^2 w\right]_{\alpha;B_1^+} \leq C \left(|f|_{0,B_2^+} + R^\alpha\, [f]_{\alpha;B_2^+} \right),$$

with a constant $C = C(n, \alpha) > 0$.

Proof. First, by applying Lemma 1.3 with

$$\Omega := B_1^+, \quad \Omega_0 := B_2^+,$$

we obtain that

$$D_i D_j w(x) = \int_{B_2^+} D_i D_j \varGamma(x-y)\,(f(y)-f(x))\,dy \\ - f(x) \int_{\partial B_2^+} D_i \varGamma(x-y)\,\nu_j(y)\,d\sigma(y) \quad \text{in } B_1^+.$$

However, it follows from an application of the divergence theorem that

$$\int_{\partial B_2^+} D_i \varGamma(x-y)\,\nu_j(y)\,d\sigma(y) = \int_{B_2^+} D_j D_i \varGamma(x-y) \\ = \int_{B_2^+} D_i D_j \varGamma(x-y) \\ = \int_{\partial B_2^+} D_j \varGamma(x-y)\,\nu_i(y)\,d\sigma(y).$$

Since we have the formula

$$\boldsymbol{\nu} = (0, \ldots, 0, -1) \quad \text{on } T,$$

we have, for $i \neq n$ or $j \neq n$,

$$D_i D_j w(x) = \int_{B_2^+} D_i D_j \varGamma(x-y)\,(f(y)-f(x))\,dy \\ - f(x) \int_{\partial B_2^+ \setminus T} D_i \varGamma(x-y)\,\nu_j(y)\,d\sigma(y) \quad \text{in } B_1^+.$$

We remark that (see Figure A1.9)

$$R \leq |x-y| \leq 3R \quad \text{for } x \in B_1^+ \text{ and } y \in \partial B_2^+ \setminus T.$$

Hence we can estimate the derivatives $D_i D_j w$ for $i \neq n$ or $j \neq n$ as follows:

$$|D_i D_j w|'_{0,\alpha;B_1^+} \leq C\, |f|'_{0,\alpha;B_2^+}.$$

Moreover, it follows from an application of lemma 1.3 that the function

$$w(x) = \int_{B_2^+} \Gamma(x-y)\,f(y)\,dy$$

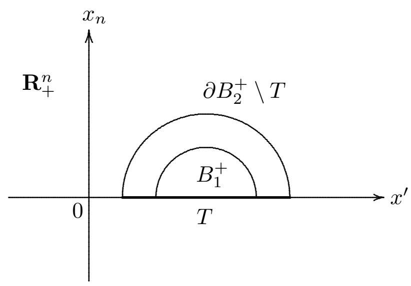

Fig. A1.9. The domain B_1^+ and the boundary $\partial B_2^+ \setminus T$

satisfies the equation

$$D_n D_n w(x) = f(x) - \sum_{k=1}^{n-1} D_k D_k w(x) \quad \text{in } B_1^+.$$

Therefore, we can estimate the derivatives $D_n D_n w$ as follows:

$$\begin{aligned} |D_n D_n w|'_{0,\alpha;B_1^+} &\leq |f|'_{0,\alpha;B_2^+} + \sum_{k=1}^{n-1} |D_k D_k w|'_{0,\alpha;B_1^+} \\ &\leq C\ |f|'_{0,\alpha;B_2^+}. \end{aligned}$$

The proof of Lemma 1.8 is complete. □

The next theorem is a generalization of Theorem 1.5 to the half space $\mathbf{R}^n_+$ ([33, Chapter 4, Theorem 4.11]):

Theorem 1.9. *Assume that a function $u \in C^2(B_2^+) \cap C(\overline{B_2^+})$ is a solution of the Dirichlet problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } T \end{cases}$$

for a function $f \in C^\alpha\left(\overline{B_2^+}\right)$. Then it follows that

$$u \in C^{2+\alpha}(\overline{B_2^+}),$$

and we have the a priori *estimate*

$$|u|'_{2,\alpha;B_1^+} \leq C\left(|u|_{0;B_2^+} + R^2\, |f|'_{0,\alpha;B_2^+}\right) \tag{A.1.25}$$

with a constant $C = C(n,\alpha) > 0$.

Proof. The proof of Theorem 1.9 is divided into five steps.

Step (I): In the proof we make use of a method based on *reflection*. Namely, for each point $x = (x', x_n) \in B_2^+$ we define the point (see Figure A1.10)

$$x^* = (x', -x_n) \in B_2^- := B(x_0, 2R) \cap \mathbf{R}^n_-.$$

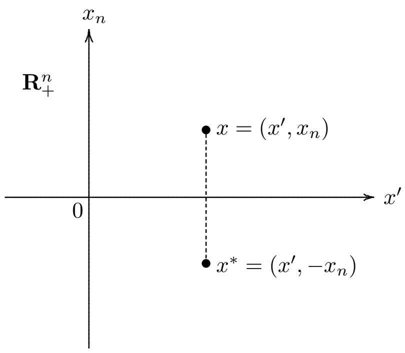

Fig. A1.10. The mapping $x \longmapsto x^*$

Then we have the following:

Claim 1.6. For a function $f \in C^\alpha\left(\overline{B_2^+}\right)$, we consider a function $f^*(x)$ defined on the domain (see Figure A1.11)

$$D = B_2^+ \cup B_2^- \cup (B_2 \cap T)$$

as follows:

$$f^{ast}(x) = f^*(x', x_n) = \begin{cases} f(x', x_n) & \text{for } x_n > 0, \\ f(x', -x_n) & \text{for } x_n < 0. \end{cases}$$

Then it follows that

$$f^* \in C^\alpha\left(\overline{D}\right),$$

and we have the estimate

$$|f^*|'_{0,\alpha;D} \leq 4\,|f|'_{0,\alpha;B_2^+}\,.$$

Proof. Indeed, it suffices to note that

$$\begin{aligned} |f^*|'_{0,\alpha;D} &= |f^*|'_{0;D} + (\text{diam } D)^\alpha\,[f^*]_{0,\alpha;D} \\ &= |f^*|_{0;D} + (\text{diam } D)^\alpha\,[f^*]_{\alpha;D} \end{aligned}$$

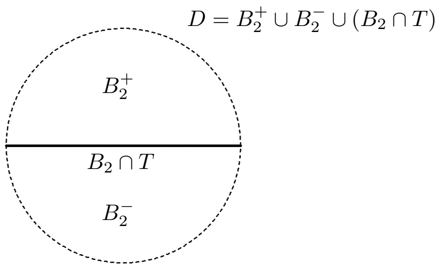

Fig. A1.11. The domains $B_2^{\pm}$, T and D in $\mathbf{R}^n$

$$\begin{aligned}
&\leq |f|_{0;B_2^+} + \left(2 \text{ diam } B_2^+\right)^{\alpha} 2\, [f^*]_{\alpha;B_2^+} \\
&\leq 4 \left(|f|_{0;B_2^+} + \left(\text{diam } B_2^+\right)^{\alpha} [f^*]_{\alpha;B_2^+}\right) \\
&= 4\, |f|'_{0,\alpha;B_2^+} .
\end{aligned}$$

The proof of Claim 1.6 is complete. □

Stpe (II): We let

$$w(x) := \int_{B_2^+} \left(\Gamma(x-y) - \Gamma(x^* - y)\right) f(y)\, dy.$$

Since we have the formula

$$|x^* - y| = |(x' - y', x_n + y_n)| = |x - y^*|\,,$$

it follows that

$$w(x) = \int_{B_2^+} \left(\Gamma(x-y) - \Gamma(x - y^*)\right) f(y)\, dy.$$

Moreover, we have the following:

Claim 1.7. The function $w(x)$ is a solution of the Dirichlet problem

$$\begin{cases} \Delta w = f & \text{in } B_2^+, \\ w = 0 & \text{in } T. \end{cases}$$

Proof. (1) First, we have the assertion

$$\begin{aligned}
&\left(\Gamma(x-y) - \Gamma(x-y^*)\right) |f(y)| \\
&= \frac{1}{(n-2)\omega_n} \left(\frac{1}{|x-y|^n} - \frac{1}{|x^*-y|^n}\right) |f(x)| \longrightarrow 0 \quad \text{as } x_n \downarrow 0.
\end{aligned}$$

Moreover, we have the inequality

$$\int_{B_2^+} (\Gamma(x-y) - \Gamma(x-y^*)) \, |f(y)| \, dy$$
$$\leq \sup_{z \in B_2^+} |f(z)| \left[\int_{B_2^+} |\Gamma(x-y)| \, dy + \int_{B_2^+} |\Gamma(x^*-y)| \, dy \right].$$

However, we have, for $x, y \in B_2^+$,

$$|x-y| \leq 4R, \quad |x^*-y| \leq 8R.$$

Hence we have the inequality

$$\int_{B_2^+} (\Gamma(x-y) - \Gamma(x-y^*)) \, |f(y)| dy$$
$$\leq \frac{1}{(n-2)\omega_n} \int_{B_2^+} \left(\frac{1}{|x-y|^n} + \frac{1}{|x^*-y|^n} \right) |f(y)| \, dy$$
$$\leq \frac{1}{(n-2)\omega_n} \sup_{z \in B_2^+} |f(z)| \left[\int_0^{4R} r^{n-2} \, r^{n-1} \, dr + \int_0^{8R} r^{n-2} \, r^{n-1} \, dr \right] \omega_n$$
$$= \frac{40R^2}{n-2} \, |f|_{0;B_2^+} .$$

Therefore, by applying Beppo-Levi's theorem we obtain that

$$\limsup_{x_n \downarrow 0} \left| \int_{B_2^+} (\Gamma(x-y) - \Gamma(x-y^*)) \, f(y) \, dy \right|$$
$$\leq \lim_{x_n \downarrow 0} \int_{B_2^+} (\Gamma(x^*-y) - \Gamma(x-y)) \, |f(x)| \, dy = 0.$$

This proves that

$$w(x) = \int_{B_2^+} (\Gamma(x-y) - \Gamma(x-y^*)) \, f(y) \, dy = 0 \quad \text{on } T = \{x_n = 0\}.$$

(2) Secondly, we show that

$$\Delta_x \Gamma(x^*-y) = 0 \quad \text{for } x, \, y \in B_2^+.$$

Indeed, we have the formulas

$$\frac{\partial^2}{\partial x_n^2} \left(|x^*-y|^{2-n} \right)$$
$$= (n-2) \left[n \left(x_n^2 + y_n^2 \right) - |x^*-y|^2 \right] |x^*-y|^{-n-2} ,$$

and

$$\frac{\partial^2}{\partial x_j^2}\left(|x^* - y|^{2-n}\right)$$
$$= (n-2)\left[n\,(x_j - y_j)^2 - |x^* - y|^2\right]|x^* - y|^{-n-2}, \quad 1 \le j \le n-1.$$

This proves that

$$\Delta_x \Gamma(x^* - y)$$
$$= (n-2)\left[n\,|x^* - y|^2 - n\,|x^* - y|^2\right]|x^* - y|^{-n-2}$$
$$= 0 \quad \text{for } x,\, y \in B_2^+,$$

since we have the formula

$$|x^* - y|^2 = \sum_{j=1}^{n-1}(x_j - y_j)^2 + \left(x_n^2 + y_n^2\right).$$

(3) Summing up, we find from Lemma 1.3 that

$$\Delta w(x)$$
$$= \Delta_x\left(\int_{B_2^+}\Gamma(x-y)\,f(y)\,dy\right) - \Delta_x\left(\int_{B_2^+}\Gamma(x-y^*)\,f(y)\,dy\right)$$
$$= \Delta_x\left(\int_{B_2^+}\Gamma(x-y)\,f(y)\,dy\right)$$
$$= f(x) \quad \text{in } B_2^+.$$

The proof of Claim 1.7 is complete. □

Step (III): Since we have the assertion

$$y \in B_2^+ \Longleftrightarrow y^* \in B_2^-, \quad f(y) = f^*(y^*),$$

we have the formula

$$w(x)$$
$$= \int_{B_2^+}\left(\Gamma(x-y) - \Gamma(x^* - y)\right) f(y)\,dy$$
$$= 2\int_{B_2^+}\Gamma(x-y)\,f(y)\,dy$$
$$- \left(\int_{B_2^+}\Gamma(x-y)\,f^*(y)\,dy + \int_{B_2^-}\Gamma(x-y)\,f(y)\,dy\right)$$

$$= 2\,(\Gamma * f)\,(x) - \int_D \Gamma(x-y)\, f^*(y)\, dy.$$

However, it follows from Claim 1.6 that

$$f^* \in C^\alpha(\overline{D}).$$

If we let

$$w^*(x) := \int_D \Gamma(x-y)\, f^*(y)\, dy,$$

then it follows from an application of Lemma 1.4 that (see Figure A1.12)

$$\left|D^2 w^*\right|'_{0,\alpha;B_1^+} \leq C\, |f|'_{0,\alpha;D} \leq 2C\, |f|'_{0,\alpha;B_2^+}.$$

Therefore, by combining this inequality with Lemma 1.8 we obtain that

$$\begin{aligned} |w|'_{2,\alpha;B_1^+} &\leq 2\,|\Gamma * f|'_{2,\alpha;B_1^+} + |w^*|'_{2,\alpha;B_1^+} \\ &\leq C\left(|f|_{0,B_2^+} + R^\alpha\,[f]_{\alpha;B_2^+}\right) \\ &= C\,|f|'_{0,\alpha;B_2^+}. \end{aligned} \tag{A.1.26}$$

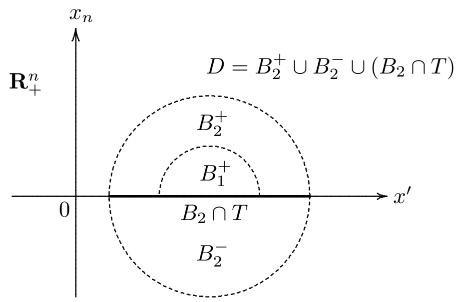

Fig. A1.12. The domains B_2^+, B_1^+ and D

Step (IV): If we let

$$v(x) := u(x) - w(x),$$

it follows that v is a solution of the Dirichlet problem

$$\begin{cases} \Delta v = \Delta u - \Delta w = 0 & \text{in } B_2^+, \\ v = 0 & \text{on } T. \end{cases}$$

Moreover, we have the following:

Claim 1.8. The function

$$V(x) = V(x', x_n) = \begin{cases} v(x', x_n) & \text{for } x_n \geq 0, \\ -v(x', -x_n) & \text{for } x_n \leq 0. \end{cases}$$

is *harmonic* in D:

$$\Delta V = 0 \quad \text{in } D.$$

Moreover, we have the estimate

$$|V|_{0;D} \leq |v|_{0;B_2^+}.$$

Proof. First, it follows that

$$\begin{cases} \Delta V = 0 & \text{in } B_2^+, \\ \Delta V = 0 & \text{in } B_2^-, \end{cases}$$

and that

$$V = v = 0 \quad \text{on } B_2 \cap T.$$

Hence we have the assertion

$$V \in C(\overline{D}) \cap C^\infty(B_2^+) \cap C^\infty(B_2^-).$$

Secondly, we show that V is harmonic near T in D. To do so, let

$$B_r := B\left((x', 0), r\right) \Subset D$$

be an arbitrary small open ball of radius r about $(x', 0) \in T$. Then we have the formula

$$\begin{aligned} &\int_{\partial B_r} V(y)\, d\sigma(y) \\ &= \int_{\partial B_r^+} V(y', y_n)\, d\sigma(y) - \int_{\partial B_r^-} V(z', -z_n)\, d\sigma(z) \\ &= \int_{\partial B_r^+} V(y', y_n)\, d\sigma(y) - \int_{\partial B_r^+} V(y', y_n)\, d\sigma(y) \\ &= 0. \end{aligned}$$

Hence there exists a constant $\varepsilon > 0$ such that

$$\int_{\partial B_r} V(y)\, d\sigma(y) = 0 \quad \text{for all } 0 < r < \varepsilon.$$

Therefore, by using Green's identity (5.4a) with $\Omega := B_r$ and $v := 1$ we obtain that

$$0 = \frac{d}{dr}\left(\frac{1}{r^{n-1}\,\omega_n} \int_{\partial B_r} V(y)\, d\sigma(y)\right)$$

$$
\begin{aligned}
&= \frac{d}{dr}\left(\frac{1}{\omega_n}\int_{S(0,1)} V\left((x',0)+rz\right)\,d\sigma(z)\right)\\
&= \frac{1}{\omega_n}\int_{S(0,1)} z\cdot\nabla V\left((x',0)+rz\right)\,d\sigma(z)\\
&= \frac{1}{\omega_n}\int_{S(0,r)} \frac{y}{r}\cdot\nabla V\left((x',0)+y\right)\,r^{1-n}d\sigma(y)\\
&= \frac{1}{r^{n-1}\,\omega_n}\int_{S(0,r)} \frac{y}{r}\cdot\nabla V\left((x',0)+y\right)\,d\sigma(y)\\
&= \frac{1}{r^{n-1}\,\omega_n}\int_{S(0,r)} \frac{\partial V}{\partial \boldsymbol{\nu}}\left((x',0)+y\right)d\sigma(y)\\
&= \frac{1}{r^{n-1}\,\omega_n}\int_{B_r} \Delta V(x)\,dx,
\end{aligned}
$$

where

$$S(0,r) = \{z\in\mathbf{R}^n : |z| = r\}\,.$$

Since the integral of ΔV over any ball near T in D vanishes, it follows that

$$\Delta V = 0 \quad \text{near } T \text{ in } D.$$

Summing up, we have proved that V is harmonic in D.

The proof of Claim 1.8 is complete. □

Step (V): It remains to prove estimate (A.1.25).

(1) For the function $v = u - w$, we have the estimate

$$
\begin{aligned}
|v|_{0;B_1^+} &\le |u|_{0;B_1^+} + |w|_{0;B_1^+}\\
&\le |u|_{0;B_2^+} + C\,R^2\,|f|_{0;B_2^+}\,.
\end{aligned}
$$

(2) By Theorem 1.1, we have the estimate

$$
\begin{aligned}
|Dv|_{0;B_1^+} &\le \frac{n}{R}\,|V|_{0;B_2} \le \frac{n}{R}\,|v|_{0;B_2^+}\,.\\
\left|D^2 v\right|_{0;B_1^+} &\le \left(\frac{2n}{R}\right)^2 |V|_{0;B_2} \le \left(\frac{2n}{R}\right)^2 |v|_{0;B_2^+}\,.
\end{aligned}
$$

(3) By Theorem 1.5, we have the estimate

$$\left[D^2 v\right]_{\alpha;B_1^+} \le C\,R^{-2-\alpha}\,|V|_{0;B_2} \le C\,R^{-2-\alpha}\,|v|_{0;B_2^+}\,.$$

(4) By Theorem 1.5, we have the estimate

$$|w|_{0;B_2^+} \le C\,R^2\,|f|_{0;B_2^+}\,.$$

For the function $v = u - w$, we have the estimate

$$\begin{aligned} |v|_{0;B_2^+} &\le |u|_{0;B_2^+} + |w|_{0;B_2^+} \\ &\le |u|_{0;B_2^+} + C\,R^2\,|f|_{0;B_2^+} . \end{aligned}$$

Summing up, we obtain the estimate

$$\begin{aligned} |v|'_{2,\alpha;B_1^+} &= |v|'_{2;B_1^+} + \left(\text{diam } B_1^+\right)^{2+\alpha} \left[D^2 v\right]_{\alpha;B_1^+} \\ &\le C\,|v|_{0;B_2^+} \\ &\le C\left(|u|_{0;B_2^+} + R^2\,|f|_{0;B_2^+}\right) \\ &\le C\left(|u|_{0;B_2^+} + R^2\,|f|'_{0,\alpha;B_2^+}\right). \end{aligned} \tag{A.1.27}$$

Therefore, the desired estimate (A.1.25) follows by combining estimates (A.1.26) and (A.1.27).

The proof of Theorem 1.9 is complete. □

Let Ω be an open set in $\mathbf{R}^n_+$ with open boundary portion T on the boundary

$$\{(x', x_n) \in \mathbf{R}^n : x_n = 0\}$$

(see Figure A1.13). For $x, y \in \Omega$, we let

$$\begin{aligned} \overline{d}_x &= \text{dist}\,(x, \partial\Omega \setminus T), \\ \overline{d}_{x,y} &= \min\left(\overline{d}_x, \overline{d}_y\right). \end{aligned}$$

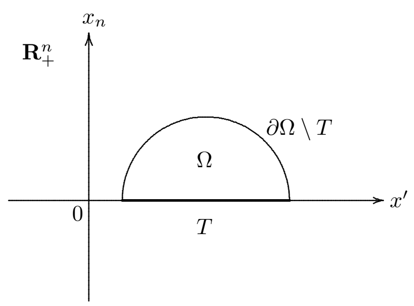

Fig. A1.13. The open set Ω with an open boundary portion T

If k is a non-negative integer and $0 < \alpha < 1$, then we introduce various interior seminorms and norms on the Hölder spaces $C^k(\Omega)$ and $C^{k+\alpha}(\Omega)$ as follows:

$$[u]^*_{k,0;\Omega\cup T} = [u]^*_{0;\Omega\cup T} = \sup_{x\in\Omega}\sup_{|\beta|=k} \overline{d}_x^k \left|D^\beta u(x)\right|, \tag{A.1.28a}$$

$$|u|^*_{k;\Omega\cup T} = \sum_{j=0}^{k} [u]_{j;\Omega\cup T}\,, \tag{A.1.28b}$$

$$[u]^*_{k,\alpha;\Omega\cup T} = \sup_{x,y\in\Omega}\sup_{|\beta|=k} \overline{d}_{x,y}^{k+\alpha}\frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}, \tag{A.1.28c}$$

$$|u|^*_{k,\alpha;\Omega\cup T} = |u|^*_{k;\Omega\cup T} + [u]^*_{k,\alpha;\Omega\cup T}\,, \tag{A.1.28d}$$

and

$$|u|^{(k)}_{0,\alpha;\Omega\cup T} = \sup_{x\in\Omega}\overline{d}_x^k\,|u(x)| + \sup_{x,y\in\Omega}\overline{d}_{x,y}^{k+\alpha}\frac{|u(x)-u(y)|}{|x-y|^\alpha}. \tag{A.1.28e}$$

The next theorem is a generalization of Theorem 1.6 to the half space $\mathbf{R}^n_+$ ([33, Chapter 4, Theorem 4.12]:

Theorem 1.10. *Let Ω be an open set in $\mathbf{R}^n_+$ with a boundary portion T on $\{(x', x_n) \in \mathbf{R}^n : x_n = 0\}$. Assume that a function $u \in C^2(\Omega) \cap C(\Omega \cup T)$ is a solution of the Dirichlet problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } T \end{cases}$$

for a function $f \in C^\alpha(\Omega \cup T)$. Then we have the boundary estimate

$$|u|^*_{2,\alpha;\Omega\cup T} \leq C\left(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega\cup T}\right), \tag{A.1.29}$$

with a constant $C = C(n,\alpha) > 0$.

Proof. The proof of this theorem is similar to that of Theorem 1.6.

Step 1: For each point x of Ω, we let (see Figure A1.14)

$$R := \frac{1}{3}\overline{d}_x,$$
$$B_1 := B(x,R),$$
$$B_2 := B(x,2R),$$

and

$$B_1^+ := B(x,R) \cap \mathbf{R}^n_+,$$
$$B_2^+ := B(x,2R) \cap \mathbf{R}^n_+.$$

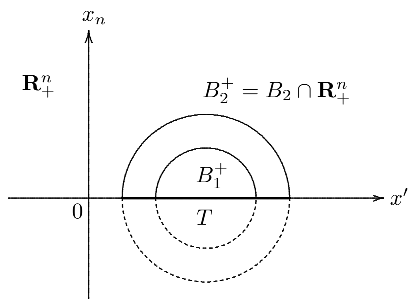

Fig. A1.14. The domains B_1 and B_2 in $\mathbf{R}^n_+$

Then we have, by estimate (A.1.25),

$$\overline{d}_x \, |Du(x)| + \overline{d}_x^2 \left|D^2u(x)\right| \leq 3R \, |Du|_{0;B_1} + (3R)^2 \left|D^2u\right|_{0,B_1} \leq C \left(|u|_{0;B_2} + R^2 \, |f|'_{0,\alpha;B_2} \right). \tag{A.1.30}$$

Moreover, if we assume that $\overline{d}_x \leq \overline{d}_y$ for $x, y \in \Omega$, then it follows that

$$\overline{d}_x = \overline{d}_{x,y} = 3R \quad \text{for } x,\, y \in \Omega.$$

Hence we have the inequalities

$$R^2 \, |f(x)| = \frac{1}{9} \overline{d}_x^2 \, |f(x)| \leq \frac{1}{9} \, |f|^{(2)}_{0,\alpha;\Omega\cup T},$$

and

$$R^2 \overline{d}_x^\alpha \frac{|f(x) - f(y)|}{|x-y|^\alpha} = \frac{1}{9} \overline{d}_{x,y}^{2+\alpha} \frac{|f(x) - f(y)|}{|x-y|^\alpha} \leq \frac{1}{9} \, |f|^{(2)}_{0,\alpha;\Omega\cup T}.$$

This proves that

$$R^2 \, |f|'_{0,\alpha;B_2} \leq C \, |f|^{(2)}_{0,\alpha;\Omega\cup T}. \tag{A.1.31}$$

Therefore, by combining estimates (A.1.30) and (A.1.31) we obtain that

$$|u|^*_{2;\Omega\cup T} \leq C \left(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega\cup T} \right), \tag{A.1.32}$$

Step 2: Similarly, we assume that $\overline{d}_x \leq \overline{d}_y$ for $x, y \in \Omega$, so that

$$\overline{d}_x = \overline{d}_{x,y} = 3R \quad \text{for } x,\, y \in \Omega.$$

Then it follows that

$$\frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \leq \begin{cases} \left[D^2u\right]_{\alpha;B_1} & \text{if } y \in B_1, \\ \frac{1}{R^\alpha}\left(\left|D^2u(x)\right| + \left|D^2u(y)\right|\right) & \text{if } y \in \Omega \setminus B_1. \end{cases}$$

Hence we have, for x, $y \in \Omega$,

$$\begin{aligned} &\overline{d}_{x,y}^{2+\alpha}\frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \qquad \text{(A.1.33)} \\ &\leq (3R)^{2+\alpha}\left[D^2u\right]_{\alpha;B_1^+} + 3^\alpha (3R)^2\left(\left|D^2u(x)\right| + \left|D^2u(y)\right|\right) \\ &\leq (3R)^{2+\alpha}\left[D^2u\right]_{\alpha;B_1^+} + 6\,|u|^*_{2;\Omega\cup T}\,. \end{aligned}$$

Moreover, by using estimates (A.1.25) and (A.1.32) we obtain that

$$\begin{aligned} \overline{d}_{x,y}^{2+\alpha}\frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} &\leq C\left(|u|_{0;B_2} + R^2\,|f|'_{0,\alpha;B_2}\right) + 6\,|u|^*_{2;\Omega\cup T} \\ &\leq C\left(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega\cup T}\right) + 6\,|u|^*_{2;\Omega\cup T}\,, \end{aligned}$$

so that

$$\begin{aligned} &\sup_{x,y\in\Omega\cup T} \overline{d}_{x,y}^{2+\alpha}\frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \qquad \text{(A.1.34)} \\ &\leq C\left(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;B_2}\right) + 6\,|u|^*_{2;\Omega\cup T}\,. \end{aligned}$$

Therefore, the desired boundary estimate (A.1.29) follows by combining estimates (A.1.32), (A.1.33) and (A.1.34).

The proof of Theorem 1.10 is complete. □

Let Ω be an open set in $\mathbf{R}^n$ with $C^{2+\alpha}$ boundary portion T. For x, $y \in \Omega$, we let (see Figure A1.15)

$$\begin{aligned} \overline{d}_x &= \operatorname{dist}(x, \partial\Omega \setminus T)\,, \\ \overline{d}_{x,y} &= \min\left(\overline{d}_x, \overline{d}_y\right). \end{aligned}$$

If k is a non-negative integer and $0 < \alpha < 1$, then we introduce various boundary seminorms and norms on the Hölder spaces $C^k(\Omega \cup T)$ and $C^{k+\alpha}(\Omega \cup T)$ as follows:

$$[u]^*_{k,0;\Omega\cup T} = [u]^*_{0;\Omega\cup T} = \sup_{x\in\Omega}\sup_{|\beta|=k} \overline{d}_x^k\left|D^\beta u(x)\right|, \qquad \text{(A.1.35a)}$$

$$|u|^*_{k;\Omega\cup T} = \sum_{j=0}^k [u]_{j;\Omega\cup T}\,, \qquad \text{(A.1.35b)}$$

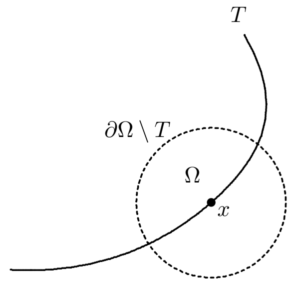

Fig. A1.15. The open set Ω with an open boundary portion T

$$[u]^*_{k,\alpha;\Omega\cup T} = \sup_{x,y\in\Omega} \sup_{|\beta|=k} \overline{d}^{\,k+\alpha}_{x,y} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}, \tag{A.1.35c}$$

$$|u|^*_{k,\alpha;\Omega\cup T} = |u|^*_{k;\Omega\cup T} + [u]^*_{k,\alpha;\Omega\cup T}\,, \tag{A.1.35d}$$

and

$$|u|^{(k)}_{0,\alpha;\Omega\cup T} = \sup_{x\in\Omega} \overline{d}^{\,k}_x\, |u(x)| + \sup_{x,y\in\Omega} \overline{d}^{\,k+\alpha}_{x,y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}. \tag{A.1.35e}$$

Then we can prove the following *Schauder local boundary estimate* for solutions of the Dirichlet problem for curved boundaries ([33, Chapter 6, Lemma 6.5]):

Lemma 1.11. *Let Ω be a $C^{2,\alpha}$ domain in $\mathbf{R}^n$ with boundary ∂Ω. Assume that a function $u \in C^{2+\alpha}(\overline{\Omega})$ is a solution of the Dirichlet problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for a function $f \in C^\alpha(\overline{\Omega})$. Then, at each boundary point $x_0 \in \partial\Omega$ there is a ball $B = B(x_0, \delta)$ of radius $\delta > 0$, independent of x_0, such that we have the boundary *estimate*

$$|u|_{2,\alpha;B\cap\Omega} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right), \tag{A.1.36}$$

with a constant $C = C(n,\alpha) > 0$.

Proof. The proof of Lemma 1.11 is divided into three steps.

Step (1): First, we consider the case where

$$x \in B'(x_0) = B(x_0,\rho) \cap \Omega.$$

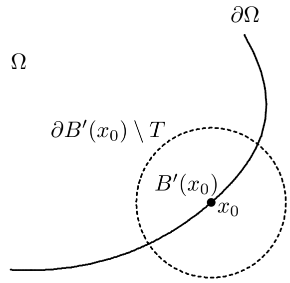

Fig. A1.16. The domain $B'(x_0)$ and the boundary $\partial B'(x_0) \setminus T$

Since we have the inequality (see Figure A1.16)

$$\overline{d}_x = \mathrm{dist}\,(x, \partial B'(x_0) \setminus T) \leq \mathrm{diam}\ \Omega,$$

it follows that

$$\begin{aligned} & |f|^{(2)}_{2,\alpha;B'(x_0)\cup T} \qquad\qquad \text{(A.1.37)} \\ &= \sup_{x \in B'(x_0)} \overline{d}_x^2 \, |f(x)| + \sup_{x,y \in B'(x_0)} \overline{d}_{x,y}^{2+\alpha} \frac{|f(x) - f(y)|}{|x-y|^\alpha} \\ &\leq C \sup_{x \in B'(x_0)} |f(x)| + \sup_{x,y \in B'(x_0)} \frac{|f(x) - f(y)|}{|x-y|^\alpha} \leq C \, |f|_{0,\alpha;B'(x_0)} \\ &\leq C \, |f|_{0,\alpha;\Omega} \,. \end{aligned}$$

On the other hand, by applying Theorem 1.10 we obtain that

$$\begin{aligned} |u|^*_{2,\alpha;B'(x_0)\cup T} &\leq C \left(|u|_{0;B'(x_0)} + |f|^{(2)}_{2,\alpha;B'(x_0)\cup T} \right) \\ &\leq C \left(|u|_{0;\Omega} + |f|^{(2)}_{2,\alpha;B'(x_0)\cup T} \right). \end{aligned}$$

Therefore, we have, by inequality A.1.37,

$$|u|^*_{2,\alpha;B'(x_0)\cup T} \leq C_1 \left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} \right), \qquad \text{(A.1.38)}$$

with a positive constant

$$C_1 = C_1 \left(n, \alpha, B'(x_0) \right).$$

Step (2): Secondly, we consider the case where

$$x \in B''(x_0) = B\left(x_0, \rho/2\right) \cap \Omega.$$

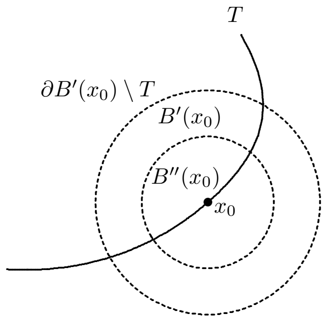

Fig. A1.17. The open neighborhoods $B'(x_0)$ and $B''(x_0)$ of x_0 in Ω

However, we remark that (see Figure A1.17)

$$\overline{d}_x = \operatorname{dist}(x, \partial B'(x_0) \setminus T) \geq \frac{\rho}{2} \quad \text{for all } x \in B''(x_0),$$
$$\overline{d}_{x,y} \geq \frac{\rho}{2} \quad \text{for all } x,\, y \in B''(x_0).$$

Hence we have the inequality

$$\begin{aligned}
&|u|^*_{2,\alpha;B'(x_0)\cup T} &\text{(A.1.39)}\\
&= \sup_{x\in B'(x_0)} |u(x)| + \sup_{x\in B'(x_0)} \overline{d}_x\, |Du(x)| + \sup_{x\in B'(x_0)} \overline{d}_x^{\,2}\, |D_2 u(x)| \\
&\quad + \sup_{x,y\in B'(x_0)} \overline{d}_{x,y}^{\,2+\alpha} \frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \\
&\geq \min\left(1, \frac{\rho}{2}, \left(\frac{\rho}{2}\right)^{2+\alpha}\right) \\
&\quad \times \left(\sup_{x\in B'(x_0)} |u(x)| + \sup_{x\in B'(x_0)} |Du(x)| + \sup_{x\in B'(x_0)} |D_2u(x)|\right) \\
&= \min\left(1, \frac{\rho}{2}, \left(\frac{\rho}{2}\right)^{2+\alpha}\right) |u|_{0,\alpha;B''(x_0)}\,.
\end{aligned}$$

Therefore, by combining inequalities (A.1.38) and (A.1.39) we obtain that This proves that

$$|u|_{0,\alpha;B''(x_0)} \leq C_2 \left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right), \qquad \text{(A.1.40)}$$

with a positive constant

$$C_2 = C_2\left(n, \alpha, B'(x_0), B''(x_0)\right).$$

Step (3): Since the boundary $\partial\Omega$ is compact, we can find a finite number of boundary points $\{x_i\}_{i=1}^N$ and positive numbers $\{\rho_i\}_{i=1}^N$ such that (see Figure A1.18)

$$\partial\Omega \subset \cup_{i=1}^N B\left(x_i, \rho_i/4\right).$$

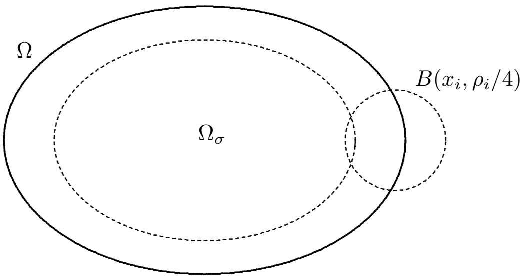

Fig. A1.18. The subdomain Ω_σ and the ball $B(x_i, \rho_i/4)$ about x_i

We let

$$\delta := \min_{1\le i\le N} \frac{\rho_i}{4},$$

and

$$C := \max_{1\le i\le N} C_2\left(n, \alpha, B'(x_i), B''(x_i)\right).$$

then, for each boundary point $x_0 \in \partial\Omega$ we can find some ball $B(x_i, \rho_i/4)$ such that

$$x_0 \in B\left(x_i, \rho_i/4\right).$$

Hence we have the inequality

$$|x - x_i| \le |x - x_0| + |x_0 - x_i| \le \delta + \frac{\rho_i}{4} < \frac{\rho_i}{2} \quad \text{for all } x \in B,$$

and so

$$B \cap \Omega \subset B(x_i, \rho_i/2) \cap \Omega = B''(x_i).$$

By using inequality (A.1.40), we obtain that

$$|u|_{0,\alpha;B\cap\Omega} \le |u|_{0,\alpha;B''(x_i)} \le C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right).$$

This proves the the desired estimate (A.1.36).

The proof of Lemma 1.11 is complete. □

Then, by using Lemma 1.11 we can obtain a *Schauder global estimate* for a general domain Ω ([33, Chapter 6, Theorem 6.6]):

Theorem 1.12. *Let Ω be a $C^{2+\alpha}$ domain in $\mathbf{R}^n$ with boundary $\partial\Omega$. For given functions $f \in C^{\alpha}(\Omega)$ and $\varphi \in C^{2+\alpha}(\Omega)$, assume that a function $u \in C^{2+\alpha}(\overline{\Omega})$ is a solution of the Dirichlet problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Then we have the global estimate

$$|u|_{2,\alpha;\Omega} \leq C \left(|u|_{0;\Omega} + |\varphi|_{2,\alpha;\Omega} + |f|_{0,\alpha;\Omega} \right), \tag{A.1.41}$$

with a constant $C = C(n, \alpha) > 0$.

Proof. The proof of Theorem 1.12 is divided into two steps.

Step I: The homogeneous case where $\varphi = 0$. We show that every solution $u \in C^{2+\alpha}(\overline{\Omega})$ of the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies the global estimate

$$|u|_{2,\alpha;\Omega} \leq C \left(|u|_{0,\alpha;\Omega} + |f|_{0,\alpha;\Omega} \right), \tag{A.1.42}$$

with a positive constant $C = C(n, \alpha)$.

(1) First, we consider the case where

$$x \in B(x_0, \delta) \cap \Omega.$$

Here δ is the positive constant in Lemma 1.11. Then it follows from an application of Lemma 1.11 that

$$\begin{aligned} |Du(x)| + \left|D^2u(x)\right| &\leq |u|_{2,\alpha;B(x_0,\delta)\cap\Omega} \\ &\leq C_\delta \left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} \right) \quad \text{for all } x \in B(x_0, \delta) \cap \Omega, \end{aligned} \tag{A.1.43}$$

with a positive constant $C_\delta = C(n, \alpha, \delta)$.

(2) Secondly, we consider the case where

$$x \in \Omega_\sigma := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \sigma\}, \quad \sigma := \frac{\delta}{2}.$$

Then it follows from an application of estimate (A.1.23) that

$$\begin{aligned} \sigma |Du|_{0;\Omega_\sigma} + \sigma^2 \left|D^2u\right|_{0;\Omega_\sigma} &\leq C \left(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega} \right) \\ &\leq C \left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} \right). \end{aligned}$$

This proves that

$$|Du(x)| + \left|D^2u(x)\right| \leq C_\sigma \left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right) \quad \text{for all } x \in \Omega_\sigma, \tag{A.1.44}$$

with a positive constant $C_\sigma = C(n, \alpha, \sigma)$.

Therefore, by combining estimates (A.1.43) and (A.1.44) we obtain that

$$|u|_{2;\Omega} \leq C_{\delta,\sigma} \left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right), \tag{A.1.45}$$

with a positive constant $C_{\delta,\sigma} = C(n, \alpha, \delta, \sigma)$.

(3) It remains to estimate the quantity $\left[D^2u\right]_{\alpha;\Omega}$.

(a) First, we consider the case where

$$x, y \in B(x_0, \delta) \cap \Omega.$$

Then it follows from an application of estimate (A.1.36) that

$$\frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \leq |u|_{2,\alpha;B(x_0,\delta)\cap\Omega} \leq C_1 \left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right). \tag{A.1.46}$$

(b) Secondly, we consider the case where

$$x, y \in \Omega_\sigma.$$

It follows from an application of estimate (A.1.23) that

$$\begin{aligned} \sigma^{2+\alpha} \frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} &\leq C \left(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}\right) \\ &\leq C_2 \left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right). \end{aligned} \tag{A.1.47}$$

(c) Finally, we consider the case where

$$|x - y| > \sigma, \quad \text{either } x \notin \Omega_\sigma \text{ or } y \notin \Omega_\sigma.$$

Then it follows from estimate (A.1.45) that

$$\begin{aligned} \frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} &\leq \sigma^{-\alpha} \left(\left|D^2u(x)\right| + \left|D^2u(y)\right|\right) \leq \sigma^{-\alpha}\, 2\, |u|_{2;\Omega} \\ &\leq C_3 \left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right). \end{aligned} \tag{A.1.48}$$

Therefore, we obtain from estimates (A.1.46), (A.1.47) and (A.1.48) that

$$\left[D^2u\right]_{\alpha;\Omega} = \sup_{x,y\in\Omega} \frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \tag{A.1.49}$$

$$\leq \left(C_1 + \frac{C_2}{\sigma^{2+\alpha}} + \frac{C_3}{\sigma^{\alpha}} \right) \left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} \right), \quad \sigma = \frac{\delta}{2}.$$

The desired global estimate (A.1.42) follows by combining estimates (A.1.45) and (A.1.49).

Step II: The non-homogeneous case where $\varphi \in C^{2+\alpha}(\overline{\Omega})$. Assume that a function $u \in C^{2+\alpha}(\overline{\Omega})$ is a solution of the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

If we let

$$v := u - \varphi \in C^{2+\alpha}(\Omega),$$

then the function v is a solution of the homogeneous Dirichlet problem

$$\begin{cases} \Delta v = f - \Delta\varphi & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, by applying estimate (A.1.42) to the solution $v = u - \varphi$ we obtain that

$$|u - \varphi|_{2,\alpha;\Omega} \leq C \left(|u - \varphi|_{0;\Omega} + |f - \Delta\varphi|_{0,\alpha;\Omega} \right). \tag{A.1.50}$$

However, we have the inequalities

$$|\Delta\varphi|_{0,\alpha;\Omega} \leq C\, |\varphi|_{2,\alpha;\Omega},$$

and

$$|u - \varphi|_{0,\alpha;\Omega} \leq |u|_{0,\alpha;\Omega} + |\varphi|_{0,\alpha;\Omega} \leq |u|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}.$$

Therefore, we obtain from estimate (A.1.50) that

$$\begin{aligned} |u - \varphi|_{2,\alpha;\Omega} &\leq C \left(|u - \varphi|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega} \right) \\ &\leq C \left(|u|_{0,\alpha;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega} \right). \end{aligned}$$

This proves that

$$\begin{aligned} |u|_{2,\alpha;\Omega} &\leq |u - \varphi|_{2,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega} \\ &\leq C \left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega} \right). \end{aligned}$$

Now the proof of Theorem 1.12 is complete. □