Part III

Theory of Singular Integral Operators

9
Elements of Singular Integrals

The Calderón–Zygmund theory of singular integral operators is a very refined mathematical tool whose full power is yet to be exploited. This chapter is devoted to a careful and accessible exposition of the most elementary part of the Calderón–Zygmund theory. We present a straightforward treatment of the Calderón–Zygmund theory necessary for the study of elliptic boundary value problems, assuming only basic knowledge of real analysis and functional analysis.

In Section 9.1 we formulate precisely the notion of singular integrals of Calderón and Zygmund (Lemma 9.1). In Section 9.2 we study the case where the integral kernel $K(x)$ is a bounded function that satisfies assumption 9.2 (Theorem 9.2). In Section 9.3 we study the case where the integral kernel $K(x)$ is a continuous function that satisfies assumption 9.3 (Theorem 9.5). The proof of Theorem 9.5 is based on a version of the Calderón–Zygmund decomposition adapted to the present context (Lemma 9.3). In Section 9.5, we introduce a non-negative, measurable function $f^*(t)$ defined on the interval $[0, \infty)$ for a given non-negative, measurable function $f(x)$ on $\mathbf{R}^n$ such that

$$\int_{\mathbf{R}^n} f(x)^p \, dx = \int_0^\infty f^*(t)^p \, dt.$$

The function $f^*(t)$ is called an *equimeasurable function* of $f(x)$ (Lemmas 9.8 and 9.9).

Sections 9.4 and 9.6 are devoted to the basic theory of the Hilbert transform H that is a special case of the singular integral of a single independent variable (Theorems 9.6 and 9.14):

$$Hf(x) := \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{|x-t|>\varepsilon} \frac{f(t)}{x-t} \, dt = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{|s|>\varepsilon} f(x-s) \, \frac{ds}{s}.$$

The proofs of Theorems 9.6 and 9.14 are flowcharted in two diagrams below.

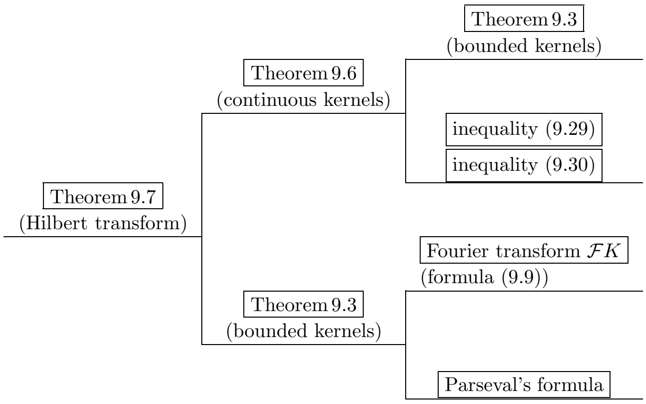

Table 9.1. *A flowchart for the proof of Theorem 9.6*

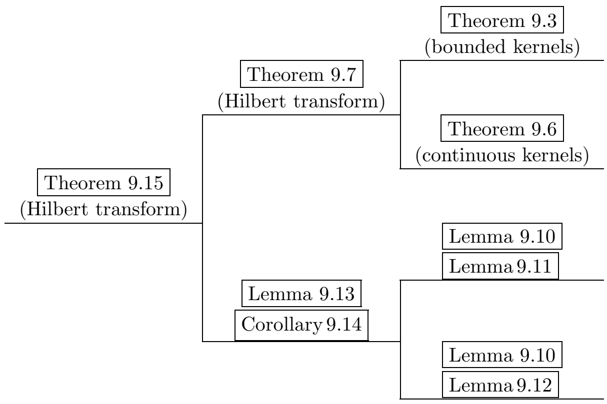

Table 9.2. *A flowchart for the proof of Theorem 9.14*

In Section 9.7 we study the case where the integral kernel $K(x)$ is an *odd* function that satisfies assumption 9.4 (Theorem 9.15). The proof of Theorem 9.15 can be reduced to the study of the Hilbert transform H (Theorem 9.14). More precisely, we have the following formula for the odd kernel $K(x)$ (see formula (9.83)):

$$K * f(x) := \lim_{\varepsilon \downarrow 0} \int_{|x-y|>\varepsilon} K(x-y)\, f(y)\, dy$$

$$= \frac{1}{2} \int_{\Sigma_{n-1}} K(\sigma) \left(\lim_{\varepsilon \downarrow 0} \int_{|t|>\varepsilon} f(x - t\sigma) \frac{dt}{t} \right) d\sigma.$$

Section 9.8 is devoted to the study of Riesz kernels (Theorem 9.16)

$$R_j(x) = -\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{x_j}{|x|^{n+1}}, \quad 1 \leq j \leq n.$$

It is easy to see that the Riesz kernels $R_j(x)$ satisfy assumption 9.3 and assumption 9.4.

In Section 9.9 we study the case where the integral kernel $K(x)$ is an *even* function that satisfies assumption 9.5 (Theorem 9.24). The case of even kernels can be reduced to the case of odd kernels. In fact, we have the following decomposition formula:

$$K * f = -\sum_{j=1}^{n} R_j * (R_j * K) * f,$$

where the operators R_j and $R_j * K$ have respectively the *odd kernels*

$$R_j(x) = -\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{x_j}{|x|^{n+1}},$$
$$R_j * K(x) = -\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{x_j}{|x|^{n+1}} * K(x).$$

This decomposition formula may be rephrased as follows:

$$\{\text{Even kernels}\} = \{\text{Riesz kernels}\} * \{\text{Odd kernels}\}.$$

In the final Section 9.10 we prove the existence of the singular integral in the general case (Theorem 9.25). The proof of Theorem 9.25 is flowcharted as follows:

9.1 Singular Integrals of Calderón and Zygmund

Let $K(x)$ be a real-valued, Lebesgue measurable function defined on $\mathbf{R}^n$. If $f(x) \in L^p(\mathbf{R}^n)$ with $1 < p < \infty$, then it follows that the usual Lebesgue integral

$$\int_{\mathbf{R}^n} K(x-y) f(y)\, dy \tag{9.1}$$

does not exist. However, under suitable conditions on the integral kernel $K(x)$, we can prove that the principal value of the integral (9.1)

$$\text{v.p.} \int_{\mathbf{R}^n} K(x-y) f(y)\, dy = \lim_{\varepsilon \downarrow 0} \int_{|x-y|>\varepsilon} K(x-y) f(y)\, dy \tag{9.2}$$

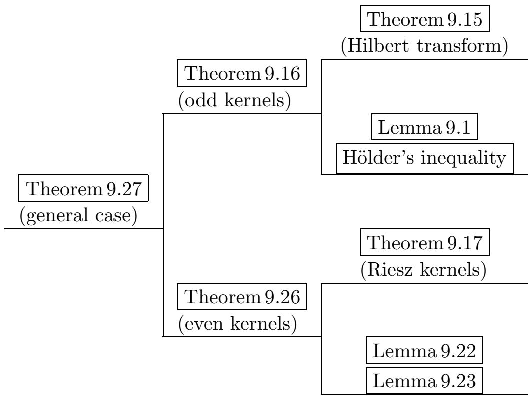

Table 9.3. *A flowchart for the proof of Theorem 9.25*

exist. The integrals defined as formula (9.2) are called *singular integrals*. In what follows we shall use the notation

$$K * f(x) = \lim_{\varepsilon \downarrow 0} \int_{|x-y|>\varepsilon} K(x-y) f(y)\, dy.$$

Throughout this chapter, we consider the case where the integral kernel $K(x)$ in formula (9.2) satisfies the following assumption:

Assumption 9.1. The integral kernel $K(x)$ is positively homogeneous of degree $-n$, that is,

$$K(\lambda x) = \lambda^{-n} K(x) \quad \text{for all } \lambda > 0 \text{ and all } x \in \mathbf{R}^n, \tag{9.3}$$

and further it is integrable on the unit sphere

$$\Sigma_{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}$$

and satisfies the *cancellation property*

$$\int_{\Sigma_{n-1}} K(\sigma)\, d\sigma = 0, \tag{9.4}$$

where $d\sigma$ is the surface measure on Σ_{n-1}.

In this chapter we shall denote the Lebesgue measure of a subset A of $\mathbf{R}^n$ by $|A|$ and the norm of the space $L^p(\mathbf{R}^n)$ by $\|\cdot\|_p$, respectively.

We begin by proving the following fundamental result:

Lemma 9.1. *Let $K(x)$ be an integral kernel satisfying Assumption 9.1.*

If $f(x) \in L^p(\mathbf{R}^n)$ for $1 < p < \infty$ and $\varepsilon > 0$, then the Lebesgue integral

$$\widetilde{f_\varepsilon}(x) := \int_{|x-y|>\varepsilon} K(x-y) f(y)\, dy \tag{9.5}$$

exists for almost all $x \in \mathbf{R}^n$.

Proof. First, we have, by a suitable change of variables,

$$\begin{aligned}\widetilde{f_\varepsilon}(x) &= \int_{|y|>\varepsilon} K(y) f(x-y)\, dy \\ &= \int_{\Sigma_{n-1}} \left(\int_\varepsilon^\infty K(t\sigma) f(x-t\sigma) t^{n-1}\, dt \right) d\sigma \\ &= \int_{\Sigma_{n-1}} K(\sigma) \left(\int_\varepsilon^\infty f(x-t\sigma) \frac{dt}{t} \right) d\sigma.\end{aligned}$$

Hence it suffices to show that, for any bounded measurable subset S of $\mathbf{R}^n$, the integral

$$\int_S \left(\int_{\Sigma_{n-1}} |K(\sigma)| \left(\int_\varepsilon^\infty |f(x-t\sigma)| \frac{dt}{t} \right) d\sigma \right) dx$$

is finite. More precisely, we show that there exists a positive constant $C_{\varepsilon,S}$ such that

$$\begin{aligned}&\int_S \left(\int_{\Sigma_{n-1}} |K(\sigma)| \left(\int_\varepsilon^\infty |f(x-t\sigma)| \frac{dt}{t} \right) d\sigma \right) dx \\ &\le C_{\varepsilon,S} \, \|f\|_p \left(\int_{\Sigma_{n-1}} |K(\sigma)|\, d\sigma \right).\end{aligned} \tag{9.6}$$

By using Hölder's inequality (Theorem 3.14), we obtain that

$$\begin{aligned}&\int_S \left(\int_\varepsilon^\infty |f(x-t\sigma)| \frac{dt}{t} \right) dx \\ &\le C_{\varepsilon,p} \int_S \left(\int_{-\infty}^\infty |f(x-t\sigma)|^p\, dt \right)^{1/p} dx \\ &\le C_{\varepsilon,p} |S|^{1-1/p} \left(\int_S \int_{-\infty}^\infty |f(x-t\sigma)|^p\, dt\, dx \right)^{1/p},\end{aligned} \tag{9.7}$$

where $C_{\varepsilon,p}$ is a positive constant given by the formula

$$C_{\varepsilon,p} := (p-1)^{(p-1)/p} \varepsilon^{-1/p}.$$

For each $\sigma \in \Sigma_{n-1}$, we make a change of variables

$$x := z - s\,\sigma, \quad z \in \mathbf{R}^n, \; s \in \mathbf{R},$$

with

$$\langle z, \sigma \rangle = z_1\sigma_1 + \ldots + z_n\sigma_n = 0.$$

Moreover, we choose a subset S' of $\mathbf{R}^{n-1}$ and two real numbers a, b such that

$$S \subset \{x = z - s\sigma : z \in S', a < s < b\}.$$

Then the last term on inequality (9.7) can be estimated as follows:

$$\begin{aligned}
& C_{\varepsilon,p}|S|^{1-1/p} \left(\int_S \int_{-\infty}^{\infty} |f(x - t\,\sigma)|^p \, dt \, dx \right)^{1/p} \\
& \le C_{\varepsilon,p}|S|^{1-1/p} \left(\int_a^b \int_{S'} \int_{-\infty}^{\infty} |f(z - (t+s)\sigma)|^p \, dt \, dz \, ds \right)^{1-p} \\
& \le C_{\varepsilon,p}|S|^{1-1/p} \left((b-a) \int_{\mathbf{R}^n} |f(x)|^p \, dx \right)^{1/p} \\
& = C_{\varepsilon,p} \, |S|^{1-1/p} \, (b-a)^{1/p} \, \|f\|_p \, .
\end{aligned}$$

Here it should be noticed that the two numbers a, b can be chosen independent of $\sigma \in \Sigma_{n-1}$.

Therefore, we obtain that

$$\begin{aligned}
& \int_S \left(\int_{\Sigma_{n-1}} |K(\sigma)| \left(\int_\varepsilon^\infty |f(x - t\sigma)| \, \frac{dt}{t} \right) d\sigma \right) dx \\
& \le C_{\varepsilon,p} \, |S|^{1-1/p} \, (b-a)^{1/p} \, \|f\|_p \left(\int_{\Sigma_{n-1}} |K(\sigma)| \, d\sigma \right).
\end{aligned}$$

This proves the desired inequality (9.6) with

$$C_{\varepsilon,S} := C_{\varepsilon,p}|S|^{1-1/p}(b-a)^{1/p}.$$

The proof of Lemma 9.1 is complete. □

Remark 9.1. We remark that the cancellation property (9.4) is not used in the proof of Lemma 9.1.

9.2 The Case of Bounded Kernels

In this section we consider the case where the integral kernel $K(x)$ satisfies the following assumption:

Assumption 9.2. The integral kernel $K(x)$ satisfies Assumption 9.1 and is *bounded* on the unit sphere Σ_{n-1}.

The purpose of this section is to prove the existence of the singular integral (9.2) in the space $L^2(\mathbf{R}^n)$ in the case of *bounded kernels*:

Theorem 9.2. *Let $K(x)$ be an integral kernel satisfying Assumption 9.2. If $f(x) \in L^2(\mathbf{R}^n)$ and if $\varepsilon > 0$, we let*

$$\widetilde{f}_\varepsilon(x) := \int_{|x-y|>\varepsilon} K(x-y) f(y)\, dy.$$

Then we have the following four assertions (i) through (iv):

(i) There exists a constant C, independent of ε, such that

$$\left\| \widetilde{f}_\varepsilon \right\|_2 \leq C \left\| f \right\|_2.$$

(ii) The sequence $\widetilde{f}_\varepsilon$ converges strongly to a function $K * f$ in the space $L^2(\mathbf{R}^n)$ as $\varepsilon \downarrow 0$. Namely, the singular integral

$$K * f(x) = \lim_{\varepsilon \downarrow 0} \int_{|x-y|>\varepsilon} K(x-y) f(y)\, dy$$

exists in the strong topology of $L^2(\mathbf{R})$.

(iii) The mapping $f \mapsto K * f$ is a bounded linear operator from $L^2(\mathbf{R}^n)$ into itself.

(iv) The Fourier transform $\mathcal{F}(K * f)$ of $K * f$ is given by the formula

$$\mathcal{F}(K * f)(\xi) = \mathcal{F}K(\xi) \cdot \mathcal{F}f(\xi), \tag{9.8}$$

where $\mathcal{F}K(\xi)$ is an essentially bounded, measurable function defined by the formula

$$\mathcal{F}K(\xi) := \int_{\Sigma_{n-1}} K(\sigma) \left(\int_0^\infty \left(e^{-is\frac{\sigma \cdot \xi}{|\xi|}} - e^{-s} \right) \frac{ds}{s} \right) d\sigma \quad \text{for } \xi \neq 0, \tag{9.9}$$

and is a positively homogeneous function of degree 0.

Proof. For $0 < \varepsilon < \mu$, we introduce two functions

$$K_{\varepsilon,\mu}(x) := \begin{cases} K(x) & \text{if } \varepsilon < |x| < \mu, \\ 0 & \text{otherwise,} \end{cases} \tag{9.10}$$

and

$$K_\varepsilon(x) := \begin{cases} K(x) & \text{if } |x| > \varepsilon, \\ 0 & \text{if } |x| \leq \varepsilon. \end{cases} \tag{9.11}$$

Then it follows that

$$\widetilde{f_\varepsilon}(x) = \int_{\mathbf{R}^n} K_\varepsilon(x-y)\, f(y)\, dy = (K_\varepsilon * f)(x).$$

The proof of Theorem 9.2 is divided into four steps.

Step (I): First, we show that the sequence $K_{\varepsilon,\mu}$ converges strongly to the function K_ε in $L^2(\mathbf{R}^n)$ as $\mu \uparrow \infty$. Indeed, it suffices to note that

$$\begin{aligned}
&\int_{\mathbf{R}^n} |K_{\varepsilon,\mu}(x) - K_\varepsilon(x)|^2\, dx \\
&= \int_{|x|>\mu} |K(x)|^2\, dx \\
&= \int_\mu^\infty \int_{\Sigma_{n-1}} |K(r\sigma)|^2\, r^{n-1}\, dr\, d\sigma \\
&= \left(\int_{\Sigma_{n-1}} |K(\sigma)|^2\, d\sigma \right) \int_\mu^\infty \frac{1}{r^{n+1}}\, dr \\
&= \frac{1}{n} \left(\int_{\Sigma_{n-1}} |K(\sigma)|^2\, d\sigma \right) \frac{1}{\mu^n} \longrightarrow 0 \quad \text{as } \mu \uparrow \infty.
\end{aligned}$$

Therefore, it follows from an application of Parseval's formula that the sequence $\mathcal{F}K_{\varepsilon,\mu}$ converges strongly to the function $\mathcal{F}K_\varepsilon$ in $L^2(\mathbf{R}^n)$ as $\mu \uparrow \infty$:

$$\lim_{\mu\uparrow\infty} \mathcal{F}K_{\varepsilon,\mu} = \mathcal{F}K_\varepsilon \quad \text{in } L^2(\mathbf{R}^n). \tag{9.12}$$

Step (II): Secondly, we calculate explicitly the Fourier transform of $K_{\varepsilon,\mu}(x)$, which is an essential step in the proof of Theorem 9.2.

To do this, we remark that $K_{\varepsilon,\mu}(x) \in L^1(\mathbf{R}^n)$. We write, for $x \in \mathbf{R}^n$ and $\xi \in \mathbf{R}^n$,

$$\begin{aligned}
&x = r\sigma, \quad |x| = r, \quad \sigma \in \Sigma_{n-1}, \\
&|\xi| = \rho, \quad x \cdot \xi = \langle x, \xi \rangle = r\rho\cos\phi, \quad \xi \neq 0.
\end{aligned}$$

Then, by using two conditions (9.3) and (9.4) (the cancellation property) we obtain from formula (9.10) that

$$\begin{aligned}
&(\mathcal{F}K_{\varepsilon,\mu})(\xi) \\
&= \int_{\mathbf{R}^n} K_{\varepsilon,\mu}(x) e^{-ix\cdot\xi}\, dx = \int_{\varepsilon<|x|<\mu} K(x) e^{-ix\cdot\xi}\, dx \\
&= \int_\varepsilon^\mu \left(\int_{\Sigma_{n-1}} K(\sigma) e^{-ir\rho\cos\phi}\, d\sigma \right) \frac{dr}{r}
\end{aligned} \tag{9.13}$$

$$
\begin{aligned}
&= \int_{\varepsilon\rho}^{\mu\rho} \left(\int_{\Sigma_{n-1}} K(\sigma) e^{-is\cos\phi}\, d\sigma \right) \frac{ds}{s} \\
&= \int_{\varepsilon\rho}^{\mu\rho} \left(\int_{\Sigma_{n-1}} K(\sigma) \left(e^{-is\cos\phi} - e^{-s} \right) d\sigma \right) \frac{ds}{s} \\
&= \int_{\Sigma_{n-1}} K(\sigma) \left(\int_{\varepsilon\rho}^{\mu\rho} \left(e^{-is\cos\phi} - e^{-s} \right) \frac{ds}{s} \right) d\sigma, \quad \rho = |\xi| \neq 0.
\end{aligned}
$$

We decompose the integral

$$
\int_{\varepsilon\rho}^{\mu\rho} \left(e^{-is\cos\phi} - e^{-s} \right) \frac{ds}{s} \quad (\rho \neq 0)
$$

into the two terms:

$$
\int_{\varepsilon\rho}^{\mu\rho} \left(e^{-is\cos\phi} - e^{-s} \right) \frac{ds}{s} := \mathrm{I}_1(\varepsilon,\mu) + i\,\mathrm{I}_2(\varepsilon,\mu), \tag{9.14}
$$

where

$$
\begin{aligned}
\mathrm{I}_1(\varepsilon,\mu) &:= \int_{\varepsilon\rho}^{\mu\rho} \left(\cos\left(s\cos\phi\right) - e^{-s} \right) \frac{ds}{s}, \\
\mathrm{I}_2(\varepsilon,\mu) &:= -\int_{\varepsilon\rho}^{\mu\rho} \sin\left(s\cos\phi\right) \frac{ds}{s}.
\end{aligned}
$$

Step (1): First, we have, for $0 < a < b$,

$$
\begin{aligned}
\int_a^b \frac{\cos s}{s}\, ds &= \int_{a+\pi/2}^{b+\pi/2} \frac{\cos(s-\pi/2)}{s-\pi/2}\, ds = \int_{a+\pi/2}^{b+\pi/2} \frac{\sin s}{s-\pi/2}\, ds \qquad (9.15) \\
&= \int_{a+\pi/2}^{b+\pi/2} \frac{\sin s}{s}\, ds + \int_{a+\pi/2}^{b+\pi/2} \left(\frac{1}{s-\pi/2} - \frac{1}{s} \right) \sin s\, ds.
\end{aligned}
$$

Since the improper integral

$$
\lim_{b\uparrow\infty} \int_0^b \frac{\sin x}{x}\, dx
$$

exists, it follows from formula (9.15) that the improper integral

$$
\lim_{b\uparrow\infty} \int_a^b \frac{\cos s}{s}\, ds
$$

exists. Hence, we find that the improper integral

$$
\lim_{\varepsilon\downarrow 0,\mu\uparrow\infty} \int_{\varepsilon\rho}^{\mu\rho} \left(e^{-is\cos\phi} - e^{-s} \right) \frac{ds}{s} \quad (\rho \neq 0)
$$

exists for $\cos\phi \neq 0$.

Step (2): Secondly, we prove an estimate on integral (9.14) independent of ε and μ.

To do this, we choose a positive constant A such that we have, for all $a > 0$,

$$\left| \int_0^a \frac{\sin x}{x}\, dx \right| \leq A. \tag{9.16}$$

For example, we may take

$$A := \sup_{a>0} \left| \int_0^a \frac{\sin x}{x}\, dx \right|.$$

Then it is easy to see that

$$|\mathrm{I}_2(\varepsilon, \mu)| = \left| \int_{\varepsilon\rho}^{\mu\rho} \sin(s\cos\phi)\, \frac{ds}{s} \right| = \left| \int_{\varepsilon\rho\cos\phi}^{\mu\rho\cos\phi} \frac{\sin t}{t}\, dt \right| \leq 2A.$$

On the other hand, by combining formula (9.15) and inequality (9.16) we obtain that

$$\begin{aligned} \left| \int_a^b \frac{\cos s}{s}\, ds \right| &\leq 2A + \int_{a+\pi/2}^{b+\pi/2} \left(\frac{1}{s-\pi/2} - \frac{1}{s} \right) ds \qquad (9.17) \\ &= 2A + \log \frac{b(a+\pi/2)}{a(b+\pi/2)} \\ &\leq 2A + \log\left(1 + \frac{\pi}{2a}\right). \end{aligned}$$

Step (2-a): We consider the case where $\cos\phi > 0$. In this case, we have the inequality

$$\begin{aligned} &\int_0^1 \left|\cos(s\cos\phi) - e^{-s}\right| \frac{ds}{s} \qquad (9.18) \\ &\leq \int_0^1 (1 - \cos(s\cos\phi))\, \frac{ds}{s} + \int_0^1 \left(1 - e^{-s}\right) \frac{ds}{s} \\ &\leq \int_0^1 (1 - \cos s)\, \frac{ds}{s} + \int_0^1 \left(1 - e^{-s}\right) \frac{ds}{s}. \end{aligned}$$

(i) If $\mu\rho < 1$, it follows that

$$|\mathrm{I}_1(\varepsilon, \mu)| = \left| \int_{\varepsilon\rho}^{\mu\rho} \left(\cos(s\cos\phi) - e^{-s}\right) \frac{ds}{s} \right| \leq B,$$

where

$$B := \int_0^1 (1 - \cos s)\, \frac{ds}{s} + \int_0^1 \left(1 - e^{-s}\right) \frac{ds}{s}.$$

(ii) If $\varepsilon\rho > 1$, it follows from inequality (9.17) that

$$\begin{aligned}|\mathrm{I}_1(\varepsilon,\mu)| &= \left| \int_{\varepsilon\rho\cos\phi}^{\mu\rho\cos\phi} \frac{\cos s}{s}\, ds - \int_{\varepsilon\rho}^{\mu\rho} e^{-s}\, \frac{ds}{s} \right| \\ &\le 2A + \log\left(1 + \frac{\pi}{2\varepsilon\rho\cos\phi}\right) + \int_1^\infty e^{-s}\, \frac{ds}{s} \\ &\le 2A + \log\left(1 + \frac{\pi}{2\cos\phi}\right) + \int_1^\infty e^{-s}\, \frac{ds}{s}.\end{aligned}$$

(iii) If $\varepsilon\rho < 1 < \mu\rho$, then, by combining two inequalities (9.17) and (9.18) we obtain that

$$\begin{aligned}|\mathrm{I}_1(\varepsilon,\mu)| &= \left| \left(\int_{\varepsilon\rho}^{1} + \int_1^{\varepsilon\rho} \right) \left(\cos(s\cos\phi) - e^{-s}\right) \frac{ds}{s} \right| \\ &\le B + \left| \int_{\cos\phi}^{\mu\rho\cos\phi} \frac{\cos s}{s}\, ds \right| + \int_1^\infty e^{-s}\, \frac{ds}{s} \\ &\le B + 2A + \log\left(1 + \frac{\pi}{2\cos\phi}\right) + \int_1^\infty e^{-s}\, \frac{ds}{s}.\end{aligned}$$

Step (2-b): The case where $\cos\phi < 0$ can be estimated similarly.

Summing up, we can find a positive constant C, independent of ε and μ, such that

$$\begin{aligned}\left| \int_{\varepsilon|\xi|}^{\mu|\xi|} \left(e^{-is\cos\phi} - e^{-s}\right) \frac{ds}{s} \right| &\le C + \log\left(\frac{1}{|\cos\phi|}\right) \\ &= C + \log\left(\frac{|\xi|}{|\sigma\cdot\xi|}\right), \quad \xi \neq 0.\end{aligned} \tag{9.19}$$

Therefore, by combining formula (9.13) and inequality (9.19) we obtain that

$$\begin{aligned}&|(\mathcal{F}K_{\varepsilon,\mu})(\xi)| \\ &\le \int_{\Sigma_{n-1}} \left| K(\sigma) \int_{\varepsilon|\xi|}^{\mu|\xi|} \left(e^{-is\cos\phi} - e^{-s}\right) \frac{ds}{s} \right| d\sigma \\ &\le \left(\sup_{\sigma\in\Sigma_{n-1}} |K(\sigma)| \right) \int_{\Sigma_{n-1}} \left(C + \log\left(\frac{|\xi|}{|\sigma\cdot\xi|}\right)\right) d\sigma, \quad \xi \neq 0.\end{aligned}$$

This proves that there exists a positive constant M such that we have, for $0 < \varepsilon < \mu$,

$$|(\mathcal{F}K_{\varepsilon,\mu})(\xi)| \le M, \quad \xi \neq 0. \tag{9.20}$$

On the other hand, it follows from formula (9.13) and inequality (9.19) that we have, for all $\xi \neq 0$,

$$\lim_{\mu\uparrow\infty} (\mathcal{F}K_{\varepsilon,\mu})(\xi) = \int_{\Sigma_{n-1}} K(\sigma) \left(\int_{\varepsilon|\xi|}^{\infty} \left(e^{-is\cos\phi} - e^{-s}\right) \frac{ds}{s} \right) d\sigma,$$

and

$$\left| \int_{\varepsilon|\xi|}^{\infty} \left(e^{-is\cos\phi} - e^{-s}\right) \frac{ds}{s} \right| \leq C + \log\left(\frac{|\xi|}{|\sigma\cdot\xi|} \right), \quad \xi \neq 0. \tag{9.21}$$

By assertion (9.12), this proves that we have, for almost all $\xi \in \mathbf{R}^n \setminus \{0\}$,

$$\begin{aligned}(\mathcal{F}K_{\varepsilon})(\xi) &= \lim_{\mu\uparrow\infty} (\mathcal{F}K_{\varepsilon,\mu})(\xi) \qquad (9.22)\\ &= \int_{\Sigma_{n-1}} K(\sigma) \left(\int_{\varepsilon|\xi|}^{\infty} \left(e^{-is\frac{\sigma\cdot\xi}{|\xi|}} - e^{-s}\right) \frac{ds}{s} \right) d\sigma.\end{aligned}$$

Moreover, it follows from inequality (9.21) and formulas (9.22) and (9.9) that we have, for almost all $\xi \in \mathbf{R}^n \setminus \{0\}$,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} (\mathcal{F}K_{\varepsilon})(\xi) &= \int_{\Sigma_{n-1}} K(\sigma) \left(\int_{0}^{\infty} \left(e^{-is\frac{\sigma\cdot\xi}{|\xi|}} - e^{-s}\right) \frac{ds}{s} \right) d\sigma. \qquad (9.23)\\ &= \mathcal{F}K(\xi).\end{aligned}$$

It should be noticed that the Fourier transform $\mathcal{F}K(\xi)$ of the kernel $K(x)$ is a positively homogeneous function of degree 0 and that we have, by inequality (9.18),

$$|(\mathcal{F}K_{\varepsilon})(\xi)| \leq M, \quad \xi \neq 0, \tag{9.24a}$$

$$|\mathcal{F}K(\xi)| \leq M, \quad \xi \neq 0. \tag{9.24b}$$

Step (III): If $f(x) \in L^2(\mathbf{R}^n)$, then it follows from an application of Theorem 3.23 with $p := 1$ and $q = r := 2$ that

$$K_{\varepsilon,\mu} * f \in L^2(\mathbf{R}^n),$$

since $K_{\varepsilon,\mu}(x) \in L^1(\mathbf{R}^n)$. However, we have, by formula (9.22),

$$\mathcal{F}(K_{\varepsilon,\mu}(\xi) \longrightarrow \mathcal{F}K_{\varepsilon}(\xi) \quad \text{for almost all } \xi \in \mathbf{R}^n \setminus \{0\} \text{ as } \mu \uparrow \infty,$$

and also, by inequalities (9.20) and (9.24),

$$\begin{aligned}&|\mathcal{F}\left(K_{\varepsilon,\mu}(\xi) - \mathcal{F}K_{\varepsilon}(\xi)\right)\mathcal{F}f(\xi)|^2\\ &\leq 4\,M^2\,|\mathcal{F}f(\xi)|^2 \quad \text{for } \xi \neq 0.\end{aligned}$$

Therefore, by applying the Lebesgue dominated convergence theorem (Theorem 3.8) we obtain from Parseval's formula that

$$\begin{aligned}
&\|K_{\varepsilon,\mu} * f - K_\varepsilon * f\|_2 \\
&= \frac{1}{(2\pi)^{n/2}} \|\mathcal{F}(K_{\varepsilon,\mu} * f) - \mathcal{F}(K_\varepsilon * f)\|_2 \\
&= \frac{1}{(2\pi)^{n/2}} \|(\mathcal{F}(K_{\varepsilon,\mu}) - \mathcal{F}(K_\varepsilon))\, \mathcal{F}f\|_2 \longrightarrow 0 \quad \text{as } \mu \uparrow \infty.
\end{aligned}$$

Step (IV): Moreover, by using formulas (9.23) and (9.9) we find that the sequence $K_\varepsilon * f$ is a Cauchy sequence in the space $L^2(\mathbf{R}^n)$ as $\varepsilon \downarrow 0$. This implies that $K_\varepsilon * f$ converges to some function g, denoted by $g := K * f$, in $L^2(\mathbf{R}^n)$ as $\varepsilon \downarrow 0$:

$$K * f := \lim_{\varepsilon \downarrow 0} K_\varepsilon * f \quad \text{in } L^2(\mathbf{R}^n).$$

Hence, by Parseval's formula and formula (9.21) it follows that

$$\begin{aligned}
\mathcal{F}(K * f)(\xi) &= \lim_{\varepsilon \downarrow 0} \mathcal{F}(K_\varepsilon * f)(\xi) = \lim_{\varepsilon \downarrow 0} \mathcal{F}(K_\varepsilon)(\xi) \cdot \mathcal{F}f(\xi) \\
&= \mathcal{F}K(\xi) \cdot \mathcal{F}f(\xi), \quad \xi \neq 0.
\end{aligned}$$

Therefore, by using the Parseval formula we obtain from inequality (9.24b) that

$$\begin{aligned}
\|K * f\|_2 &= \frac{1}{(2\pi)^{n/2}} \|\mathcal{F}(K * f)\|_2 = \frac{1}{(2\pi)^{n/2}} \|\mathcal{F}K \cdot \mathcal{F}f\|_2 \\
&\leq \frac{M}{(2\pi)^{n/2}} \|\mathcal{F}f\|_2 = M \|f\|_2 \quad \text{for all } f \in L^2(\mathbf{R}^n).
\end{aligned}$$

Now the proof of Theorem 9.2 is complete. □

9.3 The Case of Continuous Kernels

In this section, we make the following assumption:

Assumption 9.3. In addition to Assumption 9.2, the integral kernel $K(x)$ satisfies the following conditions (i) and (ii):

(i) There exists a positively homogeneous function $\Omega(x)$ of degree 0 defined on $\mathbf{R}^n \setminus \{0\}$ such that

$$K(x) = \Omega\left(\frac{x}{|x|}\right) \frac{1}{|x|^n} \quad \text{for all } x \in \mathbf{R}^n \setminus \{0\}. \tag{9.25}$$

(ii) The function $\Omega(x)$ satisfies the condition

$$|\Omega(x) - \Omega(y)| \leq \omega(|x - y|) \quad \text{for all } x,\, y \in \Sigma_{n-1}, \tag{9.26}$$

where $\omega(t)$ is a non-negative, increasing continuous function defined on the interval $[0, \infty)$ that satisfies the conditions

$$\int_0^1 \frac{\omega(t)}{t}\, dt < \infty, \tag{9.27a}$$

$$\frac{\omega(t)}{t} \geq c_0 \quad \text{for all } t \geq 0, \tag{9.27b}$$

with some constant $c_0 > 0$.

The next lemma is another version of the *Calderón–Zygmund decomposition* adapted to the present context (cf. Theorem 4.10):

Lemma 9.3 (the Calderón–Zygmund decomposition)**.** *Let $f(x)$ be an arbitrary non-negative function in the space $L^p(\mathbf{R}^n)$ with $1 \leq p < \infty$. Then, for any positive number s, there exists a sequence of non-overlapping cubes $\{I_k\}$ such that*

$$s \leq \frac{1}{|I_k|} \int_{I_k} f(x)\, dx < 2^n s. \tag{9.28}$$

If we let

$$D_s := \bigcup_{k=1}^{\infty} I_k,$$

then it follows that

$$f(x) \leq s \quad \textit{almost everywhere outside the set } D_s,$$

and that

$$s \leq \frac{1}{|D_s|} \int_{D_s} f(x)\, dx < 2^n s. \tag{9.29}$$

Proof. Let I be a cube. Since we have, as $|I| \to \infty$,

$$\frac{1}{|I|} \int_I f(x)\, dx \leq \frac{\|f\|_p}{|I|^{1/p}} \longrightarrow 0,$$

we can find a positive number ν such that, if $|I| \geq \nu$, we have the inequality

$$\frac{1}{|I|} \int_I f(x)\, dx < s.$$

We let

$$\mathbf{R}^n = \bigcup_{k=1}^{\infty} I_{0,k}$$

to be the decomposition of $\mathbf{R}^n$ into a sum of cubes $I_{0,k}$ of volume ν. Then we have, for each k,

$$\frac{1}{|I_{0,k}|} \int_{I_{0,k}} f(x)\, dx < s.$$

First, we denote by $\{I_{1,k}\}$ the sequence of cubes obtained by dividing each $I_{0,k}$ into 2^n equal parts. We classify the cubes $\{I_{1,k}\}$ as follows:

$$\{I_{1,k}\} = \{I'_{1,k}\} \cup \{1''_{1,k}\},$$
$$\frac{1}{|I'_{1,k}|} \int_{I'_{1,k}} f(x)\, dx \geq s,$$
$$\frac{1}{|I''_{1,k}|} \int_{I''_{1,k}} f(x)\, dx < s.$$

Secondly, we denote by $\{I_{2,k}\}$ the sequence of cubes obtained by dividing each $I''_{1,k}$ into 2^n equal parts, and we classify the cubes $\{I_{2,k}\}$ as follows:

$$\{I_{2,k}\} = \{I'_{2,k}\} \cup \{I''_{2,k}\},$$
$$\frac{1}{|I'_{2,k}|} \int_{I'_{2,k}} f(x)\, dx \geq s,$$
$$\frac{1}{|I''_{2,k}|} \int_{I''_{2,k}} f(x)\, dx < s.$$

Repeating this process, we obtain a sequence of cubes $\{I_{m,k}\}$ such that we have, for each integer m,

$$\{I_{m,k}\} = \{I'_{m,k}\} \cup \{I''_{m,k}\},$$
$$\frac{1}{|I'_{m,k}|} \int_{I'_{m,k}} f(x)\, dx \geq s,$$
$$\frac{1}{|I''_{m,k}|} \int_{I''_{m,k}} f(x)\, dx < s.$$

Then we have the decomposition

$$\mathbf{R}^n = \left(\cup I'_{1,k}\right) \cup \left(\cup I''_{1,k}\right) \cup \cdots \cup \left(\cup I'_{m,k}\right) \cup \left(\cup I''_{m,k}\right).$$

Moreover, if $m \geq 1$, then each cube $I'_{m,k}$ is obtained by dividing some

$I''_{m-1,l}$, where $I''_{0,l} = I_{0,l}$. This implies that

$$|I'_{m,k}| = \frac{1}{2^n} |I''_{m-1,l}|.$$

Hence we have the inequalities

$$\frac{1}{|I'_{m,k}|} \int_{I'_{m,k}} f(x)\,dx \leq \frac{2^n}{|I''_{m-1,l}|} \int_{I''_{m-1,l}} f(x)\,dx < 2^n s.$$

Now, we let $\{I_k\}$ be the totality of the cubes $I'_{m,k}$, $m = 1, 2, \ldots$, $k = 1, 2, \ldots$, and let

$$D_s = \bigcup_{k=1}^{\infty} I_k.$$

Then it follows that inequalities (9.28) hold true for all k:

$$s \leq \frac{1}{|I_k|} \int_{I_k} f(x)\,dx < 2^n s.$$

Since we have, by Hölder's inequality (Theorem 3.14),

$$|I_k| \leq \frac{1}{s} \int_{I_k} f(x)\,dx \leq \frac{1}{s} |I_k|^{1-1/p} \left(\int_{I_k} f(x)^p\,dx \right)^{1/p},$$

we obtain that

$$|I_k| \leq \frac{1}{s^p} \int_{I_k} f(x)^p\,dx,$$

so that

$$|D_s| = \sum_{k=1}^{\infty} |I_k| \leq \frac{1}{s^p} \int_{D_s} f(x)^p\,dx < \infty.$$

Moreover, it is easy to verify that inequalities (9.29) hold true. Indeed, we have, by inequalities (9.28),

$$|D_s| = \sum_{k=1}^{\infty} |I_k| \leq \frac{1}{s} \sum_{k=1}^{\infty} \int_{I_k} f(x)\,dx = \frac{1}{s} \int_{D_s} f(x)\,dx.$$

Similarly, we have the inequality

$$\int_{D_s} f(x)\,dx = \sum_{k=1}^{\infty} \int_{I_k} f(x)\,dx \leq \sum_{k=1}^{\infty} |I_k|\, 2^n\, s = 2^n\, s\, |D_s|\,.$$

Finally, let x_0 be an arbitrary Lebesgue point of the function $f(x)$ which does not belong to the set D_s (see Definition 4.1). Then, for any integer $m \geq 1$, there exists an integer $k(m)$ such that $x_0 \in I''_{m,k(m)}$.

Since $\{I''_{m,k(m)}\}$ is a sequence of regular closed sets converging to x_0, it follows from an application of Corollary 4.6 that

$$s > \frac{1}{|I''_{m,k}(m)|} \int_{I''_{m,k(m)}} f(x)\, dx \longrightarrow f(x_0),$$

so that

$$f(x_0) \le s.$$

Therefore, we have proved that $f(x) \le s$ almost everywhere outside the set D_s.

The proof of Lemma 9.3 is complete. □

The next lemma is an essential step for the proof of existence of the singular integral (9.2) in the space $L^p(\mathbf{R}^n)$ for $1 < p < \infty$:

Lemma 9.4. *Let $K(x)$ be an integral kernel satisfying Assumption 9.3. Assume that $f(x)$ is a non-negative function in $L^p(\mathbf{R}^n)$ for $1 \le p \le 2$. Let $K_\varepsilon(x)$ be the function defined by formula (9.11) and let $\widetilde{f_\varepsilon}(x) := K_\varepsilon * f(x)$ be the function defined by formula (9.5). Moreover, if s is a positive number, we let*

$$E_s := \{x : |\widetilde{f_\varepsilon}(x)| > s\}.$$

Then we can find two positive constants C_1 and C_2, independent of ε and s, such that

$$|E_s| \le \frac{C_1}{s^2} \int_{\mathbf{R}^n} [f(x)]_s^2\, dx + C_2 |D_s|, \tag{9.30}$$

where D_s is the set as in Lemma 9.3, and $[f(x)]_s$ is a function defined by the formula

$$[f(x)]_s := \begin{cases} f(x) & \text{if } f(x) \le s, \\ s & \text{if } f(x) > s. \end{cases}$$

Proof. In the following we shall denote by C a generic positive constant, independent of ε and s.

Let $\{I_k\}$ be the sequence of cubes as in Lemma 9.3, and let $g(x)$ and $h(x)$ be functions defined respectively by the formulas

$$h(x) := \begin{cases} \frac{1}{|I_k|} \int_{I_k} f(y)\, dy & \text{if } x \in I_k, \\ f(x) & \text{if } x \notin D_s, \end{cases}$$

and

$$g(x) := f(x) - h(x).$$

Then it follows that

$$s \le h(x) \le 2^n s \quad \text{in } D_s, \tag{9.31a}$$
$$h(x) = f(x) = [f(x)]_s \quad \text{outside } D_s, \tag{9.31b}$$
$$g(x) = 0 \quad \text{outside } D_s. \tag{9.31c}$$

We remark that we have, for each $k \in \mathbf{N}$,

$$\int_{I_k} g(y)\,dy = \int_{I_k} f(y)\,dy - \int_{I_k} h(y)\,dy = 0. \tag{9.32}$$

If we let

$$\widetilde{h}_\varepsilon(x) := K_\varepsilon * h(x),$$
$$\widetilde{g}_\varepsilon(x) := K_\varepsilon * g(x),$$

and

$$E_1 := \left\{ x : \widetilde{h}_\varepsilon(x) \ge \frac{s}{2} \right\},$$
$$E_2 := \left\{ x : \widetilde{g}_\varepsilon(x) \ge \frac{s}{2} \right\},$$

then we have the inclusion

$$E_s \subset E_1 \cup E_2. \tag{9.33}$$

However, by arguing just as in the proof of Theorem 9.2 we obtain from Parseval's formula and inequality (9.24a) that

$$\begin{aligned}
\int_{\mathbf{R}^n} |\widetilde{h}_\varepsilon(x)|^2\,dx &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} |(\mathcal{F}(K_\varepsilon * h)(\xi)|^2\,d\xi \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} |(\mathcal{F}K_\varepsilon)(\xi) \cdot (\mathcal{F}h)(\xi)|^2\,d\xi \\
&\le \frac{M^2}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} |(\mathcal{F}h)(\xi)|^2\,d\xi = M^2 \int_{\mathbf{R}^n} h(x)^2\,dx.
\end{aligned}$$

This proves that

$$\begin{aligned}
\frac{s^2}{4}|E_1| &\le \int_{E_1} |\widetilde{h}_\varepsilon(x)|^2\,dx \le \int_{\mathbf{R}^n} |\widetilde{h}_\varepsilon(x)|^2\,dx \\
&\le M^2 \int_{\mathbf{R}^n} h(x)^2\,dx.
\end{aligned} \tag{9.34}$$

On the other hand, it follows from inequalities (9.31a) and formula (9.31b) that

$$\int_{\mathbf{R}^n} h(x)^2\,dx = \int_{D_s} h(x)^2\,dx + \int_{\mathbf{R}^n \setminus D_s} h(x)^2\,dx$$

$$\leq 2^{2n} s^2 |D_s| + \int_{\mathbf{R}^n} [f(x)]_s^2 \, dx.$$

Therefore, by combining this inequality with inequality (9.34) we obtain that

$$|E_1| \leq 2^{2n+2} M^2 |D_s| + \frac{4M^2}{s^2} \int_{\mathbf{R}^n} [f(x)]_s^2 \, dx. \tag{9.35}$$

If S_k is the ball centered at the center of I_k and of radius the diameter of I_k, we let

$$\widetilde{D}_s := \bigcup_{k=1}^{\infty} S_k,$$

and also

$$\widetilde{D}_s^c := \mathbf{R}^n \setminus \widetilde{D}_s,$$
$$S_k^c := \mathbf{R}^n \setminus S_k.$$

Then it follows that

$$|\widetilde{D}_s| \leq C |D_s|. \tag{9.36}$$

Since we have, by assertion (9.31c),

$$\widetilde{g}_\varepsilon(x) = \int_{D_s} K_\varepsilon(x-y) g(y) \, dy = \sum_{k=1}^{\infty} \int_{I_k} K_\varepsilon(x-y) g(y) \, dy,$$

we obtain that

$$\begin{aligned} \int_{\widetilde{D}_s^c} |\widetilde{g}_\varepsilon(x)| \, dx &\leq \sum_{k=1}^{\infty} \int_{\widetilde{D}_s^c} \left| \int_{I_k} K_\varepsilon(x-y) g(y) \, dy \right| dx \qquad (9.37) \\ &\leq \sum_{k=1}^{\infty} \int_{S_k^c} \left| \int_{I_k} K_\varepsilon(x-y) g(y) \, dy \right| dx. \end{aligned}$$

Now, let x be a fixed point of $S_k^c = \mathbf{R}^n \setminus S_k$, and denote the center of I_k by y_k. We consider the following two cases:

(I) The set $\{y : |y-x| \leq \varepsilon\} \cap I_k$ is empty.
(II) The set $\{y : |y-x| \leq \varepsilon\} \cap I_k$ is not empty.

Case I: First, we consider the case where $\{y : |y-x| \leq \varepsilon\} \cap I_k = \emptyset$, In this case, it follows from formula (9.32) that

$$\int_{I_k} K_\varepsilon(x-y) g(y) \, dy = \int_{I_k} K(x-y) g(y) \, dy \tag{9.38}$$

$$= \int_{I_k} (K(x-y) - K(x-y_k))\, g(y)\, dy.$$

However, we have, by formula (9.25),

$$\begin{aligned} &|K(x-y) - K(x-y_k)| \\ &= \left| \frac{1}{|x-y|^n} \Omega\left(\frac{x-y}{|x-y|}\right) - \frac{1}{|x-y_k|^n} \Omega\left(\frac{x-y_k}{|x-y_k|}\right) \right| \\ &\le \left| \frac{1}{|x-y|^n} - \frac{1}{|x-y_k|^n} \right| \left| \Omega\left(\frac{x-y}{|x-y|}\right) \right| \\ &\quad + \frac{1}{|x-y_k|^n} \left| \Omega\left(\frac{x-y}{|x-y|}\right) - \Omega\left(\frac{x-y_k}{|x-y_k|}\right) \right|. \end{aligned} \tag{9.39}$$

We estimate each term on the right hand side of inequality (9.39).

(a) Since we have the inequality

$$|y - y_k| \le \frac{|x-y_k|}{2} \quad \text{for all } y \in I_k,$$

it follows that

$$\frac{|x-y_k|}{2} \le |x-y| \le \frac{3|x-y_k|}{2}.$$

Hence we obtain that

$$\begin{aligned} &\left| \frac{1}{|x-y|^n} - \frac{1}{|x-y_k|^n} \right| \\ &= \left| \frac{1}{|x-y|} - \frac{1}{|x-y_k|} \right| \sum_{i=0}^{n-1} \frac{1}{|x-y|^{n-1-i} |x-y_k|^i} \\ &\le \frac{|y-y_k|}{|x-y|\,|x-y_k|} \sum_{i=0}^{n-1} \frac{1}{|x-y|^{n-1-i}\, |x-y_k|^i} \\ &\le \frac{C\,|y-y_k|}{|x-y_k|^{n+1}} \le \frac{C\,|I_k|^{1/n}}{|x-y_k|^{n+1}}. \end{aligned} \tag{9.40}$$

(b) On the other hand, since we have the inequality

$$\begin{aligned} &\left| \frac{x-y}{|x-y|} - \frac{x-y_k}{|x-y_k|} \right| \\ &\le \left| \frac{x-y}{|x-y|} - \frac{x-y}{|x-y_k|} \right| + \left| \frac{x-y}{|x-y_k|} - \frac{x-y_k}{|x-y_k|} \right| \\ &\le \frac{2|y-y_k|}{|x-y_k|} \le \frac{C\,|I_k|^{1/n}}{|x-y_k|}, \end{aligned}$$

it follows from condition (9.26) that

$$\left|\Omega\left(\frac{x-y}{|x-y|}\right)-\Omega\left(\frac{x-y_k}{|x-y_k|}\right)\right| \leq \omega\left(\frac{C\,|I_k|^{1/n}}{|x-y_k|}\right). \tag{9.41}$$

By combining three inequalities (9.39), (9.40) and (9.41) and by using condition (9.27b), we obtain that

$$\begin{aligned}
&|K(x-y)-K(x-y_k)| \\
&\leq \frac{C\,|I_k|^{1/n}}{|x-y_k|^{n+1}} + \frac{1}{|x-y_k|^n}\,\omega\left(\frac{C\,|I_k|^{1/n}}{|x-y_k|}\right) \\
&\leq \frac{1}{|x-y_k|^n}\,c_0\,\omega\left(\frac{C\,|I_k|^{1/n}}{|x-y_k|}\right) + \frac{1}{|x-y_k|^n}\,\omega\left(\frac{C\,|I_k|^{1/n}}{|x-y_k|}\right) \\
&\leq \frac{C}{|x-y_k|^n}\,\omega\left(\frac{C\,|I_k|^{1/n}}{|x-y_k|}\right).
\end{aligned}$$

Therefore, it follows from this inequality and formula (9.38) that

$$\begin{aligned}
&\left|\int_{I_k} K_\varepsilon(x-y)g(y)\,dy\right| \qquad (9.42)\\
&\leq \int_{I_k} |K(x-y)-K(x-y_k)|\,g(y)\,dy \\
&\leq \frac{C}{|x-y_k|^n}\,\omega\left(\frac{C\,|I_k|^{1/n}}{|x-y_k|}\right)\int_{I_k}|g(y)|\,dy.
\end{aligned}$$

Case II: Secondly, we consider the case where $\{y : |y-x| \leq \varepsilon\} \cap I_k \neq \emptyset$. It should be noticed that, in this case, we have the inclusion

$$I_k \subset \{y : |y-x| \leq 3\varepsilon\}. \tag{9.43}$$

Let $\gamma(t)$ be the characteristic function of the interval $[0,3]$:

$$\gamma(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

If $y \in I_k$, then it follows from condition (9.43) that

$$\gamma\left(\frac{|x-y|}{\varepsilon}\right) = 1.$$

Moreover, since we have the inequality

$$|K_\varepsilon(x-y)| \leq \frac{C}{\varepsilon^n},$$

it follows that

$$\left|\int_{I_k} K_\varepsilon(x-y)g(y)\,dy\right| \le \frac{C}{\varepsilon^n}\int_{I_k}\gamma\left(\frac{|x-y|}{\varepsilon}\right)|g(y)|\,dy. \tag{9.44}$$

Therefore, by combining the inequalities (9.42) and (9.44) we obtain that, for $x \in S_k^c = \mathbf{R}^n \setminus S_k$,

$$\begin{aligned}\left|\int_{I_k} K_\varepsilon(x-y)g(y)\,dy\right| \le{}& \frac{C}{|x-y_k|^n}\,\omega\left(\frac{C\,|I_k|^{1/n}}{|x-y_k|}\right)\int_{I_k}|g(y)|\,dy \\ &+ \frac{C}{\varepsilon^n}\int_{I_k}\gamma\left(\frac{|x-y|}{\varepsilon}\right)|g(y)|\,dy.\end{aligned} \tag{9.45}$$

By inequalities (9.37) and (9.45), it follows that

$$\begin{aligned}&\int_{\widetilde{D}_s^c}|\widetilde{g}_\varepsilon(x)|\,dx \\ &\le \sum_{k=1}^{\infty}\int_{S_k^c}\frac{C}{|x-y_k|^n}\,\omega\left(\frac{C\,|I_k|^{1/n}}{|x-y_k|}\right)\left(\int_{I_k}|g(y)|\,dy\right)dx \\ &\quad + \sum_{k=1}^{\infty}\int_{S_k^c}\frac{C}{\varepsilon^n}\left(\int_{I_k}\gamma\left(\frac{|x-y|}{\varepsilon}\right)|g(y)|\,dy\right)dx \\ &= \sum_{k=1}^{\infty}\left(\int_{S_k^c}\frac{C}{|x-y_k|^n}\,\omega\left(\frac{C\,|I_k|^{1/n}}{|x-y_k|}\right)dx\right)\int_{I_k}|g(y)|\,dy \\ &\quad + C\sum_{k=1}^{\infty}\int_{I_k}|g(y)|\left(\frac{1}{\varepsilon^n}\int_{S_k^c}\gamma\left(\frac{|x-y|}{\varepsilon}\right)dx\right)dy.\end{aligned} \tag{9.46}$$

However, since we have, by condition (9.27a),

$$\begin{aligned}&\int_{S_k^c}\frac{C}{|x-y_k|^n}\omega\left(\frac{C\,|I_k|^{1/n}}{|x-y_k|}\right)dx \\ &= \int_{\Sigma_{n-1}}\left(\int_{C\,|I_k|^{1/n}}^{\infty}\frac{C}{r^n}\omega\left(\frac{C\,|I_k|^{1/n}}{r}\right)r^{n-1}\,dr\right)d\sigma \\ &= \int_{\Sigma_{n-1}}\left(\int_{C\,|I_k|^{1/n}}^{\infty}\frac{C}{r}\omega\left(\frac{C\,|I_k|^{1/n}}{r}\right)dr\right)d\sigma \\ &= C\int_{\Sigma_{n-1}}d\sigma\cdot\int_0^1\frac{\omega(t)}{t}\,dt \\ &= C\,\omega_n\int_0^1\frac{\omega(t)}{t}\,dt < \infty,\end{aligned}$$

and the inequality

$$\frac{1}{\varepsilon^n} \int_{S_k^c} \gamma \left(\frac{|x-y|}{\varepsilon} \right) dx \leq \frac{1}{\varepsilon^n} \int_{\mathbf{R}^n} \gamma \left(\frac{|x|}{\varepsilon} \right) dx = \int_{\mathbf{R}^n} \gamma(|x|)\, dx = \frac{3^n\, \omega_n}{n},$$

it follows from inequality (9.46) that

$$\begin{aligned} \int_{\widetilde{D}_s^c} |\widetilde{g}_\varepsilon(x)|\, dx &\leq C\, \omega_n \left(\int_0^1 \frac{\omega(t)}{t}\, dt + \frac{3^n}{n} \right) \sum_{k=1}^\infty \int_{I_k} |g(y)|\, dy \qquad (9.47) \\ &\leq C \int_{D_s} |g(y)|\, dy. \end{aligned}$$

Here

$$\omega_n = |\Sigma_{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

is the surface area of the unit sphere Σ_{n-1} in $\mathbf{R}^n$.

Moreover, we have the inequality

$$\begin{aligned} \int_{D_s} |g(y)|\, dy &\leq \int_{D_s} f(y)\, dy + \int_{D_s} h(y)\, dy = 2 \int_{D_s} f(y)\, dy \\ &= 2 \sum_{k=1}^\infty \int_{I_k} f(x)\, dx \leq 2 \left(\sum_{k=1}^\infty |I_k| \right) 2^n\, s \\ &= 2^{n+1}\, s\, |D_s|\,. \end{aligned}$$

By combining this inequality with inequality (9.47), we obtain that

$$\int_{\widetilde{D}_s^c} |\widetilde{g}_\varepsilon(x)|\, dx \leq C \int_{D_s} |g(y)|\, dy \leq Cs|D_s|.$$

This implies that

$$\frac{s}{2} \left| \widetilde{D}_s^c \cap E_2 \right| \leq \int_{\widetilde{D}_s^c \cap E_2} |\widetilde{g}_\varepsilon(x)|\, dx \leq \int_{\widetilde{D}_s^c} |\widetilde{g}_\varepsilon(x)|\, dx \leq Cs|D_s|,$$

so that

$$\left| \widetilde{D}_s^c \cap E_2 \right| \leq C|D_s|.$$

Hence, we have, by this inequality and inequality (9.36),

$$|E_2| \leq |\widetilde{D}_s^c \cap E_2| + |\widetilde{D}_s \cap E_2| \leq C|D_s|. \qquad (9.48)$$

Therefore, by combining two inequalities (9.35) and (9.48) we obtain from inclusion (9.33) that

$$|E_s| \leq |E_1| + |E_2| \leq 2^{2n+2} M^2 |D_s| + \frac{4M^2}{s^2} \int_{\mathbf{R}^n} [f(x)]_s^2\, dx + C|D_s|$$

$$= \frac{4M^2}{s^2} \int_{\mathbf{R}^n} [f(x)]_s^2 \, dx + \left(2^{2n+2} M^2 + C\right) |D_s|.$$

This proves the desired inequality (9.30) with

$$C_1 := 4M^2, \qquad C_2 := 2^{2n+2} M^2 + C.$$

Now the proof of Lemma 9.4 is complete. □

The next theorem asserts the existence of the singular integral (9.2) in the space $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ in the case of *continuous kernels*:

Theorem 9.5. *Assume that $K(x)$ is an integral kernel satisfying Assumption 9.3. Let $f(x) \in L^p(\mathbf{R}^n)$ for $1 < p < \infty$. If $\varepsilon > 0$, we let $\widetilde{f_\varepsilon}(x)$ be the function defined by formula* (9.5)

$$\widetilde{f_\varepsilon}(x) := \int_{|x-y|>\varepsilon} K(x-y) f(y) \, dy.$$

Then we have the following three assertions (i), (ii) and (iii):

(i) There exists a positive constant C_p, independent of ε, such that

$$\left\| \widetilde{f_\varepsilon} \right\|_p \le C_p \left\| f \right\|_p . \tag{9.49}$$

(ii) The sequence $\widetilde{f_\varepsilon}$ converges strongly to a function $K * f$ in the space $L^p(\mathbf{R}^n)$ as $\varepsilon \downarrow 0$. Namely, the singular integral

$$K * f(x) = \lim_{\varepsilon \downarrow 0} \int_{|x-y|>\varepsilon} K(x-y) f(y) \, dy$$

exists in the strong topology of $L^p(\mathbf{R}^n)$.

(iii) The mapping $f \mapsto K * f$ is a bounded linear operator from $L^p(\mathbf{R}^n)$ into itself. More precisely, we have the inequality

$$\left\| K * f \right\|_p \le C_p \left\| f \right\|_p .$$

Proof. We write, for $\varepsilon > 0$,

$$T_\varepsilon f := \widetilde{f_\varepsilon}.$$

By applying Theorem 9.2 to our situation, we can find a positive constant C, independent of ε, such that

$$\left\| T_\varepsilon f \right\|_2 \le C \left\| f \right\|_2 \quad \text{for all } f \in L^2(\mathbf{R}^n).$$

This proves that the operator T_ε is of type $(2,2)$ uniformly with respect to ε.

(1) If $f(x)$ is a non-negative function in $L^1(\mathbf{R}^n)$, then we have the inequality

$$[f(x)]_s^2 \leq s\, f(x) \quad \text{for all } s > 0.$$

Hence it follows from two inequalities (9.29) and (9.30) that

$$|\{x : |(T_\varepsilon f)(x)| > s\}| \leq \frac{C\,\|f\|_1}{s} \quad \text{for all } s > 0.$$

If $f(x)$ is an arbitrary function in $L^1(\mathbf{R}^n)$, then, by decomposing it into the positive part $f_1(x)$ and the negative part $f_2(x)$ as

$$\begin{aligned} f(x) &= f_1(x) - f_2(x), \\ f_1(x) &:= \max\{f(x), 0\}, \\ f_2(x) &:= \max\{-f(x), 0\}, \end{aligned}$$

we obtain that

$$\begin{aligned} &|\{x : |(T_\varepsilon f)(x)| > s\}| \\ &\leq |\{x : |(T_\varepsilon f_1)(x)| > s/2\}| + |\{x : |(T_\varepsilon f_2)(x)| > s/2\}| \\ &\leq \frac{2C\,\|f_1\|_1}{s} + \frac{2C\,\|f_2\|_1}{s} = \frac{2C\,\|f\|_1}{s} \quad \text{for all } s > 0. \end{aligned}$$

This proves that the operator T_ε is of weak type $(1,1)$ uniformly in ε. Therefore, by arguing just as in the proof of Marcinkiewicz's interpolation theorem (Theorem 3.30) we can find a positive constant C_p such that the desired inequality (9.49) holds true for $1 < p \leq 2$:

$$\|T_\varepsilon f\|_p \leq C_p\,\|f\|_p \quad (1 < p \leq 2).$$

Furthermore, by passing to the adjoint operator T_ε^* we obtain that the desired inequality (9.49) holds true for $2 < p < \infty$:

$$\|T_\varepsilon f\|_p \leq C_p\,\|f\|_p \quad (2 < p < \infty).$$

(2) Secondly, let $f(x) \in L^p(\mathbf{R}^n)$ for $1 < p < \infty$. If $g(x) \in C_0^1(\mathbf{R}^n)$, i.e., if $g(x)$ is a continuously differentiable function with compact support in $\mathbf{R}^n$ and if ε, $\delta > 0$, then we have, by inequality (9.49),

$$\begin{aligned} &\|T_\varepsilon f - T_\delta f\|_p \qquad (9.50) \\ &\leq \|T_\varepsilon f - T_\varepsilon g\|_p + \|T_\delta g - T_\delta f\|_p + \|T_\varepsilon g - T_\delta g\|_p \\ &\leq 2C_p\,\|f - g\|_p + \|T_\varepsilon g - T_\delta g\|_p \quad (1 < p < \infty). \end{aligned}$$

If $0 < \varepsilon < 1$, it follows from the cancellation property (9.4) that

$$(T_\varepsilon g)(x) = \int_{|x-y|>1} K(x-y)\, g(y)\, dy + \int_{\varepsilon<|x-y|<1} K(x-y)\,(g(y)-g(x))\; dy. \tag{9.51}$$

Since $K(x) \in L^p(\{x : |x| > 1\})$ and $g(x) \in L^1(\mathbf{R}^n)$, it follows from an application of Theorem 3.23 with $q := 1$ and $r = p$ that the first term on the right hand side of formula (9.51) belongs to $L^p(\mathbf{R}^n)$:

$$\int_{|x-y|>1} K(x-y)\, g(y)\, dy \in L^p(\mathbf{R}^n).$$

On the other hand, we remark that the second term on the right hand side of formula (9.51) is a function with support contained in a fixed compact set, so that it converges uniformly as $\varepsilon \downarrow 0$, since $g(x)$ is uniformly Lipschitz continuous. Hence we find that the second term on the right hand side of formula (9.51) converges strongly in the space $L^p(\mathbf{R}^n)$:

$$\lim_{\varepsilon\downarrow 0} \int_{\varepsilon<|x-y|<1} K(x-y)\,(g(y)-g(x))\; dy \in L^p(\mathbf{R}^n).$$

This implies that

$$\|T_\varepsilon g - T_\delta g\|_p \longrightarrow 0 \quad \text{as } \varepsilon,\, \delta \downarrow 0.$$

However, Corollary 3.27 tells us that $C_0^1(\mathbf{R}^n)$ is dense in $L^p(\mathbf{R}^n)$. Hence, we obtain from inequality (9.50) that the sequence $T_\varepsilon f = \widetilde{f_\varepsilon}$ converges strongly to some function $K * f$ in the space $L^p(\mathbf{R}^n)$ as $\varepsilon \downarrow 0$:

$$\lim_{\varepsilon\downarrow 0} T_\varepsilon f = \lim_{\varepsilon\downarrow 0} \widetilde{f_\varepsilon} = K * f \in L^p(\mathbf{R}^n).$$

Therefore, by letting $\varepsilon \downarrow 0$ in inequality (9.49) we obtain from Lebesgue's dominated convergence theorem (Theorem 3.8) that

$$\|K * f\|_p \le C_p \|f\|_p \,.$$

This proves that the mapping $f \mapsto K * f$ is bounded from $L^p(\mathbf{R}^n)$ into itself.

The proof of Theorem 9.5 is complete. □

9.4 The Hilbert Transform

In the case where $n = 1$, we can apply Theorem 9.5 with

$$K(x) := \frac{1}{\pi x}$$

to obtain that the singular integral $K * f$ reduces to the following formula:

$$K * f(x) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{|x-t|>\varepsilon} \frac{f(t)}{x-t}\, dt.$$

Moreover, it should be noticed that this formula is also expressed as

$$\frac{1}{\pi}\left(\text{v.p.}\,\frac{1}{x}\right) * f,$$

where v.p.$(1/x)$ is a distribution defined by the formula

$$\left\langle \text{v.p.}\,\frac{1}{x}, \varphi \right\rangle = \lim_{\varepsilon \downarrow 0} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\, dx \quad \text{for all } \varphi \in C_0^1(\mathbf{R}).$$

The function $Hf(x)$, defined by the formula

$$\begin{aligned} Hf(x) &= \frac{1}{\pi}\left(\text{v.p.}\,\frac{1}{x}\right) * f = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{|x-t|>\varepsilon} \frac{f(t)}{x-t}\, dt \\ &= \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{|s|>\varepsilon} f(x-s)\, \frac{ds}{s} \end{aligned} \tag{9.52}$$

is called the *Hilbert transform* of $f(x)$.

The purpose of this section is to prove the following basic results for the Hilbert transform:

Theorem 9.6. *Let H be the Hilbert transform defined by formula* (9.52). *Then we have the following three assertions (i), (ii) and (iii):*

(i) The Hilbert transform H is a bounded linear operator from $L^p(\mathbf{R})$ into itself for $1 < p < \infty$.

(ii) If $g = Hf$ is the Hilbert transform of $f \in L^p(\mathbf{R})$ for $1 < p < \infty$, then its inverse transform is given by the formula

$$f(x) = -h * g(x) = -\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{|x-t|>\varepsilon} \frac{g(t)}{x-t}\, dt. \tag{9.53}$$

Furthermore, the singular integrals (9.52) *and* (9.53) *exist in the strong topology of $L^p(\mathbf{R})$.*

(iii) If $f \in L^p(\mathbf{R})$ and $\widehat{f} \in L^{p'}(\mathbf{R})$ with $p' = p/(p-1)$ and if $g = Hf$ and $\widehat{g} = H\widehat{f}$ are the Hilbert transforms of f and $\widehat{f}$, respectively, then the formula

$$\int_{-\infty}^{\infty} f(x) \cdot \widehat{f}(x)\, dx = \int_{-\infty}^{\infty} Hf(x) \cdot H\widehat{f}(x)\, dx \qquad (9.54)$$
$$= \int_{-\infty}^{\infty} g(x) \cdot \widehat{g}(x)\, dx$$

holds true. In particular, the Hilbert transform H is a unitary operator *in the Hilbert space $L^2(\mathbf{R})$, and the formula*

$$\|g\|_2 = \|Hf\|_2 = \|f\|_2 \qquad (9.55)$$

holds true if $g = Hf$ is the Hilbert transform of $f \in L^2(\mathbf{R})$.

Proof. The proof of Theorem 9.6 is divided into four steps.

Step (I): First, we consider the case where $p = 2$. Following the proof of Theorem 9.2, we calculate explicitly the Fourier transform of the distribution (formula (9.57) below)

$$h(x) := \frac{1}{\pi} \text{ v.p. } \frac{1}{x}.$$

To do this, we let, for $0 < \varepsilon < \mu$,

$$h_{\varepsilon,\mu}(x) := \begin{cases} \frac{1}{\pi x} & \text{if } \varepsilon < |x| < \mu, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$h_{\varepsilon}(x) := \begin{cases} \frac{1}{\pi x} & \text{if } |x| > \varepsilon, \\ 0 & \text{if } |x| \leq \varepsilon. \end{cases}$$

Then it follows that

$$\begin{aligned} (\mathcal{F}h_{\varepsilon,\mu})(\xi) &= \int_{\varepsilon<|x|<\mu} \frac{e^{-ix\cdot\xi}}{\pi x}\, dx \\ &= -\frac{2i}{\pi} \int_{\varepsilon}^{\mu} \frac{\sin(x\cdot\xi)}{x}\, dx \\ &= \begin{cases} -\frac{2i}{\pi} \int_{\varepsilon\xi}^{\mu\xi} \frac{\sin x}{x}\, dx & \text{if } \xi > 0, \\ \frac{2i}{\pi} \int_{-\varepsilon\xi}^{-\mu\xi} \frac{\sin x}{x}\, dx & \text{if } \xi < 0 \end{cases} \\ &= -\frac{2i}{\pi} \int_{\varepsilon|\xi|}^{\mu|\xi|} \frac{\sin x}{x}\, dx \cdot \operatorname{sign}\xi \quad \text{for } \xi \neq 0, \end{aligned}$$

so that we have, as $\mu \uparrow \infty$,

$$(\mathcal{F}h_\varepsilon)(\xi) = -\frac{2i}{\pi} \int_{\varepsilon|\xi|}^{\infty} \frac{\sin x}{x}\, dx \cdot \operatorname{sign} \xi \quad \text{for } \xi \neq 0. \tag{9.56}$$

Therefore, by letting $\varepsilon \downarrow 0$ in formula (9.56) we obtain that

$$\begin{aligned}(\mathcal{F}h)(\xi) &= -\frac{2i}{\pi} \left(\int_0^{\infty} \frac{\sin x}{x}\, dx \right) \operatorname{sign} \xi = -\frac{2i}{\pi} \frac{\pi}{2} \operatorname{sign} \xi \\ &= -i \operatorname{sign} \xi \quad \text{for } \xi \neq 0.\end{aligned} \tag{9.57}$$

If $f(x) \in L^2(\mathbf{R})$, then we have, by formula (9.56) and inequality (9.16),

$$|(\mathcal{F}h_\varepsilon)(\xi)| \leq \frac{4A}{\pi},$$

and so, by Parseval's formula,

$$\begin{aligned}\|h_\varepsilon * f\|_2 &= \frac{1}{(2\pi)^{1/2}} \|\mathcal{F}(h_\varepsilon * f)\|_2 = \frac{1}{(2\pi)^{1/2}} \|\mathcal{F}h_\varepsilon \cdot \mathcal{F}f\|_2 \\ &\leq \frac{1}{(2\pi)^{1/2}} \frac{4A}{\pi} \|\mathcal{F}f\|_2 = \frac{4A}{\pi} \|f\|_2 .\end{aligned}$$

Step (II): On the other hand, Theorem 9.2 asserts that the singular integral

$$g(x) = Hf(x) = \frac{1}{\pi} \left(\text{v.p.} \frac{1}{x} \right) * f = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{|x-t|>\varepsilon} \frac{f(t)}{x-t}\, dt \tag{9.58}$$

exists in the strong topology of $L^2(\mathbf{R})$ ($p = 2$):

$$H : L^2(\mathbf{R}) \longrightarrow L^2(\mathbf{R}).$$

By combining two formulas (9.8) and (9.57), we obtain that

$$(\mathcal{F}g)(\xi) = (\mathcal{F}h)(\xi) \cdot (\mathcal{F}f)(\xi) = -i \operatorname{sign} \xi \cdot (\mathcal{F}f)(\xi) \quad \text{for } \xi \neq 0. \tag{9.59}$$

Therefore, by Parseval's formula we have the desired formula (9.55)

$$\|g\|_2 = \|Hf\|_2 = \|f\|_2 ,$$

and also the formula

$$(\mathcal{F}f)(\xi) = i \operatorname{sign} \xi \cdot (\mathcal{F}g)(\xi) = -(\mathcal{F}h)(\xi) \cdot (\mathcal{F}g)(\xi) \quad \text{for } \xi \neq 0. \tag{9.60}$$

By formulas (9.58) and (9.60), it follows from an application of the Fourier inversion formula that the singular integral (9.53) exists in the strong topology of $L^2(\mathbf{R})$ ($p = 2$).

Step (III): We consider the general case where $1 < p < \infty$. By applying Theorem 9.5, we obtain that the Hilbert transform H is a bounded linear operator from $L^p(\mathbf{R})$ into itself:

$$H : L^p(\mathbf{R}) \longrightarrow L^p(\mathbf{R}) \quad (1 < p < \infty).$$

Let $f \in L^p(\mathbf{R})$ and let $g = Hf$ be its Hilbert transform. If we take a sequence $f_n \in L^p(\mathbf{R}) \cap L^2(\mathbf{R})$ such that

$$f_n \longrightarrow f \quad \text{in } L^p(\mathbf{R}) \text{ as } n \to \infty,$$

then it follows that the Hilbert transform $g_n = Hf_n$ of f_n belongs to $L^p(\mathbf{R}) \cap L^2(\mathbf{R})$ and further that

$$g_n = Hf_n \longrightarrow g = Hf \quad \text{in } L^p(\mathbf{R}) \text{ as } n \to \infty.$$

By assertion (9.53), we remark that

$$-f_n(x) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{|x-t|>\varepsilon} \frac{g_n(t)}{x-t}\, dt$$

in the strong topology of $L^2(\mathbf{R})$ and further that the right hand side of this formula is the Hilbert transform of g_n. Hence it follows from an application of Theorem 9.5 that the above limit exists also in the strong topology of $L^p(\mathbf{R})$. Moreover, by letting $n \to \infty$ we obtain from Theorem 9.5 that formula (9.53) holds true also in the strong topology of $L^p(\mathbf{R})$:

$$f(x) = -h * g(x) = -\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{|x-t|>\varepsilon} \frac{g(t)}{x-t}\, dt \quad (1 < p < \infty).$$

Step (IV): It remains to prove formula (9.54).

Now, let $f \in L^p(\mathbf{R})$ and $\widehat{f} \in L^{p'}(\mathbf{R})$ with $p' = p/(p-1)$, and let $g = Hf$ and $\widehat{g} = H\widehat{f}$ be the Hilbert transforms of f and $\widehat{f}$, respectively. By using Friedrichs' mollifiers (Subsection 3.7.2), we can choose two sequences $f_n \in L^p(\mathbf{R}) \cap L^2(\mathbf{R})$ and $\widehat{f_n} \in L^{p'}(\mathbf{R}) \cap L^2(\mathbf{R})$ such that

$$\begin{aligned} f_n &\longrightarrow f \quad \text{in } L^p(\mathbf{R}) \text{ as } n \to \infty, \\ \widehat{f_n} &\longrightarrow \widehat{f} \quad \text{in } L^{p'}(\mathbf{R}) \text{ as } n \to \infty. \end{aligned}$$

Then we have the assertions

$$\begin{aligned} g_n &= Hf_n \in L^p(\mathbf{R}) \cap L^2(\mathbf{R}), \\ \widehat{g_n} &= H\widehat{f_n} \in L^{p'}(\mathbf{R}) \cap L^2(\mathbf{R}), \end{aligned}$$

and

$$g_n = Hf_n \longrightarrow g = Hf \quad \text{in } L^p(\mathbf{R}),$$

$$\widehat{g}_n = H\widehat{f}_n \longrightarrow \widehat{g} = H\widehat{f} \quad \text{in } L^{p'}(\mathbf{R}).$$

However, it follows from formula (9.55) that the Hilbert transform H is a unitary operator in $L^2(\mathbf{R})$. Hence we have the formulas

$$\begin{aligned}\int_{-\infty}^{\infty} f_n(x) \cdot \widehat{f}_n(x)\, dx &= \int_{-\infty}^{\infty} Hf_n(x) \cdot H\widehat{f}_n(x)\, dx \\ &= \int_{-\infty}^{\infty} g_n(x) \cdot \widehat{g}_n(x)\, dx.\end{aligned}$$

Therefore, by letting $n \to \infty$ in these formulas we obtain the desired formula (9.54)

$$\int_{-\infty}^{\infty} f(x) \cdot \widehat{f}(x)\, dx = \int_{-\infty}^{\infty} Hf(x) \cdot H\widehat{f}(x)\, dx = \int_{-\infty}^{\infty} g(x) \cdot \widehat{g}(x)\, dx.$$

The proof of Theorem 9.6 is now complete. □

9.5 Equimeasurable Functions

Let $f(x)$ be a non-negative, measurable function defined on $\mathbf{R}^n$. For each $\tau \geq 0$, we let

$$m(\tau) := |\{x \in \mathbf{R}^n : f(x) > \tau\}| . \tag{9.61}$$

In this section we consider only functions such that $m(\tau) < \infty$ for any $\tau > 0$ and that

$$\lim_{\tau \uparrow \infty} m(\tau) = 0.$$

It should be emphasized that non-zero functions $f(x)$ in $L^p(\mathbf{R}^n)$ with $1 \leq p < \infty$ satisfy this condition.

Let $f(x)$ be a non-negative, measurable function defined on $\mathbf{R}^n$. A non-negative, measurable function $f^*(t)$ defined on the interval $[0, \infty)$ is called an *equimeasurable function* of $f(x)$ if it satisfies the condition

$$m^*(\tau) := |\{t : f^*(t) > \tau\}| = m(\tau) \quad \text{for all } \tau \geq 0. \tag{9.62}$$

We begin by proving the following elementary result:

Lemma 9.7. *The function* $m(\tau)$ *defined by formula* (9.61) *is monotone decreasing and right-continuous.*

Proof. It is clear that $m(\tau)$ is monotone decreasing. If $\tau_k \downarrow \tau$, then it follows that

$$\{x : f(x) > \tau\} = \bigcup_{k=1}^{\infty} \{x : f(x) > \tau_k\} ,$$

so that $m(\tau) = \lim_{k\to\infty} m(\tau_k)$. This proves the right-continuity of $m(\tau)$.

The proof of Lemma 9.7 is complete. □

We let

$$t_0 := m(0), \quad \tau_0 := \sup\{\tau : m(\tau) > 0\}.$$

It is easy to see that $0 < t_0 \leq \infty$ and $0 < \tau_0 \leq \infty$.

The next lemma proves the existence and uniqueness of the equimeasurable function:

Lemma 9.8. *The function $f^*(t)$, defined by the formula*

$$f^*(t) := \inf\{\tau : t \geq m(\tau)\}, \quad f^*(0) := \tau_0,$$

is a unique monotone decreasing, right-continuous equimeasurable function of $f(x)$.

Proof. The proof of Lemma 9.8 is divided into two steps.

Step (1): We prove that the non-negative function $f^*(t)$ is a monotone decreasing, right-continuous equimeasurable function of $f(x)$.

(a) First, we show that

$$\tau \geq f^*(t) \Longleftrightarrow t \geq m(\tau).$$

Indeed, we remark that $t \geq m(\tau)$ implies that $\tau \geq f^*(t)$. Conversely, if $\tau > f^*(t)$, then, by the definition of f^* it follows that $t \geq m(\tau)$. Assume that $\tau = f^*(t)$. If $\tau_k \downarrow \tau$, then it follows that $t \geq m(\tau_k)$. Since the function $m(\tau)$ is right-continuous, by letting $k \to \infty$ we obtain that

$$t \geq m(\tau).$$

Therefore, we have, for all $\tau > 0$,

$$\{t : f^*(t) > \tau\} = [0, m(\tau)).$$

Hence, if $m^*(\tau)$ is the function defined by formula (9.62), then it follows that $m^*(\tau) = m(\tau)$. Namely, the function $f^*(t)$ satisfies the condition (9.62).

(b) Secondly, it is clear that $f^*(t)$ is *monotone decreasing.*

(c) Thirdly, we show that $f^*(t)$ is *right-continuous.* Since either $t_0 = m(0) \geq m(\tau)$ or $\tau \geq f^*(t_0)$ for all $\tau > 0$, it follows that $f^*(t_0) = 0$. If $t_0 < \infty$, then $f^*(t) = 0$ for all $t \geq t_0$.

We show that $f^*(t) > 0$ for $0 < t < t_0$ for all $t < t_0 = m(0)$. Indeed, there exists a number $\tau > 0$ such that $t < m(\tau)$. This proves that $0 < \tau < f^*(t)$. Let $\{t_k\}$ be an arbitrary sequence such that $t_k \downarrow t$. For any given number $\tau < f^*(t)$, it follows that $t < m(\tau)$. In particular, we have the assertion

$$t_k < m(\tau) \quad \text{if } k \text{ is sufficiently large,}$$

or equivalently

$$\tau < f^*(t_k) \leq f^*(t) \quad \text{if } k \text{ is sufficiently large.}$$

This proves that

$$f^*(t) = \lim_{k \to \infty} f^*(t_k),$$

since $\tau < f^*(t)$ is arbitrary.

Therefore, we have proved the right-continuity of $f^*(t)$.

Step (2): We prove the *uniqueness* of the equimeasurable function.

Let $g^*(t)$ be another monotone decreasing, right-continuous equimeasurable function of $f(x)$. If t_1 is a positive number, then we let $\tau := f^*(t_1)$ and

$$t_2 := \min\{t : f^*(t) = \tau\} = \inf\{t : f^*(t) = \tau\}.$$

We recall that $f^*(t)$ is right-continuous. Since we have the formula

$$\{t : f^*(t) > \tau\} = [0, t_2),$$

it follows that

$$|\{t : g^*(t) > \tau\}| = m(\tau) = |\{t : f^*(t) > \tau\}| = t_2.$$

Moreover, since $g^*(t)$ is monotone decreasing and right-continuous, it follows that $\{t : g^*(t) > \tau\} = [0, t_2)$. This proves that $g^*(t_2) \leq \tau = f^*(t_1)$. Hence we have, for $t_2 \leq t_1$,

$$g^*(t_1) \leq g^*(t_2) \leq \tau = f^*(t_1).$$

Similarly, we can prove that $f^*(t_1) \leq g^*(t_1)$ for any positive number t_1.

Summing up, we obtain that $f^*(t_1) = g^*(t_1)$ for any positive number t_1. This proves the uniqueness of the equimeasurable function of $f(x)$.

The proof of Lemma 9.8 is now complete. □

Lemma 9.9. *If $f^*(t)$ is an equimeasurable function of $f(x)$, then we have, for $1 \leq p < \infty$,*

$$\int_{\mathbf{R}^n} f(x)^p \, dx = \int_0^\infty f^*(t)^p \, dt.$$

Proof. By formula (9.61), it follows that

$$\int_{\mathbf{R}^n} f(x)^p\,dx = -\int_0^\infty \tau^p\,dm(\tau) = -\int_0^\infty \tau^p\,dm^*(\tau) = \int_0^\infty f^*(t)^p\,dt.$$

The proof of Lemma 9.9 is complete. □

Lemma 9.10. *Let $f(x)$ be a non-negative, measurable function defined on $\mathbf{R}^n$. If A is a measurable subset of $\mathbf{R}^n$ of finite measure, then we have the inequality*

$$\int_A f(x)\,dx \le \int_0^{|A|} f^*(t)\,dt.$$

Proof. We let

$$f_1(x) := \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Since $f_1(x) \le f(x)$, it follows that

$$m_1(\tau) := |\{x : f_1(x) > \tau\}| \le |\{x : f(x) > \tau\}| = m(\tau).$$

Hence, if $f_1^*(t)$ is the monotone decreasing, right-continuous equimeasurable function of $f_1(x)$, then we have the inequality

$$f_1^*(t) = \inf\{\tau : t \ge m_1(\tau)\} \le f^*(t).$$

Since we have, for all $\tau > 0$,

$$\{x : f_1(x) > \tau\} \subset \{x : f_1(x) > 0\} \subset A,$$

it follows that $m_1(\tau) \le |A|$. This proves that $f_1^*(t) = 0$ for $t \ge |A|$.

Therefore, we obtain from Lemma 9.9 with $p := 1$ that

$$\begin{aligned}\int_A f(x)\,dx &= \int_{\mathbf{R}^n} f_1(x)\,dx = \int_0^\infty f_1^*(t)\,dt \le \int_0^{|A|} f_1^*(t)\,dt \\ &\le \int_0^{|A|} f^*(t)\,dt.\end{aligned}$$

The proof of Lemma 9.10 is complete. □

From now on, we assume that $f(x)$ is a non-negative function in $L^p(\mathbf{R}^n)$ for $1 \le p < \infty$, and let $f^*(t)$ be the monotone decreasing, right-continuous equimeasurable function of $f(x)$. If $t > 0$, we let

$$\beta_f(t) := \frac{1}{t}\int_0^t f^*(s)\,ds. \tag{9.63}$$

By Lemma 9.9, it follows that $f^*(t) \in L^p(0,\infty)$ for $1 \leq p < \infty$. Hence it follows from an application of Hölder's inequality (Theorem 3.14) that the right hand side of formula (9.63) is finite. Moreover, we remark that

$$\frac{d}{dt}\beta_f(t) = \frac{1}{t^2}\int_0^t (f^*(t) - f^*(s))\,ds \leq 0.$$

If $\beta_f'(t_1) = 0$ for some $t_1 > 0$, then it follows that $f^*(t) = f^*(t_1)$ in the interval $(0, t_1)$, and so $\beta_f(t) = f^*(t_1)$ there. Hence it is easy to see that $f^*(t) \to 0$ as $t \uparrow \infty$. This proves that $\beta_{f(t)} \to 0$ as $t \uparrow \infty$.

Therefore, we obtain that either

(a) $\beta_f(t)$ is strictly decreasing in $(0,\infty)$

or

(b) there exists a positive number t_1 such that $\beta_f(t)$ is constant in $(0, t_1]$ and strictly decreasing in $[t_1, \infty)$.

In the case (a), we let $t_1 := 0$, and let $\beta_f(0) := s_1$ if $s_1 = \lim_{t\to 0} \beta_f(t)$. Then it is easy to see that $0 < s_1 \leq \infty$ and that $s_1 = \beta_f(t_1) = \beta_f(0)$.

We denote the inverse function of $s = \beta_f(t)$ by $t = \beta^f(s)$.

(1) If $t_1 = 0$ and $s_1 = \infty$, then it follows that $\beta^f(s)$ is uniquely determined in $0 < s < \infty$.
(2) If $t_1 = 0$ and $s_1 < \infty$, then we let $\beta^f(s) := 0$ for $s > s_1$.
(3) If $t_1 > 0$, then we let $\beta^f(s_1) := t_1$ and $\beta^f(s) := 0$ for $s > s_1$.

Then we have the following lemma:

Lemma 9.11. *If $f(x) \in L^p(\mathbf{R}^n)$ for $1 < p < \infty$, then we have the inequality*

$$\left(\int_0^\infty \beta_f(t)^p\,dt\right)^{1/p} \leq \frac{p}{p-1}\left(\int_{\mathbf{R}^n} f(x)^p\,dx\right)^{1/p}. \tag{9.64}$$

Proof. By Lemma 9.9, it follows that $f^*(t) \in L^p(0,\infty)$. Let $0 < a < b$. Then we have, by integration by parts,

$$\begin{aligned}\int_a^b \beta_f(t)^p\,dt &= \int_a^b t^{-p}\left(\int_0^t f^*(s)\,ds\right)^p dt \\ &\leq \frac{a^{1-p}}{p-1}\left(\int_0^a f^*(s)\,ds\right)^p \\ &\quad + \frac{p}{p-1}\int_a^b t^{1-p} f^*(t)\left(\int_0^t f^*(s)\,ds\right)^{p-1} dt.\end{aligned} \tag{9.65}$$

However, we obtain that the first term on the right hand side of inequality (9.65) tends to 0 as $a \downarrow 0$. Indeed, it suffices to note that we have, by Hölder's inequality (Theorem 3.14),

$$\frac{a^{1-p}}{p-1}\left(\int_0^a f^*(s)\,ds\right)^p \leq \frac{1}{p-1}\int_0^a f^*(s)^p\,ds.$$

Therefore, by letting $a \downarrow 0$ and $b \uparrow \infty$ in inequality (9.65) and by using Hölder's inequality (Theorem 3.14) we obtain that

$$\begin{aligned}\int_0^\infty \beta_f(t)^p\,dt &\leq \frac{p}{p-1}\int_0^\infty f^*(t)\,\beta_f(t)^{p-1}\,dt \\ &\leq \frac{p}{p-1}\left(\int_0^\infty f^*(t)^p\,dt\right)^{1/p}\left(\int_0^\infty \beta_f(t)^p\,dt\right)^{1-1/p}.\end{aligned}$$

This proves the desired inequality (9.63).

The proof of Lemma 9.11 is complete. □

The main purpose of this section is to prove the following lemma:

Lemma 9.12. *Assume that $f(x)$ is a non-negative function in $L^p(\mathbf{R})$ for $1 < p < \infty$. If $\varepsilon > 0$, we let*

$$F_\varepsilon(x) := \frac{1}{\varepsilon}\int_0^\varepsilon f(x+y)\,dy, \quad G(x) := \sup_{\varepsilon>0} F_\varepsilon(x). \tag{9.66}$$

Then we have the inequality

$$\left(\int_{-\infty}^\infty G(x)^p\,dx\right)^{1/p} \leq \frac{p}{p-1}\left(\int_{-\infty}^\infty f(x)^p\,dx\right)^{1/p}. \tag{9.67}$$

Proof. First, we remark that the function $G(x)$ is measurable, since the supremum of the right-hand side of the second equality of formula (9.66) may be taken only over the rational $\varepsilon \in \mathbf{Q}$. If $\tau > 0$, we let

$$H(\tau) := \{x : G(x) > \tau\}.$$

Since we have the formula

$$G(x) = \sup_{y>x}\frac{1}{y-x}\int_x^y f(t)\,dt,$$

it is easy to see that $x \in H(\tau)$ if and only if the inequality

$$\int_0^y f(t)\,dt - y\tau > \int_0^x f(t)\,dt - x\tau$$

holds true for some $y > x$. Hence, if we let

$$F(x) := \int_0^x f(t)\,dt - x\tau,$$

then we have the formula

$$H(\tau) = \{x : F(y) > F(x) \quad \text{for some } y > x\}.$$

This proves that the set $H(\tau)$ is open, so that it is the sum of disjoint open intervals, that is,

$$H(\tau) = \bigcup_{k=1}^{\infty} (a_k, b_k)\,.$$

We have, by Hölder's inequality (Theorem 3.14),

$$\lim_{x \to \pm\infty} F(x) = \mp\infty. \tag{9.68}$$

Step (1): We show that, for any $k \in \mathbf{N}$,

$$-\infty < a_k < b_k < \infty.$$

Our proof is based on a reduction to absurdity. We assume, to the contrary, that $a_k = -\infty$ for some k. Then we have the inclusion

$$(-\infty, b_k) \subset H(\tau).$$

By assertion (9.68), it follows that $F(c) > F(b_k)$ for some $c < b_k$. Let c_1 be a real number in the interval $[c, b_k]$ such that

$$F(c_1) = \max_{c \leq x \leq b_k} F(x),$$

Then it follows that $c \leq c_1 < b_k$, since we have the inequalities

$$F(c_1) \geq F(c) > F(b_k).$$

This implies that $c_1 \in (-\infty, b_k) \subset H(\tau)$. Furthermore, it follows that

$$F(x) \leq F(c_1) \quad \text{for } c_1 < x \leq b_k.$$

On the other hand, since we have the assertion $b_k \notin H(\tau)$, it follows that

$$F(x) \leq F(b_k) < F(c_1) \quad \text{for all } x > b_k.$$

Hence we have proved that

$$F(x) \leq F(c_1) \quad \text{for all } x > c_1.$$

This contradicts the condition that $c_1 \in H(\tau)$.

Next, we assume, to the contrary, that $a_k > -\infty$ and $b_k = \infty$. Then we have the inclusion

$$(a_k, \infty) \subset H(\tau).$$

Let c be an arbitrary number satisfying the condition $c > a_k$. By assertion (9.68), it follows that

$$F(c_1) = \max_{x \geq c} F(x) \quad \text{for some } c_1 \geq c.$$

This implies that $c_1 \in (a_k, \infty) \subset H(\tau)$. However, we have the inequality

$$F(x) \leq F(c_1) \quad \text{for all } x > c_1.$$

This is also a contradiction.

Summing up, we have proved that $-\infty < a_k < b_k < \infty$.

Step (2): Secondly, we show that $F(a_k) = F(b_k)$ for all k.

Since $a_k \notin H(\tau)$ and $b_k > a_k$, it follows that

$$F(b_k) \leq F(a_k).$$

Our proof is based on a reduction to absurdity. Assume, to the contrary, that $F(b_k) < F(a_k)$ for some k. Then it follows that

$$F(b_k) < F(c) \quad \text{for some } c \in (a_k, b_k).$$

Let c_1 be a real number in the interval $[c, b_k]$ such that

$$F(c_1) = \max_{c \leq x \leq b_k} F(x).$$

Since $F(c_1) \geq F(c) > F(b_k)$, it follows that $c \leq c_1 < b_k$, so that

$$c_1 \in (a_k, b_k) \subset H(\tau).$$

Hence we have the inequality

$$F(x) \leq F(c_1) \quad \text{for all } c_1 < x \leq b_k.$$

On the other hand, since $b_k \notin H(\tau)$, it follows that

$$F(x) \leq F(c_1) \quad \text{for all } x > c_1.$$

This contradicts the condition that $c_1 \in H(\tau)$.

Summing up, we have proved that $F(a_k) = F(b_k)$ for all k.

Step (3) Therefore, we have, by the definition of $F(x)$ and by Hölder's inequality (Theorem 3.14),

$$(b_k - a_k)\tau = \int_{a_k}^{b_k} f(t)\,dt \leq (b_k - a_k)^{1-1/p} \left(\int_{a_k}^{b_k} f(t)^p\,dt \right)^{1/p},$$

and so

$$(b_k - a_k)^{1/p}\tau \le \left(\int_{a_k}^{b_k} f(t)^p \, dt\right)^{1/p}.$$

Hence, it follows that

$$|H(\tau)|\tau^p = \sum_{k=1}^{\infty}(b_k - a_k)\tau^p \le \sum_{k=1}^{\infty}\int_{a_k}^{b_k} f(t)^p \, dt < \infty.$$

By applying Lemma 9.10 to our situation, we obtain that

$$\begin{aligned}|H(\tau)|\tau &= \sum_{k=1}^{\infty}(b_k - a_k)\tau = \sum_{k=1}^{\infty}\int_{a_k}^{b_k} f(t)\, dt = \int_{H(\tau)} f(t)\, dt \\ &\le \int_0^{|H(\tau)|} f^*(t)\, dt,\end{aligned}$$

so that

$$\tau \le \frac{1}{|H(\tau)|}\int_0^{|H(\tau)|} f^*(t)\, dt = \beta_f(|H(\tau)|).$$

This proves that

$$|H(\tau)| \le \beta^f(\tau).$$

Hence, we have, by this inequality and Theorem 3.29,

$$\begin{aligned}\int_{-\infty}^{\infty} G(x)^p \, dx &= p\int_0^{\infty} \tau^{p-1}|H(\tau)|\, d\tau \qquad (9.69) \\ &\le p\int_0^{\infty} \tau^{p-1}\beta^f(\tau)\, d\tau = \int_0^{s_1} \beta_f(\tau)\, d\tau^p.\end{aligned}$$

Here we recall that $s_1 = \lim_{t\to 0}\beta_f(t)$.

However, just as in the proof of Lemma 9.11 it follows that

$$\lim_{t\to 0} t\beta_f(t)^p = \lim_{t\to 0} t^{1-p}\left(\int_0^t f^*(s)\, ds\right)^p = 0.$$

Hence, by making the change of the independent variable $t := \beta^f(\tau)$ or $\tau := \beta_f(t)$ in the last integral of (9.69) we obtain that

$$\begin{aligned}\int_{-\infty}^{\infty} G(x)^p \, dx &\le \int_0^{s_1} \beta_f(\tau)\, d\tau^p = -\int_{t_1}^{\infty} t\, d\beta_f(t)^p \\ &= -\left[t\beta_f(t)\right]_{t_1}^{\infty} + \int_{t_1}^{\infty} \beta_f(t)^p\, dt \\ &\le t_1\beta_f(t_1)^p + \int_{t_1}^{\infty} \beta_f(t)^p\, dt = t_1\, s_1^p + \int_{t_1}^{\infty} \beta_f(t)^p\, dt\end{aligned}$$

$$= \int_0^{t_1} \beta_f(t)^p \, dt + \int_{t_1}^{\infty} \beta_f(t)^p \, dt = \int_0^{\infty} \beta_f(t)^p \, dt.$$

Therefore, we have, by this inequality and Lemma 9.11,

$$\int_{-\infty}^{\infty} G(x)^p \, dx \leq \int_0^{\infty} \beta_f(t)^p \, dt \leq \left(\frac{p}{p-1} \right)^p \int_{\mathbf{R}^n} f(x)^p \, dx.$$

This proves the desired inequality (9.67).

Now the proof of Lemma 9.12 is complete. □

Corollary 9.13. *Under the assumptions of Lemma 9.12, we let*

$$\overline{f}(x) := \sup_{\varepsilon > 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} f(x+y) \, dy.$$

Then we have the inequality

$$\left(\int_{-\infty}^{\infty} \overline{f}(x)^p \, dx \right)^{1/p} \leq \frac{2^{1/p} p}{p-1} \left(\int_{-\infty}^{\infty} f(x)^p \, dx \right)^{1/p}. \tag{9.70}$$

Proof. If $\tau > 0$, we let

$$\overline{E}(\tau) := \{x : \overline{f}(x) > \tau\},$$

and

$$F_\varepsilon^-(x) := \frac{1}{\varepsilon} \int_0^{\varepsilon} f(x-y) \, dy, \qquad G^-(x) := \sup_{\varepsilon > 0} F_\varepsilon^-(x).$$

Moreover, we let

$$H^-(\tau) := \{x : G^-(x) > \tau\}.$$

Recall that

$$F_\varepsilon(x) = \frac{1}{\varepsilon} \int_0^{\varepsilon} f(x+y) \, dy, \quad G(x) = \sup_{\varepsilon > 0} F_\varepsilon(x),$$

and that

$$H(\tau) = \{x : G(x) > \tau\}.$$

Then, just as in the proof of Lemma 9.12 we obtain that

$$|H^-(\tau)| \leq \beta^f(\tau).$$

On the other hand, since $\overline{f}(x) \leq (G(x) + G^-(x))/2$, it follows that

$$\overline{E}(\tau) \subset H(\tau) \cup H^-(\tau).$$

Hence we have the inequality

$$|\overline{E}(\tau)| \leq |H(\tau)| + |H^-(\tau)| \leq 2\beta^f(\tau).$$

Therefore, just as in the proof of Lemma 9.12 we can obtain the desired inequality (9.70).

The proof of Corollary 9.13 is complete. □

9.6 The Hilbert Transform (Continued)

Let $\phi(x)$ be an even function in $C^1(\mathbf{R})$ such that

$$\phi(x) = \begin{cases} 0 & \text{if } |x| \leq 1/4, \\ 1 & \text{if } |x| \geq 3/4, \end{cases}$$

and that $0 \leq \phi(x) \leq 1$ on $\mathbf{R}$. We denote by $\psi(x)$ the Hilbert transform of $\phi(x)/x$:

$$\psi(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{|x-t|>\varepsilon} \frac{\phi(t)}{t} \frac{dt}{x-t}.$$

First, since we have the formula

$$\begin{aligned} \frac{d}{dx}\left(\int_{|x-t|>\varepsilon} \frac{\phi(t)}{t} \frac{dt}{x-t} \right) &= \frac{d}{dx}\left(\int_{|t|>\varepsilon} \frac{\phi(x-t)}{x-t} \frac{dt}{t} \right) \\ &= \int_{|t|>\varepsilon} \frac{\partial}{\partial x}\left(\frac{\phi(x-t)}{x-t} \right) \frac{dt}{t} \\ &= \int_{|x-t|>\varepsilon} \frac{d}{dt} \frac{\phi(t)}{t} \cdot \frac{dt}{x-t}, \end{aligned}$$

it follows that that the derivative

$$\frac{d}{dx}\psi(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{|x-t|>\varepsilon} \frac{d}{dt} \frac{\phi(t)}{t} \cdot \frac{dt}{x-t}$$

exists and is continuous, that is, $\psi(x)$ is of class C^1. It is easy to see that $\psi(x)$ is an even function.

Secondly, we show that, as $x \to \pm\infty$,

$$\psi(x) = O(x^{-2}), \quad \psi'(x) = O(|x|^{-3}). \tag{9.71}$$

If $x > 1$, we let

$$\pi\psi(x) := \lim_{N \to \infty, \varepsilon \downarrow 0} \left(\int_{-N}^{-1} + \int_{-1}^{0} + \int_{0}^{1} + \int_{1}^{x-\varepsilon} + \int_{x+\varepsilon}^{N} \right) \frac{\phi(t)}{t} \frac{dt}{x-t}.$$

Then, since we have the four formulas

$$\int_{-N}^{-1} \frac{\phi(t)}{t} \frac{dt}{x-t} = \int_{-N}^{-1} \frac{dt}{t(x-t)} = -\frac{1}{x} \left[\log \frac{N}{x+N} + \log(x+1) \right],$$

$$\int_{-1}^{0} \frac{\phi(t)}{t} \frac{dt}{x-t} = -\int_{0}^{1} \frac{\phi(t)}{t} \frac{dt}{x+t},$$
$$\int_{1}^{x-\varepsilon} \frac{\phi(t)}{t} \frac{dt}{x-t} = \int_{1}^{x-\varepsilon} \frac{dt}{t(x-t)} = \frac{1}{x}\left[\log\frac{x-\varepsilon}{\varepsilon} + \log(x-1)\right],$$
$$\int_{x+\varepsilon}^{N} \frac{\phi(t)}{t} \frac{dt}{x-t} = \int_{x+\varepsilon}^{N} \frac{dt}{t(x-t)} = \frac{1}{x}\left(\log\frac{N}{N-x} - \log\frac{x+\varepsilon}{\varepsilon}\right),$$

it follows that we have, for $x > 1$,

$$\pi\psi(x) = \frac{1}{x}\log\frac{x-1}{x+1} + \int_{0}^{1} \frac{2\phi(t)}{(x-t)(x+t)}\,dt.$$

Therefore, the desired assertions (9.71) follow immediately from this formula, by using de L'Hôpital's rule.

The next theorem asserts the existence of the Hilbert transform (9.52) in the space $L^p(\mathbf{R}^n)$ for $1 < p < \infty$, refining Theorem 9.6:

Theorem 9.14. *Assume that $f(x) \in L^p(\mathbf{R})$ for $1 < p < \infty$. If $\varepsilon > 0$, we let*

$$\widetilde{f}_\varepsilon(x) := \frac{1}{\pi}\int_{|x-t|>\varepsilon} \frac{f(t)}{x-t}\,dt.$$

Then we have the following three assertions:

(i) There exists a constant $C_p > 0$, independent of ε, such that

$$\left(\int_{-\infty}^{\infty} \sup_{\varepsilon>0}\left|\widetilde{f}_\varepsilon(x)\right|^p dx\right)^{1/p} \le C_p\left(\int_{-\infty}^{\infty} |f(x)|^p\,dx\right)^{1/p}. \qquad (9.72)$$

(ii) The sequence $\widetilde{f}_\varepsilon$ converges almost everywhere in $\mathbf{R}$ and in the strong topology of $L^p(\mathbf{R})$ as $\varepsilon \downarrow 0$. Namely, the singular integral

$$Hf(x) = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_{|x-t|>\varepsilon} \frac{f(t)}{x-t}\,dt$$

exists for almost all $x \in \mathbf{R}$ and in the strong topology of $L^p(\mathbf{R})$.

(iii) The Hilbert transform H is a bounded linear operator from $L^p(\mathbf{R})$ into itself. More precisely, we have the inequality

$$\|Hf\|_p \le C_p\,\|f\|_p\,.$$

Proof. Let $f(x)$ be an arbitrary function in $L^p(\mathbf{R})$ with $1 < p < \infty$. The proof of Theorem 9.14 is divided into three steps.

Step (1): First, we prove inequality (9.72). If $\phi(x)$ and $\psi(x)$ are the two functions defined as above, then we have the formula

$$\widetilde{f}_\varepsilon(x) \qquad (9.73)$$

$$
\begin{aligned}
&= \frac{1}{\pi} \int_{|x-t|>\varepsilon} \phi\left(\frac{x-t}{\varepsilon}\right) \frac{f(t)}{x-t}\, dt \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \phi\left(\frac{x-t}{t}\right) \frac{f(t)}{x-t}\, dt - \frac{1}{\pi} \int_{|x-t|<\varepsilon} \phi\left(\frac{x-t}{\varepsilon}\right) \frac{f(t)}{x-t}\, dt \\
&:= \widetilde{f}_{1,\varepsilon}(x) + \widetilde{f}_{2,\varepsilon}(x).
\end{aligned}
$$

The Hilbert transform of the function

$$
t \longmapsto \phi\left(\frac{x-t}{\varepsilon}\right) \frac{1}{x-t} \qquad \text{(for a fixed } x\text{)}
$$

is equal to the following:

$$
\frac{1}{\varepsilon}\, \psi\left(\frac{x-t}{\varepsilon}\right).
$$

Indeed, by the change of variables

$$
s = \frac{x-\sigma}{\varepsilon},
$$

we find that

$$
\begin{aligned}
&\frac{1}{\pi}\ \text{v.p.} \int_{\mathbf{R}} \phi\left(\frac{x-\sigma}{\varepsilon}\right) \frac{1}{x-\sigma}\, \frac{1}{t-\sigma}\, d\sigma \\
&= \frac{1}{\varepsilon}\, \frac{1}{\pi}\ \text{v.p.} \int_{\mathbf{R}} \frac{\phi(s)}{s}\, \frac{1}{(x-t)/\varepsilon - s}\, ds = \frac{1}{\varepsilon}\, \psi\left(\frac{x-t}{\varepsilon}\right).
\end{aligned}
$$

Hence, if we denote the Hilbert transform of f by $g = Hf$, then it follows from an application of Theorem 9.6 (formula (9.54)) that

$$
\begin{aligned}
\widetilde{f}_{1,\varepsilon}(x) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \phi\left(\frac{x-t}{t}\right) \frac{1}{x-t} \cdot f(t)\, dt \qquad (9.74) \\
&= \frac{1}{\pi\varepsilon} \int_{-\infty}^{\infty} \psi\left(\frac{x-t}{\varepsilon}\right) \cdot g(t)\, dt \\
&= \frac{1}{\pi\varepsilon} \int_{-\infty}^{\infty} \psi\left(\frac{s}{\varepsilon}\right) g(x-s)\, ds \\
&= \frac{1}{\pi\varepsilon} \left(\int_{0}^{\infty} \psi\left(\frac{s}{\varepsilon}\right) g(x-s)\, ds + \int_{-\infty}^{0} \psi\left(\frac{s}{\varepsilon}\right) g(x-s)\, ds \right) \\
&= \frac{1}{\pi\varepsilon} \int_{0}^{\infty} \psi\left(\frac{s}{\varepsilon}\right) \left(g(x-t) + g(x+t)\right)\, dt.
\end{aligned}
$$

Here we have used the fact that the function $\psi(x)$ is even.

If $t > 0$ and $-\infty < x < \infty$, we let

$$
I(x : t) := \int_{-t}^{t} g(x+y)\, dy,
$$

$$\overline{g}(x) := \sup_{\varepsilon>0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |g(x+y)| \, dy.$$

Then we have, by Corollary 9.13,

$$\overline{g}(x) \in L^p(\mathbf{R}).$$

Let x be a point such that $\overline{g}(x) < \infty$. We remark that

$$|I(t:x)| \leq \int_{-t}^{t} |g(x+y)| \, dy \leq 2t\,\overline{g}(x) \quad \text{for all } t > 0, \tag{9.75}$$

and further that

$$dI(t:x) = (g(x-t) + g(x+t))\, dt. \tag{9.76}$$

By combining two formulas (9.74) and (9.76), we obtain that

$$\begin{aligned} \widetilde{f}_{1,\varepsilon}(x) &= \frac{1}{\pi\varepsilon} \int_0^\infty \psi\left(\frac{t}{\varepsilon}\right) dI(t:x) \\ &= \frac{1}{\pi\varepsilon} \left[\psi\left(\frac{t}{\varepsilon}\right) I(x:t)\right]_{t=0}^{t=\infty} - \frac{1}{\pi\varepsilon^2} \int_0^\infty \psi'\left(\frac{t}{\varepsilon}\right) I(t:x)\, dt. \end{aligned}$$

However, it follows from assertions (9.71) that

$$\begin{aligned} \psi\left(\frac{t}{\varepsilon}\right) &= O\left(\frac{\varepsilon^2}{t^2}\right) \quad \text{as } t \to \infty, \\ \psi'\left(\frac{t}{\varepsilon}\right) &= O\left(\frac{\varepsilon^3}{t^3}\right) \quad \text{as } t \to \infty. \end{aligned}$$

Hence we find from inequality (9.75) that the first term on the right hand side equals 0, while the second term on the right hand side converges. Summing up, we have proved the formula

$$\widetilde{f}_{1,\varepsilon}(x) = -\frac{1}{\pi\varepsilon^2} \int_0^\infty \psi'\left(\frac{t}{\varepsilon}\right) I(x:t)\, dt. \tag{9.77}$$

Therefore, by combining inequality (9.75) and formula (9.77) we obtain from formula (9.74) that

$$\begin{aligned} |\widetilde{f}_{1,\varepsilon}(x)| &\leq \frac{1}{\pi\varepsilon^2} \int_0^\infty \left|\psi'\left(\frac{t}{\varepsilon}\right)\right| 2t\,\overline{g}(x)\, dt \qquad (9.78) \\ &= \frac{1}{\pi\varepsilon^2} \int_0^\infty |\psi'(s)|\, 2(\varepsilon\, s)\, \varepsilon\, ds \cdot \overline{g}(x) = \frac{2}{\pi} \int_0^\infty t|\psi'(t)|\, dt \cdot \overline{g}(x). \end{aligned}$$

On the other hand, we have the inequality

$$|\widetilde{f}_{2,\varepsilon}(x)|$$

$$\begin{aligned}
&\le \frac{1}{\pi} \int_{\varepsilon/4<|x-t|<\varepsilon} \phi\left(\frac{x-t}{\varepsilon}\right) \left|\frac{f(t)}{x-t}\right| dt \\
&\le \frac{1}{\pi} \int_{\varepsilon/4<|x-t|<\varepsilon} \left|\frac{f(t)}{x-t}\right| dt \\
&\le \frac{4}{\pi\,\varepsilon} \int_{|x-t|<\varepsilon} |f(t)|\, dt = \frac{4}{\pi\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} |f(t)|\, dt \\
&= \frac{4}{\pi\varepsilon} \int_{-\varepsilon}^{\varepsilon} |f(x+y)|\, dy.
\end{aligned}$$

If we let

$$\overline{f}(x) := \sup_{\varepsilon>0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |f(x+y)|\, dy,$$

then it follows that

$$|\widetilde{f}_{2,\varepsilon}(x)| \le \frac{8}{\pi}\, \overline{f}(x). \tag{9.79}$$

Therefore, we obtain from formula (9.73) and two inequalities (9.78) and (9.79) that

$$\begin{aligned}
\sup_{\varepsilon>0} |\widetilde{f}_\varepsilon(x)| &\le |\widetilde{f}_{1,\varepsilon}(x)| + |\widetilde{f}_{2,\varepsilon}(x)| \\
&\le \frac{2}{\pi} \int_0^\infty t|\psi'(t)|\, dt \cdot \overline{g}(x) + \frac{8}{\pi}\, \overline{f}(x).
\end{aligned}$$

The desired inequality (9.72) follows by applying Theorem 9.6 and Corollary 9.13 to the right hand side of the above inequality:

$$\begin{aligned}
\int_{-\infty}^{\infty} \sup_{\varepsilon>0} \left|\widetilde{f}_\varepsilon(x)\right|^p dx &\le D_p \left(\int_{-\infty}^{\infty} |\overline{g}(x)|^p\, dx + \int_{-\infty}^{\infty} |\overline{f}(x)|^p\, dx \right) \\
&\le D_p' \left(\int_{-\infty}^{\infty} |Hf(x)|^p\, dx + \int_{-\infty}^{\infty} |f(x)|^p\, dx \right) \\
&\le D_p'' \int_{-\infty}^{\infty} |f(x)|^p\, dx.
\end{aligned}$$

Here D_p, D_p', D_p'' are positive constants independent of ε.

Step (2): We have already proved that, as $\varepsilon \downarrow 0$, the sequence $\widetilde{f}_\varepsilon$ converges in the strong topology of $L^p(\mathbf{R})$.

Step (3): Finally, we prove that the sequence $\widetilde{f}_\varepsilon$ converges almost everywhere in $\mathbf{R}$, as $\varepsilon \downarrow 0$.

To do this, we take a function $g(x)$ in $C_0^1(\mathbf{R})$ and let

$$h(x) := f(x) - g(x).$$

If we define two functions $\widetilde{g}_\varepsilon(x)$ and $\widetilde{h}_\varepsilon(x)$ as we defined the function

$\widetilde{f}_\varepsilon(x)$, then, by arguing just as in the proof of Theorem 9.5 we obtain that the sequence $\widetilde{g}_\varepsilon$ converges uniformly as $\varepsilon \downarrow 0$. Hence we have the inequality

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0} \widetilde{f}_\varepsilon(x) - \liminf_{\varepsilon\downarrow 0} \widetilde{f}_\varepsilon(x) \\
&= \limsup_{\varepsilon\downarrow 0} \left(\widetilde{h}_\varepsilon(x) + \widetilde{g}_\varepsilon(x)\right) - \liminf_{\varepsilon\downarrow 0} \left(\widetilde{h}_\varepsilon(x) + \widetilde{g}_\varepsilon(x)\right) \\
&= \limsup_{\varepsilon\downarrow 0} \widetilde{h}_\varepsilon(x) - \liminf_{\varepsilon\downarrow 0} \widetilde{h}_\varepsilon(x) \\
&\leq 2 \sup_{\varepsilon>0} |\widetilde{h}_\varepsilon(x)|.
\end{aligned}$$

By combining this inequality and inequality (9.72) with $f(x) := h(x)$, we obtain that

$$\int_{-\infty}^{\infty} \left| \limsup_{\varepsilon\downarrow 0} \widetilde{f}_\varepsilon(x) - \liminf_{\varepsilon\downarrow 0} \widetilde{f}_\varepsilon(x) \right|^p dx \leq 2^p \int_{-\infty}^{\infty} |\widetilde{h}_\varepsilon(x)|^p \, dx \tag{9.80}$$
$$\leq 2^p \, C_p^p \int_{-\infty}^{\infty} |h(x)|^p \, dx.$$

However, Corollary 3.27 tells us that $C_0^1(\mathbf{R})$ is dense in $L^p(\mathbf{R})$. Hence, the right hand side of inequality (9.80) can be made arbitrarily small if we choose the function $g(x)$ appropriately. This proves that the integrand of the left hand side of inequality (9.80) vanishes almost everywhere in $\mathbf{R}$:

$$\limsup_{\varepsilon\downarrow 0} \widetilde{f}_\varepsilon(x) = \liminf_{\varepsilon\downarrow 0} \widetilde{f}_\varepsilon(x).$$

Namely, the sequence $\widetilde{f}_\varepsilon$ converges almost everywhere in $\mathbf{R}$ as $\varepsilon \downarrow 0$.

Now the proof of Theorem 9.14 is complete. □

9.7 The Case of Odd Kernels

In this section we consider the case where the integral kernel $K(x)$ is an odd function, and make the following assumption:

Assumption 9.4. The integral kernel $K(x)$ is an *odd function* satisfying Assumption 9.1.

It should be noticed that the odd integral kernel $K(x)$ satisfies condition (9.2).

In the case of *odd kernels*, the existence of the singular integral (9.2) in the space $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ can be reduced to the study of the

Hilbert transform (Theorem 9.14). In fact, we can prove the following theorem:

Theorem 9.15. *Assume that the integral kernel $K(x)$ satisfies Assumption 9.4. If $f(x) \in L^p(\mathbf{R}^n)$ for $1 < p < \infty$ and $\varepsilon > 0$, we let*

$$\widetilde{f}_\varepsilon(x) := \int_{|x-y|>\varepsilon} K(x-y) f(y)\, dy.$$

Then we have the following three assertions (i), (ii) and (iii):

(i) The inequality

$$\begin{aligned} &\left(\int_{\mathbf{R}^n} \sup_{\varepsilon>0} |\widetilde{f}_\varepsilon(x)|^p \, dx \right)^{1/p} \\ &\leq \frac{\pi C_p}{2} \left(\int_{\Sigma_{n-1}} |K(\sigma)| \, d\sigma \right) \left(\int_{\mathbf{R}^n} |f(x)|^p \, dx \right)^{1/p} \end{aligned} \tag{9.81}$$

holds true where C_p is the positive constant given in Theorem 9.14.

(ii) The sequence $\widetilde{f}_\varepsilon$ converges almost everywhere in $\mathbf{R}^n$ and in the strong topology of $L^p(\mathbf{R}^n)$ as $\varepsilon \downarrow 0$. Namely, the singular integral

$$K * f(x) = \lim_{\varepsilon \downarrow 0} \int_{|x-y|>\varepsilon} K(x-y)\, f(y)\, dy$$

exists for almost all $x \in \mathbf{R}^n$ and in the strong topology of $L^p(\mathbf{R}^n)$.

(iii) The mapping $f \mapsto K*f$ is a bounded linear operator from $L^p(\mathbf{R}^n)$ into itself. More precisely, we have the inequality

$$\|K * f\|_p \leq \frac{\pi C_p}{2} \left(\int_{\Sigma_{n-1}} |K(\sigma)|\, d\sigma \right) \|f\|_p \,. \tag{9.82}$$

Proof. The proof of Theorem 9.15 is divided into three steps.

Step (1): As is shown in the proof of Lemma 9.1, we have the formula

$$\widetilde{f}_\varepsilon(x) = \int_{|y|>\varepsilon} K(y) f(x-y)\, dy = \int_{\Sigma_{n-1}} K(\sigma) \left(\int_\varepsilon^\infty f(x-t\sigma) \frac{dt}{t} \right) d\sigma.$$

However, since the kernel $K(x)$ is odd, it follows that

$$\int_{\Sigma_{n-1}} K(\sigma) \left(\int_\varepsilon^\infty f(x-t\sigma)\, \frac{dt}{t} \right) d\sigma$$

$$= -\int_{\Sigma_{n-1}} K(\sigma) \left(\int_{\varepsilon}^{\infty} f(x + t\sigma) \, \frac{dt}{t} \right) d\sigma$$
$$= \int_{\Sigma_{n-1}} K(\sigma) \left(\int_{-\infty}^{-\varepsilon} f(x - t\sigma) \, \frac{dt}{t} \right) d\sigma.$$

Hence we have the formula

$$\widetilde{f}_{\varepsilon}(x) = \frac{1}{2} \int_{\Sigma_{n-1}} K(\sigma) \left(\int_{|t|>\varepsilon} f(x - t\sigma) \frac{dt}{t} \right) d\sigma. \tag{9.83}$$

By using Hölder's inequality (Theorem 3.14), we obtain that

$$\begin{aligned} &|\widetilde{f}_{\varepsilon}(x)| \\ &\le \frac{1}{2} \int_{\Sigma_{n-1}} |K(\sigma)|^{1-1/p} |K(\sigma)|^{1/p} \left| \int_{|t|>\varepsilon} f(x - t\sigma) \frac{dt}{t} \right| d\sigma \\ &\le \frac{1}{2} \left(\int_{\Sigma_{n-1}} |K(\sigma)| \, d\sigma \right)^{1-1/p} \\ &\quad \times \left(\int_{\Sigma_{n-1}} |K(\sigma)| \left| \int_{|t|>\varepsilon} f(x - t\sigma) \frac{dt}{t} \right|^p d\sigma \right)^{1/p}. \end{aligned}$$

Hence we have the inequality

$$\begin{aligned} &\sup_{\varepsilon>0} |\widetilde{f}_{\varepsilon}(x)|^p \\ &\le \frac{1}{2^p} \left(\int_{\Sigma_{n-1}} |K(\sigma)| \, d\sigma \right)^{p-1} \int_{\Sigma_{n-1}} |K(\sigma)| \sup_{\varepsilon>0} \left| \int_{|t|>\varepsilon} f(x - t\sigma) \frac{dt}{t} \right|^p d\sigma. \end{aligned}$$

Moreover, by integrating the both sides of this inequality over $\mathbf{R}^n$ we obtain that

$$\begin{aligned} &\int_{\mathbf{R}^n} \sup_{\varepsilon>0} |\widetilde{f}_{\varepsilon}(x)|^p \, dx \\ &\le \frac{1}{2^p} \left(\int_{\Sigma_{n-1}} |K(\sigma)| \, d\sigma \right)^{p-1} \\ &\quad \times \int_{\Sigma_{n-1}} |K(\sigma)| \left(\int_{\mathbf{R}^n} \sup_{\varepsilon>0} \left| \int_{|t|>\varepsilon} f(x - t\sigma) \frac{dt}{t} \right|^p dx \right) d\sigma. \end{aligned} \tag{9.84}$$

For each $\sigma \in \Sigma_{n-1}$, we make a change of variables

$$x := y + s\,\sigma, \quad y \in \mathbf{R}^n, \; s \in \mathbf{R},$$

with

$$\langle y, \sigma \rangle = y_1\sigma_1 + \ldots + y_n\sigma_n = 0.$$

Then we have the formula

$$\int_{\mathbf{R}^n} \sup_{\varepsilon>0} \left| \int_{|t|>\varepsilon} f(x - t\,\sigma)\, \frac{dt}{t} \right|^p dx$$
$$= \int_{\mathbf{R}^{n-1}} \left(\int_{-\infty}^{\infty} \sup_{\varepsilon>0} \left| \int_{|t|>\varepsilon} f(y + (s-t)\sigma)\, \frac{dt}{t} \right|^p ds \right) dy.$$

Here it should be noticed that

$$\int_{|t|>\varepsilon} f\,(y + (s-t)\sigma)\, \frac{dt}{t} = - \int_{|s-\tau|>\varepsilon} \frac{f(y+\tau\sigma)}{s-\tau}\, d\tau.$$

Hence, by applying Theorem 9.14 with

$$f(t) := f(y + t\sigma),$$

we obtain that

$$\int_{\mathbf{R}^n} \sup_{\varepsilon>0} \left| \int_{|t|>\varepsilon} f(x - t\sigma) \frac{dt}{t} \right|^p dx$$
$$= \pi^p \int_{\mathbf{R}^{n-1}} \left(\int_{-\infty}^{\infty} \sup_{\varepsilon>0} \left| \frac{1}{\pi} \int_{|s-\tau|>\varepsilon} \frac{f(y+\tau\sigma)}{s-\tau}\, d\tau \right|^p ds \right) dy$$
$$\leq (\pi C_p)^p \int_{\mathbf{R}^{n-1}} \left(\int_{-\infty}^{\infty} |f(y+s\sigma)|^p\, ds \right) dy = (\pi C_p)^p \int_{\mathbf{R}^n} |f(x)|^p\, dx.$$

By combining this inequality with inequality (9.84), we have the inequality

$$\int_{\mathbf{R}^n} \sup_{\varepsilon>0} |\widetilde{f_\varepsilon}(x)|^p\, dx \leq \frac{1}{2^p} \left(\int_{\Sigma_{n-1}} |K(\sigma)|\, d\sigma \right)^{p-1}$$
$$\times \left(\int_{\Sigma_{n-1}} |K(\sigma)| \left((\pi C_p)^p \int_{\mathbf{R}^n} |f(x)|^p\, dx \right) d\sigma \right)$$
$$= \left(\frac{\pi C_p}{2} \right)^p \left(\int_{\Sigma_{n-1}} |K(\sigma)|\, d\sigma \right)^p \int_{\mathbf{R}^n} |f(x)|^p\, dx.$$

This proves the desired inequality (9.81).

Step (2): We consider the case where $f(x)$ is an arbitrary function in the space $C_0^1(\mathbf{R}^n)$.

We remark that if $f(x) = 0$ for $|x| > M$ and $0 < \varepsilon < 1$, then it is easy to see that the function $\widetilde{f}_\varepsilon(x)$ is independent of ε, for $|x| \geq M+1$. If $|x| < M+1$, then we have the formula

$$\begin{aligned}
&\widetilde{f}_\varepsilon(x) \\
&= \int_{\substack{|x-y|>\varepsilon \\ |x|<M+1, |y|<M}} K(x-y) f(y)\, dy = \int_{\varepsilon<|x-y|<2M+1} K(x-y) f(y)\, dy \\
&= \int_{\varepsilon<|x-y|<2M+1} K(x-y)(f(y)-f(x))\, dy.
\end{aligned}$$

However, by letting

$$L := \max_{x \in \mathbf{R}^n} |\nabla f(x)|\,,$$

we obtain that

$$\begin{aligned}
&\int_{|x-y|<\varepsilon} |K(x-y)(f(y)-f(x))|\, dy \\
&\leq L \int_{|x-y|<\varepsilon} |K(x-y)|\, |x-y|\, dy \leq \varepsilon L \int_{|x-y|>\varepsilon} |K(x-y)|\, dy \\
&= \varepsilon L \int_{\Sigma_{n-1}} |K(\sigma)|\, d\sigma \quad \text{for all } |x| < M+1.
\end{aligned}$$

This proves that the sequence $\widetilde{f}_\varepsilon$ converges uniformly to some function $\widetilde{f}$ in the open ball $\{|x| < M+1\}$, as $\varepsilon \downarrow 0$. Hence we have, by inequality (9.81),

$$\begin{aligned}
\int_{\mathbf{R}^n} |\widetilde{f}(x)|^p\, dx = \lim_{\varepsilon \downarrow 0} \int_{\mathbf{R}^n} |\widetilde{f}_\varepsilon(x)|^p\, dx &\leq \int_{\mathbf{R}^n} \sup_{\varepsilon>0} |\widetilde{f}_\varepsilon(x)|^p\, dx \\
&\leq \left(\frac{\pi C_p}{2}\right)^p \left(\int_{\Sigma_{n-1}} |K(\sigma)|\, d\sigma\right)^p \cdot \int_{\mathbf{R}^n} |f(x)|^p\, dx,
\end{aligned}$$

so that

$$\widetilde{f}(x) \in L^p(\mathbf{R}^n).$$

Therefore, by applying the Lebesgue dominated convergence theorem (Theorem 3.8) we obtain that

$$\|\widetilde{f}_\varepsilon - \widetilde{f}\|_p^p = \int_{|x|<M+1} |\widetilde{f}_\varepsilon(x) - \widetilde{f}(x)|^p\, dx \longrightarrow 0 \quad \text{as } \varepsilon \downarrow 0.$$

Step (3): We consider the general case where $f(x)$ is an arbitrary function in the space $L^p(\mathbf{R}^n)$.

We take a function $g(x) \in C_0^1(\mathbf{R}^n)$ and let

$$h(x) := f(x) - g(x).$$

Just as in the proof of Theorem 9.14, we can prove that the sequence $\widetilde{f}_\varepsilon$ converges almost everywhere in $\mathbf{R}^n$ to some function $K * f$ as $\varepsilon \downarrow 0$:

$$\widetilde{f}_\varepsilon \longrightarrow K * f \quad \text{almost everywhere in } \mathbf{R}^n \text{ as } \varepsilon \downarrow 0.$$

Moreover, we show that

$$\widetilde{f}_\varepsilon \longrightarrow K * f \quad \text{in } L^p(\mathbf{R}^n) \text{ as } \varepsilon \downarrow 0.$$

To do this, we remark that

$$\begin{aligned}\left\|\widetilde{f}_\varepsilon - \widetilde{f}_\delta\right\|_p &= \left\|\left(\widetilde{h}_\varepsilon + \widetilde{g}_\varepsilon\right) - \left(\widetilde{h}_\delta + \widetilde{g}_\varepsilon\right)\right\|_p \\ &\leq \|\widetilde{g}_\varepsilon - \widetilde{g}_\delta\|_p + \left\|\widetilde{h}_\varepsilon - \widetilde{h}_\delta\right\|_p.\end{aligned}$$

Since g is uniformly Lipschitz continuous, it follows that the first term on the right hand side of the inequality tends to 0 as ε, $\delta \downarrow 0$:

$$\lim_{\varepsilon,\delta\downarrow 0} \|\widetilde{g}_\varepsilon - \widetilde{g}_\delta\|_p = 0.$$

On the other hand, we have, by inequality (9.81),

$$\begin{aligned}\left\|\widetilde{h}_\varepsilon - \widetilde{h}_\delta\right\|_p &\leq 2\left(\int_{\mathbf{R}^n} \sup_{\varepsilon>0} |\widetilde{h}_\varepsilon(x)|^p \, dx\right)^{1/p} \\ &\leq \pi C_p \left(\int_{\Sigma_{n-1}} |K(\sigma)| \, d\sigma\right) \|h\|_p.\end{aligned}$$

However, we remark that the last term on this inequality can be made arbitrarily small if we choose the function $g(x)$ appropriately.

In this way, we find that the sequence $\widetilde{f}_\varepsilon$ converges strongly to the function $\widetilde{f}$ in $L^p(\mathbf{R}^n)$ as $\varepsilon \downarrow 0$:

$$\widetilde{f}_\varepsilon \longrightarrow \widetilde{f} \quad \text{in } L^p(\mathbf{R}^n) \text{ as } \varepsilon \downarrow 0.$$

This proves that

$$K * f = \widetilde{f} \in L^p(\mathbf{R}^n),$$

so that

$$\widetilde{f}_\varepsilon \longrightarrow K * f \quad \text{in } L^p(\mathbf{R}^n) \text{ as } \varepsilon \downarrow 0.$$

Finally, by letting $\varepsilon \downarrow 0$ in inequality (9.81) we obtain from Lebesgue's dominated convergence theorem (Theorem 3.8) that

$$\|K * f\|_p \leq \frac{\pi C_p}{2} \left(\int_{\Sigma_{n-1}} |K(\sigma)| \, d\sigma\right) \|f\|_p.$$

This proves that the mapping $f \mapsto K * f$ is a bounded linear operator from $L^p(\mathbf{R}^n)$ into itself.

Now the proof of Theorem 9.15 is complete. $\square$

9.8 Riesz Kernels

The *Riesz kernels* are functions defined by the formulas

$$R_j(x) = -\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{x_j}{|x|^{n+1}} \quad \text{for } 1 \leq j \leq n. \tag{9.85}$$

It is easy to see that the Riesz kernels $R_j(x)$ satisfy Assumption 9.3 and Assumption 9.4 with $\omega(t) := t$. Moreover, it should be emphasized that the Riesz kernels transform even kernels to *odd* kernels. Therefore, by making use of Theorem 9.15 we can prove the existence of singular integral operators with even kernel. In fact, this will be carried out in the next Section 9.9.

In this section we calculate explicitly the Fourier transforms $(\mathcal{F}R_j)(\xi)$ of the Riesz kernels $R_j(x)$ (see formula (9.97)).

To do this, we let $0 < \varepsilon < 1 < \mu$. Then we have, by the divergence theorem,

$$\begin{aligned}
&(1-n) \int_{\varepsilon<|x|<1} e^{-ix\cdot\xi} \frac{x_j}{|x|^{n+1}}\, dx \qquad (9.86)\\
&= \int_{\varepsilon<|x|<1} e^{-ix\cdot\xi} \frac{\partial}{\partial x_j} \left(|x|^{1-n}\right) dx \\
&= \int_{|x|=1} e^{-ix\cdot\xi} \frac{x_j}{|x|^n}\, d\sigma - \int_{|x|=\varepsilon} e^{-ix\cdot\xi} \frac{x_j}{|x|^n}\, dS \\
&= \int_{|x|=1} e^{-ix\cdot\xi} \frac{x_j}{|x|^n}\, d\sigma + \int_{\varepsilon<|x|<1} i\xi j e^{-ix\cdot\xi} |x|^{1-n}\, dx \\
&= \int_{|x|=1} x_j e^{-ix\cdot\xi}\, d\sigma - \frac{1}{\varepsilon^n} \int_{|x|=\varepsilon} x_j e^{-ix\cdot\xi}\, dS \\
&= \int_{|x|=1} x_j e^{-ix\cdot\xi}\, d\sigma + i\xi_j \int_{\varepsilon<|x|<1} e^{-ix\cdot\xi} |x|^{1-n}\, dx.
\end{aligned}$$

However, it follows that

$$\begin{aligned}
\int_{|x|=1} x_j e^{-ix\cdot\xi}\, d\sigma &= i \frac{\partial}{\partial \xi_j} \left(\int_{|x|=1} e^{-ix\cdot\xi}\, d\sigma \right) \\
&= i \frac{\partial}{\partial \xi_j} \left(\int_{|x|=1} \cos(x\cdot\xi)\, d\sigma \right).
\end{aligned}$$

If we make an orthogonal transformation which maps ξ to $(|\xi|, 0, \dots, 0)$, then we obtain that

$$\begin{aligned}\int_{|x|=1} x_j e^{-ix\cdot\xi}\, d\sigma &= i\, \frac{\partial}{\partial \xi_j} \left(\int_{|y|=1} \cos\left(|\xi| y_1\right) d\sigma \right) \\ &= -i\, \frac{\xi_j}{|\xi|} \int_{|y|=1} y_1 \, \sin\left(|\xi| y_1\right) d\sigma.\end{aligned} \tag{9.87}$$

Moreover, it is easy to verify that

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^n} \int_{|x|=\varepsilon} x_j e^{-ix\cdot\xi}\, dS \\ &= \lim_{\varepsilon\downarrow 0} \int_{|x|=1} \sigma_j e^{-i\varepsilon\sigma\xi}\, d\sigma = \int_{\Sigma_{n-1}} \sigma_j \, d\sigma \\ &= 0,\end{aligned} \tag{9.88}$$

and further that

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} \int_{\varepsilon<|x|<1} e^{-ix\cdot\xi} |x|^{1-n}\, dx &= \int_{|x|<1} e^{-ix\cdot\xi} |x|^{1-n}\, dx \\ &= \int_{|x|<1} |x|^{1-n} \cos\left(x\cdot\xi\right) dx \\ &= \int_{|y|<1} |y|^{1-n} \cos\left(|\xi| y_1\right) dy.\end{aligned} \tag{9.89}$$

By combining four formulas (9.86), (9.87), (9.88) and (9.89), we obtain that

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0} \int_{\varepsilon<|x|<1} e^{-ix\cdot\xi} \frac{x_j}{|x|^{n+1}}\, dx \\ &= \frac{1}{1-n} \Bigg(-i \frac{\xi_j}{|\xi|} \int_{|y|=1} y_1 \sin\left(|\xi| y_1\right) d\sigma \\ &\quad + i\,\xi_j \int_{|y|<1} |y|^{1-n} \cos\left(|\xi| y_1\right) dy \Bigg).\end{aligned}$$

Hence, if we let

$$c_1 := \frac{i}{1-n} \left(-\int_{|y|=1} y_1 \sin y_1 \, d\sigma + \int_{|y|<1} |y|^{1-n} \cos y_1 \, dy \right),$$

then we have, for $|\xi| = 1$,

$$\lim_{\varepsilon\downarrow 0} \int_{\varepsilon<|x|<1} e^{-ix\cdot\xi} \frac{x_j}{|x|^{n+1}}\, dx = c_1 \xi_j. \tag{9.90}$$

Similarly, we have the formula

$$\int_{1<|x|<\mu} e^{-ix\cdot\xi} \frac{x_j}{|x|^{n+1}}\, dx = i\, \frac{\partial}{\partial \xi_j} \left(\int_{1<|x|<\mu} \frac{e^{-ix\cdot\xi}}{|x|^{n+1}}\, dx \right). \tag{9.91}$$

It is easy to see that the limit

$$\lim_{\mu\uparrow\infty} \int_{1<|x|<\mu} \frac{e^{-ix\cdot\xi}}{|x|^{n+1}}\, dx = \int_{|x|>1} \frac{e^{-ix\cdot\xi}}{|x|^{n+1}}\, dx \tag{9.92}$$

exists uniformly in $\mathbf{R}^n$. If $\xi_j \neq 0$, then it follows from an application of the divergence theorem that

$$\begin{aligned}
&\int_{1<|x|<\mu} e^{-ix\cdot\xi} \frac{x_j}{|x|^{n+1}}\, dx \\
&= \frac{i}{\xi_j} \int_{1<|x|<\mu} \frac{x_j}{|x|^{n+1}} \frac{\partial}{\partial x_j} \left(e^{-ix\cdot\xi}\right) dx \\
&= \frac{i}{\xi_j} \left(\int_{|x|=\mu} e^{-ix\cdot\xi} \frac{x_j}{|x|^{n+1}} \frac{x_j}{|x|}\, dS + \int_{|x|=1} e^{-ix\cdot\xi} \frac{x_j}{|x|^{n+1}} \frac{-x_j}{|x|}\, d\sigma \right.\\
&\quad \left. - \int_{1<|x|<\mu} e^{-ix\cdot\xi} \frac{\partial}{\partial x_j} \left(\frac{x_j}{|x|^{n+1}} \right) dx \right) \\
&= \frac{i}{\xi_j} \left(\int_{|x|=\mu} e^{-ix\cdot\xi} \frac{x_j^2}{|x|^{n+2}}\, dS - \int_{|x|=1} e^{-ix\cdot\xi} \frac{x_j^2}{|x|^{n+2}}\, d\sigma \right.\\
&\quad \left. - \int_{1<|x|<\mu} e^{-ix\cdot\xi} \frac{\partial}{\partial x_j} \left(\frac{x_j}{|x|^{n+1}} \right) dx \right) \\
&= \frac{i}{\xi_j} \left(\frac{1}{\mu} \int_{|x|=1} x_j^2 e^{-i\mu x\cdot\xi}\, d\sigma - \int_{|x|=1} x_j^2 e^{-ix\cdot\xi}\, d\sigma \right.\\
&\quad \left. - \int_{1<|x|<\mu} e^{-ix\cdot\xi} \frac{\partial}{\partial x_j} \left(\frac{x_j}{|x|^{n+1}} \right) dx \right),
\end{aligned}$$

where $dS = \mu^{n-1}\, d\sigma$ is the surface measure on the sphere $\{|x| = \mu\}$.

Hence we find that the limit

$$\begin{aligned}
&\lim_{\mu\uparrow\infty} \int_{1<|x|<\mu} e^{-ix\cdot\xi} \frac{x_j}{|x|^{x+1}}\, dx \qquad (9.93)\\
&= -\frac{i}{\xi_j} \left(\int_{|x|=1} x_j^2 e^{-ix\cdot\xi}\, d\sigma + \int_{|x|>1} e^{-ix\cdot\xi} \frac{\partial}{\partial x_j} \left(\frac{x_j}{|x|^{n+1}} \right) dx \right)
\end{aligned}$$

exists uniformly on any compact subset of the space $\{\xi \in \mathbf{R}^n : \xi_j \neq 0\}$.

By combining three formulas (9.91), (9.92) and (9.93), we have proved that

$$\int_{|x|>1} \frac{e^{-ix\cdot\xi}}{|x|^{n+1}}\,dx \in C^1(\{\xi \in \mathbf{R}^n : \xi_1 \cdots \xi_n \neq 0\}) \quad \text{for all } 1 \leq j \leq n, \tag{9.94}$$

and that

$$\begin{aligned} & i\,\frac{\partial}{\partial \xi_j}\left(\int_{|x|>1} \frac{e^{-ix\cdot\xi}}{|x|^{n+1}}\,dx\right) = i \lim_{\mu\uparrow\infty} \frac{\partial}{\partial \xi_j}\left(\int_{1<|x|<\mu} \frac{e^{-ix\cdot\xi}}{|x|^{n+1}}\,dx\right) \\ &= \lim_{\mu\uparrow\infty} \int_{1<|x|<\mu} e^{-ix\cdot\xi}\,\frac{x_j}{|x|^{n+1}}\,dx \\ &= -\frac{i}{\xi_j}\left(\int_{|x|=1} x_j^2 e^{-ix\cdot\xi}\,d\sigma + \int_{|x|>1} e^{-ix\cdot\xi}\,\frac{\partial}{\partial x_j}\left(\frac{x_j}{|x|^{n+1}}\right) dx\right) \\ &\quad \text{for } \xi_j \neq 0. \end{aligned} \tag{9.95}$$

If we let (cf. formula (9.87))

$$\begin{aligned} & f(|\xi|) \\ &:= \int_{|x|>1} \frac{e^{-ix\cdot\xi}}{|x|^{n+1}}\,dx = \int_{|x|>1} \frac{\cos(x\cdot\xi)}{|x|^{n+1}}\,dx - i\int_{|x|>1} \frac{\sin(x\cdot\xi)}{|x|^{n+1}}\,dx \\ &= \int_{|y|>1} \frac{\cos(y_1|\xi|)}{|y|^{n+1}}\,dy, \end{aligned}$$

then it follows from assertion (9.94) that the function $f(|\xi|)$ is of class C^1 in the space $\{\xi \in \mathbf{R}^n : \xi_1 \cdots \xi_n \neq 0\}$. Hence we obtain that

$$f(t) = \int_{|y|>1} \frac{\cos(y_1 t)}{|y|^{n+1}}\,dy \in C^1(0,\infty),$$

and further that $f(|\xi|)$ is continuously differentiable in the space $\mathbf{R}^n\setminus\{0\}$. By formula (9.95), it follows that

$$\lim_{\mu\uparrow\infty} \int_{1<|x|<\mu} e^{-ix\cdot\xi}\,\frac{x_j}{|x|^{n+1}}\,dx = i\,\frac{\partial}{\partial \xi_j}\left(f(|\xi|)\right) = i\,\frac{\xi_j}{|\xi|}\,f'(|\xi|).$$

Hence, if we let

$$c_2 := if'(1) = -i\int_{|y|>1} \frac{y_1 \sin y_1}{|x|^{n+1}}\,dy,$$

then we have, for $|\xi| = 1$,

$$\lim_{\mu\uparrow\infty} \int_{1<|x|<\mu} e^{-ix\cdot\xi}\,\frac{x_j}{|x|^{n+1}}\,dx = i\xi_j\,f'(1) = c_2\,\xi_j. \tag{9.96}$$

By two formulas (9.90) and (9.96), it follows that we have, for $|\xi| = 1$,

$$\lim_{\varepsilon \downarrow 0, \mu \uparrow \infty} \int_{\varepsilon < |x| < \mu} e^{-ix\cdot\xi} \frac{x_j}{|x|^{n+1}}\, dx = c_1\, \xi_j + c_2\, \xi_j := c_3\, \xi_j, \tag{9.97}$$

where

$$c_3 := c_1 + c_2.$$

However, we remark that, since the left hand side of formula (9.97) is a positively homogeneous function of ξ degree 0, we have, for all $\xi \neq 0$,

$$\lim_{\varepsilon \downarrow 0, \mu \uparrow \infty} \int_{\varepsilon < |x| < \mu} e^{-ix\cdot\xi} \frac{x_j}{|x|^{n+1}}\, dx = c_3\, \frac{\xi_j}{|\xi|}. \tag{9.98}$$

We calculate explicitly the value of the constant c_3. To do this, we may choose $j = 1$ and $\xi = (1, 0, \ldots, 0)$. Then it follows from formula (9.98) that

$$\lim_{\varepsilon \downarrow 0, \mu \uparrow \infty} \int_{\varepsilon < |x| < \mu} e^{-ix_1} \frac{x_1}{|x|^{n+1}}\, dx = c_3, \tag{9.99}$$

and further that

$$\begin{aligned} &\int_{\varepsilon < |x| < \mu} e^{-ix_1} \frac{x_1}{|x|^{n+1}}\, dx \qquad (9.100) \\ &= \int_{\varepsilon < |x| < \mu} \frac{x_1 \cos x_1}{|x|^{n+1}}\, dx - i \int_{\varepsilon < |x| < \mu} \frac{x_1 \sin x_1}{|x|^{n+1}}\, dx \\ &= -i \int_{\Sigma_{n-1}} \sigma_1 \left(\int_\varepsilon^\mu \frac{\sin(r\sigma_1)}{r}\, dr \right) d\sigma. \end{aligned}$$

However, since we have the formula

$$\lim_{\varepsilon \downarrow 0, \mu \uparrow \infty} \int_\varepsilon^\mu \frac{\sin(r\sigma_1)}{r}\, dr = \frac{\pi}{2} \operatorname{sign} \sigma_1$$

and the inequality

$$\left| \int_\varepsilon^\mu \frac{\sin(r\sigma_1)}{r}\, dr \right| \leq 2A,$$

by applying Lebesgue's dominated convergence theorem (Theorem 3.8) we obtain from two formulas (9.99) and (9.100) that

$$\begin{aligned} c_3 &= \lim_{\varepsilon \downarrow 0, \mu \uparrow \infty} \int_{\varepsilon < |x| < \mu} e^{-ix_1} \frac{x_1}{|x|^{n+1}}\, dx \\ &= -i \lim_{\varepsilon \downarrow 0, \mu \uparrow \infty} \int_{\Sigma_{n-1}} \sigma_1 \left(\int_\varepsilon^\mu \frac{\sin(r\sigma_1)}{r}\, dr \right) d\sigma \end{aligned}$$

$$= -\frac{\pi}{2}\, i \left(\int_{\Sigma_{n-1}} \sigma_1 \operatorname{sign} \sigma_1 \, d\sigma \right) = -\pi i \int_{\sigma_1 > 0} \sigma_1 \, d\sigma.$$

Moreover, we have the formula

$$\begin{aligned}\int_{\sigma_1>0} \sigma_1 \, d\sigma &= \omega_{n-1} \int_0^1 t(1-t^2)^{(n-3)/2} \, dt \\ &= \frac{1}{2}\, \omega_{n-1} \int_0^1 (1-s)^{(n-3)/2} \, ds \\ &= \frac{\omega_{n-1}}{n-1},\end{aligned}$$

where

$$\omega_{n-1} := |\Sigma_{n-2}| = \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)}$$

is the surface area of the unit sphere Σ_{n-2} in $\mathbf{R}^{n-1}$.

Therefore, we obtain that

$$c_3 = -\pi i \, \frac{\omega_n}{n-1} = -\frac{\pi i}{n-1} \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)} = -\frac{i\pi^{(n+1)/2}}{\Gamma((n+1)/2)}.$$

Summing up, we find from formula (9.98) that the Fourier transforms $(\mathcal{F}R_j)(\xi)$ of the Riesz kernels $R_j(x)$ are given by the formulas

$$\begin{aligned}(\mathcal{F}R_j)(\xi) &= \lim_{\varepsilon\downarrow 0, \mu\uparrow\infty} \int_{\varepsilon<|x|<\mu} e^{-ix\cdot\xi} R_j(x) \, dx \qquad (9.101) \\ &= -\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \lim_{\varepsilon\downarrow 0, \mu\uparrow\infty} \int_{\varepsilon<|x|<\mu} e^{-ix\cdot\xi} \frac{x_j}{|x|^{n+1}} \, dx \\ &= -\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} c_3 \frac{\xi_j}{|\xi|} \\ &= \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{i\pi^{(n+1)/2}}{\Gamma((n+1)/2)} \frac{\xi_j}{|\xi|} = i\, \frac{\xi_j}{|\xi|} \quad \text{for } 1 \le j \le n.\end{aligned}$$

Now, we let $f(x) \in L^2(\mathbf{R}^n)$ and

$$g_j(x) := -\lim_{\varepsilon\downarrow 0} \int_{|x-y|>\varepsilon} R_j(x-y) f(y) \, dy = -(R_j * f)(x).$$

Then we have, by formula (9.101),

$$-\mathcal{F}\left(\sum_{j=1}^n R_j * (R_j * f) \right)(\xi) = \mathcal{F}\left(\sum_{j=1}^n R_j * g_j \right)(\xi)$$

$$= \sum_{j=1}^{n} (\mathcal{F}R_j)(\xi) \cdot (\mathcal{F}g_j)(\xi)$$
$$= -\sum_{j=1}^{n} \left((\mathcal{F}R_j)(\xi)\right)^2 \cdot (\mathcal{F}f)(\xi) = -\sum_{j=1}^{n} \frac{(-\xi_j^2)}{|\xi|^2} \cdot (\mathcal{F}f)(\xi) = (\mathcal{F}f)(\xi).$$

By the Fourier inversion formula, this proves that we have, for all $f \in L^2(\mathbf{R}^n)$,

$$-\sum_{j=1}^{n} R_j * (R_j * f) = f. \tag{9.102}$$

Moreover, by arguing just as in the proof of Theorem 9.5 we obtain that formula (9.98) holds true for all $f(x) \in L^p(\mathbf{R}^n)$ with $1 < p < \infty$.

Therefore, we have proved the following fundamental theorem for the *Riesz kernels*:

Theorem 9.16. *The Fourier transforms* $(\mathcal{F}R_j)(\xi)$ *of the Riesz kernels* $R_j(x)$ *are given by formulas* (9.101). *Moreover, formula* (9.102) *holds true for all* $f \in L^p(\mathbf{R}^n)$ *with* $1 < p < \infty$.

Remark 9.2. By virtue of formula (9.101), we have, for all $1 \leq j, k \leq n$,

$$\frac{\partial^2 f}{\partial x_j \partial x_k} = -R_j * (R_k * (\Delta f)) \quad \text{for all } f \in C_0^\infty(\mathbf{R}^n).$$

Therefore, by applying Theorem 9.15 (inequality (9.82)) we find that

$$\left\| \frac{\partial^2 f}{\partial x_j \partial x_k} \right\|_{L^p(\mathbf{R}^n)} \leq C_p' \, \|\Delta f\|_{L^p(\mathbf{R}^n)} \quad \text{for all } 1 \leq j, k \leq n,$$

where $C_p' > 0$ is a constant. In other words, the Laplacian Δ controls all second-order partial derivatives in the L^p-norm for $1 < p < \infty$.

9.9 The Case of Even Kernels

In this section we consider the case where the integral kernel $K(x)$ is an even function, and make the following assumption:

Assumption 9.5. The integral kernel $K(x)$ is an even function satisfying Assumption 9.1 and the condition

$$\int_{\Sigma_{n-1}} |K(\sigma)| \log^+ |K(\sigma)| \, d\sigma < \infty. \tag{9.103}$$

Here $\log^+ x$ is a function defined by the formula

$$\log^+ x := \max\{\log x, 0\} = \begin{cases} \log x & \text{if } x > 1, \\ 0 & \text{if } 0 < x \leq 1. \end{cases}$$

The case of even kernels can be reduced to the case of odd kernels. Indeed, we have, by formula (9.102) with $f := K * f$,

$$K * f = -\sum_{j=1}^{n} R_j * (R_j * K) * f.$$

However, it should be noticed that the operators R_j and $R_j * K$ have respectively the odd kernels

$$R_j(x) = -\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{x_j}{|x|^{n+1}},$$
$$R_j * K(x) = -\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{x_j}{|x|^{n+1}} * K(x)$$

of positively homogeneous of degree $-n$. In other words, a singular integral operator with even kernel can be expressed as a finite sum of products of singular integral operators with odd kernel.

We begin by proving the following elementary result:

Lemma 9.17. *For all $a > 0$ and $b > 0$, we have the inequality*

$$ab \leq a \log^+ a + e^{b-1}. \tag{9.104}$$

Proof. Let $\varphi(x)$ and $\psi(y)$ be two functions defined respectively by the formulas

$$\varphi(x) := \begin{cases} \log x + 1 & \text{if } x > 1, \\ 0 & \text{if } 0 \leq x \leq 1, \end{cases}$$

and

$$\psi(y) := \begin{cases} e^{y-1} & \text{if } y > 1, \\ 1 & \text{if } 0 \leq y \leq 1. \end{cases}$$

Then it follows that $\psi(y)$ is the inverse function of $\varphi(x)$ in the region $\{y > 1\}$ (or $\{x > 1\}$). Hence we obtain that

$$ab \leq \int_0^a \varphi(x)\,dx + \int_0^b \psi(y)\,dy. \tag{9.105}$$

However, we have the formulas

$$\int_0^a \varphi(x)\,dx = \begin{cases} 0 & \text{if } 0 \le a \le 1, \\ a\log a & \text{if } a > 1, \end{cases}$$

and

$$\int_0^b \psi(y)\,dy = \begin{cases} 0 & \text{if } 0 \le b \le 1, \\ e^{b-1} - 1 & \text{if } b > 1. \end{cases}$$

Therefore, the desired inequality (9.104) follows immediately from inequality (9.105).

The proof of Lemma 9.17 is complete. □

Lemma 9.18. *Let $K(x)$ be an integral kernel satisfying Assumption 9.3. Assume that $f(x)$ is a measurable function with compact support in $\mathbf{R}^n$ which satisfies the condition*

$$\int_{\mathbf{R}^n} |f(x)| \log^+ |f(x)|\, dx < \infty. \tag{9.106}$$

If S is a measurable set of $\mathbf{R}^n$ with $|\operatorname{supp} f| < |S| < \infty$, where $\operatorname{supp} f$ denotes the support of $f(x)$, then there exist positive constants C_3 and C_4 such that we have, for all $\varepsilon > 0$,

$$\int_S \left|\widetilde{f}_\varepsilon(x)\right| dx \le C_3 \int_{\mathbf{R}^n} |f(x)| \log^+ |f(x)|\, dx + C_4\,|S|, \tag{9.107}$$

where $\widetilde{f}_\varepsilon(x)$ is the function defined by formula (9.5):

$$\widetilde{f}_\varepsilon(x) = \int_{|x-y|>\varepsilon} K(x-y)\, f(y)\, dy.$$

Proof. Since we have the inequality

$$\begin{aligned} \int_{\mathbf{R}^n} |f(x)|\,dx &= \int_{|f|>e} |f(x)|\,dx + \int_{|f|\le e} |f(x)|\,dx \\ &\le \int_{\mathbf{R}^n} |f(x)| \log^+ |f(x)|\, dx + e\,|\operatorname{supp} f|, \end{aligned} \tag{9.108}$$

it follows from condition (9.106) that

$$f(x) \in L^1(\mathbf{R}^n).$$

Step (1): First, we consider the case where $f(x)$ is non-negative: We let

$$E_s := \left\{x : \left|\widetilde{f}_\varepsilon(x)\right| > s\right\},$$

$$E'_s := E_s \cap S.$$

Let $\beta_f(t)$ and $\beta^f(t)$ be the two functions defined in Section 9.5, and $s_0 := \beta_f(|S|)$. Then we have, by Theorem 3.29,

$$\int_S \left|\widetilde{f}_\varepsilon(x)\right| dx = \int_0^\infty |E'_s|\, ds \leq |S| s_0 + \int_{s_0}^\infty |E_s|\, ds. \tag{9.109}$$

However, by Lemma 9.9 it follows that

$$|S| s_0 = |S| \beta_f(|S|) = \int_0^{|S|} f^*(s)\, ds \leq \int_{\mathbf{R}^n} f(x)\, dx. \tag{9.110}$$

Moreover, by combining Theorem 3.29 and Lemma 9.10 we obtain that

$$|D_s| \leq \beta^f(s).$$

Hence, we have, by Lemma 9.4,

$$\int_{s_0}^\infty |E_s|\, ds \leq C_1 \int_0^\infty \frac{1}{s^2} \int_{\mathbf{R}^n} [f(x)]_s^2\, dx\, ds + C_2 \int_{s_0}^\infty \beta^f(s)\, ds, \tag{9.111}$$

where

$$[f(x)]_s = \begin{cases} f(x) & \text{if } f(x) \leq s, \\ s & \text{if } f(x) > s. \end{cases}$$

Hence we have, by Fubini's theorem (Theorem 3.10),

$$\int_0^\infty \frac{1}{s^2} \left(\int_{\mathbf{R}^n} [f(x)]_s^2\, dx \right) ds = \int_{\mathbf{R}^n} \left(\int_0^\infty \frac{1}{s^2} [f(x)]_s^2\, ds \right) dx \tag{9.112}$$

$$= \int_{\mathbf{R}^n} \left[\int_0^{f(x)} \frac{1}{s^2} \cdot s^2\, ds + \int_{f(x)}^\infty \frac{1}{s^2} f(x)^2\, ds \right] dx$$

$$= \int_{\mathbf{R}^n} \left[\int_0^{f(x)} ds + f(x)^2 \cdot \frac{1}{f(x)} \right] dx = 2 \int_{\mathbf{R}^n} f(x)\, dx.$$

Let $s_1 = \beta_f(t_1)$ be the number defined in Section 9.5. If we make the change of the variable $t = \beta^f(s)$ or $s = \beta_f(t)$, then it is easy to see that $\beta^f(s_0) = |S|$ and $\beta^f(s_1) = t_1$. Hence we have the formula

$$\int_{s_0}^\infty \beta^f(s)\, ds$$

$$= \int_{s_0}^{s_1} \beta^f(s)\, ds = \int_{|S|}^{t_1} t d\beta_f(t)$$

$$= [t\beta_f(t)]_{|S|}^{t_1} + \int_{t_1}^{|S|} \beta_f(t)\, dt = \left[\int_0^t f^*(s)\, ds \right]_{|S|}^{t_1} + \int_{t_1}^{|S|} \beta_f(t)\, dt$$

$$= \int_0^{t_1} f^*(s)\,ds - \int_0^{|S|} f^*(s)\,ds + \int_{t_1}^{|S|} \beta_f(t)\,dt$$
$$:= A.$$

Here it should be noticed that $t_1 \le |S|$, since $|S| > |\text{supp } f|$. Moreover, since $f^*(s) = \beta_f(s) = \beta_f(t_1)$ for $0 < s \le t_1$, we can estimate the last term on the above formula as follows:

$$\begin{aligned}
A &:= \int_0^{t_1} f^*(s)\,ds - \int_0^{|S|} f^*(s)\,ds + \int_{t_1}^{|S|} \beta_f(t)\,dt \\
&= \int_0^{|S|} \beta_f(t)\,dt - \int_0^{|S|} f^*(s)\,ds \\
&\le \int_0^{|S|} \beta_f(t)\,dt = \int_0^{|S|} \frac{1}{t}\int_0^t f^*(s)\,ds\,dt \\
&= \int_0^{|S|} f^*(s)\left(\int_s^{|S|} \frac{dt}{t}\right) ds = \int_0^{|S|} f^*(t)\log\frac{|S|}{t}\,dt \\
&= 2\int_0^{|S|} f^*(t)\log\left(\frac{|S|}{t}\right)^{1/2} dt.
\end{aligned}$$

However, by applying Lemma 9.17 with

$$a := f^*(t), \qquad b := \log\left(\frac{|S|}{t}\right)^{1/2},$$

we obtain that

$$\int_{s_0}^{\infty} \beta^f(s)\,ds = A \le 2\int_0^{|S|} f^*(t)\log\left(\frac{|S|}{t}\right)^{1/2} dt \tag{9.113}$$
$$\begin{aligned}
&\le 2\int_0^{|S|}\left[f^*(t)\log^+ f^*(t) + \exp\left(\log\left(\frac{|S|}{t}\right)^{1/2} - 1\right)\right] dt \\
&\le 2\int_0^{\infty} f^*(t)\,\log^+ f^*(t)\,dt + \frac{2C_2}{e}\int_0^{|S|}\left(\frac{|S|}{t}\right)^{1/2} dt \\
&= 2\int_{\mathbf{R}^n} f(x)\,\log^+ f(x)\,dx + \frac{4C_2}{e}|S|.
\end{aligned}$$

Therefore, by combining inequality (9.109) with inequalities (9.110) and (9.111) and formulas (9.112) and (9.113) and then using inequality (9.108), we obtain that

$$\int_S \left|\widetilde{f}_\varepsilon(x)\right| dx$$

$$
\begin{aligned}
&\le |S|s_0 + \int_{s_0}^{\infty} |E_s|\,ds \\
&\le \int_{\mathbf{R}^n} f(x)\,dx + C_1 \int_0^{\infty} \frac{1}{s^2}\left(\int_{\mathbf{R}^n} [f(x)]_s^2\,dx\right) ds + C_2 \int_{s_0}^{\infty} \beta^f(s)\,ds \\
&\le (1+2C_1)\int_{\mathbf{R}^n} f(x)\,dx + C_2 \int_{s_0}^{\infty} \beta^f(s)\,ds \\
&\le (1+2C_1)\left(\int_{\mathbf{R}^n} f(x)\,\log^+ f(x)\,dx + e\,|S|\right) \\
&\quad + 2C_2 \int_{\mathbf{R}^n} f(x)\log^+\, f(x)\,dx + \frac{4C_2}{e}|S| \\
&= (1+2C_1+2C_2)\int_{\mathbf{R}^n} f(x)\,\log^+ f(x)\,dx + \left((1+2C_1)e + \frac{4C_2}{e}\right)|S|.
\end{aligned}
$$

This proves the desired inequality (9.107) in the case where $f(x)$ is non-negative, with

$$C_3 := 1 + 2C_1 + 2C_2, \qquad C_4 := (1+2C_1)e + \frac{4C_2}{e}.$$

Step (2): In the general case, by decomposing a function $f(x)$ into its positive part $f^+(x)$ and its negative part $f^-(x)$ as

$$
\begin{aligned}
&f(x) = f^+(x) - f^-(x), \\
&f^+(x) := \max\{f(x), 0\}, \\
&f^-(x) := \max\{-f(x), 0\},
\end{aligned}
$$

we can easily prove that the desired inequality (9.107) holds true with

$$C_3 := 1 + 2C_1 + 2C_2, \qquad C_4 := 2(1+2C_1)e + \frac{8C_2}{e}.$$

The proof of Lemma 9.18 is now complete. □

We need the following elementary inequalities for the function $\log^+ x$:

Lemma 9.19. *For all $\alpha > 0$ and $\beta > 0$, we have the two inequalities*

$$\log^+(\alpha\beta) \le \log^+ \alpha + \log^+ \beta, \tag{9.114}$$

and

$$\frac{\alpha+\beta}{2}\,\log^+\left(\frac{\alpha+\beta}{2}\right) \le \frac{1}{2}\left(\alpha\log^+\alpha + \beta\log^+\beta\right). \tag{9.115}$$

Proof. (1) If $\alpha\beta \le 1$, it follows that $\log^+(\alpha\beta) = 0$. If $\alpha\beta > 1$, then it follows that

$$\log^+(\alpha\beta) = \log(\alpha\beta) = \log\alpha + \log\beta \le \log^+\alpha + \log^+\beta.$$

so that the first inequality (9.114) follows.

(2) Next, we remark that the function $f(x) := x \log^+ x$ is convex for $x > 1$ and $f(x) = 0$ for $0 < x \leq 1$. Hence we have the inequality

$$f\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{2}\left(f(\alpha)+f(\beta)\right).$$

This proves the second inequality (9.115).

The proof of Lemma 9.19 is complete. □

Now we can prove the following fundamental result:

Lemma 9.20. *Assume that the integral kernel $K(x)$ satisfies Assumption 9.3. If $f(x)$ is a measurable function with compact support in $\mathbf{R}^n$ that satisfies condition* (9.106)*, then the sequence $\widetilde{f_\varepsilon}$ converges in the space $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ as $\varepsilon \downarrow 0$.*

Proof. Let S be an arbitrary bounded measurable set such that

$$|\operatorname{supp} f| < |S|.$$

It suffices to show that

$$\lim_{\varepsilon,\varepsilon' \downarrow 0} \int_S \left|\widetilde{f_\varepsilon}(x) - \widetilde{f_{\varepsilon'}}(x)\right| dx = 0.$$

If k is a positive integer, we let

$$[f(x)]_k := \begin{cases} f(x) & \text{if } |f(x)| \leq k, \\ k\,\frac{f(x)}{|f(x)|} & \text{if } |f(x)| > k. \end{cases}$$

For any given $0 < \delta < 2$, we can choose the integer k so large that

$$\int_{\mathbf{R}^n} |f(x) - [f(x)]_k| \cdot \log^+ |f(x) - [f(x)]_k|\, dx < \frac{\delta}{4}, \tag{9.116}$$

and that

$$\int_{\mathbf{R}^n} |f(x) - [f(x)]_k|\, dx < \frac{\delta}{4}\,\frac{1}{\log^+(2/\delta)}. \tag{9.117}$$

Indeed, we have, by condition (9.106),

$$\begin{aligned} &\lim_{k\to\infty} \int_{\mathbf{R}^n} |f(x) - [f(x)]_k| \cdot \log^+ |f(x) - [f(x)]_k|\, dx \\ &= \lim_{k\to\infty} \int_{|f(x)|>k+1} (|f(x)| - k) \cdot \log^+ (|f(x)| - k)\, dx \\ &\leq \lim_{k\to\infty} \int_{|f(x)|>k+1} |f(x)| \cdot \log^+ |f(x)|\, dx = 0. \end{aligned}$$

Similarly, since $f \in L^1(\mathbf{R}^n)$, it follows that

$$\lim_{k\to\infty}\int_{\mathbf{R}^n} |f(x)-[f(x)]_k|\,dx = \lim_{k\to\infty}\int_{|f(x)|>k} |f(x)-k|\,dx \le \lim_{k\to\infty}\int_{|f(x)|>k} |f(x)|\,dx = 0.$$

By applying inequality (9.114) with

$$\alpha := |f(x)-[f(x)]_k|, \qquad \beta := \frac{2}{\delta},$$

we obtain from two inequalities (9.116) and (9.117) that

$$\begin{aligned}
&\int_{\mathbf{R}^n} \frac{|f(x)-[f(x)]_k|}{\delta/2} \log^+\left(\frac{|f(x)-[f(x)]_k|}{\delta/2}\right) dx \qquad (9.118)\\
&\le \int_{\mathbf{R}^n} \frac{|f(x)-[f(x)]_k|}{\delta/2} \log^+\left(|f(x)-[f(x)]_k|\right) dx\\
&\quad + \int_{\mathbf{R}^n} \frac{|f(x)-[f(x)]_k|}{\delta/2} \log^+\left(\frac{2}{\delta}\right) dx\\
&< 1.
\end{aligned}$$

We take a function $g(x)$ in $C_0^1(\mathbf{R}^n)$ that satisfies the conditions

$$|\operatorname{supp} g| <| S|,$$
$$\int_{\mathbf{R}^n} |g(x)-[f(x)]_k|^2\, dx < \left(\frac{\delta}{2}\right)^2.$$

Then, since we have the inequality

$$\alpha \log^+ \alpha \le \alpha^2 \quad \text{for all } \alpha > 0,$$

it follows that

$$\begin{aligned}
&\int_{\mathbf{R}^n} \frac{|g(x)-[f(x)]_k|}{\delta/2} \log^+\left(\frac{|g(x)-[f(x)]_k|}{\delta/2}\right) dx \qquad (9.119)\\
&\le \int_{\mathbf{R}^n} \left(\frac{|g(x)-[f(x)]_k|}{\delta/2}\right)^2 dx\\
&< 1.
\end{aligned}$$

Moreover, by letting

$$\alpha := \frac{|f(x)-[f(x)]_k|}{\delta/2}, \qquad \beta := \frac{|[f(x)]_k - g(x)|}{\delta/2},$$

we have the inequality

$$\frac{|f(x)-g(x)|}{\delta/2} \leq \frac{|f(x)-[f(x)]_k|}{\delta/2} + \frac{|[f(x)]_k - g(x)|}{\delta/2} = \alpha + \beta.$$

Hence it follows from an application of inequality (9.115) that

$$\begin{aligned} & 2\,\frac{|f(x)-g(x)|}{\delta} \log^+\left(\frac{|f(x)-g(x)|}{\delta}\right) \qquad (9.120) \\ & \leq (\alpha+\beta)\log^+\left(\frac{\alpha+\beta}{2}\right) \leq \alpha \log^+ \alpha + \beta \log^+ \beta \\ & = 2\,\frac{|f(x)-[f(x)]_k|}{\delta} \log^+\left(\frac{|f(x)-[f(x)]_k|}{\delta/2}\right) \\ & \quad + 2\,\frac{|[f(x)]_k - g(x)|}{\delta} \log^+\left(\frac{|[f(x)]_k - g(x)|}{\delta/2}\right). \end{aligned}$$

If we let

$$h(x) := f(x) - g(x),$$

then we obtain from inequalities (9.118), (9.119) and (9.120) that

$$\begin{aligned} & \int_{\mathbf{R}^n} \frac{|h(x)|}{\delta} \log^+\left(\frac{|h(x)|}{\delta}\right) dx \qquad (9.121) \\ & \leq \int_{\mathbf{R}^n} \frac{|f(x)-[f(x)]_k|}{\delta} \log^+\left(\frac{|f(x)-[f(x)]_k|}{\delta/2}\right) dx \\ & \quad + \int_{\mathbf{R}^n} \frac{|g(x)-[f(x)]_k|}{\delta} \log^+\left(\frac{|g(x)-[f(x)]_k|}{\delta/2}\right) dx \\ & < \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

On the other hand, since $|\operatorname{supp} h| < |S|$, by applying inequality (9.107) (Lemma 9.18) with

$$f(x) := \frac{h(x)}{\delta},$$

we obtain from inequality (9.121) that

$$\begin{aligned} \int_S \frac{|\widetilde{h}_\varepsilon(x)|}{\delta}\, dx & \leq C_3 \int_{\mathbf{R}^n} \frac{|h(x)|}{\delta} \log^+\left(\frac{|h(x)|}{\delta}\right) dx + C_4|S| \\ & \leq C_3 + C_4|S|. \end{aligned}$$

Hence we have, for all $\varepsilon > 0$ and $\varepsilon' > 0$,

$$\int_S \left|\widetilde{f}_\varepsilon(x) - \widetilde{f}_{\varepsilon'}(x)\right| dx$$

$$= \int_S \left| \left(\widetilde{g}_\varepsilon(x) - \widetilde{h}_\varepsilon(x) \right) - \left(\widetilde{g}_{\varepsilon'}(x) - \widetilde{h}_{\varepsilon'}(x) \right) \right| dx$$
$$\leq \int_S |\widetilde{g}_\varepsilon(x) - \widetilde{g}_{\varepsilon'}(x)| \, dx + \int_S \left| \widetilde{h}_\varepsilon(x) \right| dx + \int_S \left| \widetilde{h}_{\varepsilon'}(x) \right| dx$$
$$\leq \int_S |\widetilde{g}_\varepsilon(x) - \widetilde{g}_{\varepsilon'}(x)| \, dx + 2\delta(C_3 + C_4|S|).$$

However, the sequence $\widetilde{g}_\varepsilon(x)$ converges uniformly as $\varepsilon \downarrow 0$, as is shown in the proof of Theorem 9.5. Therefore, we obtain that

$$\limsup_{\varepsilon, \varepsilon' \downarrow 0} \int_S \left| \widetilde{f}_\varepsilon(x) - \widetilde{f}_{\varepsilon'}(x) \right| dx \leq 2\delta(C_3 + C_4|S|).$$

This proves the desired assertion

$$\lim_{\varepsilon, \varepsilon' \downarrow 0} \int_S \left| \widetilde{f}_\varepsilon(x) - \widetilde{f}_{\varepsilon'}(x) \right| dx = 0,$$

since $\delta > 0$ is arbitrary.

The proof of Lemma 9.20 is complete. □

Now we take a function $\phi(t)$ in $C^1(\mathbf{R})$ such that

$$\phi(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1/4, \\ 1 & \text{if } t > 3/4. \end{cases}$$

If the $R_j(x)$ are Riesz kernels

$$R_j(x) = -\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{x_j}{|x|^{n+1}} \quad \text{for } 1 \leq j \leq n,$$

then we define the integral kernels

$$K_j^{(1)}(x) := \lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \int_{|x-y|>\varepsilon, |y|>\delta} R_j(x-y) \, K(y) \, dy, \tag{9.122}$$

and

$$K_j^{(2)}(x) := \lim_{\varepsilon \downarrow 0} \int_{|x-y|>\varepsilon} R_j(x-y) \, K(y) \, \phi(|y|) \, dy. \tag{9.123}$$

The first purpose of this section is to prove the following lemma:

Lemma 9.21. *(i) The right hand side of formula* (9.122) *converges in* $L^1_{\mathrm{loc}}(\mathbf{R}^n \setminus \{0\})$

$$K_j^{(1)}(x) \in L^1_{\mathrm{loc}}(\mathbf{R}^n \setminus \{0\}) \quad \text{for } 1 \leq j \leq n,$$

while the right hand side of formula (9.123) *converges in* $L^1_{\mathrm{loc}}(\mathbf{R}^n)$

$$K_j^{(2)}(x) \in L^1_{\mathrm{loc}}(\mathbf{R}^n) \quad \text{for } 1 \leq j \leq n.$$

The integral kernels $K_j^{(1)}(x)$ *and* $K_j^{(2)}(x)$ *are both* odd *functions, and* $K_1(x)$ *is positively homogeneous of degree* $-n$.

(ii) If $K(\sigma) \in L^q(\Sigma_{n-1})$ *for some* $1 < q < \infty$, *then it follows that*

$$K_j^{(1)}(\sigma) \in L^q(\Sigma_{n-1}) \quad \text{for } 1 \leq j \leq n,$$

and further that we have, for some positive constant C_q,

$$\left(\int_{\Sigma_{n-1}} \left| K_j^{(1)}(\sigma) \right|^q d\sigma \right)^{1/q} \leq C_q \left(\int_{\Sigma_{n-1}} |K(\sigma)|^q \, d\sigma \right)^{1/q} \quad \text{for } 1 \leq j \leq n. \tag{9.124}$$

Proof. The proof of Lemma 9.21 is divided into four steps. In the following we shall denote by C a generic positive constant.

Step (1): First, we assume that $1/2 \leq |x| \leq 1$ and that $0 < \varepsilon < 1/4$. Then we have the formula

$$\begin{aligned} &\int_{|x-y|>\varepsilon} R_j(x-y)\, K(y)\, dy \\ &= \lim_{\delta \downarrow 0} \int_{|x-y|>\varepsilon, |y|>\delta} R_j(x-y)\, K(y)\, dy \\ &= \lim_{\delta \downarrow 0} \int_{\delta<|y|<1/4} R_j(x-y)\, K(y)\, dy \\ &\quad + \int_{|x-y|>\varepsilon, 1/4<|y|<2} R_j(x-y)\, K(y)\, dy + \int_{|y|>2} R_j(x-y)\, K(y)\, dy \\ &:= \mathrm{I}_j^{(1)}(x) + \mathrm{I}_j^{(2)}(x;\varepsilon) + \mathrm{I}_j^{3}(x) \quad \text{for } 1 \leq j \leq n. \end{aligned} \tag{9.125}$$

Here it should be noticed that

$$K_j^{(1)}(x) = \mathrm{I}_j^{(1)}(x) + \lim_{\varepsilon \downarrow 0} \mathrm{I}_j^{(2)}(x;\varepsilon) + \mathrm{I}_j^{(3)}(x) \quad \text{for } 1 \leq j \leq n. \tag{9.126}$$

We estimate the three terms $\mathrm{I}_j^{(1)}(x)$, $\mathrm{I}_j^{(2)}(x;\varepsilon)$ and $\mathrm{I}_j^{(3)}(x)$ on the right hand side of formula (9.125). In the following we shall denote by C a generic positive constant.

Step (1-a): We recall that the Riesz kernels $R_j(x)$ satisfy Assumption 9.3 with $\omega(t) := t$. By using inequality (9.40) with $y_k := 0$, we can find a constant $C > 0$ such that we have, for all $|x| \geq 1/2$ and $|y| < 1/4$,

$$|R_j(x-y) - R_j(x)| \leq \frac{C}{|x|^{n+1}}\, |y|.$$

Hence we have the inequality

$$\left| \mathrm{I}_j^{(1)}(x) \right| \tag{9.127}$$

$$
\begin{aligned}
&= \left| \lim_{\delta \downarrow 0} \int_{\delta<|y|<1/4} (R_j(x-y) - R_j(x)) K(y)\, dy \right| \\
&\le \frac{C}{|x|^{n+1}} \int_{0<|y|<1/4} |y|\, |K(y)|\; dy \\
&= \frac{C}{|x|^{n+1}} \int_{\Sigma_{n-1}} \left(\int_0^{1/4} |t\sigma|\, |K(t\sigma)|\, t^{n-1}\, dt \right) d\sigma \\
&= \frac{C}{4\,|x|^{n+1}} \int_{\Sigma_{n-1}} |K(\sigma)|\, d\sigma \quad \text{for all } 1/2 \le |x| \le 1 \text{ and } 1 \le j \le n.
\end{aligned}
$$

This proves that

$$\mathrm{I}_j^{(1)}(x) \in L^1(\{1/2 \le |x| \le 1\}) \quad \text{for } 1 \le j \le n.$$

Step (1-b): If $1/2 \le |x| \le 1$ and $|y| > 2$, then it follows that

$$|x| \le 1 < \frac{|y|}{2},$$

so that

$$|x-y| \ge |y| - |x| > \frac{|y|}{2}.$$

Hence we have the inequality

$$
\begin{aligned}
\left| \mathrm{I}_j^{(3)}(x) \right| &= \left| \int_{|y|>2} R_j(x-y)\, K(y)\, dy \right| \qquad (9.128) \\
&\le \int_{|y|>2} \frac{C}{|x-y|^n} |K(y)|\, dy \le 2^n\, C \int_{|y|>2} \frac{|K(y)|}{|y|^n}\, dy \\
&= 2^n\, C \int_{\Sigma_{n-1}} \left(\int_2^{\infty} \frac{1}{|t\sigma|^n}\, |K(t\sigma)|\, t^{n-1}\, dt \right) d\sigma \\
&= \frac{C}{n} \int_{\Sigma_{n-1}} |K(\sigma)|\, d\sigma \quad \text{for all } 1/2 \le |x| \le 1 \text{ and } 1 \le j \le n.
\end{aligned}
$$

This proves that

$$\mathrm{I}_j^{(3)}(x) \in L^1(\{1/2 \le |x| \le 1\}) \quad \text{for } 1 \le j \le n.$$

Step (1-c): In order to estimate the middle term $\mathrm{I}_j^{(2)}(x;\varepsilon)$, we introduce a function $f(x)$ by the formula

$$f(x) := \begin{cases} K(x) & \text{if } 1/4 < |x| < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the inequality

$$\begin{aligned}
&\int_{\mathbf{R}^n} |f(x)| \log^+ |f(x)| \, dx \\
&= \int_{1/4<|x|<2} |K(x)| \log^+ |K(x)| \, dx \\
&= \int_{\Sigma_{n-1}} \left(\int_{1/4}^{2} |K(t\sigma)| \log^+ |K(t\sigma)| \, t^{n-1} \, dt \right) d\sigma \\
&= \int_{\Sigma_{n-1}} \left(\int_{1/4}^{2} |K(\sigma)| \log^+ (t^{-n} |K(\sigma)|) \, \frac{dt}{t} \right) d\sigma \\
&\leq \int_{\Sigma_{n-1}} \left(\int_{1/4}^{2} |K(\sigma)| \left(\log^+ |K(\sigma) + \log^+ (t^{-n}) \right) \frac{dt}{t} \right) d\sigma \\
&= \int_{\Sigma_{n-1}} |K(\sigma)| \log^+ |K(\sigma)| \, d\sigma \cdot \int_{1/4}^{2} \frac{dt}{t} \\
&\quad + \int_{\Sigma_{n-1}} |K(\sigma)| \, d\sigma \cdot \int_{1/4}^{2} \log^+ (t^{-n}) \, \frac{dt}{t} \\
&\leq C \left(\int_{\Sigma_{n-1}} |K(\sigma)| \log^+ |K(\sigma)| \, d\sigma + \int_{\Sigma_{n-1}} |K(\sigma)| \, d\sigma \right) \\
&< \infty.
\end{aligned}$$

This proves that the function $f(x)$ satisfies condition (9.106). Hence, by applying Lemma 9.20 with

$$K(x) := R_j(x) = -\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{x_j}{|x|^{n+1}} \quad \text{for } 1 \leq j \leq n,$$

we find that the term

$$\begin{aligned}
\mathrm{I}_j^{(2)}(x;\varepsilon) &= \int_{|x-y|>\varepsilon, 1/4<|y|<2} R_j(x-y) \, K(y) \, dy \\
&= \int_{|x-y|>\varepsilon} R_j(x-y) \, f(y) \, dy \quad \text{for } 1 \leq j \leq n,
\end{aligned}$$

converges in the space $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ as $\varepsilon \downarrow 0$.

Therefore, we have proved that formula (9.126) converges in the space $L^1(\{1/2 \leq |x| \leq 1\})$.

Step (2): We assume that $1 \leq |x| \leq 2$. Then we have the formula

$$\int_{|x-y|>\varepsilon} R_j(x-y) \, K(y) \, dy = \int_{|x-2z|>\varepsilon/2} R_j \, (x-2z) \, K(2z) \, 2^n \, dz$$

$$= \frac{1}{2^n} \int_{|x/2-z|>\varepsilon/2} R_j\left(\frac{x}{2} - z\right) K(z)\, dz.$$

Hence it follows that formula (9.126) converges in the space $L^1(\{1 \leq |x| \leq 2\})$.

Similarly, we have, for $1/2^2 \leq |x| \leq 1/2$,

$$\begin{aligned}
&\int_{|x-y|>\varepsilon} R_j(x-y)\, K(y)\, dy \\
&= \int_{|x-z/2|>\varepsilon/2} R_j\left(x - \frac{z}{2}\right) K\left(\frac{z}{2}\right) \frac{1}{2^n}\, dz \\
&= 2^n \int_{|2x-z|>2\varepsilon} R_j\,(2x-z)\, K(z)\, dz \quad \text{for } 1 \leq j \leq n.
\end{aligned}$$

Hence it follows that formula (9.126) converges in the space $L^1(\{1/2^2 \leq |x| \leq 1/2\})$.

Repeating this process, we obtain that formula (9.126) (and hence formula (9.122)) converges in the space $L^1_{\mathrm{loc}}(\mathbf{R}^n \setminus \{0\})$:

$$K_j^{(1)}(x) = \mathrm{I}_j^{(1)}(x) + \lim_{\varepsilon \downarrow 0} \mathrm{I}_j^{(2)}(x;\varepsilon) + \mathrm{I}_j^{(3)}(x) \quad \text{for } 1 \leq j \leq n.$$

Moreover, it is easy to see that the function $K_j^{(1)}(x)$ is positively homogeneous of degree $-n$.

Step (3): Now we assume that $K(\sigma) \in L^q(\Sigma_{n-1})$ for $1 < q < \infty$. By applying Hölder's inequality (Theorem 3.14), we obtain from inequalities (9.127) and (9.128) that

$$\int_{1/2<|x|<1} \left|\mathrm{I}_j^{(1)}(x)\right|^q dx \leq C \int_{\Sigma_{n-1}} |K(\sigma)|^q\, d\sigma \quad \text{for } 1 \leq j \leq n, \tag{9.129a}$$

$$\int_{1/2<|x|<1} \left|\mathrm{I}_j^{(3)}(x)\right|^q dx \leq C \int_{\Sigma_{n-1}} |K(\sigma)|^q\, d\sigma \quad \text{for } 1 \leq j \leq n. \tag{9.129b}$$

We let

$$g(y) := \begin{cases} K(y) & \text{if } 1/4 < || < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by applying Theorem 9.15 with

$$K(x) := R_j(x) = -\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{x_j}{|x|^{n+1}} \quad \text{for } 1 \leq j \leq n,$$

we find that

$$
\begin{aligned}
&\int_{\mathbf{R}^n} \left| \mathrm{I}_j^{(2)}(x;\varepsilon) \right|^q \, dx \\
&\le \left(\frac{\pi C_q}{2} \right)^q \left(\int_{\Sigma_{n-1}} |R_j(\sigma)| \, d\sigma \right)^q \int_{1/4<|x|<2} |K(x)|^q \, dx \\
&\le C \int_{1/4<|x|<2} |K(x)|^q \, dx \\
&= C \int_{\Sigma_{n-1}} \left(\int_{1/4}^{2} |K(t\,\sigma)|^q \, t^{n-1} \, dt \right) d\sigma \\
&\le C \, \frac{4^{n(q-1)}}{n(q-1)} \int_{\Sigma_{n-1}} |K(\sigma)|^q \, d\sigma.
\end{aligned}
$$

By combining this inequality with inequalities (9.129), we obtain that

$$
\int_{1/2<|x|<1} |K_j^{(1)}(x)|^q \, dx \le C \int_{\Sigma_{n-1}} |K(\sigma)|^q \, d\sigma \quad \text{for } 1 \le j \le n. \tag{9.130}
$$

Therefore, the desired inequality (9.124) follows from the inequality (9.130) and the homogeneity of $K_j^{(1)}(x)$. Indeed, it suffices to note that

$$
\begin{aligned}
\int_{1/2<|x|<1} \left| K_j^{(1)}(x) \right|^q \, dx &= \int_{\Sigma_{n-1}} \left(\int_{1/2}^{1} \left| K_j^{(1)}(t\,\sigma) \right|^q t^{n-1} \, dt \right) d\sigma \\
&= \frac{1}{n(q-1)} \left(2^{n(q-1)} - 1 \right) \int_{\Sigma_{n-1}} \left| K_j^{(1)}(\sigma) \right|^q \, d\sigma.
\end{aligned}
$$

Step (4): Finally, we consider the convergence of formula (9.123). Let N be an arbitrary positive integer. If $|x| < N$ and $0 < \varepsilon < 1$, it follows that

$$
\begin{aligned}
&\int_{|x-y|>\varepsilon} R_j(x-y) \, K(y) \, \phi(|y|) \, dy \\
&= \int_{|x-y|>\varepsilon, |y|<N+1} R_j(x-y) \, K(y) \, \phi(|y|) \, dy \\
&\quad + \int_{|y|>N+1} R_j(x-y) \, K(y) \, dy \\
&:= \mathrm{I}_j^{(4)}(x;\varepsilon) + \mathrm{I}_j^{(5)}(x).
\end{aligned}
$$

Here it should be noticed that

$$
K_j^{(2)}(x) = \lim_{\varepsilon \downarrow 0} \mathrm{I}_j^{(4)}(x;\varepsilon) + \mathrm{I}_j^{(5)}(x;\varepsilon) \quad \text{for } 1 \le j \le n. \tag{9.131}
$$

We study the convergence of the two terms $\mathrm{I}_j^{(4)}(x;\varepsilon)$ and $\mathrm{I}_j^{(5)}(x)$ on the right hand side of the above formula in the space $L^1_{\mathrm{loc}}(\mathbf{R}^n)$.

Step (4-a): Since the function $h(y)$, defined by the formula

$$h(y) = \begin{cases} K(y)\phi(|y|) & \text{if } |y| < N+1, \\ 0 & \text{if } |y| > N+1 \end{cases}$$
$$= \begin{cases} K(y)\phi(|y|) & \text{if } 1/4 < |y| < N+1, \\ 0 & \text{otherwise}, \end{cases}$$

satisfies condition (9.102) (see **Step 1**), we can apply Lemma 9.20 to obtain that the term

$$\mathrm{I}_j^{(4)}(x;\varepsilon) = \int_{|x-y|>\varepsilon,\, 1/4<|y|<N+1} R_j(x-y)\, K(y)\, \phi(|y|)\, dy$$
$$= \int_{|x-y|>\varepsilon} R_j(x-y)\, h(y)\, dy \quad \text{for } 1 \leq j \leq n,$$

converges in the space $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ as $\varepsilon \downarrow 0$.

Step (4-b): On the other hand, we have the inequality

$$\begin{aligned}
\left|\mathrm{I}_j^{(5)}(x)\right| &= \int_{|y|>N+1} R_j(x-y)\, K(y)\, dy \\
&\leq C \int_{|y|>N+1} \frac{1}{|x-y|^n} \, |K(y)| \, dy \\
&\leq C \int_{|y|>N+1} \frac{1}{(|y|-N)^n} \, |K(y)| \, dy \\
&\leq C \int_{|y|>N+1} \frac{(N+1)^n}{|y|^n} \, |K(y)| \, dy \\
&= C(N+1)^n \left(\int_{N+1}^{\infty} \frac{1}{|t\sigma|^n} \int_{\Sigma_{n-1}} |K(t\,\sigma)| \, t^{n-1} \, dt \right) d\sigma \\
&= \frac{C}{n} \int_{\Sigma_{n-1}} |K(\sigma)| \, d\sigma.
\end{aligned}$$

This proves that

$$\mathrm{I}_j^{(5)}(x) \in L^1_{\mathrm{loc}}(\mathbf{R}^n) \quad \text{for } 1 \leq j \leq n.$$

Therefore, we find that formula (9.131) (and hence formula (9.123)) converges in the space $L^1_{\mathrm{loc}}(\mathbf{R}^n)$:

$$K_j^{(2)}(x) = \lim_{\varepsilon \downarrow 0} \mathrm{I}_j^{(4)}(x;\varepsilon) + \mathrm{I}_j^{(5)}(x) \quad \text{for } 1 \leq j \leq n.$$

The proof of Lemma 9.21 is now complete. □

The second purpose of this section is to prove the following lemma:

Lemma 9.22. *We can find positive constants C, C_1 and a non-negative, positively homogeneous function $G_j(x)$ of degree 0 such that*

$$|K_j^{(1)}(x) - K_j^{(2)}(x)| \leq \frac{C}{|x|^{n+1}} \int_{\Sigma_{n-1}} |K(\sigma)|\, d\sigma \quad \text{for all } |x| \geq 1 \tag{9.132}$$

and

$$\int_{\Sigma_{n-1}} G_j(\sigma)\, d\sigma < \infty,$$

and further that

$$|K_j^{(2)}(x)| \leq G_j(x) \quad \text{for all } |x| \leq 1. \tag{9.133}$$

Moreover, if $K(\sigma) \in L^q(\Sigma_{n-1})$ for $1 < q < \infty$, then we have, for some constant $C_q > 0$,

$$\left(\int_{\Sigma_{n-1}} G_j(\sigma)^q\, d\sigma \right)^{1/q} \leq C_q \left(\int_{\Sigma_{n-1}} |K(\sigma)|^q\, d\sigma \right)^{1/q} \quad \text{for } 1 \leq j \leq n. \tag{9.134}$$

This inequality implies that formula (9.123)

$$K_j^{(2)}(x) = \lim_{\varepsilon \downarrow 0} \int_{|x-y|>\varepsilon} R_j(x-y)\, K(y)\, \phi(|y|)\, dy, \quad 1 \leq j \leq n,$$

holds true in the strong topology of $L^q(\mathbf{R}^n)$.

Proof. By Lemma 9.21, it follows that

$$K_j^{(1)}(x) \in L^1_{\text{loc}}(\mathbf{R}^n \setminus \{0\}) \quad \text{for } 1 \leq j \leq n,$$
$$K_j^{(2)}(x) \in L^1_{\text{loc}}(\mathbf{R}^n) \quad \text{for } 1 \leq j \leq n,$$

and further that we have, in the space $L^1_{\text{loc}}(\mathbf{R}^n \setminus \{0\})$,

$$K_j^{(1)}(x) - K_j^{(2)}(x) = \int_{|x-y|>1/4} R_j(x-y)\, K(y)\, (1 - \phi(|y|))\, dy \quad \text{for } 1 \leq j \leq n. \tag{9.135}$$

In the following we shall denote by C a generic positive constant.

Step (1): We may assume that $0 < \varepsilon < 1/16$. If $|x| \geq 1$ and $|y| < 3/4$, we have, by inequality (9.40) with $y_k := 0$,

$$|R_j(x-y) - R_j(x)| \leq \frac{C}{|x|^{n+1}}\, |y| \quad \text{for } |x-y| > \tfrac{1}{4}.$$

Hence we obtain from formula (9.135) that

$$\begin{aligned}
\left|K_j^{(1)}(x) - K_j^{(2)}(x)\right| &= \left|\int_{\mathbf{R}^n} R_j(x-y)\,K(y)\,(1-\phi(|y|))\;dy\right| \\
&= \left|\int_{\mathbf{R}^n} (R_j(x-y) - R_j(x))\,K(y)\,(1-\phi(|y|))\;dy\right| \\
&\le \frac{C}{|x|^{n+1}} \int_{|y|<3/4} |y|\,|K(y)|\;dy \\
&= \frac{C}{|x|^{n+1}} \int_{\Sigma_{n-1}} \left(\int_0^{3/4} |t\sigma|\,|K(t\sigma)|\,t^{n-1}\,dt\right) d\sigma \\
&\le \frac{3C}{4|x|^{n+1}} \int_{\Sigma_{n-1}} |K(\sigma)|\,d\sigma \quad \text{for all } |x| \ge 1.
\end{aligned}$$

This proves the desired inequality (9.132).

Step (2): If $|x| \le 1/8$ and $|y| > 1/4$, it follows that

$$|y| > \frac{1}{4} \ge 2|x|,$$

so that

$$|x-y| \ge |y| - |x| > \frac{|y|}{2}.$$

Hence we have the inequality

$$\begin{aligned}
\left|K_j^{(2)}(x)\right| &= \left|\int_{|y|>1/4} R_j(x-y)\,K(y)\,\phi(|y|)\,dy\right| \\
&\le C \int_{|y|>1/4} \frac{|K(y)|}{|x-y|^n}\,dy \le 2^n\,C \int_{|y|>1/4} \frac{|K(y)|}{|y|^n}\,dy \\
&= 2^n\,C \int_{\Sigma_{n-1}} \left(\int_{1/4}^{\infty} \frac{1}{|t\sigma|^n}\,|K(t\sigma)|\,t^{n-1}\,dt\right) d\sigma \\
&\le \frac{8^n\,C}{n} \int_{\Sigma_{n-1}} |K(\sigma)|\,d\sigma \quad \text{for all } |x| \le 1/8.
\end{aligned} \tag{9.136}$$

Step (3): If $1/8 \le |x| \le 1$, we consider the characteristic function $\chi(t)$ of the interval $[0, 1/16]$:

$$\chi(t) = \begin{cases} 1 & \text{if } 0 \le t \le \frac{1}{16}, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the formula

$$\int_{|x-y|>\varepsilon} R_j(x-y)\,K(y)\,\phi(|y|)\,dy \tag{9.137}$$

$$
\begin{aligned}
&= \phi(|x|) \int_{|x-y|>\varepsilon} R_j(x-y)\, K(y)\, dy \\
&\quad + \int_{|x-y|>\varepsilon} R_j(x-y)\, K(y)\, (\phi(|y|) - \phi(|x|))\, dy \\
&= \phi(|x|) \int_{|x-y|>\varepsilon} R_j(x-y)\, K(y)\, dy \\
&\quad + \int_{|x-y|>\varepsilon} (R_j(x-y) - \chi(|y|) R_j(x))\, K(y)\, (\phi(|y|) - \phi(|x|))\, dy \\
&\quad + R_j(x) \int_{|x-y|>\varepsilon} \chi(|y|)\, K(y)\, (\phi(|y|) - \phi(|x|))\, dy \quad \text{for } 1 \le j \le n.
\end{aligned}
$$

However, since $\phi(t) = 0$ for $t \le 1/4$ it follows from the cancellation property (9.4) that the last integral of formula (9.137) tends to zero as $\varepsilon \downarrow 0$:

$$
\begin{aligned}
&\int_{|x-y|>1/16>\varepsilon,\, |y|<1/16} \chi(|y|)\, K(y)\, (\phi(|y|) - \phi(|x|))\, dy \\
&= -\int_{0<|y|<1/16} K(y) \phi(|x|)\, dy = -\int_{0<|y|<1/16} K(y)\, dy \cdot \phi(|x|) \\
&= 0.
\end{aligned}
$$

Hence, by letting $\varepsilon \downarrow 0$ in formula (9.137) we obtain that

$$
\begin{aligned}
&K_j^{(2)}(x) \\
&= \phi(|x|)\, K_j^{(1)}(x) \\
&\quad + \int_{\mathbf{R}^n} (R_j(x-y) - \chi(|y|) R_j(x))\, K(y)\, (\phi(|y|) - \phi(|x|))\; dy \\
&\quad \text{for } 1/8 \le |x| \le 1 \text{ and } 1 \le j \le n.
\end{aligned}
\tag{9.138}
$$

Here it should be noticed that the function $\phi(|x|)$ is uniformly Lipschitz continuous.

Step (3-a): If $1/8 \le |x| \le 1$ and if $|y| \le 1/16$, then it follows that

$$
2\,|x-y| \le 2\,|x| + 2\,|y| \le 2\,|x| + \frac{1}{16} \le 3\,|x|.
$$

Hence we have the inequality

$$
\begin{aligned}
&|R_j(x-y) - \chi(|y|)\, R_j(x)| = |R_j(x-y) - R_j(x)| \le \frac{C}{|x-y|^n} + \frac{C}{|x|^n} \\
&\le \left(1 + \left(\frac{3}{2}\right)^n\right) \frac{C}{|x-y|^n}.
\end{aligned}
$$

On the other hand, we have, by inequality (9.40) with $y_k := 0$,

$$\begin{aligned}|R_j(x-y) - \chi(|y|)R_j(x)| &= |R_j(x-y) - R_j(x)| \\ &\leq \frac{C}{|x-y|^{n+1}}\,|y|.\end{aligned}$$

Therefore, by combining these two inequalities we obtain that

$$\begin{aligned}&|R_j(x-y) - \chi(|y|)\,R_j(x)| \qquad (9.139)\\ &\leq C\left(\frac{|y|}{|x-y|^{n+1}}\right)^{1/2} \cdot \left(\frac{1}{|x-y|^n}\right)^{1/2}\\ &= \frac{C}{|x-y|^{n+1/2}}\,|y|^{1/2} \quad \text{for all } 1/8 \leq |x| \leq 1 \text{ and } |y| \leq 1/16.\end{aligned}$$

Step (3-b): If $1/8 \leq |x| \leq 1$ and if $|y| > 1/16$, then it follows that

$$\begin{aligned}|x-y| &\leq |x| + |y| \leq 1 + |y| \leq 16\,|y| + |y| \\ &= 17\,|y|.\end{aligned}$$

Hence we have the inequality

$$\begin{aligned}&|R_j(x-y) - \chi(|y|)\,R_j(x)| = |R_j(x-y)| \qquad (9.140)\\ &\leq \frac{C}{|x-y|^n} \leq \left(\frac{17\,|y|}{|x-y|}\right)^{1/2} \frac{C}{|x-y|^n}\\ &= \frac{C}{|x-y|^{n+1/2}}\,|y|^{1/2}\\ &\qquad \text{for all } 1/8 \leq |x| \leq 1 \text{ and } |y| > 1/16 \text{ and } 1 \leq j \leq n.\end{aligned}$$

By using two inequalities (9.139) and (9.140), we have, for $1/8 \leq |x| \leq 1$,

$$|R_j(x-y) - \chi(|y|)\,R_j(x)| \leq \frac{C}{|x-y|^{n+1/2}}\,|y|^{1/2} \quad \text{on } \mathbf{R}^n.$$

Therefore, by carrying this inequality into formula (9.138) we obtain that

$$\begin{aligned}&\left|K_j^{(2)}(x)\right| \qquad (9.141)\\ &\leq |K_j^{(1)}(x)| + C\int_{\mathbf{R}^n} |y|^{1/2}\,|K(y)|\,|x-y|^{-n+1/2}\,dy\\ &\leq 8^n|x|^n|K_1(x)| + 8^{n-1}C|x|^{n-1}\int_{\mathbf{R}^n} |y|^{1/2}\,|K(y)|\,|x-y|^{-n+1/2}\,dy\\ &\qquad \text{for all } 1/8 \leq |x| \leq 1 \text{ and } 1 \leq j \leq n.\end{aligned}$$

Step (4): By combining two inequalities (9.136) and (9.141), we can choose a positive constant C_0 such that the function $G_j(x)$, defined by the formula

$$G_j(x) := C_0\left(\int_{\Sigma_{n-1}} |K(\sigma)|\, d\sigma + |x|^n \left|K_j^{(1)}(x)\right| + |x|^{n-1} \int_{\mathbf{R}^n} |y|^{1/2}\, |K(y)|\, |x-y|^{-n+1/2}\, dy\right) \quad \text{for } 1 \leq j \leq n, \tag{9.142}$$

satisfies the inequality

$$\left|K_j^{(2)}(x)\right| \leq G_j(x) \quad \text{for all } |x| \leq 1 \text{ and } 1 \leq j \leq n.$$

Here it should be noticed that the functions $G_j(x)$ are positively homogeneous of degree 0.

Next, we show that

$$G_j(\sigma) \in L^1(\Sigma_{n-1}) \quad \text{for } 1 \leq j \leq n.$$

To do this, since we have the formula

$$\begin{aligned}\int_{1/2<|x|<3/2} G_j(x)\, dx &= \int_{\Sigma_{n-1}} \int_{1/2}^{3/2} G_j(t\,\sigma)\, t^{n-1}\, dt\, d\sigma \\ &= \frac{1}{n}\left(\left(\frac{3}{2}\right)^n - \left(\frac{1}{2}\right)^n\right) \int_{\Sigma_{n-1}} G_j(\sigma)\, d\sigma \\ &\quad \text{for } 1 \leq j \leq n,\end{aligned}$$

it suffices to prove that

$$\int_{1/2<|x|<3/2} G_j(x)\, dx < \infty \quad \text{for } 1 \leq j \leq n. \tag{9.143}$$

However, we remark that

$$\begin{aligned}\int_{1/2<|x|<3/2} |x|^n \left|K_j^{(1)}(x)\right| dx &\leq \left(\frac{3}{2}\right)^n \int_{1/2<|x|<3/2} \left|K_j^{(1)}(x)\right| dx \\ &< \infty \quad \text{for } 1 \leq j \leq n,\end{aligned}$$

since the functions $K_j^{(1)}(x)$ are locally integrable on $\mathbf{R}^n \setminus \{0\}$.

Therefore, we have only to show that

$$\int_{1/2<|x|<3/2} \left(\int_{\mathbf{R}^n} |y|^{1/2} |K(y)|\, |x-y|^{-n+1/2}\, dy\right) dx \leq C \int_{\Sigma_{n-1}} |K(\sigma)|\, d\sigma. \tag{9.144}$$

To do this, we remark the following three inequalities:

$$\bullet \int_{1/2<|x|<3/2} \left(\int_{1/4<|y|<3} |y|^{1/2} |K(y)| \, |x-y|^{-n+1/2} \, dy \right) dx \tag{9.145}$$

$$\leq \int_{1/4<|y|<3} |y|^{1/2} |K(y)| \left(\int_{|x-y|<9/2} |x-y|^{-n+1/2} \, dx \right) dy$$

$$= 3\sqrt{2}\omega_n \int_{1/4<|y|<3} |y|^{1/2} |K(y)| \, dy$$

$$= C \int_{\Sigma_{n-1}} |K(\sigma)| \, d\sigma,$$

where

$$\omega_n = |\Sigma_{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

is the surface area of the unit sphere Σ_{n-1} in $\mathbf{R}^n$.

$$\bullet \int_{1/2<|x|<3/2} \left(\int_{0<|y|<1/4} |y|^{1/2} |K(y)| \, |x-y|^{-n+1/2} \, dy \right) dx \tag{9.146}$$

$$\leq 4^{n-1/2} \int_{1/2<|x|<3/2} \left(\int_{|y|<1/4} |y|^{1/2} |K(y)| \, dy \right) dx$$

$$= C \int_{\Sigma_{n-1}} |K(\sigma)| \, d\sigma.$$

$$\bullet \int_{1/2<|x|<3/2} \left(\int_{|y|>3} |y|^{1/2} |K(y)| \, |x-y|^{-n+1/2} \, dy \right) dx \tag{9.147}$$

$$\leq \left(\frac{2}{3} \right)^{n-1/2} \int_{1/2<|x|<3/2} \left(\int_{|y|>3} |y|^{1-n} |K(y)| \, dy \right) dx$$

$$= C \int_{\Sigma_{n-1}} |K(\sigma)| \, d\sigma.$$

The desired inequality (9.144) follows by combining inequalities (9.145), (9.146) and (9.147).

Furthermore, we obtain from inequality (9.144) that the last integral of inequality (9.141) is finite for almost all x:

$$\int_{\mathbf{R}^n} |y|^{1/2} \, |K(y)| \, |x-y|^{-n+1/2} \, dy < \infty \quad \text{for almost all } x \in \mathbf{R}^n.$$

Step (5): Finally, we consider the case where $K(\sigma) \in L^q(\Sigma_{n-1})$ for $1 < q < \infty$.

The first two terms in formula (9.142) can be estimated as follows:

(1) We have, by Hölder's inequality (Theorem 3.14),

$$\int_{\Sigma_{n-1}} |K(\sigma)|\, d\sigma \leq \omega_n^{1-1/q} \left(\int_{\Sigma_{n-1}} |K(\sigma)|^q \, d\sigma \right)^{1/q}.$$

(2) We have already proved inequality (9.124):

$$\left(\int_{\Sigma_{n-1}} |K_1(\sigma)|^q \, d\sigma \right)^{1/q} \leq C_q \left(\int_{\Sigma_{n-1}} |K(\sigma)|^q \, d\sigma \right)^{1/q}.$$

Therefore, in order to prove inequality (9.134) it remains to prove the following inequality for the last term in formula (9.142):

$$\int_{1/2<|x|<3/2} \left(\int_{\mathbf{R}^n} |y|^{1/2} |K(y)| \, |x-y|^{-n+1/2} \, dy \right)^q dx \qquad (9.148)$$
$$\leq C \int_{\Sigma_{n-1}} |K(\sigma)|^q \, d\sigma.$$

Step (5-a): First, by Hölder's inequality (Theorem 3.14) it follows that

$$\left(\int_{|y|<1/4} |y|^{1/2} |K(y)| \, |x-y|^{-n+1/2} \, dy \right)^q$$
$$= \left(\int_{|y|<1/4} \left(|y|^{1/2} |K(y)| \right)^{1-1/q} \left(|y|^{1/2} |K(y)| \right)^{1/q} |x-y|^{-n+1/2} \, dy \right)^q$$
$$\leq \left(\int_{|y|<1/4} |y|^{1/2} |K(y)| \, dy \right)^{q-1}$$
$$\times \left(\int_{|y|<1/4} |y|^{1/2} |K(y)| \, |x-y|^{(-n+1/2)q} \, dy \right).$$

Hence, by integrating the both sides of this inequality over the annular region

$$\left\{ x \in \mathbf{R}^n : \frac{1}{2} < |x| < \frac{3}{2} \right\},$$

we obtain from Fubini's theorem (Theorem 3.10) and Hölder's inequality (Theorem 3.14) that

$$\int_{1/2<|x|<3/2} \left(\int_{|y|<1/4} |y|^{1/2} |K(y)| \, |x-y|^{-n+1/2} \, dy \right)^q dx \qquad (9.149)$$

$$\leq \left(\int_{|y|<1/4} |y|^{1/2} |K(y)|\, dy \right)^{q-1}$$
$$\times \int_{1/2<|x|<3/2} \left(\int_{|y|<1/4} |y|^{1/2} |K(y)|\, |x-y|^{(-n+1/2)q}\, dy \right) dx$$
$$= \left(\int_{|y|<1/4} |y|^{1/2} |K(y)|\, dy \right)^{q-1}$$
$$\times \int_{|y|<1/4} |y|^{1/2} |K(y)| \left(\int_{1/2<|x|<3/2} |x-y|^{(-n+1/2)q}\, dx \right) dy$$
$$\leq \left(\int_{|y|<1/4} |y|^{1/2}\, |K(y)|\, dy \right)^{q} \left(\int_{1/4<|z|<7/4} |z|^{(-n+1/2)q}\, dz \right)$$
$$\leq C \left(\int_{\Sigma_{n-1}} |K(\sigma)|\, d\sigma \right)^{q}$$
$$\leq C\, \omega_n^{q-1} \int_{\Sigma_{n-1}} |K(\sigma)|^q\, d\sigma.$$

Step (5-b): Secondly, since we have, for $1/2 < |x| < 3/2$ and $1/4 < |y| < 3$,

$$\left(\int_{1/4<|y|<3} |y|^{1/2} |K(y)|\, |x-y|^{-n+1/2}\, dy \right)^{q}$$
$$\leq \int_{1/4<|y|<3} \left(|y|^{1/2} |K(y)| \right)^{q} |x-y|^{-n+1/2}\, dy$$
$$\times \left(\int_{1/4<|y|<3} |x-y|^{-n+1/2}\, dy \right)^{q-1}$$
$$\leq \left(\int_{1/4<|y|<3} \left(|y|^{1/2} |K(y)| \right)^{q} |x-y|^{-n+1/2}\, dy \right)$$
$$\times \left(\int_{|z|<9/2} |z|^{-n+1/2}\, dz \right)^{q-1},$$

it follows from an application of Fubini's theorem (Theorem 3.10) that we have, for $1/4 < |y| < 3$,

$$\int_{1/2<|x|<3/2} \left(\int_{1/4<|y|<3} |y|^{1/2} |K(y)|\, |x-y|^{-n+1/2}\, dy \right)^{q} dx \quad (9.150)$$

$$\leq \int_{1/2<|x|<3/2} \left(\int_{1/4<|y|<3} \left(|y|^{1/2} |K(y)| \right)^q |x-y|^{-n+1/2} \, dy \right) dx$$
$$\times \left(\int_{|z|<9/2} |z|^{-n+1/2} \, dz \right)^{q-1}$$
$$\leq \left(\int_{1/4<|y|<3} \left(|y|^{1/2} |K(y)| \right)^q \, dy \right) \left(\int_{|y|<9/2} |y|^{-n+1/2} \, dy \right)^q$$
$$\leq C \int_{\Sigma_{n-1}} |K(\sigma)|^q \, d\sigma.$$

Step (5-c): Thirdly, by Hölder's inequality (Theorem 3.14) it follows that

$$\int_{1/2<|x|<3/2} \left(\int_{|y|>3} |y|^{1/2} |K(y)| \, |x-y|^{-n+1/2} \, dy \right)^q dx \tag{9.151}$$
$$\leq C \int_{1/2<|x|<3/2} \left(\int_{|y|>3} |y|^{1-n} |K(y)| \, dy \right)^q dx$$
$$\leq C \left(\int_{\Sigma_{n-1}} |K(\sigma)| \, d\sigma \right)^q$$
$$\leq C \, \omega_n^{q-1} \int_{\Sigma_{n-1}} |K(\sigma)|^q \, d\sigma.$$

Therefore, by combining inequalities (9.149), (9.150) and (9.151) we can obtain inequality (9.148) and hence the desired inequality (9.134).

The proof of Lemma 9.22 is now complete. □

The third purpose of this section is to prove the following lemma:

Lemma 9.23. *If $f(x)$ is a non-negative function in $L^p(\mathbf{R})$ for $1 < p < \infty$, then we have the inequalities*

$$\left(\int_{-\infty}^{\infty} \left(\sup_{\varepsilon>0} \frac{1}{\varepsilon^n} \int_0^{\varepsilon} f(t+s) s^{n-1} \, ds \right)^p dt \right)^{1/p} \tag{9.152}$$
$$\leq \frac{p}{p-1} \left(\int_{-\infty}^{\infty} f(t)^p \, dt \right)^{1/p},$$

and

$$\left(\int_{-\infty}^{\infty} \left(\sup_{\varepsilon>0} \varepsilon \int_{\varepsilon}^{\infty} f(t+s) s^{-2} \, ds \right)^p dt \right)^{1/p} \tag{9.153}$$

$$\leq \frac{2p}{p-1} \left(\int_{-\infty}^{\infty} f(t)^p \, dt \right)^{1/p}.$$

Proof. (1) If $\varepsilon > 0$, we define two functions $F_\varepsilon(t)$ and $G(t)$ by formulas (9.66):

$$F_\varepsilon(t) := \frac{1}{\varepsilon} \int_0^\varepsilon f(t+s) \, ds,$$
$$G(t) := \sup_{\varepsilon > 0} F_\varepsilon(t).$$

Then it follows from inequality (9.67) (Lemma 9.12) that

$$\int_{-\infty}^{\infty} G(t)^p \, dt \leq \left(\frac{p}{p-1} \right)^p \int_{-\infty}^{\infty} f(t)^p \, dt. \tag{9.154}$$

However, we have the inequality

$$\begin{aligned}
\sup_{\varepsilon>0} \left(\frac{1}{\varepsilon^n} \int_0^\varepsilon f(t+s) \, s^{n-1} \, ds \right) &= \sup_{\varepsilon>0} \left(\frac{1}{\varepsilon} \int_0^\varepsilon f(t+s) \left(\frac{s}{\varepsilon} \right)^{n-1} ds \right) \\
&\leq \sup_{\varepsilon>0} \left(\frac{1}{\varepsilon} \int_0^\varepsilon f(t+s) \, ds \right) = \sup_{\varepsilon>0} F_\varepsilon(t) \\
&= G(t).
\end{aligned}$$

Therefore, the desired inequality (9.152) follows from inequality (9.154).

(2) First, we have, by Hölder's inequality (Theorem 3.14),

$$\begin{aligned}
\varepsilon F_\varepsilon(t) &= \int_0^\varepsilon f(t+s) \, ds \leq \left(\int_0^\varepsilon ds \right)^{1-1/p} \left(\int_0^\varepsilon f(t+s)^p \, ds \right)^{1/p} \\
&\leq \varepsilon^{1-1/p} \, \|f\|_p \,,
\end{aligned}$$

and so

$$\lim_{\varepsilon \uparrow \infty} \frac{F_\varepsilon(t)}{\varepsilon} = 0.$$

Secondly, we have the formula

$$\frac{d}{ds} \left(s F_s(t) \right) = \frac{d}{ds} \left(\int_0^s f(t+\sigma) \, d\sigma \right) = f(t+s).$$

Hence, by integration by parts it follows that

$$\begin{aligned}
\varepsilon \int_\varepsilon^\infty f(t+s) s^{-2} \, ds &= \varepsilon \int_\varepsilon^\infty \frac{d}{ds} \left(s F_s(t) \right) s^{-2} \, ds \qquad (9.155) \\
&= \varepsilon \left[s^{-1} F_s(t) \right]_{s=\varepsilon}^{s=\infty} + 2\varepsilon \int_\varepsilon^\infty s^{-2} F_s(t) \, ds
\end{aligned}$$

$$
\begin{aligned}
&= -F_\varepsilon(t) + 2\varepsilon \int_\varepsilon^\infty s^{-2} F_s(t)\, ds \\
&\le 2\varepsilon \int_\varepsilon^\infty s^{-2}\, F_s(t)\, ds \le 2\varepsilon \int_\varepsilon^\infty s^{-2}\, ds \cdot G(t) \\
&= 2G(t).
\end{aligned}
$$

Therefore, the desired inequality (9.153) follows by combining inequalities (9.155) and (9.154).

The proof of Lemma 9.23 is complete. □

Now, by using Lemmas 9.21, 9.22 and 9.23 we can prove the existence of the singular integral (9.2) in the space $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ in the case of *even kernels*:

Theorem 9.24. *Assume that the integral kernel $K(x)$ satisfies Assumption 9.5. If $f(x) \in L^p(\mathbf{R}^n)$ for $1 < p < \infty$, we let*

$$\widetilde{f}_\varepsilon(x) := \int_{|x-y|>\varepsilon} K(x-y) f(y)\, dy.$$

Then we have the following three assertions (i), (ii) and (iii):

(i) There exists a positive constant C_p, independent of ε, such that

$$\left(\int_{\mathbf{R}^n} \sup_{\varepsilon>0} |\widetilde{f}_\varepsilon(x)|^p\, dx \right)^{1/p} \le C_p \left(\int_{\mathbf{R}^n} |f(x)|^p\, dx \right)^{1/p}. \tag{9.156}$$

(ii) The sequence $\widetilde{f}_\varepsilon$ converges almost everywhere in $\mathbf{R}^n$ and in the strong topology of $L^p(\mathbf{R}^n)$ as $\varepsilon \downarrow 0$. Namely, the singular integral

$$K * f(x) = \lim_{\varepsilon \downarrow 0} \int_{|x-y|>\varepsilon} K(x-y) f(y)\, dy$$

exists for almost all $x \in \mathbf{R}^n$ and in the strong topology of $L^p(\mathbf{R}^n)$.

(iii) The mapping $f \mapsto K*f$ is a bounded linear operator from $L^p(\mathbf{R}^n)$ into itself. More precisely, we have the inequality

$$\|K * f\|_p \le C_p \|f\|_p .$$

Proof. Let $K_j^{(1)}(x)$ and $K_j^{(2)}(x)$ be the functions defined by formulas (9.122) and (9.123), respectively. If we let

$$g_j(x) := -\left(R_j * f\right)(x) = -\lim_{\varepsilon \downarrow 0} \int_{|x-y|>\varepsilon} R_j(x-y)\, f(y)\, dy$$

for $1 \leq j \leq n$,

by applying Theorem 9.15 and Theorem 9.16 we obtain from formula (9.102) that the function

$$\begin{aligned} f(x) &= \sum_{j=1}^{n} (R_j * g_j)(x) = \lim_{\varepsilon \downarrow 0} \sum_{j=1}^{n} \int_{|x-y|>\varepsilon} R_j(x-y)\, g_j(y)\, dy \\ &= -\sum_{j=1}^{n} R_j * (R_j * f) \end{aligned}$$

exists for almost all $x \in \mathbf{R}^n$ and in the strong topology of $L^p(\mathbf{R}^n)$.

The proof of Theorem 9.24 is divided into three steps.

Step (1): Now we show that

$$\begin{aligned} &\int_{\mathbf{R}^n} K(x-y)\phi\left(\frac{|x-y|}{\varepsilon}\right) f(y)\, dy \qquad (9.157) \\ &= \frac{1}{\varepsilon^n} \sum_{j=1}^{n} \int_{\mathbf{R}^n} K_j^{(2)}\left(\frac{x-y}{\varepsilon}\right) g_j(y)\, dy \\ &= -\frac{1}{\varepsilon^n} \sum_{j=1}^{n} \int_{\mathbf{R}^n} K_j^{(2)}\left(\frac{x-y}{\varepsilon}\right) (R_j * f)(y)\, dy. \end{aligned}$$

Here $\phi(t) \in C^1(\mathbf{R})$ is the function used in the definition of the functions $K_j^{(2)}(x)$:

$$\phi(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1/4, \\ 1 & \text{if } t > 3/4. \end{cases}$$

Step (1-a): First, we assume that

$$g_j(x) \in C_0^1(\mathbf{R}^n) \quad \text{for } 1 \leq j \leq n,$$

and that

$$\operatorname{supp} g_j \subset B(0, N) := \{x \in \mathbf{R}^n : |x| \leq N\}$$

for some positive integer N. If $|y| \leq N+1$, then we find that the integral

$$\sum_{j=1}^{n} \int_{|y-z|>\delta} R_j(y-z)\, g_j(z)\, dz = \sum_{j=1}^{n} \int_{|y-z|>\delta} R_j(y-z)\, (g_j(z) - g_j(y))\, dz$$

converges to the function

$$f(y) = \sum_{j=1}^{n} (R_j * g_j)(y)$$

uniformly in $y \in B(0, N)$, as $\delta \downarrow 0$. Indeed, it suffices to note that the function $g(x)$ is uniformly Lipschitz continuous.

If $|y| > N + 1$, then it follows that

$$\begin{aligned} 2(N+1)\,|y-z| &= (N+1)|y-z| + (N+1)|y-z| \\ &\geq |y-z| + (N+1) \geq |y| + (N - |z|) + 1 \\ &\geq |y| + 1 \quad \text{for all } |z| \leq N. \end{aligned}$$

Hence we have, for $0 < \delta < 1$,

$$\begin{aligned} \left| \int_{|y-z|>\delta} R_j(y-z)\, g_j(z)\, dz \right| &\leq \int_{|z|<N} \frac{C|g_j(z)|}{|y-z|^n}\, dz \\ &\leq 2^n\, (N+1)^n\, C\, \|g_j\|_1\, \frac{1}{(|y|+1)^n}. \end{aligned}$$

Moreover, since we have the inequality

$$\begin{aligned} 2\,(|x-z|+1) &\geq (|z|-|x|) + (|z|-|x|) \geq 2|x| - 2|x| + |z| \\ &= |z| \quad \text{for all } |z| > 2|x|, \end{aligned}$$

it follows that

$$\begin{aligned} &\int_{\mathbf{R}^n} |K(x-y)|\, \phi\left(\frac{|x-y|}{\varepsilon}\right) \frac{dy}{(|y|+1)^n} \\ &= \int_{\mathbf{R}^n} |K(z)| \phi\left(\frac{|z|}{\varepsilon}\right) \frac{dz}{(|x-z|+1)^n} \\ &\leq \int_{|z|>\varepsilon/4} |K(z)|\, \phi\left(\frac{|z|}{\varepsilon}\right) \frac{dz}{(|x-z|+1)^n} \\ &\leq \int_{\varepsilon/4<|z|<2|x|} |K(y)|\; dz + 2^n \int_{|z|>2|x|} \frac{|K(z)|}{|z|^n}\, dz \\ &= \left(\log\left(\frac{8|x|}{\varepsilon}\right) + \frac{1}{n\,|x|^n}\right) \int_{\Sigma_{n-1}} |K(\sigma)|\, d\sigma \\ &< \infty \quad \text{for all } x \neq 0. \end{aligned}$$

Hence it follows from an application of Lebesgue's dominated convergence theorem (Theorem 3.8) that

$$\begin{aligned} &\lim_{\delta \downarrow 0} \int_{\mathbf{R}^n} K(x-y) \phi\Big(\frac{|x-y|}{\varepsilon}\Big) \\ &\quad \times \left(\sum_{j=1}^{n} \int_{|y-z|>\delta} R_j(y-z)\, g_j(z)\, dz\right) dy \end{aligned} \tag{9.158}$$

$$= \int_{\mathbf{R}^n} K(x-y)\,\phi\left(\frac{|x-y|}{\varepsilon}\right)\left(\sum_{j=1}^{n}(R_j * g_j)(y)\right)dy$$
$$= \int_{\mathbf{R}^n} K(x-y)\,\phi\left(\frac{|x-y|}{\varepsilon}\right) f(y)\,dy.$$

Step (1-b): If $|y-z| > \delta$, then it follows that

$$2\,|y-z| = |y-z| + |y-z| \geq |y-z| + \delta \geq \delta\,(|y-z|+1)\,.$$

Hence we have the inequality

$$\left|K(x-y)\phi\left(\frac{|x-y|}{\varepsilon}\right) R_j(y-z)\,g_j(z)\right| \tag{9.159}$$
$$\leq \frac{2^n\,C}{\delta^n}\,|K(x-y)|\,\phi\left(\frac{|x-y|}{\varepsilon}\right)\frac{|g_j(z)|}{(|y-z|+1)^n}$$
$$\leq \frac{2^n\,C(N+1)^n}{\delta^n}\,|K(x-y)|\,\phi\left(\frac{|x-y|}{\varepsilon}\right)\frac{|g_j(z)|}{(|y|+1)^n} \quad \text{for all } |z| \leq N.$$

However, we find that this term is integrable in $(y,z) \in \mathbf{R}^n \times \mathbf{R}^n$ for $x \neq 0$. Indeed, it suffices to note that

$$\int_{\mathbf{R}^n} |K(x-y)|\phi\left(\frac{|x-y|}{\varepsilon}\right)\frac{dy}{(|y|+1)^n}$$
$$= \int_{\mathbf{R}^n} |K(z)|\phi\left(\frac{|z|}{\varepsilon}\right)\frac{dz}{(|x-z|+1)^n}$$
$$\leq \int_{\varepsilon/4<|z|<2|x|} |K(y)|\,dz + 2^n \int_{|z|>2|x|} \frac{|K(z)|}{|z|^n}\,dz$$
$$= \left(\log\left(\frac{8|x|}{\varepsilon}\right) + \frac{1}{n\,|x|^n}\right)\int_{\Sigma_{n-1}} |K(\sigma)|\,d\sigma$$
$$< \infty \quad \text{for all } x \neq 0.$$

Therefore, by using Fubini's theorem (Theorem 3.10) and then formula (9.123) we obtain that the left hand side of formula (9.158) is equal to the following:

$$\int_{\mathbf{R}^n} K(x-y)\,\phi\left(\frac{|x-y|}{\varepsilon}\right) f(y)\,dy$$
$$= \lim_{\delta\downarrow 0}\int_{\mathbf{R}^n} K(x-y)\phi\left(\frac{|x-y|}{\varepsilon}\right)\left(\sum_{j=1}^{n}\int_{|y-z|>\delta} R_j(y-z)\,g_j(z)\,dz\right)dy$$
$$= \lim_{\delta\downarrow 0}\sum_{j=1}^{n}\int_{\mathbf{R}^n}\left(\int_{|y-z|>\delta} K(x-y)\phi\left(\frac{|x-y|}{\varepsilon}\right) R_j(y-z)\,dy\right) g_j(z)\,dz$$

$$
\begin{aligned}
&= \lim_{\delta \downarrow 0} \int_{\mathbf{R}^n} \sum_{j=1}^n \left(\int_{|x-\varepsilon w - z|>\delta} K(\varepsilon w)\phi(|w|)R_j(x-\varepsilon w - z)\varepsilon^n \, dw \right) g_j(z)\, dz \\
&= \frac{1}{\varepsilon^n} \lim_{\delta \downarrow 0} \int_{\mathbf{R}^n} \sum_{j=1}^n \left(\int_{|(x-z)/\varepsilon - w|>\delta/\varepsilon} K(w)\phi(|w|)R_j\left(\frac{x-z}{\varepsilon} - w\right) dw \right) \\
&\qquad \times g_j(z)\, dz \\
&= \frac{1}{\varepsilon^n} \sum_{j=1}^n \int_{\mathbf{R}^n} K_j^{(2)}\left(\frac{x-z}{\varepsilon}\right) g_j(z)\, dz.
\end{aligned}
$$

Namely, the desired formula (9.157) holds true for all $f = \sum_{j=1}^n R_j * g_j$ with $g_j \in C_0^1(\mathbf{R}^n)$:

$$
\begin{aligned}
&\int_{\mathbf{R}^n} K(x-y)\, \phi\left(\frac{|x-y|}{\varepsilon}\right) f(y)\, dy \\
&= -\frac{1}{\varepsilon^n} \sum_{j=1}^n \int_{\mathbf{R}^n} K_j^{(2)}\left(\frac{x-y}{\varepsilon}\right) (R_j * f)\,(y)\, dy.
\end{aligned}
$$

Step (1-c): Now we consider the general case where $f \in L^p(\mathbf{R}^n)$ for $1 < p < \infty$.

We let

$$
g_j(x) := -\,(R_j * f)\,(x) \in L^p(\mathbf{R}^n), \quad 1 \le j \le n.
$$

Then we choose a sequence $\{g_j^k\}$ in the space $C_0^1(\mathbf{R}^n)$ such that (see Theorem 3.27)

$$
\begin{aligned}
&g_j^k - g_j \longrightarrow 0 \quad \text{in } L^p(\mathbf{R}^n) \text{ as } k \to \infty, \\
&\sum_{k=1}^\infty \left\| g_j^{k+1} - g_j^k \right\|_p < \frac{1}{2^k},
\end{aligned}
$$

and define a sequence

$$
f_k := \sum_{j=1}^n R_j * g_j^k.
$$

By applying Theorem 9.5 or Theorem 9.15 to our situation, we obtain that

$$
\begin{aligned}
\|f_k - f\|_p &\le \left\| \sum_{j=1}^n R_j * \left(g_j^k - g_j\right) \right\|_p \\
&\le C \sum_{j=1}^n \left\| g_j^k - g_j) \right\|_p \longrightarrow 0 \quad \text{as } k \to \infty,
\end{aligned}
$$

and

$$\begin{aligned}
&\sum_{k=1}^{\infty} \|f_{k+1} - f_k\|_p \\
&= \sum_{k=1}^{\infty} \left\| \sum_{j=1}^{n} R_j * \left(g_j^{k+1} - g_j^k\right) \right\|_p \\
&\le \sum_{k=1}^{\infty} \sum_{j=1}^{n} \left\| R_j * \left(g_j^{k+1} - g_j^k\right) \right\|_p \\
&\le C \sum_{k=1}^{\infty} \sum_{j=1}^{n} \left\| g_j^{k+1} - g_j^k \right\|_p = C \sum_{j=1}^{n} \left(\sum_{k=1}^{\infty} \left\| g_j^{k+1} - g_j^k \right\|_p \right) \\
&< \frac{nC}{2^k}.
\end{aligned}$$

Moreover, we find that the series

$$\overline{g}_j(x) := \left|g_j^1(x)\right| + \sum_{k=1}^{\infty} \left|g_j^{k+1}(x) - g_j^k(x)\right|, \quad 1 \le j \le n,$$

and that the series

$$\overline{f}(x) := |f_1(x)| + \sum_{k=1}^{\infty} |f_{k+1}(x) - f_k(x)|$$

converge for almost all $x \in \mathbf{R}^n$, and satisfy the following three conditions (i), (ii) and (iii):

(i) $\overline{g}_j(x) \in L^p(\mathbf{R}^n)$ and $\overline{f}(x) \in L^p(\mathbf{R}^n)$.
(ii) $\left|g_j^k(x)\right| \le \overline{g}_j(x)$ for $1 \le j \le n$ and $|f_k(x)| \le \overline{f}(x)$ almost everywhere in $\mathbf{R}^n$.
(iii) $g_j^k(x) \to g_j(x)$ and $f_k(x) \to f(x)$ in $\mathbf{R}$ for almost all $x \in \mathbf{R}^n$, as $k \to \infty$.

Moreover, since $\overline{f}(x) \in L^p(\mathbf{R}^n)$, it follows from an application of Lemma 9.1 that the integral

$$\int_{\mathbf{R}^n} |K(x-y)|\, \phi\left(\frac{|x-y|}{\varepsilon}\right) \overline{f}(y)\, dy \le \int_{|x-y|>\varepsilon/4} |K(x-y)|\, \overline{f}(y)\, dy$$

exists for almost all $x \in \mathbf{R}^n$, as $\varepsilon \downarrow 0$.

On the other hand, we have, by inequality (9.132),

$$\sum_{j=1}^{n} \int_{\mathbf{R}^n} \left| K_j^{(2)}\left(\frac{x-y}{\varepsilon}\right) \right| \overline{g}_j(y)\, dy$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \int_{|x-y|<\varepsilon} \left| K_j^{(2)} \left(\frac{x-y}{\varepsilon} \right) \right| \overline{g}_j(y)\, dy \\
&\quad + \sum_{j=1}^{n} \int_{|x-y|>\varepsilon} \left| K_j^{(2)} \left(\frac{x-y}{\varepsilon} \right) \right| \overline{g}_j(y)\, dy \\
&\leq \sum_{j=1}^{n} \int_{|x-y|<\varepsilon} \left| K_j^{(2)} \left(\frac{x-y}{\varepsilon} \right) \right| \overline{g}_j(y)\, dy \\
&\quad + \sum_{j=1}^{n} \int_{|x-y|>\varepsilon} \left| K_j^{(2)} \left(\frac{x-y}{\varepsilon} \right) - K_j^{(1)} \left(\frac{x-y}{\varepsilon} \right) \right| \overline{g}_j(y)\, dy \\
&\quad + \sum_{j=1}^{n} \int_{|x-y|>\varepsilon} \left| K_j^{(1)} \left(\frac{x-y}{\varepsilon} \right) \right| \overline{g}_j(y)\, dy \\
&\leq \sum_{j=1}^{n} \int_{|x-y|<\varepsilon} \left| K_j^{(2)} (\frac{x-y}{\varepsilon}) \right| \overline{g}_j(y)\, dy \\
&\quad + \sum_{j=1}^{n} \int_{|x-y|>\varepsilon} \left| K_j^{(1)} (\frac{x-y}{\varepsilon}) \right| \overline{g}_j(y)\, dy \\
&\quad + C\varepsilon^{n+1} \sum_{j=1}^{n} \int_{|x-y|>\varepsilon} \frac{\overline{g}_j(y)}{|x-y|^{n+1}}\, dy.
\end{aligned}
$$

By applying Theorem 3.23 with $p := 1$ and $q = r := p$, we obtain that the first term on the right hand side of this inequality is finite for almost every $x \in \mathbf{R}^n$. Indeed, we have, by inequality (9.133),

$$
\begin{aligned}
&\int_{|x|<\varepsilon} \left| K_j^{(2)} \left(\frac{x}{\varepsilon} \right) \right| dx \\
&= \varepsilon^n \int_{|y|<1} |K_j^{(2)}(y)|\, dy \\
&\leq \varepsilon^n \int_{|y|<1} G_j(y)\, dy = \varepsilon^n \int_{\Sigma_{n-1}} \left(\int_0^1 G_j(t\,\sigma)\, t^{n-1}\, dt \right) d\sigma \\
&= \frac{\varepsilon^n}{n} \int_{\Sigma_{n-1}} G_j(\sigma)\, d\sigma \quad \text{for } 1 \leq j \leq n.
\end{aligned}
$$

Similarly, the third term is also finite for almost every $x \in \mathbf{R}^n$ if we apply Theorem 3.23 with $p := 1$ and $q = r := p$. Indeed, it suffices to note the inequality

$$
\varepsilon^{n+1} \int_{|x|>\varepsilon} \frac{1}{|x|^{n+1}}\, dx
$$

$$= \varepsilon^n \int_{|y|>1} \frac{1}{|y|^{n+1}}\, dy$$
$$= \varepsilon^n \int_{\Sigma_{n-1}} \left(\int_1^\infty \frac{1}{t^{n+1}}\, t^{n-1}\, dt \right) d\sigma = \varepsilon^n\, \omega_n.$$

The second term is finite for almost every $x \in \mathbf{R}^n$ if we apply Lemma 9.1.

Therefore, by applying formula (9.157) to the sequences f_k and g_j^k and by letting $k \to \infty$ we obtain from Lebesgue's dominated convergence theorem (Theorem 3.8) that the desired formula (9.157) holds true for all $f \in L^p(\mathbf{R}^n)$ with $1 < p < \infty$:

$$\int_{\mathbf{R}^n} K(x-y)\, \phi\left(\frac{|x-y|}{\varepsilon} \right) f(y)\, dy$$
$$= \lim_{k\to\infty} \int_{\mathbf{R}^n} K(x-y)\, \phi\left(\frac{|x-y|}{\varepsilon} \right) f_k(y)\, dy$$
$$= \lim_{k\to\infty} \frac{1}{\varepsilon^n} \sum_{j=1}^n \int_{\mathbf{R}^n} K_j^{(2)} \left(\frac{x-y}{\varepsilon} \right) g_j^k(y)\, dy$$
$$= \frac{1}{\varepsilon^n} \sum_{j=1}^n \int_{\mathbf{R}^n} K_j^{(2)} \left(\frac{x-y}{\varepsilon} \right) g_j(y)\, dy$$
$$= -\frac{1}{\varepsilon^n} \sum_{j=1}^n \int_{\mathbf{R}^n} K_j^{(2)} \left(\frac{x-y}{\varepsilon} \right) (R_j * f)(y)\, dy.$$

Step (2): By formula (9.157), it follows that

$$\widetilde{f}_\varepsilon(x) = \int_{|x-y|>\varepsilon} K(x-y)\, f(y)\, dy$$
$$= \int_{|x-y|>\varepsilon} K(x-y)\phi\left(\frac{|x-y|}{\varepsilon} \right) f(y)\, dy$$
$$= \int_{\mathbf{R}^n} K(x-y)\phi\left(\frac{|x-y|}{\varepsilon} \right) f(y)\, dy$$
$$- \int_{|x-y|<\varepsilon} K(x-y)\phi\left(\frac{|x-y|}{\varepsilon} \right) f(y)\, dy$$
$$= \frac{1}{\varepsilon^n} \sum_{j=1}^\infty \int_{\mathbf{R}^n} K_j^{(2)} \left(\frac{x-y}{\varepsilon} \right) g_j(y)\, dy$$
$$- \int_{|x-y|<\varepsilon} K(x-y)\phi\left(\frac{|x-y|}{\varepsilon} \right) f(y)\, dy$$

$$
\begin{aligned}
&= \frac{1}{\varepsilon^n} \sum_{j=1}^{\infty} \int_{|x-y|>\varepsilon} K_j^{(1)} \left(\frac{x-y}{\varepsilon} \right) g_j(y)\, dy \\
&\quad + \frac{1}{\varepsilon^n} \sum_{j=1}^{\infty} \int_{|x-y|>\varepsilon} \left(K_j^{(2)} \left(\frac{x-y}{\varepsilon} \right) - K_j^{(1)} \left(\frac{x-y}{\varepsilon} \right) \right) g_j(y)\, dy \\
&\quad + \frac{1}{\varepsilon^n} \sum_{j=1}^{\infty} \int_{|x-y|<\varepsilon} K_j^{(2)} \left(\frac{x-y}{\varepsilon} \right) g_j(y)\, dy \\
&\quad - \int_{|x-y|<\varepsilon} K(x-y) \phi \left(\frac{|x-y|}{\varepsilon} \right) f(y)\, dy \\
&:= \mathrm{I}_1(x;\varepsilon) + \mathrm{I}_2(x;\varepsilon) + \mathrm{I}_3(x;\varepsilon) + \mathrm{I}_4(x;\varepsilon).
\end{aligned}
$$

We estimate the four terms $\mathrm{I}_1(x;\varepsilon)$ through $\mathrm{I}_4(x;\varepsilon)$ on the right hand side of the above formula.

Step (2-a): First, since $K_j^{(1)}(x)$ are positively homogeneous of degree $-n$, we have the formula

$$
\begin{aligned}
\mathrm{I}_1(x;\varepsilon) &= \sum_{j=1}^{\infty} \int_{|x-y|>\varepsilon} K_j^{(1)}(x-y)\, g_j(y)\, dy \\
&= -\sum_{j=1}^{\infty} \int_{|x-y|>\varepsilon} K_j^{(1)}(x-y)\, (R_j * f)(y)\, dy.
\end{aligned}
$$

Hence, by applying inequality (9.82) with

$$
K(x) := K_j^{(1)}(x), \quad f(x) := R_j * f(x) \quad \text{for } 1 \leq j \leq n,
$$

we obtain that

$$
\begin{aligned}
&\int_{\mathbf{R}^n} \sup_{\varepsilon>0} |\mathrm{I}_1(x;\varepsilon)|^p \, dx \qquad (9.160) \\
&\leq \left(\frac{\pi C_p}{2} \right)^p \left(\sum_{j=1}^{n} \int_{\Sigma_{n-1}} \left| K_j^{(1)}(\sigma) \right| d\sigma \right)^p \int_{\mathbf{R}^n} |g_j(x)|^p \, dx \\
&= \left(\frac{\pi C_p}{2} \right)^p \left(\int_{\Sigma_{n-1}} \left| K_j^{(1)}(\sigma) \right| d\sigma \right)^p \sum_{j=1}^{n} \|R_j * f\|_p^p \\
&\leq C \left(\sum_{j=1}^{n} \int_{\Sigma_{n-1}} \left| K_j^{(1)}(\sigma) \right| d\sigma \right)^p \|f\|_p^p .
\end{aligned}
$$

Here we recall that the functions $K_j^{(1)}(x)$ are integrable on Σ_{n-1}, since they are positively homogeneous of degree $-n$ and belong to the space $L^1_{\mathrm{loc}}(\mathbf{R}^n \setminus \{0\})$.

Step (2-b): Secondly, by inequality (9.132) with

$$x := \frac{x-y}{\varepsilon},$$

it follows that

$$\begin{aligned}
&|\mathrm{I}_2(x;\varepsilon)| \\
&= \left| \frac{1}{\varepsilon^n} \int_{|x-y|>\varepsilon} \left(K_j^{(2)}\left(\frac{x-y}{\varepsilon}\right) - K_j^{(1)}\left(\frac{x-y}{\varepsilon}\right)\right) g_j(y)\,dy \right| \\
&\le \frac{1}{\varepsilon^n} \int_{|x-y|>\varepsilon} \left| \left(K_j^{(2)}\left(\frac{x-y}{\varepsilon}\right) - K_j^{(1)}\left(\frac{x-y}{\varepsilon}\right)\right) \right| |g_j(y)|\,dy \\
&\le C\,\frac{1}{\varepsilon^n} \int_{|x-y|>\varepsilon} \frac{\varepsilon^{n+1}}{|x-y|^{n+1}}\,|g_j(y)|\,dy \cdot \int_{\Sigma_{n-1}} |K(\sigma)|\,d\sigma \\
&\le C \left(\varepsilon \int_{|x-y|>\varepsilon} \frac{|g_j(y)|}{|x-y|^{n+1}}\,dy \right) \int_{\Sigma_{n-1}} |K(\sigma)|\,d\sigma.
\end{aligned}$$

However, we have, by Hölder's inequality (Theorem 3.14),

$$\begin{aligned}
&\varepsilon \int_{|x-y|>\varepsilon} \frac{|g_j(y)|}{|x-y|^{n+1}}\,dy = \varepsilon \int_{|z|>\varepsilon} \frac{|g_j(x-z)|}{|z|^{n+1}}\,dz \\
&= \varepsilon \int_{\Sigma_{n-1}} \left(\int_\varepsilon^\infty \frac{|g_j(x-s\sigma)|}{|s\sigma|^{n+1}}\,s^{n-1}\,ds \right) dz \\
&= \varepsilon \int_{\Sigma_{n-1}} \left(\int_\varepsilon^\infty s^{-2}|g_j(x-s\sigma)|\,ds \right) d\sigma \\
&\le \left(\int_{\Sigma_{n-1}} d\sigma \right)^{1-1/p} \left(\int_{\Sigma_{n-1}} \left(\varepsilon \int_\varepsilon^\infty s^{-2}|g_j(x-s\sigma)|\,ds \right)^p d\sigma \right)^{1/p} \\
&= \omega_n^{1-1/p} \left(\int_{\Sigma_{n-1}} \left(\varepsilon \int_\varepsilon^\infty s^{-2}|g_j(x-s\sigma)|\,ds \right)^p d\sigma \right)^{1/p}.
\end{aligned}$$

Hence it follows from an application of Fubini's theorem (Theorem 3.10) that

$$\begin{aligned}
&\int_{\mathbf{R}^n} \left(\sup_{\varepsilon>0} \varepsilon \int_{|x-y|>\varepsilon} \frac{|g_j(y)|}{|x-y|^{n+1}}\,dy \right)^p dx \\
&\le \omega_n^{p-1} \int_{\Sigma_{n-1}} \left(\int_{\mathbf{R}^n} \left(\sup_{\varepsilon>0} \varepsilon \int_\varepsilon^\infty s^{-2}|g_j(x-s\sigma)|\,ds \right)^p dx \right) d\sigma.
\end{aligned}$$

For each $\sigma \in \Sigma_{n-1}$, we make the change of the variables

$$x := y - t\,\sigma, \quad y \in \mathbf{R}^n,\ t \in \mathbf{R},$$

with

$$\langle y, \sigma\rangle = y_1\sigma_1 + \ldots + y_n\sigma_n = 0,$$

as in the proof of Theorem 9.5. Then, by applying inequality (9.153) with

$$f(t) := |g_j(y - t\,\sigma)|,$$

we obtain that

$$\begin{aligned}
&\int_{\mathbf{R}^n} \left(\sup_{\varepsilon>0} \varepsilon \int_\varepsilon^\infty s^{-2}|g_j(x - s\,\sigma)|\,ds\right)^p dx \\
&= \int_{\mathbf{R}^{n-1}} \left(\int_{-\infty}^\infty \left(\sup_{\varepsilon>0} \varepsilon \int_\varepsilon^\infty s^{-2}|g_j(y-(t+s)\sigma)|\,ds\right)^p dt\right) dy \\
&\le \left(\frac{2p}{p-1}\right)^p \int_{\mathbf{R}^{n-1}} \left(\int_{-\infty}^\infty |g_j(y - t\sigma)|^p\,dt\right) dy \\
&= \left(\frac{2p}{p-1}\right)^p \int_{\mathbf{R}^n} |g_j(x)|^p\,dx \quad \text{for every } \sigma \in \Sigma_{n-1}.
\end{aligned}$$

This proves that

$$\begin{aligned}
&\int_{\mathbf{R}^n} \left(\sup_{\varepsilon>0} \varepsilon \int_{|x-y|>\varepsilon} \frac{|g_j(y)|}{|x-y|^{n+1}}\,dy\right)^p dx \\
&\le \omega_n^{p-1} \left(\int_{\Sigma_{n-1}} \left(\frac{2p}{p-1}\right)^p \int_{\mathbf{R}^n} |g_j(x)|^p\,dx\right) d\sigma \\
&= \left(\frac{2p}{p-1}\right)^p \omega_n^{p-1} \left(\int_{\mathbf{R}^n} |g_j(x)|^p\,dx\right) \int_{\Sigma_{n-1}} d\sigma \\
&= \left(\frac{2p\omega_n}{p-1}\right)^p \int_{\mathbf{R}^n} |g_j(x)|^p\,dx.
\end{aligned}$$

Summing up, we have proved that

$$\begin{aligned}
&\int_{\mathbf{R}^n} \sup_{\varepsilon>0} |\mathrm{I}_2(x;\varepsilon)|^p\,dx \qquad (9.161)\\
&\le \left(\frac{2pC\omega_n}{p-1}\right)^p \left(\sum_{j=1}^n \int_{\Sigma_{n-1}} |K(\sigma)|\,d\sigma\right)^p \int_{\mathbf{R}^n} |g_j(x)|^p\,dx \\
&= \left(\frac{2pC\omega_n}{p-1}\right)^p \left(\int_{\Sigma_{n-1}} |K(\sigma)|\,d\sigma\right)^p \sum_{j=1}^n \|R_j * f\|_p^p \\
&\le C \left(\int_{\Sigma_{n-1}} |K(\sigma)|\,d\sigma\right)^p \|f\|_p^p.
\end{aligned}$$

Step (2-c): Thirdly, we have, by inequality (9.133) and Hölder's inequality (Theorem 3.14),

$$
\begin{aligned}
&|\mathrm{I}_3(x;\varepsilon)| \\
&= \left| \frac{1}{\varepsilon^n} \sum_{j=1}^{n} \int_{|x-y|<\varepsilon} K_2\left(\frac{x-y}{\varepsilon}\right) g_j(y)\, dy \right| \\
&\le \frac{1}{\varepsilon^n} \sum_{j=1}^{n} \int_{|(x-y)/\varepsilon|<1} \left| K_j^{(2)}\left(\frac{x-y}{\varepsilon}\right) \right| g_j(y)\, dy \\
&\le \frac{1}{\varepsilon^n} \sum_{j=1}^{n} \int_{|x-y|<\varepsilon} G_j(x-y)\, |g_j(y)|\, dy \\
&= \frac{1}{\varepsilon^n} \sum_{j=1}^{n} \int_{|z|<\varepsilon} G_j(z) |g_j(x-z)|\, dz \\
&= \int_{\Sigma_{n-1}} G_j(\sigma) \left(\frac{1}{\varepsilon^n} \int_0^{\varepsilon} |g_j(x-s\sigma)| s^{n-1}\, ds \right) d\sigma \\
&\le \sum_{j=1}^{n} \left(\int_{\Sigma_{n-1}} G_j(\sigma)\, d\sigma \right)^{1-1/p} \\
&\quad \times \left(\int_{\Sigma_{n-1}} G_j(\sigma) \left(\frac{1}{\varepsilon^n} \int_0^{\varepsilon} |g_j(x-s\sigma)| s^{n-1}\, ds \right)^p d\sigma \right)^{1/p}.
\end{aligned}
$$

However, by applying inequality (9.152) with

$$
f(t) := |g_j(y - t\,\sigma)|,
$$

we obtain that

$$
\begin{aligned}
&\int_{\mathbf{R}^n} \left(\sup_{\varepsilon>0} \frac{1}{\varepsilon^n} \int_0^{\varepsilon} |g_j(x-s\sigma)| s^{n-1}\, ds \right)^p dx \\
&= \int_{\mathbf{R}^{n-1}} \left(\int_{-\infty}^{\infty} \left(\sup_{\varepsilon>0} \frac{1}{\varepsilon^n} \int_0^{\varepsilon} |g_j(y-(t+s)\sigma)|\, s^{n-1}\, ds \right)^p dt \right) dy \\
&\le \int_{\mathbf{R}^{n-1}} \left(\frac{p}{p-1} \right)^p \left(\int_{-\infty}^{\infty} |g_j(y-t\sigma)|^p\, dt \right) dy \\
&= \left(\frac{p}{p-1} \right)^p \int_{\mathbf{R}^{n-1}} \int_{-\infty}^{\infty} |g_j(y-t\sigma)|^p\, dt\, dy \\
&= \left(\frac{p}{p-1} \right)^p \int_{\mathbf{R}^n} |g_j(x)|^p\, dx.
\end{aligned}
$$

Therefore, by Fubini's theorem (Theorem 3.10) it follows that

$$\begin{aligned}
&\int_{\mathbf{R}^n} \sup_{\varepsilon>0} |\mathrm{I}_3(x;\varepsilon)|^p\,dx \qquad (9.162)\\
&\leq \sum_{j=1}^{n} \left(\int_{\Sigma_{n-1}} G_j(\sigma)\,d\sigma\right)^{p-1}\\
&\quad \times \int_{\mathbf{R}^n} \left(\int_{\Sigma_{n-1}} G_j(\sigma) \sup_{\varepsilon>0} \left(\frac{1}{\varepsilon^n}\int_0^{\varepsilon} |g_j(x-s\,\sigma)|\,s^{n-1}\,ds\right)^p d\sigma\right) dx\\
&= \sum_{j=1}^{n} \left(\int_{\Sigma_{n-1}} G_j(\sigma)\,d\sigma\right)^{p-1}\\
&\quad \times \int_{\Sigma_{n-1}} G_j(\sigma) \left(\int_{\mathbf{R}^n} \sup_{\varepsilon>0} \left(\frac{1}{\varepsilon^n}\int_0^{\varepsilon} |g_j(x-s\,\sigma)|\,s^{n-1}\,ds\right)^p dx\right) d\sigma\\
&\leq \sum_{j=1}^{n} \left(\int_{\Sigma_{n-1}} G_j(\sigma)\,d\sigma\right)^{p-1} \left(\frac{p}{p-1}\right)^p \int_{\mathbf{R}^n} |g_j(x)|^p\,dx\\
&\qquad \times \int_{\Sigma_{n-1}} G_j(\sigma)\,d\sigma\\
&= \sum_{j=1}^{n} \left(\frac{p}{p-1}\right)^p \left(\int_{\Sigma_{n-1}} G_j(\sigma)\,d\sigma\right)^p \int_{\mathbf{R}^n} |g_j(x)|^p\,dx\\
&\leq \sum_{j=1}^{n} \left(\frac{p}{p-1}\right)^p \left(\int_{\Sigma_{n-1}} G_j(\sigma)\,d\sigma\right)^p \|R_j * f\|_p^p\\
&\leq C\sum_{j=1}^{n} \left(\int_{\Sigma_{n-1}} G_j(\sigma)\,d\sigma\right)^p \|f\|_p^p .
\end{aligned}$$

Here we recall that the functions $G_j(\sigma)$ are integrable on Σ_{n-1}.

Step (2-d): Finally, we have, by Hölder's inequality (Theorem 3.14),

$$\begin{aligned}
&|\mathrm{I}_4(x;\varepsilon)|\\
&= \left|\int_{\varepsilon/4<|x-y|<\varepsilon} |x-y|^{-n} K\left(\frac{x-y}{|x-y|}\right) \phi\left(\frac{|x-y|}{\varepsilon}\right) f(y)\,dy\right|\\
&\leq \left(\frac{4}{\varepsilon}\right)^n \int_{|x-y|<\varepsilon} \left|K\left(\frac{x-y}{|x-y|}\right)\right| |f(y)|\;dy\\
&= \left(\frac{4}{\varepsilon}\right)^n \int_{\Sigma_{n-1}} |K(\sigma)|^{1-1/p}\,|K(\sigma)|^{1/p} \left(\int_0^{\varepsilon} |f(x-s\sigma)|s^{n-1}\,ds\right) d\sigma
\end{aligned}$$

$$\leq 4^n \left(\int_{\Sigma_{n-1}} |K(\sigma)| \, d\sigma \right)^{1-1/p}$$
$$\times \left(\int_{\Sigma_{n-1}} |K(\sigma)| \left(\frac{1}{\varepsilon^n} \int_0^\varepsilon |f(x - s\sigma)| s^{n-1} \, ds \right)^p d\sigma \right)^{1/p}.$$

Hence, by applying Fubini's theorem (Theorem 3.10) and inequality (9.141) with

$$f(t) := |f(y - t\,\sigma)|,$$

we obtain that

$$\int_{\mathbf{R}^n} \sup_{\varepsilon>0} |\mathrm{I}_4(x;\varepsilon)|^p \, dx \tag{9.163}$$
$$\leq 4^{np} \left(\int_{\Sigma_{n-1}} |K(\sigma)| \, d\sigma \right)^{p-1}$$
$$\times \int_{\Sigma_{n-1}} |K(\sigma)| \left(\int_{\mathbf{R}^n} \left(\sup_{\varepsilon>0} \frac{1}{\varepsilon^n} \int_0^\varepsilon |f(x - s\sigma)| s^{n-1} \, ds \right)^p dx \right) d\sigma$$
$$\leq 4^{np} \left(\int_{\Sigma_{n-1}} |K(\sigma)| \, d\sigma \right)^{p-1}$$
$$\times \left(\frac{p}{p-1} \right)^p \int_{\Sigma_{n-1}} |K(\sigma)| \left(\int_{\mathbf{R}^{n-1}} \int_{-\infty}^{\infty} |f(y - t\sigma)|^p \, dt \, dy \right) d\sigma$$
$$= \left(\frac{4^n p}{p-1} \right)^p \left(\int_{\Sigma_{n-1}} |K(\sigma)| \, d\sigma \right)^p \int_{\mathbf{R}^n} |f(x)|^p \, dx.$$

Therefore, the desired inequality (9.156) follows from four inequalities (9.160), (9.161), (9.162) and (9.163), if we recall the formula

$$\widetilde{f_\varepsilon}(x) = \mathrm{I}_1(x;\varepsilon) + \mathrm{I}_2(x;\varepsilon) + \mathrm{I}_3(x;\varepsilon) + \mathrm{I}_4(x;\varepsilon).$$

Step (3): The proof of the remaining part is analogous to that of Theorem 9.15.

Now the proof of Theorem 9.24 is complete. □

9.10 The General Case

The next theorem asserts the existence of the singular integral (9.2) in the space $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ in the general case:

Theorem 9.25. *Assume that the integral kernel $K(x)$ is a measurable,* *positively homogeneous function of degree $-n$ defined on $\mathbf{R}^n$ and satisfies the two conditions*

$$\int_{\Sigma_{n-1}} |K(\sigma)| \log^+ |K(\sigma)| \, d\sigma < \infty, \tag{9.164a}$$

$$\int_{\Sigma_{n-1}} K(\sigma) \, d\sigma = 0 \quad \textit{(the } \text{cancellation property}\textit{)}, \tag{9.164b}$$

where Σ_{n-1} is the unit sphere in $\mathbf{R}^n$ and $d\sigma$ is the surface measure on Σ_{n-1}. If $f(x) \in L^p(\mathbf{R}^n)$ with $1 < p < \infty$ and $\varepsilon > 0$, we let

$$\widetilde{f}_\varepsilon(x) := \int_{|x-y|>\varepsilon} K(x-y) f(y) \, dy.$$

Then we have the following three assertions (i), (ii) and (iii):

(i) There exists a positive constant C_p, independent of ε, such that

$$\left(\int_{\mathbf{R}^n} \sup_{\varepsilon>0} |\widetilde{f}_\varepsilon(x)|^p \, dx \right)^{1/p} \leq C_p \left(\int_{\mathbf{R}^n} |f(x)|^p \, dx \right)^{1/p}.$$

(ii) The sequence $\widetilde{f}_\varepsilon$ converges almost everywhere in $\mathbf{R}$ and in the strong topology of $L^p(\mathbf{R})$ as $\varepsilon \downarrow 0$. Namely, the singular integral

$$K * f(x) = \lim_{\varepsilon \downarrow 0} \int_{|x-y|>\varepsilon} K(x-y) f(y) \, dy$$

exists for almost all $x \in \mathbf{R}^n$ and in the strong topology of $L^p(\mathbf{R}^n)$.

(iii) The mapping $f \mapsto K*f$ is a bounded linear operator from $L^p(\mathbf{R}^n)$ into itself. More precisely, we have the inequality

$$\|K * f\|_p \leq C_p \|f\|_p .$$

Proof. It suffices to apply Theorem 9.15 and Theorem 9.24 to the odd part

$$K_1(x) := \frac{K(x) - K(-x)}{2}$$

and the even part

$$K_2(x) := \frac{K(x) + K(-x)}{2}$$

of the integral kernel $K(x)$, respectively.

Indeed, it is easy to see that

$$\int_{\Sigma_{n-1}} K_1(\sigma) \, d\sigma = \frac{1}{2} \left(\int_{\Sigma_{n-1}} K(\sigma) \, d\sigma - \int_{\Sigma_{n-1}} K(-\sigma) \, d\sigma \right) = 0,$$

This proves that the odd part $K_1(x)$ satisfies the cancellation property (9.4).

On the other hand, by inequality (9.115) with

$$\alpha := |K(\sigma)|, \quad \beta := |K(-\sigma)|,$$

it follows that

$$\begin{aligned} & |K_2(\sigma)| \log^+ |K_2(\sigma)| \\ & \leq \frac{|K(\sigma)| + |K(-\sigma)|}{2} \log^+ \left(\frac{|K(\sigma)| + |K(-\sigma)|}{2} \right) \\ & \leq \frac{1}{2} \left(|K(\sigma)| \log^+ |K(\sigma)| + |K(-\sigma)| \log^+ |K(-\sigma)| \right). \end{aligned}$$

Hence we have, by conditions (9.164),

$$\begin{aligned} & \int_{\Sigma_{n-1}} |K_2(\sigma)| \log^+ |K_2(\sigma)| \, d\sigma \\ & \leq \frac{1}{2} \left(\int_{\Sigma_{n-1}} |K(\sigma)| \log^+ |K(\sigma)| \, d\sigma + \int_{\Sigma_{n-1}} |K(-\sigma)| \log^+ |K(-\sigma)| \, d\sigma \right) \\ & = \frac{1}{2} \left(\int_{\Sigma_{n-1}} |K(\sigma)| \log^+ |K(\sigma)| \, d\sigma + \int_{\Sigma_{n-1}} |K(\sigma)| \log^+ |K(\sigma)| \, d\sigma \right) \\ & = \int_{\Sigma_{n-1}} |K(\sigma)| \log^+ |K(\sigma)| \, d\sigma \\ & < \infty. \end{aligned}$$

This proves that the even part $K_2(x)$ satisfies condition (9.103).

The proof of Theorem 9.25 is complete. □

9.11 Notes and Comments

The results of this chapter are adapted from Calderón–Zygmund [15] and Tanabe [89] and [90].

10

Calderón–Zygmund Kernels and their Commutators

This chapter 10 and the next chapter 11 are the heart of the subject. The Calderón–Zygmund theory of singular integrals continues to be one of the most influential works in modern history of analysis. The first main result (Theorem 10.1) asserts the existence of singular integral operators and the second main result (Theorem 10.2) concerns commutators of BMO functions and singular integral operators. It should be emphasized that singular integral operators with non-smooth kernels provide a powerful tool to deal with smoothness of solutions of partial differential equations, with minimal assumptions of regularity on the coefficients.

10.1 Calderón–Zygmund Kernels

Let $k(x)$ be a real-valued function defined on $\mathbf{R}^n \setminus \{0\}$. We say that $k(x)$ is a *Calderón–Zygmund kernel* if it satisfies the following three conditions (i), (ii) and (iii):

(i) $k \in C^\infty(\mathbf{R}^n \setminus \{0\})$.
(ii) $k(x)$ is positively homogeneous of degree $-n$, that is, $k(tx) = t^{-n}k(x)$ for all $t > 0$.
(iii) $\int_{\Sigma_{n-1}} k(\sigma)\,d\sigma = 0$ where Σ_{n-1} is the unit sphere in $\mathbf{R}^n$ and $d\sigma$ is the surface measure on Σ_{n-1}.

Example 10.1. Let $h(x)$ be a function in $C^\infty(\mathbf{R}^n \setminus \{0\})$ which is positively homogeneous of degree $1-n$. Then the derivatives

$$\frac{\partial h}{\partial x_j}, \quad 1 \leq j \leq n,$$

are Calderón–Zygmund kernels.

Proof. It is easy to see that $\partial h/\partial x_j$ is positively homogeneous of degree $-n$. Hence we have only to verify that $\partial h/\partial x_j$ has the *cancellation property*

$$\int_{\Sigma_{n-1}} \frac{\partial h}{\partial x_j}(\sigma)\, d\sigma = 0. \tag{10.1}$$

Let $\rho(t)$ be a non-negative, smooth function on $\mathbf{R}$ such that

$$\operatorname{supp}\rho \subset [1,2], \tag{10.2a}$$

$$\int_0^\infty \frac{\rho(t)}{t}\, dt = 1. \tag{10.2b}$$

First, by integration by parts it follows that

$$\int_{\mathbf{R}^n} \frac{\partial h}{\partial x_j}(x) \cdot \rho(|x|)\, dx = -\int_{\mathbf{R}^n} h(x)\rho'(|x|)\frac{x_j}{|x|}\, dx. \tag{10.3}$$

Since $h(x)$ is positively homogeneous of degree $1-n$ and since $\partial h/\partial x_j$ is positively homogeneous of degree $-n$, by introducing polar coordinates

$$x = r\,\sigma,$$
$$r = |x|,$$
$$\sigma \in \Sigma_{n-1},$$

we obtain from formula (10.3) that

$$\begin{aligned}
&\int_0^\infty \int_{\Sigma_{n-1}} r^{-n} \frac{\partial h}{\partial x_j}(\sigma)\rho(r) r^{n-1}\, d\sigma\, dr \\
&= -\int_0^\infty \int_{\Sigma_{n-1}} r^{1-n} h(\sigma)\rho'(r)\sigma_j r^{n-1}\, d\sigma\, dr,
\end{aligned}$$

so that

$$\begin{aligned}
&\left(\int_0^\infty \frac{\rho(r)}{r}\, dr\right) \int_{\Sigma_{n-1}} \frac{\partial h}{\partial x_j}(\sigma)\, d\sigma \\
&= -\left(\int_0^\infty \rho'(r)\, dr\right) \int_{\Sigma_{n-1}} h(\sigma)\sigma_j\, d\sigma.
\end{aligned} \tag{10.4}$$

However, we have, by conditions (10.2),

$$\int_0^\infty \frac{\rho(r)}{r}\, dr = 1,$$

$$\int_0^\infty \rho'(r)\, dr = 0.$$

Therefore, the desired assertion (10.1) follows from formula (10.4). $\square$

The first main result — the most important property of Calderón–Zygmund kernels — asserts the existence of singular integral operators ([15]). The next theorem is a special case of Theorem 9.2 (cf. [18, Theorem 2.5]):

Theorem 10.1. *Let $k(x)$ be a Calderón–Zygmund kernel. If $\varepsilon > 0$ and $f \in L^p(\mathbf{R}^n)$ for $1 < p < \infty$, we let*

$$K_\varepsilon f(x) := \int_{|x-y|>\varepsilon} k(x-y) f(y)\, dy.$$

Then there exists a function $Kf \in L^p(\mathbf{R}^n)$ such that

$$\lim_{\varepsilon \downarrow 0} \|K_\varepsilon f - Kf\|_{L^p(\mathbf{R}^n)} = 0.$$

Moreover, the operator K is bounded on the space $L^p(\mathbf{R}^n)$. More precisely, there exists a constant $c_1 = c_1\left(n, p, \|k\|_{L^2(\Sigma_{n-1})}\right) > 0$ such that

$$\|Kf\|_{L^p(\mathbf{R}^n)} \le c_1 \|f\|_{L^p(\mathbf{R}^n)} \quad \text{for all } f \in L^p(\mathbf{R}^n).$$

The operator K is called a *Calderón–Zygmund singular integral operator*. In what follows we shall use the notation

$$Kf(x) = (\text{v. p.}k) * f(x) = \text{v. p.} \int_{\mathbf{R}^n} k(x-y) f(y)\, dy.$$

10.2 Commutators of Calderón–Zygmund Kernels

Now we are in a position to state the second main result concerning commutators of singular integrals. Let $k(x)$ be a Calderón–Zygmund kernel and let K be its associated singular integral operator defined by the formula

$$Kf(x) = \text{v. p.} \int_{\mathbf{R}^n} k(x-y) f(y)\, dy,$$

where $f \in L^p(\mathbf{R}^n)$ for $1 < p < \infty$. If $\varphi \in \text{BMO}$, then we define the commutator $C[\varphi, K]$ of φ and K as the principal value

$$\begin{aligned} C[\varphi, K]f &:= \varphi(Kf) - K(\varphi f) \\ &= \text{v. p.} \int_{\mathbf{R}^n} k(x-y)[\varphi(x) - \varphi(y)] f(y)\, dy. \end{aligned}$$

Then we have the following (cf. [21, Theorem I], [13, Theorem 2.6], [18, Theorem 2.7]):

Theorem 10.2. *If* $\varphi \in \mathrm{BMO}$*, then the commutator of singular integrals*

$$C[\varphi, K]f = \varphi(Kf) - K(\varphi f)$$

is well-defined. Moreover, the commutator $C[\varphi, K]$ *is bounded on the space* $L^p(\mathbf{R}^n)$ *for* $1 < p < \infty$*. More precisely, there exists a constant* $c_2 = c_2\left(n, p, \|k\|_{L^2(\Sigma_{n-1})}\right) > 0$ *such that*

$$\|C[\varphi, K]f\|_{L^p(\mathbf{R}^n)} \leq c_2 \|\varphi\|_* \|f\|_{L^p(\mathbf{R}^n)} \quad \text{for all } f \in L^p(\mathbf{R}^n). \tag{10.5}$$

Proof. The idea of our proof is due to Strömberg (cf. [91, pp. 417–419]). The proof is divided into three steps.

Step 1: Let Q be a cube with sides parallel to the coordinate axes. If $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, then we define the Hardy–Littlewood maximal function (see Section 4.4)

$$Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

and the sharp function (see Section 4.6)

$$f^\sharp(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\, dy,$$

where the supremum is taken over all cubes containing x and f_Q is the integral average of f over Q

$$f_Q := \frac{1}{|Q|} \int_Q f(z)\, dz.$$

For the sake of simplicity, we write

$$Tf(x) := C[\varphi, K]f(x) \quad \text{for } x \in \mathbf{R}^n.$$

For each $r \in (1, p)$, we shall prove a pointwise estimate

$$\begin{aligned} &(Tf)^\sharp(x) \\ &\leq c(n, r) \|\varphi\|_* \left((M(|Kf|^r)(x))^{1/r} + (M(|f|^r)(x))^{1/r} \right) \quad \text{for } x \in \mathbf{R}^n, \end{aligned} \tag{10.6}$$

with a positive constant $c(n, r)$.

Step 2: Assuming estimate (10.6) for the moment, we shall prove Theorem 10.2. The proof of estimate (10.6) will be given in the next Section 10.3, due to its length.

By applying Corollary 4.15 and Remark 4.3 and estimate (10.6), we obtain that, for $1 < r < p$,

$$\|Tf\|_{L^p(\mathbf{R}^n)} \tag{10.7}$$

$$\le c_p \left\| (Tf)^\sharp \right\|_{L^p(\mathbf{R}^n)}$$
$$\le 2^{(p-1)/p} c_p c(n,r) \|\varphi\|_*$$
$$\times \left\{ \left(\int_{\mathbf{R}^n} (M(|Kf|^r)(x))^{p/r}\, dx \right)^{1/p} + \left(\int_{\mathbf{R}^n} (M(|f|^r)(x))^{p/r}\, dx \right)^{1/p} \right\}.$$

Step 2-1: However, it should be noticed that

$$\left(\int_{\mathbf{R}^n} (M(|f|^r)(x))^{p/r}\, dx \right)^{1/p} = \left(\|M(|f|^r)\|_{L^{p/r}(\mathbf{R}^n)} \right)^{1/r},$$

so that, by Theorem 4.8 with $p := p/r$ and Remark 4.1,

$$\|M(|f|^r)\|_{L^{p/r}(\mathbf{R}^n)} \le C(p,r) \, \||f|^r\|_{L^{p/r}(\mathbf{R}^n)} = C(p,r) \, \|f\|^r_{L^p(\mathbf{R}^n)}.$$

Hence, we have the inequality

$$\left(\int_{\mathbf{R}^n} (M(|f|^r)(x))^{p/r}\, dx \right)^{1/p} \le C(p,r)^{1/r} \, \|f\|_{L^p(\mathbf{R}^n)}. \tag{10.8}$$

Step 2-2: Similarly, we have the formula

$$\left(\int_{\mathbf{R}^n} (M(|Kf|^r)(x))^{p/r}\, dx \right)^{1/p} = \left(\|M(|Kf|^r)\|_{L^{p/r}(\mathbf{R}^n)} \right)^{1/r},$$

and also, by Theorem 4.8 with $p := p/r$,

$$\|M(|Kf|^r)\|_{L^{p/r}(\mathbf{R}^n)} \le C(p,r) \, \||Kf|^r\|_{L^{p/r}(\mathbf{R}^n)} = C(p,r) \, \|Kf\|^r_{L^p(\mathbf{R}^n)}.$$

Hence, it follows from an application of Theorem 10.1 that

$$\begin{aligned} \left(\int_{\mathbf{R}^n} (M(|Kf|^r)(x))^{p/r}\, dx \right)^{1/p} &\le C(p,r)^{1/r} \, \|Kf\|_{L^p(\mathbf{R}^n)} \qquad (10.9) \\ &\le c_1(n,p) C(p,r)^{1/r} \, \|f\|_{L^p(\mathbf{R}^n)}. \end{aligned}$$

Step 3: Therefore, the desired estimate (10.5) follows by combining estimates (10.7), (10.8) and (10.9), with

$$c_2(n,p) := 2^{(p-1)/p} c_p c(n,r) \, (c_1(n,p) + 1) \, C(p,r)^{1/r}.$$

The proof of Theorem 10.2 is now complete, apart from the proof of estimate (10.6). □

10.3 Proof of Estimate (10.6)

The purpose of this section is to prove estimate (10.6). If Q is a cube, then we denote by δ_Q its side length and by x_Q its center, respectively.

For each $j \in \mathbf{N}$, we denote by $2^j Q$ the cube centered at x_Q with side length $2^j \delta_Q$ (see Figure 4.3). Let Q be an arbitrary cube containing x. We write

$$Tf(x) := C[\varphi, K]f(x)$$

in the form

$$\begin{aligned} Tf(x) &= K((\varphi(\cdot) - \varphi_Q)f(\cdot)\chi_{2Q})(x) \\ &\quad + K((\varphi(\cdot) - \varphi_Q)f(\cdot)\chi_{\mathbf{R}^n \setminus 2Q})(x) - (\varphi(x) - \varphi_Q)Kf(x) \\ &:= A(x) + B(x) - C(x), \end{aligned}$$

where $\chi_A(x)$ is the characteristic function of the set A. The proof is divided into four steps.

Step 1: The estimate of the term $A(x)$. We prove that, for $1 < r < p$,

$$\frac{1}{|Q|}\int_Q |A(x) - A_Q|\,dx \leq c_1(n,r)\,\|\varphi\|_*\,(M(|f|^r)(y))^{1/r} \quad \text{for } y \in Q. \tag{10.10}$$

First, we have the inequality

$$\begin{aligned} \frac{1}{|Q|}\int_Q |A(x) - A_Q|\,dx &\leq \frac{1}{|Q|}\int_Q |A(x)|\,dx + |A_Q| \\ &= \frac{1}{|Q|}\int_Q |A(x)|\,dx + \frac{1}{|Q|}\left|\int_Q A(y)\,dy\right| \\ &\leq \frac{2}{|Q|}\int_Q |A(x)|\,dx \\ &= \frac{2}{|Q|}\int_Q |K((\varphi(\cdot) - \varphi_Q)f(\cdot)\chi_{2Q})(x)|\,dx. \end{aligned} \tag{10.11}$$

If q is a number such that $1 < q < r$, then, by Hölder's inequality (Theorem 3.14) it follows that

$$\begin{aligned} &\frac{2}{|Q|}\int_Q |K((\varphi(\cdot) - \varphi_Q)f(\cdot)\chi_{2Q})(x)|\,dx \\ &\leq \frac{2}{|Q|}\left(\int_Q |K((\varphi(\cdot) - \varphi_Q)f(\cdot)\chi_{2Q})(x)|^q\,dx\right)^{1/q}\left(\int_Q dx\right)^{1-1/q} \\ &\leq \frac{2}{|Q|^{1/q}}\left(\int_{\mathbf{R}^n} |K((\varphi(\cdot) - \varphi_Q)f(\cdot)\chi_{2Q})(x)|^q\,dx\right)^{1/q}. \end{aligned} \tag{10.12}$$

However, we have, by Theorem 10.1 and Hölder's inequality (Theorem

3.14),

$$\frac{2}{|Q|^{1/q}}\left(\int_{\mathbf{R}^n}|K((\varphi(\cdot)-\varphi_Q)f(\cdot)\chi_{2Q})(x)|^q\,dx\right)^{1/q} \tag{10.13}$$
$$\le c(n,q)\frac{2}{|Q|^{1/q}}\left(\int_{\mathbf{R}^n}|(\varphi(x)-\varphi_Q)f(x)\chi_{2Q}(x)|^q\,dx\right)^{1/q}$$
$$=2c(n,q)\left(\frac{1}{|Q|}\int_{2Q}|\varphi(x)-\varphi_Q|^q\,|f(x)|^q\,dx\right)^{1/q}$$
$$\le 2c(n,q)$$
$$\times\left[\frac{1}{|Q|}\left(\int_{2Q}|f(x)|^r\,dx\right)^{q/r}\left(\int_{2Q}|\varphi(x)-\varphi_Q|^{rq/(r-q)}\,dx\right)^{(r-q)/r}\right]^{1/q}.$$

Moreover, we have the following claim:

Claim 10.1. There exists a constant $c(n,q,r)>0$ such that we have, for $1<q<r<p$,

$$\int_{2Q}|\varphi(x)-\varphi_Q|^{rq/(r-q)}\,dx\le c(n,q,r)\,\|\varphi\|_*^{rq/(r-q)}\,|2Q|\,. \tag{10.14}$$

Proof. First, by Lemma 4.2 with $f:=\varphi$ and $j:=1$ it follows that

$$\begin{aligned}|\varphi(x)-\varphi_Q| &\le |\varphi(x)-\varphi_{2Q}|+|\varphi_{2Q}-\varphi_Q|\\ &\le |\varphi(x)-\varphi_{2Q}|+c(n)\,\|\varphi\|_*\,,\end{aligned}$$

so that

$$\int_{2Q}|\varphi(x)-\varphi_Q|^{rq/(r-q)}\,dx \tag{10.15}$$
$$\le 2^{C(q,r)}\left(\int_{2Q}|\varphi(x)-\varphi_{2Q}|^{rq/(r-q)}\,dx+\int_{2Q}(c(n)\,\|\varphi\|_*)^{rq/(r-q)}\,dx\right)$$
$$=2^{C(q,r)}\left(\int_{2Q}|\varphi(x)-\varphi_{2Q}|^{rq/(r-q)}\,dx+c(n)^{rq/(r-q)}\|\varphi\|_*^{rq/(r-q)}|2Q|\right),$$

where

$$C(q,r):=\frac{rq}{r-q}-1,\quad 1<q<r.$$

However, we have, by Theorem 4.11 with $p:=rq/(r-q)$ and Remark 4.2,

$$\int_{Q}|\varphi(x)-\varphi_{2Q}|^{rq/(r-q)}\,dx\le c_1(q,r)\,\|\varphi\|_*^{rq/(r-q)}\,|2Q|\,. \tag{10.16}$$

Therefore, by combining estimates (10.15) and (10.16) we obtain that

$$\int_{2Q} |\varphi(x) - \varphi_Q|^{rq/(r-q)}\, dx$$
$$\leq 2^{C(q,r)} \left(c_1(q,r) + c(n)^{rq/(r-q)}\right) \|\varphi\|_*^{rq/(r-q)} |2Q| .$$

This proves the desired estimate (10.14), with

$$c(n,q,r) := 2^{C(q,r)} \left(c_1(q,r) + c(n)^{rq/(r-q)}\right).$$

The proof of Claim 10.1 is complete. □

Therefore, by combining five estimates (10.11), (10.12), (10.13) and (10.14) we obtain that

$$\begin{aligned}
&\frac{1}{|Q|} \int_Q |A(x) - A_Q|\, dx \qquad (10.17)\\
&\leq 2c(n,q)\\
&\quad \times \left[\frac{1}{|Q|}\left(\int_{2Q} |f(x)|^r\, dx\right)^{q/r} \left(\int_{2Q} |\varphi(x) - \varphi_Q|^{rq/(r-q)}\, dx\right)^{(r-q)/r}\right]^{1/q}\\
&= 2^{1+n/r} c(n,q) c(n,q,r) \frac{1}{|Q|^{1/q}} \left(\frac{1}{|2Q|}\int_{2Q} |f(x)|^r\, dx\right)^{1/r} |Q|^{1/r}\\
&\quad \times \left(\|\varphi\|_*^{rq/(r-q)} |2Q|\right)^{(r-q)/rq}\\
&= 2^{1+n/r} c(n,q) c(n,q,r) \|\varphi\|_* \left(\frac{1}{|2Q|}\int_{2Q} |f(x)|^r\, dx\right)^{1/r}\\
&\leq 2^{1+n/r} c(n,q) c(n,q,r) \|\varphi\|_* \left(M(|f|^r)(y)\right)^{1/r} \quad \text{for } y \in Q,
\end{aligned}$$

since we have the estimate

$$\frac{1}{|2Q|}\int_{2Q} |f(x)|^r\, dx \leq M(|f|^r)(y) \quad \text{for } y \in Q.$$

The desired estimate (10.10) follows from estimate (10.17), with

$$c_1(n,r) := 2^{1+n/r} c(n,q) c(n,q,r).$$

Step 2: The estimate of the term $C(x)$. Similarly, we prove that, for $1 < r < p$,

$$\frac{1}{|Q|}\int_Q |C(x) - C_Q|\, dx \leq c_2(n,r) \|\varphi\|_* \left(M(|Kf|^r)(y)\right)^{1/r} \qquad (10.18)$$
$$\text{for } y \in Q.$$

First, we have the inequality

$$\begin{aligned}\frac{1}{|Q|}\int_Q |C(x)-C_Q|\,dx &\le \frac{1}{|Q|}\int_Q |C(x)|\,dx + |C_Q| \qquad (10.19)\\ &= \frac{1}{|Q|}\int_Q |C(x)|\,dx + \frac{1}{|Q|}\left|\int_Q C(y)\,dy\right|\\ &\le \frac{2}{|Q|}\int_Q |C(x)|\,dx.\end{aligned}$$

If q is a number such that $1 < q < r$, then, by Hölder's inequality (Theorem 3.14) it follows that we have, for $r' = r/(r-1)$,

$$\begin{aligned}&\frac{2}{|Q|}\int_Q |C(x)|\,dx\\ &= \frac{2}{|Q|}\int_Q |(\varphi(x)-\varphi_Q)Kf(x)|\,dx\\ &\le 2\left(\frac{1}{|Q|}\int_Q |\varphi(x)-\varphi_Q|^{r'}\,dx\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q |Kf(x)|^r\,dx\right)^{1/r}.\end{aligned}$$

However, we have, by Theorem 4.11 with $p := r'$ and Remark 4.2,

$$\frac{1}{|Q|}\int_Q |\varphi(x)-\varphi_Q|^{r'}\,dx \le c_1(r')\,\|\varphi\|_*^{r'},$$

and also

$$\frac{1}{|Q|}\int_Q |Kf(x)|^r\,dx \le M(|Kf|^r)(y) \quad \text{for } y \in Q.$$

Hence it follows that

$$\begin{aligned}\frac{2}{|Q|}\int_Q |C(x)|\,dx &\le 2\left(c_1(r')\,\|\varphi\|_*^{r'}\right)^{1/r'}(M(|Kf|^r)(y))^{1/r} \qquad (10.20)\\ &= 2c_1(r')^{1/r'}\,\|\varphi\|_*\,(M(|Kf|^r)(y))^{1/r} \quad \text{for } y \in Q.\end{aligned}$$

Therefore, the desired estimate (10.18) follows by combining estimates (10.19) and (10.20), with

$$c_2(n,r) := 2c_1(r')^{1/r'}.$$

Step 3: The estimate of the term $B(x)$. Thirdly, we prove that, for $1 < r < p$,

$$\frac{1}{|Q|}\int_Q |B(x)-B_Q|\,dx \le c_3(n,r)\,\|\varphi\|_*\,(M(|f|^r)(y))^{1/r}, \qquad (10.21)$$

for $y \in Q$.

The proof of estimate (10.21) is divided into four steps.

Step 3-1: We begin with the *pointwise Hörmander condition* for the Calderón–Zygmund kernels $k(x)$:

Lemma 10.3. *Let $k(x)$ be a Calderón–Zygmund kernel. If Q is a cube with center x_Q, then we have, for all $x \in Q$ and $y \notin (2Q)$ (see Figure 10.1 below),*

$$|k(x-y) - k(x_Q - y)| \leq c\frac{|x - x_Q|}{|x_Q - y|^{n+1}}, \tag{10.22}$$

with a positive constant c.

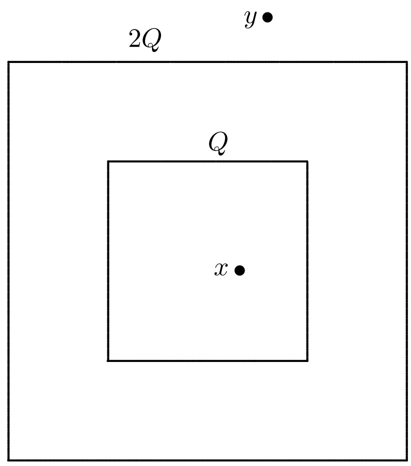

Fig. 10.1. The points $x \in Q$ and $y \in \mathbf{R}^n \setminus (2Q)$

Proof. (1) Since $k(x)$ is positively homogeneous of degree $-n$, it follows that, for $h \neq 0$,

$$\frac{k(x + h\mathbf{e}_j) - k(x)}{h} = \frac{k\left(\frac{x}{|x|} + \frac{h}{|x|}\mathbf{e}_j\right) - k\left(\frac{x}{|x|}\right)}{h} \cdot \frac{1}{|x|^n}, \quad x \in \mathbf{R}^n \setminus \{0\},$$

so that

$$\frac{k(x + h\mathbf{e}_j) - k(x)}{h} = \frac{k(\overline{x} + \delta\mathbf{e}_j) - k(\overline{x})}{\delta} \cdot \frac{1}{|x|^{n+1}}, \tag{10.23}$$

with

$$\overline{x} = \frac{x}{|x|} \in \Sigma_{n-1}, \qquad \delta = \frac{h}{|x|}.$$

Hence, by letting $h \to 0$ in formula (10.23) we obtain that

$$|\nabla k(x)| \leq \frac{C}{|x|^{n+1}} \quad \text{for all } x \in \mathbf{R}^n \setminus \{0\}, \tag{10.24}$$

where C is a positive constant given by the formula

$$C := \max_{|z|=1} |\nabla k(z)| \,.$$

(2) If we let

$$x_Q = (x_Q^1, x_Q^2, \ldots, x_Q^n),$$

then, by the mean value theorem it follows that, for $0 < \theta < 1$,

$$\begin{aligned}
&|k(x-y) - k(x_Q - y)| \qquad\qquad (10.25)\\
&= |k((x_Q - y) + (x - x_Q)) - k(x_Q - y)| \\
&\leq \left| \frac{\partial k}{\partial x_1}(x_Q^1 - y_1 + \theta(x_1 - x_Q^1), x_2 - y_2, \ldots, x_n - y_n) \right| |x_1 - x_Q^1| \\
&\quad + \left| \frac{\partial k}{\partial x_2}(x_Q^1 - y_1, x_Q^2 - y_2 + \theta(x_2 - x_Q^2), x_3 - y_3, \ldots, x_n - y_n) \right| \\
&\quad \times |x_2 - x_Q^2| \\
&\quad + \ldots + \left| \frac{\partial k}{\partial x_n}(x_Q^1 - y_1, x_Q^2 - y_2, \ldots, x_Q^n - y_n + \theta(x_n - x_Q^n)) \right| \\
&\quad \times |x_n - x_Q^n| \\
&\leq \left| \frac{\partial k}{\partial x_1}(x_Q^1 - y_1 + \theta(x_1 - x_Q^1), x_2 - y_2, \ldots, x_n - y_n) \right| |x - x_Q| \\
&\quad + \left| \frac{\partial k}{\partial x_2}(x_Q^1 - y_1, x_Q^2 - y_2 + \theta(x_2 - x_Q^2), x_3 - y_3, \ldots, x_n - y_n) \right| \\
&\quad \times |x - x_Q| \\
&\quad + \ldots + \left| \frac{\partial k}{\partial x_n}(x_Q^1 - y_1, x_Q^2 - y_2, \ldots, x_Q^n - y_n + \theta(x_n - x_Q^n)) \right| \\
&\quad \times |x - x_Q|
\end{aligned}$$

However, we have, for all $x \in Q$ and $y \notin 2Q$,

$$\begin{aligned}
\left| x_Q^j - y_j + \theta(x_j - x_Q^j) \right| &\geq \left| x_Q^j - y_j \right| - \left| x_j - x_Q^j \right| \\
&\geq \frac{1}{2} \left| y_j - x_Q^j \right|, \quad 0 < \theta < 1,
\end{aligned}$$

and also

$$|x_j - y_j| = \left| (x_Q^j - y_j) + (x_j - x_Q^j) \right| \geq \left| x_Q^j - y_j \right| - \left| x_j - x_Q^j \right|$$

$$\geq \frac{1}{2}\left|y_j - x_Q^j\right|.$$

In particular, it follows that, for $1 \leq j \leq n$,

$$\left|(x_Q^1 - y_1, \ldots, x_Q^j - y_j + \theta(x_j - x_Q^j), \ldots, x_n - y_n)\right| \geq \frac{1}{2}|y - x_Q|. \tag{10.26}$$

Therefore, by using inequality (10.24) we obtain from inequalities (10.25) and (10.26) that

$$\begin{aligned}
&|k(x-y) - k(x_Q - y)| \\
&\leq \left(\sum_{j=1}^{n}\left|\frac{\partial k}{\partial x_j}(x_Q^1 - y_1, \ldots, x_Q^j - y_j + \theta(x_j - x_Q^j), \ldots, x_Q^n - y_n)\right|^2\right)^{1/2} \\
&\times |x - x_Q| \\
&\leq 2^{n+1}\, C\, \frac{|x - x_Q|}{|y - x_Q|^{n+1}}.
\end{aligned}$$

This proves the desired inequality (10.22), with

$$c := 2^{n+1}\, C = 2^{n+1} \max_{|z|=1} |\nabla k(z)|.$$

The proof of Lemma 10.3 is complete. □

Step 3-2: First, it follows that

$$\begin{aligned}
&B(x) - B(x_Q) \\
&= K((\varphi(\cdot) - \varphi_Q) f(\cdot)\chi_{\mathbf{R}^n \setminus 2Q})(x) - K((\varphi(\cdot) - \varphi_Q) f(\cdot)\chi_{\mathbf{R}^n \setminus 2Q})(x_Q) \\
&= \int_{\mathbf{R}^n \setminus (2Q)} k(x-y)[\varphi(y) - \varphi_Q] f(y)\, dy \\
&\quad - \int_{\mathbf{R}^n \setminus (2Q)} k(x_Q - y)[\varphi(y) - \varphi_Q] f(y)\, dy,
\end{aligned}$$

so that

$$|B(x) - B(x_Q)| \leq \int_{\mathbf{R}^n \setminus (2Q)} |k(x-y) - k(x_Q - y)||\varphi(y) - \varphi_Q||f(y)|\, dy.$$

However, by Lemma 10.3 we can find a positive constant c such that

$$|k(x-y) - k(x_Q - y)| \leq c \frac{|x - x_Q|}{|x_Q - y|^{n+1}}, \quad x \in Q,\ y \notin 2Q.$$

Hence we have, by Hölder's inequality (Theorem 3.14),

$$|B(x) - B(x_Q)| \tag{10.27}$$

$$\begin{aligned}
&\le \int_{\mathbf{R}^n\setminus(2Q)} |k(x-y)-k(x_Q-y)||\varphi(y)-\varphi_Q||f(y)|\,dy \\
&\le c\int_{\mathbf{R}^n\setminus(2Q)} \frac{|x-x_Q|}{|x_Q-y|^{n+1}}|\varphi(x)-\varphi_Q||f(y)|\,dy \\
&\le c'\delta_Q \int_{\mathbf{R}^n\setminus(2Q)} \frac{|f(y)|}{|x_Q-y|^{(n+1)/r}}\,\frac{|\varphi(x)-\varphi_Q|}{|x_Q-y|^{(n+1)/r'}}\,dy \\
&\le c'\delta_Q \left(\int_{\mathbf{R}^n\setminus(2Q)} \frac{|f(y)|^r}{|x_Q-y|^{n+1}}\,dy\right)^{1/r} \\
&\quad\times \left(\int_{\mathbf{R}^n\setminus(2Q)} \frac{|\varphi(x)-\varphi_Q|^{r'}}{|x_Q-y|^{n+1}}\,dy\right)^{1/r'}.
\end{aligned}$$

Here δ_Q is the side length of Q.

Step 3-3: Now we prove the following two estimates:

$$I(x) := \int_{\mathbf{R}^n\setminus(2Q)} \frac{|f(z)|^r}{|x_Q-z|^{n+1}}\,dz \le \frac{C_1}{\delta_Q} M(|f|^r)(y), \quad y\in Q. \tag{10.28}$$

$$II(x) := \int_{\mathbf{R}^n\setminus(2Q)} \frac{|\varphi(x)-\varphi_Q|^{r'}}{|x_Q-y|^{n+1}}\,dy \le \frac{C_2}{\delta_Q}\,\|\varphi\|_*^{r'}. \tag{10.29}$$

Proof of Estimate (10.28): Indeed, we have, for all $z \in 2^jQ$ with $j\in\mathbf{N}$,

$$|x_Q-z|^{n+1} \ge (2^{j-1}\delta_Q)^{n+1} = 2^{(n+1)j}\delta_Q^{n+1}\frac{1}{2^{n+1}},$$

and also

$$2^j\delta_Q|2^jQ| = 2^{(n+1)j}\delta_Q^{n+1}.$$

Hence it follows that

$$\begin{aligned}
\int_{\mathbf{R}^n\setminus(2Q)} \frac{|f(z)|^r}{|x_Q-z|^{n+1}}\,dz &= \sum_{j=2}^{\infty}\int_{2^jQ\setminus(2^{j-1}Q)} \frac{|f(z)|^r}{|x_Q-z|^{n+1}}\,dz \\
&\le C\sum_{j=2}^{\infty}\frac{1}{2^j\delta_Q}\left(\frac{1}{|2^jQ|}\int_{2^jQ}|f(z)|^r\,dz\right) \\
&\le \frac{C}{2\delta_Q}M(|f|^r)(y) \quad \text{for } y\in Q.
\end{aligned}$$

This proves the desired estimate (10.28), with

$$C_1 := \frac{C}{2}.$$

The proof of Estimate (10.28) is complete. □

Proof of Estimate (10.29): Similarly, we have, for $r' = r/(r-1)$,

$$\begin{aligned}
&\int_{\mathbf{R}^n\setminus(2Q)} \frac{|\varphi(z)-\varphi_Q|^{r'}}{|x_Q - z|^{n+1}}\,dz \qquad (10.30)\\
&= \sum_{j=2}^{\infty} \int_{2^jQ\setminus(2^{j-1}Q)} \frac{|\varphi(z)-\varphi_Q|^{r'}}{|x_Q - z|^{n+1}}\,dz \\
&\le C \sum_{j=2}^{\infty} \frac{1}{2^j\delta_Q}\left(\frac{1}{|2^jQ|}\int_{2^jQ} |\varphi(z)-\varphi_Q|^{r'}\,dz\right).
\end{aligned}$$

However, by Lemma 4.2 with $f := \varphi$ it follows that

$$\begin{aligned}
|\varphi(z)-\varphi_Q| &\le |\varphi(z)-\varphi_{2^jQ}| + |\varphi_{2^jQ}-\varphi_Q| \\
&\le |\varphi(z)-\varphi_{2^jQ}| + c(n)j\,\|\varphi\|_*,
\end{aligned}$$

so that

$$\begin{aligned}
&\frac{1}{|2^jQ|}\int_{2^jQ} |\varphi(z)-\varphi_Q|^{r'}\,dz \qquad (10.31)\\
&\le 2^{r'-1}\left(\frac{1}{|2^jQ|}\int_{2^jQ} |\varphi(z)-\varphi_{2^jQ}|^{r'}\,dz\right) \\
&\quad + \frac{1}{|2^jQ|}\int_{2^jQ} \left(c(n)j\,\|\varphi\|_*\right)^{r'}\,dz \\
&\le 2^{r'-1}\left(\frac{1}{|2^jQ|}\int_{2^jQ} |\varphi(z)-\varphi_{2^jQ}|^{r'}\,dz\right) + 2^{r'-1}\left(c(n)j\,\|\varphi\|_*\right)^{r'}.
\end{aligned}$$

However, we have, by Theorem 4.11 with $p := r'$ and Remark 4.2,

$$\frac{1}{|2^jQ|}\int_{2^jQ} |\varphi(z)-\varphi_{2^jQ}|^{r'}\,dz \le c_1(r')\,\|\varphi\|_*^{r'}. \qquad (10.32)$$

Hence, it follows from estimates (10.31) and (10.32) that

$$\frac{1}{|2^jQ|}\int_{2^jQ} |\varphi(z)-\varphi_Q|^{r'}\,dz \le 2^{r'-1}\left(c(n)^{r'}j^{r'} + c_1(r')\right)\|\varphi\|_*^{r'}. \qquad (10.33)$$

Therefore, by combining estimates (10.30) and (10.33) we obtain that

$$\begin{aligned}
&\int_{\mathbf{R}^n\setminus(2Q)} \frac{|\varphi(z)-\varphi_Q|^{r'}}{|x_Q - z|^{n+1}}\,dz \\
&= \sum_{j=2}^{\infty} \int_{2^jQ\setminus(2^{j-1}Q)} \frac{|\varphi(z)-\varphi_Q|^{r'}}{|x_Q - z|^{n+1}}\,dz \\
&\le 2^{r'-1} C \sum_{j=2}^{\infty} \frac{1}{2^j\delta_Q}\left(c(n)^{r'}j^{r'} + c_1(r')\right)\|\varphi\|_*^{r'}
\end{aligned}$$

$$= 2^{r'-1}\frac{C}{\delta_Q}\left[c_1(r')\left(\sum_{j=2}^{\infty}\frac{1}{2^j}\right)+c(n)^{r'}\left(\sum_{j=2}^{\infty}\frac{j^{r'}}{2^j}\right)\right]\|\varphi\|_*^{r'}.$$

This proves the desired estimate (10.29), with

$$C_2 := 2^{r'-1}C\left[c_1(r')\left(\sum_{j=2}^{\infty}\frac{1}{2^j}\right)+c(n)^{r'}\left(\sum_{j=2}^{\infty}\frac{j^{r'}}{2^j}\right)\right].$$

The proof of Estimate (10.29) is complete. □

Step 3-4: Therefore, by combining estimates (10.27), (10.28) and (10.29) we obtain that

$$\begin{aligned}\frac{1}{|Q|}\int_Q |B(x)-B_Q|\,dx &\le \frac{2}{|Q|}\int_Q |B(x)-B(x_Q)|\,dx\\ &\le \frac{2}{|Q|}c(n,r)\,\|\varphi\|_*\,(M(|f|^r)(y))^{1/r}\int_Q dx\\ &= 2c(n,r)\,\|\varphi\|_*\,(M(|f|^r)(y))^{1/r} \quad \text{for all } y\in Q.\end{aligned}$$

This proves the desired estimate (10.21), with

$$c_3(n,r) := 2c(n,r).$$

The proof of estimate (10.21) is complete. □

Step 4: Finally, if we let

$$c(n,r) := \max\{c_1(n,r), c_2(n,r), c_3(n,r)\},$$

then, by combining estimates (10.10), (10.18) and (10.21) we obtain that

$$\begin{aligned}&\frac{1}{|Q|}\int_Q |Tf(x)-(Tf)_Q|\,dx\\ &\le \frac{1}{|Q|}\int_Q |A(x)-A_Q|\,dx + \frac{1}{|Q|}\int_Q |B(x)-B_Q|\,dx\\ &\quad + \frac{1}{|Q|}\int_Q |C(x)-C_Q|\,dx\\ &\le c(n,r)\,\|\varphi\|_*\left((M(|Kf|^r)(y))^{1/r}+(M(|f|^r)(y))^{1/r}\right) \quad \text{for all } y\in Q.\end{aligned}$$

This proves that

$$\begin{aligned}&(Tf)^{\sharp}f(y)\\ &\le c(n,r)\,\|\varphi\|_*\left((M(|Kf|^r)(y))^{1/r}+(M(|f|^r)(y))^{1/r}\right) \quad \text{for all } y\in \mathbf{R}^n,\end{aligned}$$

since Q is arbitrary.

Now the proof of estimate (10.6) (and hence that of Theorem 10.2) is complete. □

10.4 Notes and Comments

The results of this chapter are adapted from Coifman–Rochberg–Weiss [21] and Bramanti–Cerutti [13].

11

Calderón–Zygmund Variable Kernels and their Commutators

In this chapter we consider singular integrals with kernels depending on a *parameter*, and prove theorems about singular integrals and commutators of L^∞ functions and singular integral operators (Theorems 11.1 and 11.2), generalizing Theorems 10.1 and 10.2 in Chapter 10. The main idea of proof is to reduce the variable kernel case to the constant kernel case. This is done by expanding the kernel into a series of spherical harmonics (Theorem 4.31), each term defining a constant kernel operator treated in Chapter 10. Theorems about singular integrals and commutators are usually formulated in the whole space $\mathbf{R}^n$. However, our application to the theory of elliptic equations with discontinuous coefficients will require a local version of Theorems 11.1 and 11.2 (Theorems 11.3 and 11.4 and Corollary 11.5).

11.1 Commutators of Calderón–Zygmund Variable Kernels

Let $k(x,z)$ be a real-valued function defined on $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ that satisfies the following two conditions (i) and (ii):

(i) $k(x,\cdot)$ is a Calderón–Zygmund kernel for almost all $x \in \mathbf{R}^n$.

(ii) The quantity

$$M := \max_{|\alpha| \leq 2n} \left\| \frac{\partial^\alpha k}{\partial z^\alpha}(\cdot,\cdot) \right\|_{L^\infty(\mathbf{R}^n \times \Sigma_{n-1})}$$

is finite. Here Σ_{n-1} is the unit sphere in $\mathbf{R}^n$.

It should be emphasized that the lack of regularity in the first variable x of $k(x,z)$ prevents to apply many recent results on singular integral operators. However, the good regularity in the second variable z of $k(x,z)$ allows us to use an old argument employed by Giraud, Calderón and Zygmund based on an expansion into spherical harmonics (Theorem 4.31).

Example 11.1. Assume that the functions $a^{ij}(x) \in \mathrm{VMO} \cap L^\infty(\mathbf{R}^n)$ satisfy the following two conditions (1) and (2):

(1) $a^{ij}(x) = a^{ji}(x)$ for all $1 \leq i, j \leq n$ and for almost all $x \in \Omega$.
(2) There exists a positive constant λ such that

$$\frac{1}{\lambda}|\xi|^2 \leq \sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2$$

for almost all $x \in \Omega$ and for all $\xi \in \mathbf{R}^n$.

Let $\widetilde{\Omega}$ be the subset of Ω where conditions (1) and (2) hold true, and let

$$\Gamma(x,z) = \frac{1}{(2-n)\omega_n} \frac{1}{\sqrt{\det(a^{ij}(x))}} \left(\sum_{i,j=1}^n A_{ij}(x) z_i z_j \right)^{(2-n)/2}$$

for all $x \in \widetilde{\Omega}$ and all $z \in \mathbf{R}^n \setminus \{0\}$.

Here

$$(A_{ij}(x)) = \text{ the inverse matrix of } (a^{ij}(x)),$$

$$\omega_n := |\Sigma_{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

(the surface area of the unit sphere Σ_{n-1} in $\mathbf{R}^n$).

Then it follows that the functions

$$\begin{aligned}\Gamma_i(x,z) &= \frac{\partial \Gamma}{\partial z_i}(x,z) \\ &= \frac{1}{\omega_n} \frac{1}{\sqrt{\det(a^{ij}(x))}} \left(\sum_{i,j=1}^n A_{ij}(x) z_i z_j \right)^{-n/2} \left(\sum_{j=1}^n A_{ij}(x) z_j \right)\end{aligned}$$

for all $x \in \widetilde{\Omega}$ and all $z \in \mathbf{R}^n \setminus \{0\}$,

are positively homogeneous of degree $1-n$ with respect to the variable z ($1 \leq i \leq n$). Therefore, by applying Example 10.1 to the functions $\Gamma_i(x,z)$ we obtain that the functions

$$\Gamma_{ij}(x,z) = \frac{\partial^2 \Gamma}{\partial z_i \partial z_j}(x,z), \quad 1 \leq i, j \leq n,$$

are Calderón–Zygmund kernels in the z variable.

Then we have the following existence theorem of singular integrals (see [18, Theorem 2.10]):

Theorem 11.1. *Let $k(x, z)$ be a real-valued function defined on $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ that satisfies conditions (i) and (ii), and let $f \in L^p(\mathbf{R}^n)$ with $1 < p < \infty$. If $\varepsilon > 0$, we let*

$$K_\varepsilon f(x) = \int_{|x-y|>\varepsilon} k(x, x-y) f(y)\, dy.$$

Then there exists a function $Kf \in L^p(\mathbf{R}^n)$ such that

$$\lim_{\varepsilon \downarrow 0} \|K_\varepsilon f - Kf\|_{L^p(\mathbf{R}^n)} = 0. \tag{11.1}$$

Moreover, the operator K is bounded on $L^p(\mathbf{R}^n)$; more precisely, there exists a positive constant $c_3 = c_3(n, p, M)$ such that

$$\|Kf\|_{L^p(\mathbf{R}^n)} \leq c_3 \|f\|_{L^p(\mathbf{R}^n)} \quad \text{for all } f \in L^p(\mathbf{R}^n). \tag{11.2}$$

Proof. By a density argument, it suffices to prove Theorem 11.1 for all $f \in C_0^\infty(\mathbf{R}^n)$. The proof is divided into five steps.

Step 1: First, it should be noticed that the function

$$z \longmapsto |z|^n k(x, z)$$

belongs to $C^\infty(\mathbf{R}^n \setminus \{0\})$ for almost all $x \in \mathbf{R}^n$, and it is positively homogeneous of degree zero and satisfies the condition

$$\int_{\Sigma_{n-1}} k(x, z)\, dz = 0.$$

For each $m = 1, 2, \ldots$ and $k = 1, 2, \ldots, d(m)$, we let

$$a_{km}(x) = \int_{\Sigma_{n-1}} k(x, z) Y_{km}(z)\, dz,$$

then, by the completeness of the spherical harmonics $\{Y_{km}\}$ in $L^2(\Sigma_{n-1})$ it follows that

$$|z|^n k(x, z) = \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} a_{km}(x) Y_{km}(z) \tag{11.3}$$

Moreover, we have, by assertion (4.76) with $r := n$ and assertion (4.75) with $\alpha := 0$ of Theorem 4.31,

$$\|a_{km}\|_{L^\infty(\mathbf{R}^n)} \leq \frac{c_1(n)}{m^{2n}} M, \tag{11.4a}$$

$$\|Y_{km}\|_{L^\infty(\Sigma_{n-1})} \leq c_2(n)\, m^{(n-2)/2}, \tag{11.4b}$$

and, by assertion (4.74) of Theorem 4.31,

$$d(m) \leq c_3(n)\, m^{n-2}. \tag{11.4c}$$

Step 2: Secondly, we have, by the spherical expansion (11.3) with $z := x - y$,

$$\begin{aligned} K_\varepsilon f(x) &= \int_{|x-y|>\varepsilon} k(x, x-y) f(y)\, dy \\ &= \int_{|x-y|>\varepsilon} \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} a_{km}(x) \frac{Y_{km}(x-y)}{|x-y|^n} f(y)\, dy. \end{aligned}$$

However, we obtain from estimates (11.4) that, for almost all $x \in \mathbf{R}^n$ and all $y \in \mathbf{R}^n$ satisfying $|x - y| > \varepsilon$,

$$\begin{aligned} &\left| \sum_{m=1}^{N} \sum_{k=1}^{d(m)} a_{km}(x) \frac{Y_{km}(x-y)}{|x-y|^n} f(y) \right| \\ &\leq \frac{1}{\varepsilon^n} |f(y)| \sum_{m=1}^{N} \sum_{k=1}^{d(m)} \|a_{km}\|_{L^\infty(\Sigma_{n-1})} \|Y_{km}\|_{L^\infty(\Sigma_{n-1})} \\ &\leq \frac{1}{\varepsilon^n} |f(y)| \sum_{m=1}^{N} \sum_{k=1}^{d(m)} \frac{c_1(n) M}{m^{2n}}\, c_2(n)\, m^{(n-2)/2} \\ &\leq \frac{1}{\varepsilon^n} |f(y)| \sum_{m=1}^{\infty} \frac{c_1(n) M}{m^{2n}}\, c_2(n)\, m^{(n-2)/2}\, c_3(n)\, m^{n-2} \\ &= \frac{1}{\varepsilon^n} \left(\sum_{m=1}^{\infty} \frac{1}{m^{n/2+3}} \right) c_1(n)\, c_2(n)\, c_3(n)\, M\, |f(y)| \,. \end{aligned}$$

Therefore, it follows from an application of the Lebesgue dominated convergence theorem (Theorem 3.8) that

$$\begin{aligned} K_\varepsilon f(x) &= \int_{|x-y|>\varepsilon} \lim_{N\to\infty} \sum_{m=1}^{N} \sum_{k=1}^{d(m)} a_{km}(x) \frac{Y_{km}(x-y)}{|x-y|^n} f(y)\, dy \\ &= \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} a_{km}(x) \int_{|x-y|>\varepsilon} \frac{Y_{km}(x-y)}{|x-y|^n} f(y)\, dy \quad \text{in } L^p(\mathbf{R}^n). \end{aligned} \tag{11.5}$$

Step 3: We let

$$R_{km\varepsilon} f(x) = \int_{|x-y|>\varepsilon} \frac{Y_{km}(x-y)}{|x-y|^n} f(y)\, dy.$$

It should be noticed that the function

$$\frac{Y_{km}(z)}{|z|^n}$$

is a Calderón–Zygmund kernel and that

$$\|Y_{km}\|_{L^2(\Sigma_{n-1})} = 1.$$

Therefore, by applying Theorem 10.1 we obtain that there exists a function

$$R_{km} f \in L^p(\mathbf{R}^n)$$

such that

$$\lim_{\varepsilon \downarrow 0} \|R_{km\varepsilon} f - R_{km} f\|_{L^p(\mathbf{R}^n)} = 0. \tag{11.6}$$

Moreover, the operator R_{km} is bounded on $L^p(\mathbf{R}^n)$; more precisely, there exists a positive constant $c_1(n,p)$ such that

$$\|R_{km} f\|_{L^p(\mathbf{R}^n)} \leq c_1(n,p) \, \|f\|_{L^p(\mathbf{R}^n)} . \tag{11.7}$$

Step 4: We show that the series of operators

$$Kf(x) = \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} a_{km}(x) R_{km} f(x)$$

converges absolutely in the L^p space. More precisely, we have the following two assertions (A) and (B):

(A) $Kf \in L^p(\mathbf{R}^n)$.
(B) There exists a constant $c_3(n,p,M) > 0$ such that

$$\|Kf\|_{L^p(\mathbf{R}^n)} \leq c_3(n,p,M) \, \|f\|_{L^p(\mathbf{R}^n)} . \tag{11.2}$$

Indeed, it suffices to note that, by estimates (11.4) and inequality (11.7),

$$\begin{aligned}
&\|Kf\|_{L^p(\mathbf{R}^n)} \\
&\leq \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} \|a_{km} \, R_{km} f\|_{L^p(\mathbf{R}^n)} \\
&\leq \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} \|a_{km}\|_{L^\infty(\Sigma_{n-1})} \, \|R_{km} f\|_{L^p(\mathbf{R}^n)} \\
&\leq \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} \frac{c_1(n) M}{m^{2n}} \, c_1(n,p) \, \|f\|_{L^p(\mathbf{R}^n)}
\end{aligned}$$

$$\leq \sum_{m=1}^{\infty} \frac{c_1(n)M}{m^{2n}} \, c_1(n,p)\, c_3(n)\, m^{n-2} \, \|f\|_{L^p(\mathbf{R}^n)}$$
$$= \left(\sum_{m=1}^{\infty} \frac{1}{m^{n+2}} \right) c_1(n)\, c_1(n,p)\, c_3(n)\, M \, \|f\|_{L^p(\mathbf{R}^n)} \, .$$

This proves the desired inequality (11.2) with

$$c_3(n,p,M) := c_1(n)\, c_1(n,p)\, c_3(n)\, M \left(\sum_{m=1}^{\infty} \frac{1}{m^{n+2}} \right).$$

Step 5: Finally, we prove assertion (11.1)

$$\lim_{\varepsilon \downarrow 0} \|K_\varepsilon f - Kf\|_{L^p(\mathbf{R}^n)} = 0.$$

Indeed, we have, by estimates (11.4) and formula (11.5),

$$\|K_\varepsilon f - Kf\|_{L^p(\mathbf{R}^n)}$$
$$= \left\| \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} a_{km} \, R_{km\varepsilon} f - \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} a_{km} \, R_{km} f \right\|_{L^p(\mathbf{R}^n)}$$
$$= \left\| \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} \left(a_{km} \, R_{km\varepsilon} f - a_{km} \, R_{km} f \right) \right\|_{L^p(\mathbf{R}^n)}$$
$$\leq \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} \|a_{km}\|_{L^\infty(\Sigma_{n-1})} \, \|R_{km\varepsilon} f - R_{km} f\|_{L^p(\mathbf{R}^n)}$$
$$\leq \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} \frac{c_1(n)M}{m^{2n}} \, \|R_{km\varepsilon} f - R_{km} f\|_{L^p(\mathbf{R}^n)}$$
$$\leq \sum_{m=1}^{\infty} \frac{c_1(n)M}{m^{2n}} \, c_3(n)\, m^{n-2} \, \|R_{km\varepsilon} f - R_{km} f\|_{L^p(\mathbf{R}^n)}$$
$$= c_1(n)\, c_3(n)\, M \left(\sum_{m=1}^{\infty} \frac{1}{m^{n+2}} \right) \|R_{km\varepsilon} f - R_{km} f\|_{L^p(\mathbf{R}^n)}$$
$$\leq c_1(n)\, c_3(n)\, M \left(\sum_{m=1}^{N} \frac{1}{m^{n+2}} \right) \|R_{km\varepsilon} f - R_{km} f\|_{L^p(\mathbf{R}^n)}$$
$$+ c_1(n)\, c_3(n)\, M \left(\sum_{m=N+1}^{\infty} \frac{1}{m^{n+2}} \right) \left(\|R_{km\varepsilon} f\|_{L^p(\mathbf{R}^n)} + \|R_{km} f\|_{L^p(\mathbf{R}^n)} \right)$$

$$\leq c_1(n)\, c_3(n)\, M \left(\sum_{m=1}^{N} \frac{1}{m^{n+2}} \right) \|R_{km\varepsilon} f - R_{km} f\|_{L^p(\mathbf{R}^n)}$$
$$+ 2\, c_1(n)\, c_3(n)\, M\, c_1(n,p) \left(\sum_{m=N+1}^{\infty} \frac{1}{m^{n+2}} \right) \|f\|_{L^p(\mathbf{R}^n)}.$$

Hence, we obtain from assertion (11.6) that

$$\limsup_{\varepsilon \downarrow 0} \|K_\varepsilon f - K f\|_{L^p(\mathbf{R}^n)} \leq 2\, c_1(n)\, c_3(n)\, M\, c_1(n,p) \left(\sum_{m=N+1}^{\infty} \frac{1}{m^{n+2}} \right) \|f\|_{L^p(\mathbf{R}^n)}. \tag{11.8}$$

Therefore, the desired assertion (11.1) follows by letting $N \to \infty$ in inequality (11.8).

Now the proof of Theorem 11.1 is complete. □

Let $\varphi \in L^\infty(\mathbf{R}^n)$. If $\varepsilon > 0$ and $f \in L^p(\mathbf{R}^n)$ for $1 < p < \infty$, we define the commutator $C[\varphi, K_\varepsilon]$ by the formula

$$C[\varphi, K_\varepsilon] f := \varphi(K_\varepsilon f) - K_\varepsilon(\varphi f) = \int_{|x-y|>\varepsilon} k(x, x-y)\, [\varphi(x) - \varphi(y)]\, f(y)\, dy.$$

Then we have the following (see [18, Theorem 2.10]):

Theorem 11.2. *Let $f \in L^p(\mathbf{R}^n)$ for $1 < p < \infty$. If $\varphi \in L^\infty(\mathbf{R}^n)$, then there exists a function $C[\varphi, K] f \in L^p(\mathbf{R}^n)$ such that*

$$\lim_{\varepsilon \downarrow 0} \|C[\varphi, K_\varepsilon] f - C[\varphi, K] f\|_{L^p(\mathbf{R}^n)} = 0.$$

Furthermore, the commutator $C[\varphi, K]$ is bounded on $L^p(\mathbf{R}^n)$; more precisely, there exists a positive constant $c_4 = c_4(n,p,M)$ such that

$$\|C[\varphi, K] f\|_{L^p(\mathbf{R}^n)} \leq c_4 \|\varphi\|_* \|f\|_{L^p(\mathbf{R}^n)} \quad \text{for all } f \in L^p(\mathbf{R}^n).$$

The proof of Theorem 11.2 is essentially the same as that of Theorem 11.1 if we use Theorem 10.2 (instead of Theorem 10.1).

11.2 Local Version of Theorems 11.1 and 11.2

Theorems about singular integrals and commutators are usually formulated in the whole space $\mathbf{R}^n$. However, our application to the theory of elliptic equations with discontinuous coefficients will require a local version of Theorems 11.1 and 11.2.

Let Ω be an open subset of $\mathbf{R}^n$, and let $k(x,z)$ be a real-valued function defined on $\Omega \times (\mathbf{R}^n \setminus \{0\})$ that satisfies the following two conditions (i) and (ii):

(i) $k(x,\cdot)$ is a Calderón–Zygmund kernel for almost all $x \in \Omega$.
(ii) The quantity

$$M := \max_{|\alpha| \le 2n} \left\| \frac{\partial^\alpha k}{\partial z^\alpha}(\cdot,\cdot) \right\|_{L^\infty(\Omega \times \Sigma_{n-1})}$$

is finite.

The next existence theorem of singular integrals is a local version of Theorem 11.1 (cf. [18, Theorem 2.11]):

Theorem 11.3. *Let $k(x,z)$ be a real-valued function defined on $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ that satisfies conditions (i) and (ii), and let $f \in L^p(\Omega)$ with $1 < p < \infty$. If $\varepsilon > 0$, we let*

$$K_\varepsilon f(x) := \int_{\substack{y \in \Omega \\ |x-y| > \varepsilon}} k(x, x-y) f(y)\, dy.$$

Then there exists a function $Kf \in L^p(\Omega)$ such that

$$\lim_{\varepsilon \downarrow 0} \|K_\varepsilon f - Kf\|_{L^p(\Omega)} = 0 \quad \text{for every } f \in L^p(\Omega).$$

Moreover, the operator K is bounded on $L^p(\Omega)$; more precisely, there exists a positive constant $c_5 = c_5(n,p,M)$ such that

$$\|Kf\|_{L^p(\Omega)} \le c_5 \|f\|_{L^p(\Omega)} \quad \text{for all } f \in L^p(\Omega).$$

Proof. First, we remark that the function

$$\widetilde{k}(x,z) := \begin{cases} k(x,z) & \text{if } x \in \Omega \text{ and } z \in \mathbf{R}^n \setminus \{0\}, \\ 0 & \text{if } x \notin \Omega \text{ and } z \in \mathbf{R}^n \setminus \{0\} \end{cases}$$

satisfies all the conditions (i) and (ii) of Theorem 11.1.

(i) $\widetilde{k}(x,\cdot)$ is a Calderón–Zygmund kernel for almost all $x \in \mathbf{R}^n$.
(ii) The quantity

$$\begin{aligned} M &:= \max_{|\alpha| \le 2n} \left\| \frac{\partial^\alpha \widetilde{k}}{\partial z^\alpha}(\cdot,\cdot) \right\|_{L^\infty(\mathbf{R}^n \times \Sigma_{n-1})} \\ &= \max_{|\alpha| \le 2n} \left\| \frac{\partial^\alpha k}{\partial z^\alpha}(\cdot,\cdot) \right\|_{L^\infty(\Omega \times \Sigma_{n-1})} \end{aligned}$$

is finite.

Moreover, if $f \in L^p(\Omega)$, we let

$$\widetilde{f}(x) := \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

Then it follows that

$$\left\| \widetilde{f} \right\|_{L^p(\mathbf{R}^n)} = \| f \|_{L^p(\Omega)} .$$

By applying Theorem 11.1, we obtain that there exists a function

$$\widetilde{K} \widetilde{f} \in L^p(\mathbf{R}^n)$$

such that

$$\lim_{\varepsilon \downarrow 0} \left\| \widetilde{K}_\varepsilon \widetilde{f} - \widetilde{K} \widetilde{f} \right\|_{L^p(\mathbf{R}^n)} = 0,$$

and that

$$\left\| \widetilde{K} \widetilde{f} \right\|_{L^p(\mathbf{R}^n)} \leq c_5 \left\| \widetilde{f} \right\|_{L^p(\mathbf{R}^n)} = c_5 \| f \|_{L^p(\Omega)} ,$$

with a positive constant $c_5 = c_5(n, p, M)$.

However, it should be noticed that

$$\begin{aligned} \widetilde{K}_\varepsilon \widetilde{f}(x) &= \int_{|x-y|>\varepsilon} \widetilde{k}(x, x-y) \widetilde{f}(y) \, dy \\ &= \int_{\substack{|x-y|>\varepsilon \\ y \in \Omega}} \widetilde{k}(x, x-y) f(y) \, dy \\ &= \int_{\substack{|x-y|>\varepsilon \\ y \in \Omega}} k(x, x-y) f(y) \, dy \\ &= K_\varepsilon f(x) \quad \text{for almost all } x \in \Omega. \end{aligned}$$

Therefore, if we let

$$Kf = \widetilde{K} \widetilde{f}|_\Omega \in L^p(\Omega),$$

then we have the inequality

$$\| K_\varepsilon f - Kf \|_{L^p(\Omega)} \leq \left\| \widetilde{K}_\varepsilon \widetilde{f} - \widetilde{K} \widetilde{f} \right\|_{L^p(\mathbf{R}^n)} ,$$

and hence

$$\lim_{\varepsilon \downarrow 0} \| K_\varepsilon f - Kf \|_{L^p(\Omega)} = 0 \quad \text{for every } f \in L^p(\Omega).$$

Finally, we obtain that

$$\| Kf \|_{L^p(\Omega)} \leq \left\| \widetilde{K} \widetilde{f} \right\|_{L^p(\mathbf{R}^n)} \leq c_5 \left\| \widetilde{f} \right\|_{L^p(\mathbf{R}^n)}$$

$$= c_5 \|f\|_{L^p(\Omega)} \qquad \text{for all } f \in L^p(\Omega).$$

The proof of Theorem 11.3 is complete. □

The next theorem is a local version of Theorem 11.2 (cf. [18, Theorem 2.11]):

Theorem 11.4. *If $\varphi \in L^\infty(\mathbf{R}^n)$, then the commutator*

$$\begin{aligned} C[\varphi, K] f &:= \varphi(Kf) - K(\varphi f) = \lim_{\varepsilon \downarrow 0} C[\varphi, K_\varepsilon] f \\ &= \lim_{\varepsilon \downarrow 0} \int_{\substack{y \in \Omega \\ |x-y|>\varepsilon}} k(x, x-y) [\varphi(x) - \varphi(y)] f(y)\, dy, \end{aligned}$$

is well-defined for all $f \in L^p(\Omega)$ with $1 < p < \infty$. Moreover, the commutator $C[\varphi, K]$ is bounded on $L^p(\Omega)$; more precisely, there exists a positive constant $c_6 = c_6(n, p, M)$ such that

$$\|C[\varphi, K] f\|_{L^p(\Omega)} \le c_6 \|\varphi\|_* \|f\|_{L^p(\Omega)} \quad \textit{for all } f \in L^p(\Omega).$$

The proof of Theorem 11.4 is essentially the same as that of Theorem 11.3 if we use Theorem 11.2 (instead of Theorem 11.1).

Remark 11.1. Let φ, $\psi \in L^\infty(\mathbf{R}^n)$ such that $\varphi(x) = \psi(x)$ almost everywhere in Ω. Then we have, for all $f \in L^p(\Omega)$,

$$\begin{aligned} &\int_{\substack{y \in \Omega \\ |x-y|>\varepsilon}} k(x, x-y) [\varphi(x) - \varphi(y)] f(y)\, dy \\ &= \int_{\substack{y \in \Omega \\ |x-y|>\varepsilon}} k(x, x-y) [\psi(x) - \psi(y)] f(y)\, dy \quad \text{almost everywhere in } \Omega. \end{aligned}$$

Therefore, we have, for all $f \in L^p(\Omega)$,

$$C[\varphi, K] f = C[\psi, K] f \quad \text{almost everywhere in } \Omega.$$

The next result asserts that the norm of singular commutators can be made small if $\varphi \in \mathrm{VMO}$ (see [18, Theorem 2.13]):

Corollary 11.5. *Let $\varphi \in \mathrm{VMO} \cap L^\infty(\mathbf{R}^n)$ and η the* VMO *modulus of φ. Then, for each $\varepsilon > 0$, there exists a positive constant $\rho_0 = \rho_0(\varepsilon, \eta)$ such that, for any ball B_r of radius r, $0 < r < \rho_0$, contained in Ω, we have the inequality*

$$\|C[\varphi, K] f\|_{L^p(B_r)} \le c_7\, \varepsilon \|f\|_{L^p(B_r)} \quad \textit{for all } f \in L^p(B_r), \tag{11.9}$$

with a positive constant $c_7 = c_7(n, p, M)$.

Proof. The proof of Corollary 11.5 is divided into three steps.

Step 1: By using Theorem 4.3, for each $\varepsilon > 0$ we can find a bounded, uniformly continuous function $a(x)$ on $\mathbf{R}^n$ such that

$$\|a - \varphi\|_* < \frac{\varepsilon}{2}. \tag{11.10}$$

Let $\omega_a(r)$ be the modulus of uniform continuity of $a(x)$ defined by the formula

$$\omega_a(r) = \sup_{|x-y|\leq r} |a(x) - a(y)|,$$

and choose a constant $\rho_0 = \rho_0(\varepsilon, \eta) > 0$ such that

$$\omega_a(\rho_0) < \frac{\varepsilon}{2}. \tag{11.11}$$

If $B_r = B_r(x_0)$ is a ball of radius r about x_0, we let

$$b(x) := \begin{cases} a(x) & \text{if } x \in B_r(x_0), \\ a\left(x_0 + r\frac{x-x_0}{|x-x_0|}\right) & \text{if } x \in \mathbf{R}^n \setminus B_r(x_0). \end{cases}$$

It should be noticed that the function $b(x)$ is uniformly continuous on $\mathbf{R}^n$ and that the oscillation of $b(x)$ in $\mathbf{R}^n$ equals the oscillation of $a(x)$ in B_r.

Step 2: Now we remark that

$$\begin{aligned} C[\varphi, K]f &= \varphi(Kf) - K(\varphi f) \\ &= (\varphi - a)Kf - K((\varphi - a)f) + a(Kf) - K(af) \\ &= C[\varphi - a, K]f + C[a, K]f, \end{aligned}$$

so that

$$\|C[\varphi, K]f\|_{L^p(B_r)} \leq \|C[\varphi - a, K]f\|_{L^p(B_r)} + \|C[a, K]f\|_{L^p(B_r)}.$$

However, we have, by Theorem 11.4 with $\Omega := B_r$,

$$\|C[\varphi - a, K]f\|_{L^p(B_r)} \leq c_7 \|\varphi - a\|_* \|f\|_{L^p(B_r)}, \tag{11.12}$$

and also, by Remark 11.1,

$$\begin{aligned} \|C[a, K]f\|_{L^p(B_r)} &= \|C[b, K]f\|_{L^p(B_r)} \\ &\leq c_7 \|b\|_* \|f\|_{L^p(B_r)} \quad \text{for all } f \in L^p(B_r). \end{aligned} \tag{11.13}$$

Moreover, it is easy to see that

$$\|b\|_* \leq \omega_b(r) = \omega_a(r). \tag{11.14}$$

Hence, by combining inequalities (11.13) and (11.14) we obtain that

$$\|C\,[a,K]\,f\|_{L^p(B_r)} \le c_7\,\omega_a(r)\ \|f\|_{L^p(B_r)} \quad \text{for all } f \in L^p(B_r). \quad (11.15)$$

Step 3: Therefore, we obtain from inequalities (11.12) and (11.15) that, for all $0 < r < \rho_0$,

$$\begin{aligned}\|C\,[\varphi,K]\,f\|_{L^p(B_r)} &\le \|C\,[\varphi - a,K]\,f\|_{L^p(B_r)} + \|C\,[a,K]\,f\|_{L^p(B_r)} \\ &\le c_7\ \|\varphi - a\|_*\ \|f\|_{L^p(B_r)} + c_7\,\omega_a(r)\ \|f\|_{L^p(B_r)} \\ &\le c_7\,(\|\varphi - a\|_* + \omega_a(\rho_0))\,\|f\|_{L^p(B_r)}\,.\end{aligned}$$

By inequalities (11.10) and (11.11), this inequality proves the desired inequality (11.9).

The proof of Corollary 11.5 is complete. □

Remark 11.2. Roughly speaking, inequality (11.9) may be expressed as follows:

$$\|C\,[\varphi,K]\,f\|_{L^p(B_r)} \le c_8\,\eta(r)\ \|f\|_{L^p(B_r)} \quad \text{for all } f \in L^p(B_r), \quad (11.9')$$

with a positive constant $c_8 = c_8(n,p,M)$. Here

$$\eta(r) = \sup_{\rho \le r} \frac{1}{|B|} \int_B |\varphi(x) - \varphi_B|\,dx$$

is the VMO modulus of φ (see Section 4.2).

11.3 Notes and Comments

The results of this chapter are adapted from Chiarenza–Frasca–Longo [18].

Part IV

Dirichlet Problems for Elliptic Differential Equations with Discontinuous Coefficients

12
Dirichlet Problems in Sobolev Spaces

The purpose of this chapter is to formulate the *homogeneous* Dirichlet problem in the framework of L^p Sobolev spaces. We state interior and global *a priori* estimates for the Dirichlet problem (Theorems 12.1 and 12.2) that will play an essential role in the proof of the unique solvability theorem for the homogeneous Dirichlet problem (Theorem 15.1) in Chapter 15.

12.1 Formulation of the Dirichlet Problem

Let Ω be a bounded domain in Euclidean space $\mathbf{R}^n$, $n \geq 3$, with boundary $\partial\Omega$ of class $C^{1,1}$. If $1 < p < \infty$ and if $k = 1$ or $k = 2$, then we define the Sobolev space

$$\begin{aligned} W^{k,p}(\Omega) = & \text{ the space of (equivalence classes of) functions} \\ & u \in L^p(\Omega) \text{ whose derivatives } D^\alpha u, \ |\alpha| \leq k, \text{ in the} \\ & \text{sense of distributions are in } L^p(\Omega), \end{aligned}$$

and the boundary space

$$B^{k-1/p,p}(\partial\Omega) = \text{ the space of the traces } \gamma_0 u \text{ of functions } u \in W^{k,p}(\Omega).$$

In the space $B^{k-1/p,p}(\partial\Omega)$, we introduce a norm

$$|\varphi|_{B^{k-1/p,p}(\partial\Omega)} = \inf \left\{ \|u\|_{W^{k,p}(\Omega)} : u \in W^{k,p}(\Omega), \ \gamma_0 u = \varphi \text{ on } \partial\Omega \right\}.$$

We recall that the space $B^{k-1/p,p}(\partial\Omega)$ is a Besov space (see the trace theorem (Theorem 7.6)).

Moreover, it should be emphasized (see [2, Theorem 5.37]) that the closure $W_0^{1,p}(\Omega)$ of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$ can be characterized as follows:

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega) : \gamma_0 u = 0 \text{ on } \partial\Omega \right\}.$$

Now we consider a second-order, elliptic differential operator $\mathcal{L}$ with real *discontinuous* coefficients of the form

$$\mathcal{L}u := \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

More precisely, we assume that the coefficients $a^{ij}(x)$ satisfy the following three conditions (i), (ii) and (iii):

(i) $a^{ij}(x) \in \mathrm{VMO} \cap L^\infty(\mathbf{R}^n)$ for all $1 \leq i, j \leq n$.
(ii) $a^{ij}(x) = a^{ji}(x)$ for all $1 \leq i, j \leq n$ and for almost all $x \in \Omega$.
(iii) There exists a positive constant λ such that

$$\frac{1}{\lambda} |\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2$$

for almost all $x \in \Omega$ and for all $\xi \in \mathbf{R}^n$.

In this chapter we study the following *homogeneous* Dirichlet problem in the framework of L^p Sobolev spaces:

$$\begin{cases} \mathcal{L}u = \sum_{i,j=1}^n a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f & \text{in } \Omega, \\ \gamma_0 u = 0 & \text{on } \partial\Omega. \end{cases} \tag{12.1}$$

More precisely, for a given function $f \in L^p(\Omega)$ we find a function $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ that satisfies the equation $\mathcal{L}u = f$ almost everywhere in Ω.

If $\widetilde{\Omega}$ is the subset of Ω where conditions (ii) and (iii) hold true, then we let

$$\Gamma(x, z) := \frac{1}{(2-n)\omega_n} \frac{1}{\sqrt{\det(a^{ij}(x))}} \left(\sum_{i,j=1}^{n} A_{ij}(x) z_i z_j \right)^{(2-n)/2},$$
$$x \in \widetilde{\Omega}, \; z \in \mathbf{R}^n \setminus \{0\}.$$

Here:

$$(A_{ij}(x)) = \text{ the inverse matrix of } (a^{ij}(x)),$$
$$\omega_n := |\Sigma_{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$
(the surface area of the unit sphere Σ_{n-1} in $\mathbf{R}^n$).

Moreover, we let

$$\Gamma_i(x,z) := \frac{\partial \Gamma}{\partial z_i}(x,z) \quad \text{for } x \in \widetilde{\Omega},\ z \in \mathbf{R}^n \setminus \{0\} \text{ and } 1 \le i \le n,$$

$$\Gamma_{ij}(x,z) := \frac{\partial^2 \Gamma}{\partial z_i \partial z_j}(x,z) \quad \text{for } x \in \widetilde{\Omega},\ z \in \mathbf{R}^n \setminus \{0\} \text{ and } 1 \le i,j \le n,$$

and

$$M := \max_{1 \le i,j \le n} \max_{|\alpha| \le 2n} \left\| \frac{\partial^\alpha}{\partial z^\alpha} \Gamma_{ij}(\cdot,\cdot) \right\|_{L^\infty(\Omega \times \Sigma_{n-1})}.$$

Finally, if $\eta^{ij}(r)$ are the VMO moduli of $a^{ij}(x)$, we let

$$\eta(r) := \left(\sum_{i,j=1}^{n} \eta^{ij}(r)^2 \right)^{1/2}.$$

12.2 Statement of Main Results (Theorems 12.1 and 12.2)

The next *interior* $W^{2,p}$ estimate plays an important role in the proof of the existence theorem for the Dirichlet problem (12.1) in Chapter 15 (see [18, Theorem 4.2]):

Theorem 12.1 (the interior regularity theorem). *Let $1 < q < p < \infty$ and $f \in L^p(\Omega)$. If a function $u \in W^{2,q}_{\text{loc}}(\Omega)$ satisfies the equation*

$$\mathcal{L}u = f \quad \text{in } \Omega,$$

then it follows that $u \in W^{2,p}_{\text{loc}}(\Omega)$. Moreover, for any open subsets $\Omega' \Subset \Omega'' \Subset \Omega$ (see Figure 12.1), we have the interior a priori *estimate*

$$\|u\|_{W^{2,p}(\Omega')} \le C_1 \left(\|u\|_{L^p(\Omega'')} + \|f\|_{L^p(\Omega)} \right), \tag{12.2}$$

with a positive constant $C_1 = C_1(n, p, M, \text{dist}(\Omega', \partial\Omega''), \lambda, \eta)$.

The next *global* $W^{2,p}$ estimate plays an essential role in the proof of the existence theorem for the Dirichlet problem (12.1) in Chapter 14 (see [19, Theorem 4.2]):

Theorem 12.2 (the global regularity theorem). *Let $1 < q < p < \infty$ and $f \in L^p(\Omega)$. If a function $u \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$ satisfies the equation*

$$\mathcal{L}u = f \quad \text{in } \Omega,$$

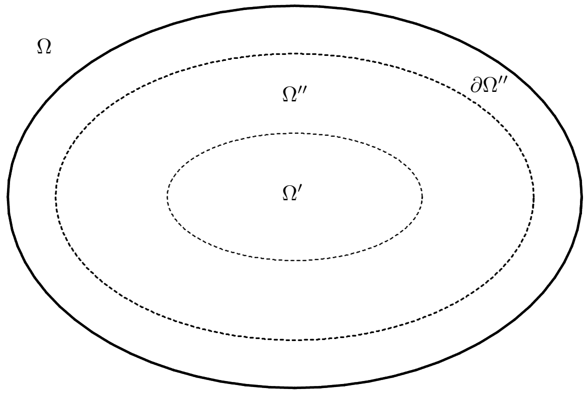

Fig. 12.1. The subsets Ω', Ω'' of Ω such that $\Omega' \Subset \Omega'' \Subset \Omega$

then it follows that $u \in W^{2,p}(\Omega)$. *Moreover, we have the global* a priori *estimate*

$$\|u\|_{W^{2,p}(\Omega)} \leq C_2 \left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} \right), \tag{12.3}$$

with a positive constant $C_2 = C_2(n, p, M, \partial\Omega, \lambda, \eta)$.

The desired interior *a priori* estimate (12.2) and global *a priori* estimate (12.3) are consequences of the following two facts (I) and (II):

(I) The explicit representation formulas for the solutions of the homogeneous Dirichlet problem (Chapters 13 and 14).
(II) An L^p boundedness of some singular integral operators appearing in those formulas (Chapter 14).

It is worthwhile pointing out here that VMO functions are invariant under $C^{1,1}$-diffeomorphisms (see [1, Proposition 1.3]).

The proofs of Theorems 12.1 and 12.2 can be visualized in the following diagrams:

Remark 12.1. There are classical results by Miranda [50] in the case of $W^{1,n}$ coefficients and $p = 2$ and by Meyers [49] and Talenti [88] in the two dimensional case ($n = 2$), respectively.

12.3 Notes and Comments

The results of this chapter are adapted from Chiarenza–Frasca–Longo [18] and [19]. Our approach can be traced back to the pioneering work of

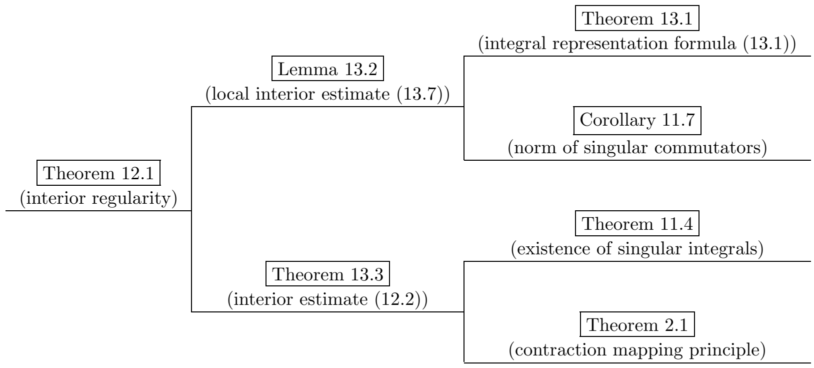

Table 12.1. *A flowchart for the proof of Theorem 12.1*

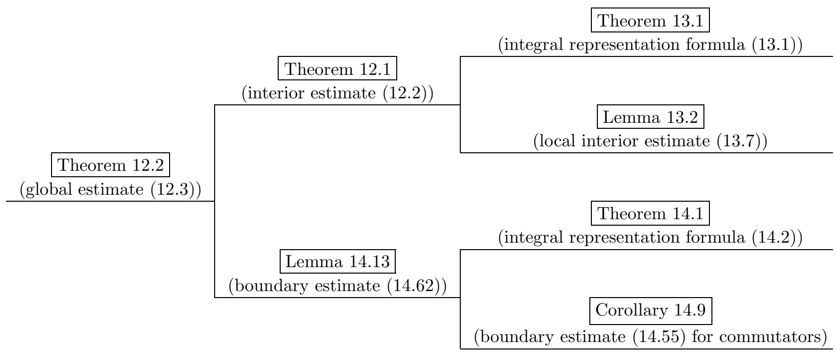

Table 12.2. *A flowchart for the proof of Theorem 12.2*

Schauder [64] and [65] on the Dirichlet problem for second-order, elliptic differential operators.

13

Calderón–Zygmund Kernels and Interior Estimates

This chapter is devoted to the proof of Theorem 12.1 (Theorem 13.3) that is based on some local interior *a priori* estimates for the solutions of the homogeneous Dirichlet problem (Lemma 13.2). The main idea of proof may be considered as an integral perturbation about the constant coefficient case, which goes back to Eugenio Elia Levi [42] (Theorem 13.1). The VMO assumption on the coefficients is of the greatest relevance in the study of an error term expressed by singular commutators (Corollary 11.5). The desired interior *a priori* estimate (12.2) follows in a standard way from Lemma 13.2 by a covering argument if we make use of the following three theorems:

(1) Sobolev's imbedding theorem (Theorem 7.3).
(2) The contraction mapping principle (Theorem 2.1).
(3) The interpolation inequality (Theorem 13.4).

13.1 Interior Representation Formula for Solutions

In this chapter we consider a second-order, elliptic differential operator with variable coefficients

$$\mathcal{L} = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}.$$

We assume that the coefficients a^{ij} satisfy the following three conditions (i), (ii) and (iii): Let B be an open ball of $\mathbf{R}^n$.

(i) $a^{ij}(x) \in \mathrm{VMO} \cap L^\infty(\mathbf{R}^n)$ for all $1 \leq i, j \leq n$.
(ii) $a^{ij}(x) = a^{ji}(x)$ for all $1 \leq i, j \leq n$ and for almost all $x \in B$.

(iii) There exists a positive constant λ such that

$$\frac{1}{\lambda}\,|\xi|^2 \le \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \le \lambda|\xi|^2$$

for almost all $x \in B$ and for all $\xi \in \mathbf{R}^n$.

If $\widetilde{B}$ is the subset of B where conditions (ii) and (iii) hold true, then we let

$$\Gamma(x,z) := \frac{1}{(2-n)\omega_n}\,\frac{1}{\sqrt{\det(a^{ij}(x))}}\left(\sum_{i,j=1}^{n} A_{ij}(x)z_iz_j\right)^{(2-n)/2},$$

for all $x \in \widetilde{B}$ and all $z \in \mathbf{R}^n \setminus \{0\}$.

Here:

$$(A_{ij}(x)) = \text{ the inverse matrix of } (a^{ij}(x)),$$

$$\omega_n := |\Sigma_{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

(the surface area of the unit sphere Σ_{n-1} in $\mathbf{R}^n$).

It should be emphasized that the function $\Gamma(x_0,z)$, $x_0 \in \widetilde{B}$, is a fundamental solution for the constant coefficients elliptic differential operator

$$\mathcal{L}_0 = \sum_{i,j=1}^{n} a^{ij}(x_0)\frac{\partial^2}{\partial x_i \partial x_j}.$$

Moreover, we recall that the functions

$$\begin{aligned}\Gamma_i(x,z) &= \frac{\partial\Gamma}{\partial z_i}(x,z)\\ &= \frac{1}{\omega_n}\,\frac{1}{\sqrt{\det(a^{ij}(x))}}\left(\sum_{i,j=1}^{n} A_{ij}(x)z_iz_j\right)^{-n/2}\left(\sum_{j=1}^{n} A_{ij}(x)z_j\right),\end{aligned}$$

for $x \in \widetilde{B}$, $z \in \mathbf{R}^n \setminus \{0\}$ and $1 \le i \le n$,

are positively homogeneous of degree $1-n$ with respect to the variable z, so that the functions

$$\Gamma_{ij}(x,z) = \frac{\partial^2\Gamma}{\partial z_i \partial z_j}(x,z) \quad \text{for } x \in \widetilde{B},\ z \in \mathbf{R}^n \setminus \{0\} \text{ and } 1 \le i,j \le n,$$

are Calderón–Zygmund kernels in the z variable (see Example 10.1).

The next theorem gives *integral representation formulas* for the second derivatives of solutions of the homogeneous Dirichlet problem for the elliptic differential operator $\mathcal{L}$ with variable coefficients (see [18, Theorem 3.1]):

Theorem 13.1 (the integral representation formula)**.** *Let B be an open ball of $\mathbf{R}^n$ and assume that $u \in W_0^{2,p}(B)$ for $1 < p < \infty$. If $\widetilde{B}$ is the subset of B where conditions (ii) and (iii) hold true, then we have, for all $x \in \widetilde{B}$,*

$$\begin{aligned}\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \text{v. p.} \int_B \Gamma_{ij}(x, x-y) & \qquad (13.1)\\ \times \left(\sum_{h,k=1}^{n} \left[a^{hk}(x) - a^{hk}(y) \right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + \mathcal{L}u(y) \right) dy & \\ + \mathcal{L}u(x) \left(\int_{|t|=1} \Gamma_i(x,t) t_j \, d\sigma_t \right) \quad \textit{for } 1 \le i, j \le n. &\end{aligned}$$

Proof. By a *density argument*, it suffices to prove formula (13.1) for all $u \in C_0^\infty(B)$. Indeed, the general case can be proved by using Theorems 11.3 and 11.4. The proof is divided into three steps.

Step 1: If x_0 is an arbitrary point of $\widetilde{B}$, we consider the constant coefficients elliptic differential operator

$$\mathcal{L}_0 = \sum_{i,j=1}^{n} a^{ij}(x_0) \frac{\partial^2}{\partial x_i \partial x_j}.$$

Then it follows that

$$\mathcal{L}_0 u(x) = \sum_{i,j=1}^{n} a^{ij}(x_0) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \in C_0^\infty(B).$$

Moreover, we have the formula

$$\begin{aligned}\mathcal{L}_0 u(y) &= \sum_{i,j=1}^{n} \left[a^{ij}(x_0) - a^{ij}(y) \right] \frac{\partial^2 u}{\partial x_i \partial x_j}(y) + \sum_{i,j=1}^{n} a^{ij}(y) \frac{\partial^2 u}{\partial x_i \partial x_j}(y) \\ &= (\mathcal{L}_0 - \mathcal{L}) u(y) + \mathcal{L}u(y) \quad \text{for } y \in B.\end{aligned}$$

This implies that

$$\begin{aligned}u(x) &= \text{v. p.} \int_B \Gamma(x_0, x-y) \mathcal{L}_0 u(y) \, dy \qquad (13.2)\\ &= \text{v. p.} \int_B \Gamma(x_0, x-y)\end{aligned}$$

$$\times \left(\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y) \right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + \mathcal{L}u(y) \right) dy.$$

We introduce a function $g(x) \in C_0^\infty(\mathbf{R}^n)$ by the formula

$$g(y) = \begin{cases} \mathcal{L}_0 u(y) = \sum_{i,j=1}^{n} a^{ij}(x_0) \frac{\partial^2 u}{\partial x_i \partial x_j}(y) & \text{if } y \in B, \\ 0 & \text{if } y \notin B. \end{cases}$$

Then it follows that

$$u(x) = \text{v. p.} \int_{\mathbf{R}^n} \Gamma(x_0, x-y) g(y)\, dy = (\Gamma * g)(x) \quad \text{for } x \in \mathbf{R}^n.$$

Hence we have, by the Fourier transform,

$$\widehat{u}(\xi) = \widehat{\Gamma}(x_0, \xi)\widehat{g}(\xi) \quad \text{for } \xi \in \mathbf{R}^n,$$

and

$$\xi_i \xi_j \widehat{u}(\xi) = \xi_i \xi_j \widehat{\Gamma}(x_0, \xi)\widehat{g}(\xi) \quad \text{for } \xi \in \mathbf{R}^n \text{ and } 1 \le i, j \le n. \tag{13.3}$$

Step 2: Secondly, we prove that

$$-\xi_i \xi_j \widehat{\Gamma}(x_0, \xi) = \widehat{\text{v. p.}\, \Gamma_{ij}}(x_0, \xi) + c_{ij}(x_0)\, \widehat{\delta}(\xi) \quad \text{for } \xi \in \mathbf{R}^n \text{ and } 1 \le i, j \le n, \tag{13.4}$$

where $\delta(x)$ is the Dirac measure at 0 on $\mathbf{R}^n$ and

$$c_{ij}(x_0) := \int_{|y|=1} \Gamma_i(x_0, y) y_j \, d\sigma_y.$$

Indeed, it follows that we have, for all $\varphi \in \mathcal{S}(\mathbf{R}^n)$,

$$\begin{aligned} -\left\langle \xi_i \xi_j \widehat{\Gamma}(x_0, \xi), \varphi(\xi) \right\rangle &= -\left\langle \Gamma(x_0, t), \widehat{\xi_i \xi_j \varphi}(t) \right\rangle \\ &= \left\langle \Gamma(x_0, t), \frac{\partial^2}{\partial t_i \partial t_j} \widehat{\varphi}(t) \right\rangle \\ &= \left\langle \frac{\partial^2}{\partial t_i \partial t_j} \Gamma(x_0, t), \widehat{\varphi}(t) \right\rangle \\ &= \langle \Gamma_{ij}(x_0, t), \widehat{\varphi}(t) \rangle \quad \text{for } 1 \le i, j \le n. \end{aligned} \tag{13.5}$$

On the other hand, we have, for all $\varphi \in \mathcal{S}(\mathbf{R}^n)$,

$$\begin{aligned} &\langle \Gamma_{ij}(x_0, t), \widehat{\varphi}(t) \rangle \\ &= \left\langle \frac{\partial^2}{\partial t_i \partial t_j} \Gamma(x_0, t), \psi(t) \right\rangle = \left\langle \frac{\partial}{\partial t_j} \Gamma_i(x_0, t), \widehat{\varphi}(t) \right\rangle \end{aligned}$$

$$= -\left\langle \Gamma_i(x_0,t), \frac{\partial \widehat{\varphi}}{\partial t_j}(t) \right\rangle = -\int_{\mathbf{R}^n} \Gamma_i(x_0,t)\frac{\partial \widehat{\varphi}}{\partial t_j}(t)\,dt$$

$$= -\lim_{\varepsilon \downarrow 0} \int_{|t|>\varepsilon} \Gamma_i(x_0,t)\frac{\partial \widehat{\varphi}}{\partial t_j}(t)\,dt$$

$$= \lim_{\varepsilon \downarrow 0} \left\{ \int_{|t|>\varepsilon} \frac{\partial}{\partial t_j}\Gamma_i(x_0,t)\widehat{\varphi}(t)\,dt + \int_{|t|=\varepsilon} \Gamma_i(x_0,t)\frac{t_j}{|t|}\widehat{\varphi}(t)\,d\sigma_y \right\}$$

$$= \langle \text{v. p.}\Gamma_{ij}(x_0,t), \widehat{\varphi}\rangle + \lim_{\varepsilon \downarrow 0} \int_{|y|=1} \Gamma_i(x_0,y) y_j \widehat{\varphi}(\varepsilon y)\,d\sigma_y$$

$$= \langle \text{v. p.}\Gamma_{ij}(x_0,t), \widehat{\varphi}\rangle + \left(\int_{|y|=1} \Gamma_i(x_0,y) y_j\,d\sigma_y \right) \widehat{\varphi}(0),$$

so that

$$\Gamma_{ij}(x_0,t) = \text{v. p.}\Gamma_{ij}(x_0,t) + c_{ij}(x_0)\delta(t) \quad \text{for } 1 \le i,j \le n. \tag{13.6}$$

Therefore, by combining formulas (13.5) and (13.6) we obtain that

$$\begin{aligned} -\left\langle \xi_i\xi_j\widehat{\Gamma}(x_0,\xi), \varphi(\xi)\right\rangle &= \langle \Gamma_{ij}(x_0,t), \widehat{\varphi}(t)\rangle \\ &= \langle \text{v. p.}\Gamma_{ij}(x_0,t) + c_{ij}(x_0)\delta(t), \widehat{\varphi}(t)\rangle \\ &= \left\langle \widehat{\text{v. p.}\Gamma}_{ij}(x_0,\xi) + c_{ij}(x_0)\widehat{\delta}(\xi), \varphi(\xi)\right\rangle \\ &\quad \text{for all } \varphi \in \mathcal{S}(\mathbf{R}^n). \end{aligned}$$

This proves the desired formula (13.4).

Step 3: By formulas (13.3) and (13.4), it follows that

$$\begin{aligned} \widehat{\frac{\partial^2 u}{\partial x_i \partial x_j}} &= -\xi_i\xi_j\widehat{u}(\xi) = -\xi_i\xi_j\widehat{\Gamma}(x_0,\xi)\widehat{g}(\xi) \\ &= \left(\widehat{\text{v. p.}\Gamma}_{ij}(x_0,\xi) + c_{ij}(x_0)\widehat{\delta}(\xi)\right)\widehat{g}(\xi) \\ &= \mathcal{F}\left(\text{v. p.}\Gamma_{ij}(x_0,\cdot) * g\right)(\xi) + c_{ij}(x_0)\widehat{\delta * g}(\xi) \\ &= \mathcal{F}\left(\text{v. p.}\Gamma_{ij}(x_0,\cdot) * g\right)(\xi) + c_{ij}(x_0)\widehat{g}(\xi). \end{aligned}$$

Hence we have, by the Fourier inversion formula,

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \lim_{\varepsilon \downarrow 0} \int_{|x-y|>\varepsilon} \Gamma_{ij}(x_0, x-y) g(y)\,dy + c_{ij}(x_0) g(x) \quad \text{for } x \in \mathbf{R}^n.$$

In particular, by taking

$$x = x_0 \in \widetilde{B} \subset B,$$

we obtain that

$$\begin{aligned}\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) &= \text{v. p.} \int_B \Gamma_{ij}(x_0, x_0 - y)\mathcal{L}_0 u(y)\, dy + c_{ij}(x_0)\mathcal{L}_0 u(x_0) \\ &= \text{v. p.} \int_B \Gamma_{ij}(x_0, x_0 - y) \\ &\quad \times \left[\sum_{i,j=1}^{n} \left[a^{ij}(x_0) - a^{ij}(y)\right] \frac{\partial^2 u}{\partial x_i \partial x_j}(y) + \mathcal{L}u(y) \right] dy \\ &\quad + c_{ij}(x_0)\mathcal{L}u(x_0) \quad \text{for } 1 \leq i, j \leq n.\end{aligned}$$

This proves the representation formula (13.1) with $x := x_0$.

The proof of Theorem 13.1 is complete. □

13.2 Local Interior Estimates

Let Ω be an open subset of $\mathbf{R}^n$, $n \geq 3$, and we assume that the functions $a^{ij}(x) \in \mathrm{VMO} \cap L^\infty(\mathbf{R}^n)$, $1 \leq i, j \leq n$, satisfy the following three conditions (i), (ii) and (iii):

(i) $a^{ij}(x) \in \mathrm{VMO} \cap L^\infty(\mathbf{R}^n)$ for all $1 \leq i, j \leq n$.

(ii) $a^{ij}(x) = a^{ji}(x)$ for all $1 \leq i, j \leq n$ and for almost all $x \in \Omega$.

(iii) There exist a positive constant λ such that

$$\frac{1}{\lambda}|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i \xi_j \leq \lambda |\xi|^2$$
$$\text{for almost all } x \in \Omega \text{ and all } \xi \in \mathbf{R}^n.$$

If $\widetilde{\Omega}$ is the subset of Ω where conditions (ii) and (iii) hold true, then we let

$$\Gamma(x, z) := \frac{1}{(2-n)\omega_n} \frac{1}{\sqrt{\det(a^{ij}(x))}} \left(\sum_{i,j=1}^{n} A_{ij}(x) z_i z_j \right)^{(2-n)/2}$$
$$\text{for all } x \in \widetilde{\Omega} \text{ and all } z \in \mathbf{R}^n \setminus \{0\},$$

and

$$M := \max_{1 \leq i,j \leq n} \max_{|\alpha| \leq 2n} \left\| \frac{\partial^\alpha}{\partial z^\alpha} \Gamma_{ij}(\cdot, \cdot) \right\|_{L^\infty(\Omega \times \Sigma_{n-1})}.$$

Moreover, if $\eta^{ij}(r)$ is the VMO modulus of $a^{ij}(x)$, we let

$$\eta(r) := \left(\sum_{i,j=1}^{n} \eta^{ij}(r)^2 \right)^{1/2}.$$

Then we can prove local interior *a priori* estimates for the second derivatives of solutions of the homogeneous Dirichlet problem for the elliptic differential operator

$$\mathcal{L} = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$$

with variable coefficients (cf. [18, Lemma 4.1]):

Lemma 13.2. *Let $1 < p < \infty$ and assume that*

$$M = \max_{1 \le i,j \le n} \max_{|\alpha| \le 2n} \left\| \frac{\partial^\alpha}{\partial z^\alpha} \Gamma_{ij}(\cdot,\cdot) \right\|_{L^\infty(\Omega \times \Sigma_{n-1})} < \infty.$$

Then there exist positive constants $C = C(n,p,M)$ and $\rho_0 = \rho_0(C,\eta)$ such that, for any $u \in W_0^{2,p}(B_r)$ in an open ball B_r of radius r with $0 < r < \rho_0$, contained in Ω (see Figure 13.1), we have the interior a priori *estimate*

$$\left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^p(B_r)} \le C \, \|\mathcal{L}u\|_{L^p(B_r)} \quad \text{for all } 1 \le i,j \le n. \tag{13.7}$$

In other words, the second-order, elliptic differential operator $\mathcal{L}$ controls all second-order partial derivatives in the L^p-norm for $1 < p < \infty$ in the open ball B_r (cf. Remark 9.2).

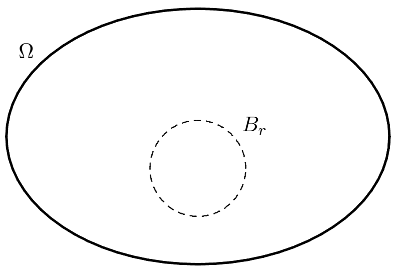

Fig. 13.1. The open ball B_r of radius r contained in Ω

Proof. By Theorem 13.1, we can represent the second derivatives

$$\frac{\partial^2 u}{\partial x_i \partial x_j}$$

of u in the following form:

$$\begin{aligned}\frac{\partial^2 u}{\partial x_i \partial x_j}(x) &= \text{v. p.} \int_{B_r} \Gamma_{ij}(x, x-y) \\ &\quad \times \left(\sum_{h,k=1}^{n} \left[a^{hk}(x) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + \mathcal{L}u(y) \right) dy \\ &\quad + \left(\int_{|t|=1} \Gamma_i(x,t) t_j \, d\sigma_t \right) \mathcal{L}u(x) \\ &:= \sum_{h,k=1}^{n} C\left[a^{hk}, K_{ij}\right] \left(\frac{\partial^2 u}{\partial x_h \partial x_k} \right)(x) + K_{ij}(\mathcal{L}u)(x) \\ &\quad + c_{ij}(x)\ \mathcal{L}u(x) \quad \text{for } 1 \le i, j \le n,\end{aligned} \tag{13.8}$$

where

$$K_{ij} f(x) = \text{v. p.} \int_{B_r} \Gamma_{ij}(x, x-y) f(y) \, dy,$$
$$C\left[\varphi, K_{ij}\right] = \text{v. p.} \int_{B_r} \Gamma_{ij}(x, x-y)[\varphi(x) - \varphi(y)] f(y) \, dy,$$

and

$$c_{ij}(x) = \int_{|t|=1} \Gamma_i(x,t) t_j \, d\sigma_t.$$

We can estimate the three terms on the right-hand side of formula (13.8) as follows:

(1) First, we have, for some positive constant c_1,

$$\|\Gamma_i(\cdot,\cdot)\|_{L^\infty(\Omega \times \Sigma_{n-1})} \le c_1 \quad \text{for } 1 \le i \le n,$$

and hence

$$\|c_{ij}\, \mathcal{L}u\|_{L^p(B_r)} \le C_1 \,\|\mathcal{L}u\|_{L^p(B_r)} \quad \text{for } 1 \le i, j \le n,$$

for some positive constant C_1.

(2) Secondly, by applying Theorem 11.3 with

$$\Omega := B_r, \quad k(x,z) := \Gamma_{ij}(x,z),$$

we obtain that

$$\|K_{ij}(\mathcal{L}u)\|_{L^p(B_r)} \le C_2 \,\|\mathcal{L}u\|_{L^p(B_r)} \quad \text{for } 1 \le i, j \le n.$$

(3) Thirdly, we show that the norm of singular commutators can be made small if the coefficients have a small integral oscillation.

Indeed, by applying Corollary 11.5 with

$$k(x,z) := \Gamma_{ij}(x,z), \quad \varphi(x) := a^{hk}(x),$$

we obtain from Remark 11.2 that

$$\left\| C\left[a^{hk}, K_{ij}\right] \frac{\partial^2 u}{\partial x_h \partial x_k} \right\|_{L^p(B_r)} \leq c_2\, \eta^{hk}(r) \left\| \frac{\partial^2 u}{\partial x_h \partial x_k} \right\|_{L^p(B_r)}.$$

Hence it follows that

$$\left\| C\left[a^{hk}, K_{ij}\right] \frac{\partial^2 u}{\partial x_h \partial x_k} \right\|_{L^p(B_r)} \leq c_3\, \eta(r) \left\| \nabla^2 u \right\|_{L^p(B_r)} \quad \text{for } 1 \leq h,k \leq n.$$

Summing up, we have proved that

$$\left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^p(B_r)} \leq c_3\, \eta(r) \left\| \nabla^2 u \right\|_{L^p(B_r)} + C_3 \left\| \mathcal{L}u \right\|_{L^p(B_r)} \quad \text{for all } u \in W_0^{2,p}(B_r) \text{ and } 1 \leq i,j \leq n.$$

This implies that

$$\begin{aligned} &\left\| \nabla^2 u \right\|_{L^p(B_r)} \\ &\leq n^2 c_3\, \eta(r) \left\| \nabla^2 u \right\|_{L^p(B_r)} + n^2\, C_3 \left\| \mathcal{L}u \right\|_{L^p(B_r)} \quad \text{for all } u \in W_0^{2,p}(B_r). \end{aligned} \tag{13.9}$$

Therefore, the desired interior *a priori* estimate (13.7) follows from estimate (13.9) if we take $\rho_0 > 0$ sufficiently small so that

$$n^2\, c_3\, \eta(r) < \frac{1}{2}, \quad 0 < r < \rho_0.$$

The proof of Lemma 13.2 is complete. □

13.3 Proof of Theorem 12.1

This section is devoted to the proof of Theorem 12.1. In fact, the next theorem proves Theorem 12.1 (see [18, Theorem 4.2]):

Theorem 13.3. *Let $1 < q < p < \infty$ and $f \in L^p_{\text{loc}}(\Omega)$. If a function $u \in W^{2,q}_{\text{loc}}(\Omega)$ satisfies the equation*

$$\mathcal{L}u = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f \quad \textit{in } \Omega,$$

then it follows that $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$. *Moreover, for any open subsets* $\Omega' \Subset \Omega'' \Subset \Omega$ *(see Figure 12.1), we have the interior* a priori *estimate*

$$\|u\|_{W^{2,p}(\Omega')} \leq C \left(\|u\|_{L^p(\Omega'')} + \|f\|_{L^p(\Omega)} \right), \tag{12.2}$$

with a positive constant $C = C(n, p, M, \mathrm{dist}(\Omega', \partial\Omega''), \lambda, \eta)$.

Proof. The proof is divided into two steps.

Step 1: Let B_ρ be an open ball such that $B_\rho \Subset \Omega$, and let $f \in L^r(B_\rho)$ for some $r \in (1, \infty)$. For i, j, h, $k = 1, 2, \ldots, n$, we introduce a linear bounded operator S_{ijhk} on $L^r(B_\rho)$ by the formula

$$S_{ijhk}(f) = \text{v. p.} \int_{B_\rho} \Gamma_{ij}(x, x - y) \left[a^{hk}(x) - a^{hk}(y) \right] f(y) \, dy.$$

By applying Corollary 11.5 with

$$k(x, z) := \Gamma_{ij}(x, z), \quad a(x) := a^{hk}(x),$$

we can find a constant $\rho_0 > 0$ such that

$$\sum_{i,j,h,k=1}^{n} \|S_{ijhk}\| < \frac{1}{2}, \quad 0 < \rho < \rho_0. \tag{13.10}$$

Here the norm of the operators S_{ijhk} is the norm in the space of bounded linear operators on $L^r(B_\rho)$ for $0 < \rho < \rho_0$.

Step 1-1: Let B be an open ball with radius less than ρ_0 such that $B \Subset \Omega$, and take a function $\beta \in C_0^\infty(B)$ such that $\beta(x) = 1$ on some ball $B' \Subset B$ (see Figure 13.2).

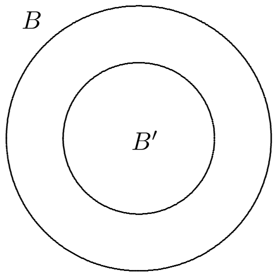

Fig. 13.2. The open balls B and B' such that $B \Subset \Omega$

Since we have the assertions

$$u \in W^{2,q}_{\mathrm{loc}}(\Omega), \quad \mathcal{L}u = f \in L^p_{\mathrm{loc}}(\Omega),$$

it follows that

$$v = \beta u \in W_0^{2,q}(B), \tag{13.11}$$

and that

$$\mathcal{L}u \in L^p(B). \tag{13.12}$$

Step 1-2: In order to estimate the term $\mathcal{L}v$, we need Sobolev's imbedding theorem (Theorem 7.3).

Since we have the assertions

$$\frac{\partial \beta}{\partial x_i}\frac{\partial u}{\partial x_j} \in W^{1,q}(B) \quad \text{for } 1 \leq i, j \leq n,$$
$$\frac{\partial^2 \beta}{\partial x_i \partial x_j} u \in W^{2,q}(B) \quad \text{for } 1 \leq i, j \leq n,$$

by applying Sobolev's imbedding theorem (Theorem 7.3) to our situation we obtain that

$$\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial \beta}{\partial x_i}\frac{\partial u}{\partial x_j} \in L^{q_1}(B), \tag{13.13}$$

$$(\mathcal{L}\beta)u = \sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 \beta}{\partial x_i \partial x_j} u \in L^{q_1}(B), \tag{13.14}$$

for some $q_1 \in (q, p]$.

Therefore, by assertions (13.12), (13.13) and (13.14) it follows that

$$\begin{aligned}\mathcal{L}v &= \sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}(\beta u) \\ &= \beta(\mathcal{L}u) + 2\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial \beta}{\partial x_i}\frac{\partial u}{\partial x_j} + (\mathcal{L}\beta)u \in L^{q_1}(B).\end{aligned}$$

Step 1-3: Now we have, by formula (13.1),

$$\begin{aligned}&\frac{\partial^2 v}{\partial x_i \partial x_j}(x) \\ &= \text{v. p.} \int_B \Gamma_{ij}(x, x-y) \\ &\quad \times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x) - a^{hk}(y)\right] \frac{\partial^2 v}{\partial x_h \partial x_k}(y) + \mathcal{L}v(y)\right] dy \\ &\quad + c_{ij}(x)\mathcal{L}v(x) \quad \text{for } 1 \leq i, j \leq n,\end{aligned} \tag{13.15}$$

with

$$c_{ij}(x) := \int_{|y|=1} \Gamma_i(x,y) y_j \, d\sigma_y \in L^\infty(\Omega).$$

If we let

$$h_{ij}(x) := \text{v. p.} \int_B \Gamma_{ij}(x, x-y) \mathcal{L}v(y) \, dy + c_{ij}(x) \mathcal{L}v(x) \quad \text{for } 1 \leq i, j \leq n,$$

then, since $\mathcal{L}v \in L^{q_1}(B)$, it follows from an application of Theorem 11.3 with

$$\Omega := B, \quad k(x,z) := \Gamma_{ij}(x,z), \quad p := q_1$$

that

$$h_{ij} \in L^{q_1}(B).$$

Therefore, if $r \in [q, q_1]$, we can introduce a mapping

$$T : (L^r(B))^{n^2} \longrightarrow (L^r(B))^{n^2}$$

by the formula

$$T\mathbf{w} = \left(\sum_{h,k=1}^{n} S_{ijhk}(w_{hk}) + h_{ij}(x) \right)_{1 \leq i,j \leq n}, \quad \mathbf{w} = (w_{ij})_{1 \leq i,j \leq n}.$$

Then it follows that T is a *contraction mapping*. Indeed, we have, by condition (13.10),

$$\begin{aligned}
&\left\| T\mathbf{w}^{(\mathbf{1})} - T\mathbf{w}^{(\mathbf{2})} \right\| \\
&= \sum_{i,j=1}^{n} \left\| \sum_{h,k=1}^{n} S_{ijhk}\left(w_{hk}^{(1)}\right) - \sum_{h,k=1}^{n} S_{ijhk}\left(w_{hk}^{(2)}\right) \right\|_{L^r(B)} \\
&\leq \sum_{i,j,h,k=1}^{n} \|S_{ijhk}\| \left\| w_{hk}^{(1)} - w_{hk}^{(2)} \right\|_{L^r(B)} \\
&\leq \sum_{h,k=1}^{n} \left(\sum_{i,j=1}^{n} \|S_{ijhk}\| \right) \left\| w_{hk}^{(1)} - w_{hk}^{(2)} \right\|_{L^r(B)} \\
&\leq \frac{1}{2} \sum_{h,k=1}^{n} \left\| w_{hk}^{(1)} - w_{hk}^{(2)} \right\|_{L^r(B)} \\
&= \frac{1}{2} \left\| \mathbf{w}^{(\mathbf{1})} - \mathbf{w}^{(\mathbf{2})} \right\|.
\end{aligned}$$

Therefore, by applying the *contraction mapping principle* (Theorem 2.1) we obtain that the mapping T has a unique fixed point

$$\mathbf{w} = (w_{ij}) \in \left(L^{q_1}(B)\right)^{n^2},$$
$$T\mathbf{w} = \mathbf{w}.$$

This implies that

$$\begin{aligned}
&\sum_{h,k=1}^{n} S_{ijhk}\,(w_{hk}) + h_{ij}(x) \\
&= \text{v. p.} \int_B \Gamma_{ij}(x, x-y) \left[\sum_{h,k=1}^{n} \left[a^{hk}(x) - a^{hk}(y)\right] w_{hk}(y) + \mathcal{L}v(y) \right] dy \\
&\quad + c_{ij}(x)\, \mathcal{L}v(x) \\
&= w_{ij}(x) \quad \text{for } 1 \le i, j \le n.
\end{aligned}$$

On the other hand, we obtain from formula (13.15) and assertion (13.11) that the Hessian matrix

$$\nabla^2 v = \left(\frac{\partial^2 v}{\partial x_i \partial x_j} \right) \in \left(L^q(B)\right)^{n^2}$$

is also a fixed point of T.

Therefore, since we have, for $q_1 > q$,

$$\left(L^{q_1}(B)\right)^{n^2} \subset \left(L^q(B)\right)^{n^2},$$

by the uniqueness of fixed points of T it follows that

$$\frac{\partial^2(\beta u)}{\partial x_i \partial x_j} = \frac{\partial^2 v}{\partial x_i \partial x_j} = w_{ij} \in L^{q_1}(B) \quad \text{for } 1 \le i, j \le n.$$

If $q_1 = p$, then we obtain that

$$u \in W^{2,p}_{\text{loc}}(\Omega),$$

since $B \Subset \Omega$ and $B' \Subset B$ are arbitrary.

If $q_1 < p$, by continuing this procedure we obtain, after a finite number of steps, that

$$u \in W^{2,p}_{\text{loc}}(\Omega).$$

Step 2: The interior *a priori* estimate (12.2) follows in a standard way from Lemma 13.2 by a covering argument.

Step 2-1: Let Ω' and Ω'' be any open subsets such that $\Omega' \Subset \Omega'' \Subset \Omega$ (see Figure 12.1). First, we cover the closure $\overline{\Omega'}$ of Ω' by a finite number of open balls $\{B_r(x_k)\}_{k=1}^N$ in each of which inequality (13.7) holds true (see Figure 13.3 below), and we take a partition of unity $\{\alpha_k\}_{k=1}^N$ subordinate to the open covering $\{B_r(x_k)\}_{k=1}^N$. Moreover, we may assume that

$$\operatorname{supp} \alpha_k \subset \Omega'', \quad 1 \leq k \leq N.$$

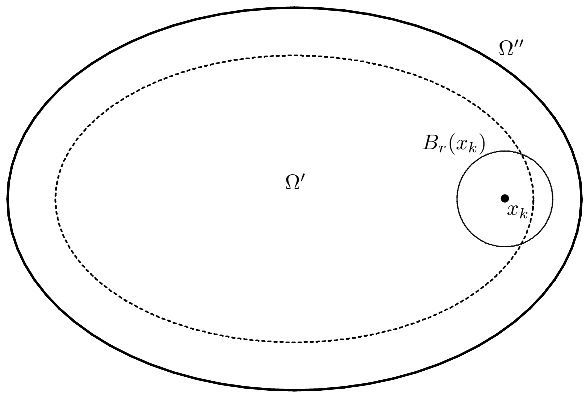

Fig. 13.3. The open subsets Ω', Ω'' such that $\Omega' \Subset \Omega''$ and the open ball $B_r(x_k)$

If $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$, then it follows that

$$\begin{aligned} &\|u\|_{W^{2,p}(\Omega')} \qquad (13.16) \\ &\leq \sum_{k=1}^{N} \|\alpha_k u\|_{W^{2,p}(\Omega)} \\ &\leq \sum_{k=1}^{N} \left(\left\|\nabla^2(\alpha_k u)\right\|_{L^p(\Omega)} + \|\nabla(\alpha_k u)\|_{L^p(\Omega)} + \|\alpha_k u\|_{L^p(\Omega)} \right) \\ &\leq \sum_{k=1}^{N} \left(\left\|\nabla^2(\alpha_k u)\right\|_{L^p(B_r(x_k))} + \|\nabla(\alpha_k u)\|_{L^p(B_r(x_k))} + \|u\|_{L^p(\Omega'')} \right). \end{aligned}$$

Step 2-2: In order to estimate the terms $\nabla(\alpha_k u)$, we need the following *interpolation inequality* (see [2, Theorem 5.2]; [33, Theorem 7.28]; [80, Theorem 2.15]):

Theorem 13.4 (the interpolation inequality)**.** *Let Ω be a $C^{1,1}$ domain in $\mathbf{R}^n$, and $1 \leq p < \infty$. Then there exists a positive constant $C =$*

$C(\Omega, p)$ such that, for any $\varepsilon > 0$ we have the inequality

$$\|\nabla v\|_{L^p(\Omega)} \leq \varepsilon \left\|\nabla^2 v\right\|_{L^p(\Omega)} + \frac{C}{\varepsilon} \; \|v\|_{L^p(\Omega)} \tag{13.17}$$

for all $v \in W^{2,p}(\Omega)$.

Since we have the formula

$$\nabla(\alpha_k u) = \alpha_k \, \nabla u + u(\nabla \alpha_k),$$

by applying inequality (13.17) to the function $\alpha_k u$ we obtain that, for some positive constants C_1 and C_2,

$$\begin{aligned}
&\|\nabla(\alpha_k u)\|_{L^p(B_r(x_k))} \\
&\leq \|\alpha_k(\nabla u)\|_{L^p(B_r(x_k))} + \|u(\nabla \alpha_k)\|_{L^p(B_r(x_k))} \\
&\leq \|\nabla u\|_{L^p(\Omega'')} + C_1 \|u\|_{L^p(\Omega'')} \\
&\leq \varepsilon \left\|\nabla^2 u\right\|_{L^p(\Omega'')} + \frac{C_2}{\varepsilon} \|u\|_{L^p(\Omega'')} + C_1 \|u\|_{L^p(\Omega'')} \quad \text{for } 1 \leq k \leq N.
\end{aligned} \tag{13.18}$$

Step 2-3: On the other hand, by applying inequality (13.7) to the functions $\alpha_k u$ we obtain that

$$\left\|\nabla^2(\alpha_k u)\right\|_{L^p(B_r(x_k))} \leq C \, \|\mathcal{L}(\alpha_k u)\|_{L^p(B_r(x_k))} \quad \text{for } 1 \leq k \leq N, \tag{13.19}$$

and also that

$$\mathcal{L}(\alpha_k u) = \alpha_k f + u \, \mathcal{L}(\alpha_k) + 2 \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial \alpha_k}{\partial x_i} \frac{\partial u}{\partial x_j} \quad \text{for } 1 \leq k \leq N. \tag{13.20}$$

Therefore, it follows from inequality (13.19) and formula (13.20) that

$$\begin{aligned}
&\left\|\nabla^2(\alpha_k u)\right\|_{L^p(B_r(x_k))} \\
&\leq C \, \|\mathcal{L}(\alpha_k u)\|_{L^p(B_r(x_k))} \\
&\leq C \Bigg(\|\alpha_k f\|_{L^p(B_r(x_k))} + \|u \, \mathcal{L}(\alpha_k)\|_{L^p(B_r(x_k))} \\
&\quad + 2 \sum_{i,j=1}^{n} \left\| a^{ij} \frac{\partial \alpha_k}{\partial x_i} \frac{\partial u}{\partial x_j} \right\|_{L^p(B_r(x_k))} \Bigg) \\
&\leq C_3 \left(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega'')} + \|\nabla u\|_{L^p(\Omega'')} \right) \\
&\leq C_3 \left(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega'')} \right) + \varepsilon \left\|\nabla^2 u\right\|_{L^p(\Omega'')} + \frac{C_4}{\varepsilon} \|u\|_{L^p(\Omega'')} .
\end{aligned} \tag{13.21}$$

By combining inequalities (13.16), (13.18) and (13.21), we obtain that

$$\begin{aligned} \|u\|_{W^{2,p}(\Omega')} &\leq 2N\varepsilon \left\|\nabla^2 u\right\|_{L^p(\Omega')} + C_5 \left(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega'')}\right) \\ &\quad + \frac{C_6}{\varepsilon} \|u\|_{L^p(\Omega'')} . \end{aligned}$$

This proves the desired interior *a priori* estimate (12.2), if we take

$$\varepsilon := \frac{1}{4N}.$$

The proof of Theorem 13.3 (and hence that of Theorem 12.1) is complete. □

13.4 Notes and Comments

The results of this chapter are adapted from Chiarenza–Frasca–Longo [18].

14

Calderón–Zygmund Kernels and Boundary Estimates

The purpose of this chapter is to prove the global *a priori* estimate (12.3) for the homogeneous Dirichlet problem stated in Theorem 12.2 (see [19, Theorem 4.2]). The desired global *a priori* estimate (12.3) is a consequence of the following two facts (I) and (II):

(I) The explicit boundary representation formula (14.2) for the solutions of the homogeneous Dirichlet problem, which is obtained from the half space Green function, involves the same integral operators as in the interior case.

(II) An L^p boundedness of the singular integral operators and boundary commutators appearing in the boundary representation formula (14.2) (Theorems 14.2 and 14.5).

The results here can be visualized in the following diagram:

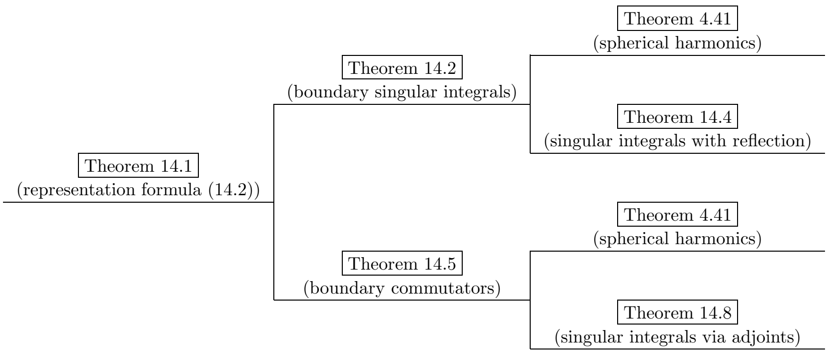

Table 14.1. *A flowchart for the proof of Theorem 14.1*

14.1 Boundary Representation Formula for Solutions

In this section we give a boundary representation formula for solutions of the homogeneous Dirichlet problem, by using the half space Green function for the elliptic differential operator

$$\mathcal{L} = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}.$$

Let B_σ be an open ball of radius σ in Euclidean space $\mathbf{R}^n$, and assume that the functions a^{ij} satisfy the following three conditions (i), (ii) and (iii):

(i) $a^{ij}(x) \in \mathrm{VMO} \cap L^\infty(\mathbf{R}^n)$ for all $1 \leq i, j \leq n$.
(ii) $a^{ij}(x) = a^{ji}(x)$ for all $1 \leq i, j \leq n$ and for almost all $x \in B_\sigma$.
(iii) There exists a positive constant λ such that

$$\frac{1}{\lambda} |\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i \xi_j \leq \lambda |\xi|^2 \tag{14.1}$$

for almost all $x \in B_\sigma$ and for all $\xi \in \mathbf{R}^n$.

If $\widetilde{B}_\sigma$ is the subset of B_σ where conditions (ii) and (iii) hold true, then we let (see Section 12.1)

$$\Gamma(x, z) := \frac{1}{(2-n)\omega_n} \frac{1}{\sqrt{\det(a^{ij}(x))}} \left(\sum_{i,j=1}^{n} A_{ij}(x) z_i z_j \right)^{(2-n)/2}$$

for $x \in \widetilde{B}_\sigma$ and $z \in \mathbf{R}^n \setminus \{0\}$.

Here:

$$(A_{ij}(x)) = \text{ the inverse matrix of } (a^{ij}(x)),$$

$$\omega_n := |\Sigma_{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

(the surface area of the unit sphere Σ_{n-1} in $\mathbf{R}^n$).

Moreover, we let

$$\Gamma_i(x, z) := \frac{\partial \Gamma}{\partial z_i}(x, z) \quad \text{for } x \in \widetilde{B}_\sigma,\ z \in \mathbf{R}^n \setminus \{0\} \text{ and } 1 \leq i \leq n,$$

$$\Gamma_{ij}(x, z) := \frac{\partial^2 \Gamma}{\partial z_i \partial z_j}(x, z) \quad \text{for } x \in \widetilde{B}_\sigma,\ z \in \mathbf{R}^n \setminus \{0\} \text{ and } 1 \leq i, j \leq n.$$

Now we let (see Figure 14.1)

$$B_\sigma^+ := B_\sigma \cap \mathbf{R}_+^n = B_\sigma \cap \{x_n > 0\},$$

and introduce two function spaces

$$C_{\gamma_0} := \left\{ u|_{B_\sigma^+} : u \in C_0^\infty(B_\sigma),\ u = 0 \text{ on } B_\sigma \cap \{x_n = 0\} \right\},$$

and

$$W^{2,p}_{\gamma_0}(B_\sigma^+) := \text{ the closure of } C_{\gamma_0} \text{ in the Sobolev space } W^{2,p}(B_\sigma^+).$$

Then it is easy to verify that $u \in W_0^{2,p}(B_\sigma)$ belongs to $W^{2,p}_{\gamma_0}(B_\sigma^+)$ if and

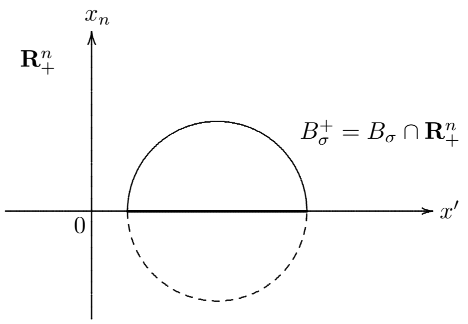

Fig. 14.1. The semi-ball B_σ^+ in the half-space $\mathbf{R}^n_+$

only if it vanishes on $B_\sigma \cap \{x_n = 0\}$ (see [2, Theorem 5.37]).

The next theorem gives *integral representation formulas* for the second derivatives of solutions of the homogeneous Dirichlet problem for the elliptic differential operator

$$\mathcal{L} = \sum_{i,j=1}^n a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$$

with variable coefficients (see [19, Theorem 3.2]):

Theorem 14.1 (the integral representation formula)**.** *Let $\widetilde{B}_\sigma$ be the subset of B_σ where conditions (ii) and (iii) hold true, and let $u \in W^{2,p}_{\gamma_0}(B_\sigma^+)$ for $1 < p < \infty$. Then we have, for all $x \in \widetilde{B}_\sigma^+ = \widetilde{B}_\sigma \cap \mathbf{R}^n_+$,*

$$\begin{aligned} &\frac{\partial^2 u}{\partial x_i \partial x_j}(x) \qquad (14.2)\\ &= \text{v. p.} \int_{B_\sigma^+} \Gamma_{ij}(x, x-y) \\ &\quad \times \left(\sum_{h,k=1}^n \left[a^{hk}(x) - a^{hk}(y) \right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + \mathcal{L}u(y) \right) dy \end{aligned}$$

$$+ \mathcal{L}u(x) \left(\int_{|t|=1} \Gamma_i(x,t) t_j \, d\sigma_t \right) - I_{ij}(x;x) \quad \textit{for } 1 \le i,j \le n.$$

Here:

- $$I_{ij}(x;z) := \int_{B_\sigma^+} \Gamma_{ij}(z, T(x;z) - y) \times \left[\sum_{h,k=1}^{n} \left[a^{hk}(z) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + \mathcal{L}u(y) \right] dy$$
  for $1 \le i,j \le n-1$;

- $$I_{in}(x;z) = I_{ni}(x;z) := \int_{B_\sigma^+} \sum_{\ell=1}^{n} \Gamma_{i\ell}(z, T(x;z) - y) A_\ell(z) \times \left[\sum_{h,k=1}^{n} \left[a^{hk}(z) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + \mathcal{L}u(y) \right] dy$$
  for $1 \le i \le n-1$;

and

- $$I_{nn}(x;z) := \int_{B_\sigma^+} \sum_{\ell,m=1}^{n} \Gamma_{\ell m}(z, T(x;z) - y) A_\ell(z) A_m(z) \times \left[\sum_{h,k=1}^{n} \left[a^{hk}(z) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + \mathcal{L}u(y) \right] dy.$$

Moreover, the generalized reflection map $T(x;z)$ *is defined by the formula*

$$T(x;z) = \begin{pmatrix} x_1 - 2x_n \frac{a^{1n}(z)}{a^{nn}(z)} \\ x_2 - 2x_n \frac{a^{2n}(z)}{a^{nn}(z)} \\ \vdots \\ -x_n \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

and the vector $A(x)$ *is defined by the formula*

$$A(x) = \begin{pmatrix} A_1(x) \\ A_2(x) \\ \vdots \\ A_n(x) \end{pmatrix} := \frac{\partial T}{\partial x_n}(x;x_0)\Big|_{x_0=x} = \begin{pmatrix} -2\frac{a^{1n}(x)}{a^{nn}(x)} \\ -2\frac{a^{2n}(x)}{a^{nn}(x)} \\ \vdots \\ -1 \end{pmatrix}.$$

Proof. By a *density argument*, it suffices to prove boundary representation formula (14.2) for all $u \in C_{\gamma_0}$. Indeed, the general case can be obtained by using Theorems 11.3 and 11.4, and Theorems 14.2 and 14.5 which will be proved in Sections 14.2 and 14.3, respectively. More precisely, we shall prove the following two assertions (i) and (ii):

(i) If we let

$$\widetilde{K}_{ij} f(x) := \int_{\mathbf{R}^n_+} \Gamma_{ij}(x, T(x) - y) f(y)\, dy \quad \text{for } 1 \leq i, j \leq n,$$

where

$$T(x) := T(x; x) = \begin{pmatrix} x_1 - 2x_n \frac{a^{1n}(x)}{a^{nn}(x)} \\ x_2 - 2x_n \frac{a^{2n}(x)}{a^{nn}(x)} \\ \vdots \\ -x_n \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

then there exists a positive constant $C_1 = C_1(n, p, \lambda, M)$ such that (Theorem 14.2)

$$\left\| \widetilde{K}_{ij} f \right\|_{L^p(\mathbf{R}^n_+)} \leq C_1 \|f\|_{L^p(\mathbf{R}^n_+)} \quad \text{for all } f \in L^p(\mathbf{R}^n_+).$$

(ii) Let $\varphi \in L^\infty(\mathbf{R}^n)$. If we let

$$\begin{aligned} &\widetilde{C}\left[\varphi, K_{ij}\right] f(x) \\ &:= \int_{\mathbf{R}^n_+} \Gamma_{ij}(x, T(x) - y) \left[\varphi(x) - \varphi(y)\right] f(y)\, dy \quad \text{for } 1 \leq i, j \leq n, \end{aligned}$$

then there exists a positive constant $C_2 = C_2(n, p, \lambda, M)$ such that (Theorem 14.5)

$$\left\| \widetilde{C}\left[\varphi, K_{ij}\right] f \right\|_{L^p(\mathbf{R}^n_+)} \leq C_2 \|\varphi\|_* \|f\|_{L^p(\mathbf{R}^n_+)} \quad \text{for all } f \in L^p(\mathbf{R}^n_+).$$

Let x_0 be an arbitrary point of $\widetilde{B}_\sigma$. First, it is easy to verify that the *half space Green function* $G(x_0, x, y)$ for the elliptic differential operator

$$\mathcal{L}_0 = \sum_{i,j=1}^n a^{ij}(x_0) \frac{\partial^2}{\partial x_i \partial x_j}$$

with *constant coefficients* is given by the formula (cf. [33, Section 4.4, formula (4.28)])

$$G(x_0, x, y) = \Gamma(x_0, x - y) - \Gamma(x_0, T(x; x_0) - y).$$

Then we have the representation formula

$$\begin{aligned} u(x) &= \int_{B_\sigma^+} G(x_0, x, y)\mathcal{L}_0 u(y)\, dy \qquad (14.3)\\ &= \int_{B_\sigma^+} \Gamma(x_0, x-y)\mathcal{L}_0 u(y)\, dy - \int_{B_\sigma^+} \Gamma(x_0, T(x;x_0)-y)\mathcal{L}_0 u(y)\, dy \\ &:= J(x;x_0) - I(x;x_0). \end{aligned}$$

We can differentiate the first term $J(x;x_0)$ in formula (14.3) twice to obtain that

$$\begin{aligned} \bullet\ & \frac{\partial^2 J}{\partial x_i \partial x_j}(x;x_0) \qquad (14.4)\\ &= \frac{\partial^2}{\partial x_i \partial x_j}\left(\int_{B_\sigma^+} \Gamma(x_0, x-y)\mathcal{L}_0 u(y)\, dy\right) \\ &= \text{v. p.} \int_{B_\sigma^+} \Gamma_{ij}(x_0, x-y) \\ &\quad \times \left(\sum_{h,k=1}^n \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + \mathcal{L}u(y)\right) dy \\ &\quad + \mathcal{L}u(x)\left(\int_{|t|=1} \Gamma_i(x_0, t) t_j \, d\sigma_t\right) \\ &= J_{ij}(x;x_0) \quad \text{for } 1 \le i, j \le n, \end{aligned}$$

since we have the formula

$$\begin{aligned} \mathcal{L}_0 u(y) &= (\mathcal{L}_0 - \mathcal{L})u(y) + \mathcal{L}u(y) \\ &= \sum_{i,j=1}^n \left[a^{ij}(x_0) - a^{ij}(y)\right] \frac{\partial^2 u}{\partial x_i \partial x_j}(y) + \sum_{i,j=1}^n a^{ij}(y) \frac{\partial^2 u}{\partial x_i \partial x_j}(y). \end{aligned}$$

As for the second term $I(x;x_0)$ in formula (14.3), we can differentiate it under the integral sign to obtain that

$$\begin{aligned} \bullet\ & \frac{\partial^2 I}{\partial x_i \partial x_j}(x;x_0) \qquad (14.5a)\\ &= \frac{\partial^2}{\partial x_i \partial x_j}\left(\int_{B_\sigma^+} \Gamma(x_0, T(x;x_0)-y)\mathcal{L}_0 u(y)\, dy\right) \\ &= \int_{B_\sigma^+} \Gamma_{ij}(x_0, T(x;x_0)-y) \end{aligned}$$

$$\times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + \mathcal{L}u(y) \right] dy$$

$$= I_{ij}(x; x_0), \quad 1 \leq i, j \leq n-1;$$

- $$\frac{\partial^2 I}{\partial x_i \partial x_n}(x; x_0) \tag{14.5b}$$

$$= \frac{\partial^2}{\partial x_i \partial x_n} \left(\int_{B_\sigma^+} \Gamma(x_0, T(x; x_0) - y) \mathcal{L}_0 u(y)\, dy \right)$$

$$= \int_{B_\sigma^+} \sum_{\ell=1}^{n} \Gamma_{i\ell}(x_0, T(x; x_0) - y) \frac{\partial T_\ell}{\partial x_n}(x; x_0)$$

$$\times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + \mathcal{L}u(y) \right] dy$$

$$= I_{in}(x; x_0), \quad 1 \leq i \leq n-1;$$

and

- $$\frac{\partial^2 I}{\partial x_n^2}(x; x_0) \tag{14.5c}$$

$$= \frac{\partial^2}{\partial x_n^2} \left(\int_{B_\sigma^+} \Gamma(x_0, T(x; x_0) - y) \mathcal{L}_0 u(y)\, dy \right)$$

$$= \int_{B_\sigma^+} \sum_{\ell,m=1}^{n} \Gamma_{\ell m}(x_0, T(x; x_0) - y) \frac{\partial T_\ell}{\partial x_n}(x; x_0) \frac{\partial T_m}{\partial x_n}(x; x_0)$$

$$\times \left[\sum_{h,k=1}^{n} \left[a^{hk}(x_0) - a^{hk}(y)\right] \frac{\partial^2 u}{\partial x_h \partial x_k}(y) + \mathcal{L}u(y) \right] dy$$

$$= I_{nn}(x; x_0).$$

Therefore, the desired boundary representation formula (14.2) for $x := x_0$ follows by setting $x = x_0$ in formulas (14.4) and (14.5).

The proof of Theorem 14.1 is complete. □

14.2 L^p Boundedness of Boundary Singular Integral Operators

This section is devoted to the proof of L^p boundedness of boundary singular integral operators appearing in boundary representation formula (14.2).

14.2.1 Boundary Singular Integral Operators

Let $k(x, z)$ be a real-valued function defined on $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ that satisfies the following two conditions (i) and (ii):

(i) $k(x, \cdot)$ is a Calderón–Zygmund kernel for almost all $x \in \mathbf{R}^n$.

(ii) The quantity

$$M := \max_{|\alpha| \le 2n} \left\| \frac{\partial^\alpha k}{\partial z^\alpha}(\cdot, \cdot) \right\|_{L^\infty(\mathbf{R}^n \times \Sigma_{n-1})}$$

is finite, where Σ_{n-1} is the unit sphere in $\mathbf{R}^n$. Moreover, we recall that

$$T(x) := T(x; x) = \begin{pmatrix} x_1 - 2x_n \frac{a^{1n}(x)}{a^{nn}(x)} \\ x_2 - 2x_n \frac{a^{2n}(x)}{a^{nn}(x)} \\ \vdots \\ -x_n \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

and

$$A(x) := \begin{pmatrix} A_1(x) \\ A_2(x) \\ \vdots \\ A_n(x) \end{pmatrix} = \begin{pmatrix} -2\frac{a^{1n}(x)}{a^{nn}(x)} \\ -2\frac{a^{2n}(x)}{a^{nn}(x)} \\ \vdots \\ -1 \end{pmatrix}.$$

Then we have the following L^p boundedness of boundary singular integral operators (cf. [13, Theorem 3.1]):

Theorem 14.2. *If $f \in L^p(\mathbf{R}^n_+)$ for $1 < p < \infty$, we let*

$$\widetilde{K} f(x) := \int_{\mathbf{R}^n_+} k(x, T(x) - y) f(y)\, dy.$$

Then there exists a positive constant $c = c(n, p, \lambda, M)$ such that

$$\left\| \widetilde{K} f \right\|_{L^p(\mathbf{R}^n_+)} \le c \, \|f\|_{L^p(\mathbf{R}^n_+)} \quad \text{for all } f \in L^p(\mathbf{R}^n_+). \tag{14.6}$$

The proof of Theorem 14.2 will be given in the next Subsection 14.2.2, due to its length.

Before the proof of Theorem 14.2, we need the following geometric properties of the map $T(x) = T(x; x)$ (cf. [19, Lemma 3.1], [13, Lemma 3.2]):

Lemma 14.3. *For the map $T(x) := T(x;x)$, there exist positive constants c_1, c_2 such that, for all $y \in \mathbf{R}^n_+$ and all $x \in \mathbf{R}^n_+$ for which $T(x)$ is defined, we have the inequalities*

$$c_1|\widetilde{x} - y| \leq |T(x) - y| \leq c_2\,|\widetilde{x} - y|\,, \tag{14.7}$$

where $\widetilde{x} = (x', -x_n)$ for $x = (x', x_n) \in \mathbf{R}^n_+$.

Proof. (1) First, we have, for all $y = (y', y_n) = (y_1, \ldots, y_{n-1}, y_n) \in \mathbf{R}^n_+$,

$$T(x) - y = \begin{pmatrix} T_1(x) - y_1 \\ T_2(x) - y_2 \\ \vdots \\ T_{n-1}(x) - y_{n-1} \\ -x_n - y_n \end{pmatrix} := \begin{pmatrix} T(x)' - y' \\ -x_n - y_n \end{pmatrix}, \tag{14.8}$$

and so

$$|T(x) - y| \geq x_n + y_n \geq x_n.$$

Hence it follows from (14.8) that

$$\begin{aligned} \frac{|T(x) - \widetilde{x}|}{|T(x) - y|} &\leq \frac{|T(x) - \widetilde{x}|}{x_n} \\ &= \frac{1}{x_n}\,|(x', 0) + x_n A(x) - (x', -x_n)| \\ &= |(0', 1) + A(x)| \\ &= \frac{2}{a^{nn}(x)}\,\sqrt{\left(a^{1n}(x)\right)^2 + \ldots + \left(a^{n-1\,n}(x)\right)^2} \end{aligned}$$

for all $y \in \mathbf{R}^n_+$ and all $x \in \mathbf{R}^n_+$ for which $T(x)$ is defined.

However, since the functions $a^{ij} \in L^\infty(\mathbf{R}^n)$ satisfy condition (14.1), we have the inequality

$$|(0', 1) + A(x)| \leq C_1(n, \lambda),$$

where

$$C_1(n\lambda) = 2\,\lambda\left(\left\|a^{1n}\right\|_\infty + \ldots + \left\|a^{n-1\,n}\right\|_\infty\right).$$

Hence we have, for all $y \in \mathbf{R}^n_+$ and all $x \in \mathbf{R}^n_+$ for which $T(x)$ is defined,

$$\frac{|T(x) - \widetilde{x}|}{|T(x) - y|} \leq C_1(n, \lambda). \tag{14.9}$$

Therefore, we obtain from inequality (14.9) that, for all $y \in \mathbf{R}^n_+$ and for all $x \in \mathbf{R}^n_+$ for which $T(x)$ is defined,

$$|\widetilde{x} - y| \leq |T(x) - \widetilde{x}| + |T(x) - y|$$

$$\begin{aligned}
&\leq |T(x) - y| \left(1 + \frac{|T(x) - \widetilde{x}|}{|T(x) - y|}\right) \\
&\leq (1 + C_1(n, \lambda)|T(x) - y|.
\end{aligned}$$

This proves the desired inequality (14.7) with

$$c_1 := \frac{1}{1 + C_1(n, \lambda)}.$$

(2) On the other hand, it follows that, for all $y = (y', y_n) \in \mathbf{R}^n_+$ and for all $x = (x', x_n) \in \mathbf{R}^n_+$,

$$\begin{aligned}
|\widetilde{x} - y| &= |(x' - y', -x_n - y_n)| = \sqrt{|x' - y'|^2 + (x_n + y_n)^2}, \\
|T(x) - y| &= \sqrt{|T(x)' - y'|^2 + (x_n + y_n)^2},
\end{aligned}$$

where

$$T(x)' - y' = \begin{pmatrix} x_1 - y_1 - 2x_n \frac{a^{1n}(x)}{a^{nn}(x)} \\ x_2 - y_2 - 2x_n \frac{a^{2n}(x)}{a^{nn}(x)} \\ \vdots \\ x_{n-1} - y_{n-1} - 2x_n \frac{a^{n-1n}(x)}{a^{nn}(x)} \end{pmatrix}.$$

We remark that

$$\begin{aligned}
|T(x)' - y'| &\leq |x' - y'| + C_1(n, \lambda)\, x_n \\
&\leq |x' - y'| + C_1(n, \mu)\, (x_n + y_n)\,.
\end{aligned}$$

Hence we have, for all $y \in \mathbf{R}^n_+$ and for all $x \in \mathbf{R}^n_+$ for which $T(x)$ is defined,

$$\begin{aligned}
|T(x) - y| &\leq |T(x)' - y'| + (x_n + y_n) \\
&\leq |x' - y'| + (1 + C_1(n, \lambda))\, (x_n + y_n) \\
&\leq (C_1(n, \lambda) + 1)\, (|x' - y| + (x_n + y_n)) \\
&\leq \sqrt{2}\, (C_1(n, \lambda) + 1)\, \sqrt{|x' - y|^2 + (x_n + y_n)^2} \\
&= \sqrt{2}\, (C_1(n, \lambda) + 1)\, |\widetilde{x} - y|\,.
\end{aligned}$$

This proves the desired inequality (14.7) with

$$c_2 := \sqrt{2}\, (C_1(n, \lambda) + 1)\,.$$

The proof of Lemma 14.3 is complete. □

Moreover, we have the following L^p boundedness theorem for singular intgrals with reflection (cf. [19, Theorem 2.5], [13, Lemma 3.3]):

Theorem 14.4. *If $f \in L^p(\mathbf{R}^n_+)$ for $1 < p < \infty$, we let*

$$\widetilde{K} f(x) := \int_{\mathbf{R}^n_+} \frac{f(y)}{|\widetilde{x} - y|^n} \, dy,$$

where $\widetilde{x} = (x', -x_n)$. Then there exists a positive constant $c(n,p)$ such that

$$\left\| \widetilde{K} f \right\|_{L^p(\mathbf{R}^n_+)} \leq c(n,p) \, \|f\|_{L^p(\mathbf{R}^n_+)} \quad \text{for all } f \in L^p(\mathbf{R}^n_+). \tag{14.10}$$

For example, we may take

$$c(n,p) := B\left(\frac{1}{p}, 1 - \frac{1}{p}\right) \int_{\mathbf{R}^{n-1}} \frac{1}{(|z'|^2 + 1)^{n/2}} \, dz'.$$

Proof. The proof of Theorem 14.4 is divided into three steps.

Step (1): First, we have the inequality

$$\begin{aligned} \left\| \widetilde{K} f(x) \right\|^p_{L^p(\mathbf{R}^n_+)} &\leq \int_{\mathbf{R}^n_+} \left(\int_{\mathbf{R}^n} \frac{|f(y)|}{|\widetilde{x} - y|^n} \, dy \right)^p dx \\ &= \int_0^\infty \int_{\mathbf{R}^{n-1}} \left(\int_{\mathbf{R}^n_+} \frac{|f(y)|}{|\widetilde{x} - y|^n} \, dy \right)^p dx' dx_n \\ &= \int_0^\infty I(x_n) \, dx_n, \end{aligned} \tag{14.11}$$

where

$$I(x_n) := \int_{\mathbf{R}^{n-1}} \left(\int_0^\infty \int_{\mathbf{R}^{n-1}} \frac{|f(y', y_n)|}{\left(|x' - y'|^2 + (x_n + y_n)^2\right)^{n/2}} \, dy' dy_n \right)^p dx'.$$

Thus we are reduced to the estimate of the function $I(x_n)$.

Step (2): We prove that

$$I(x_n)^{1/p} \leq C_n \int_0^\infty \frac{1}{x_n + y_n} \left(\int_{\mathbf{R}^{n-1}} |f(y', y_n)|^p \, dy' \right)^{1/p} dy_n, \tag{14.12}$$

where C_n is a positive constant given by the formula

$$C_n := \int_{\mathbf{R}^{n-1}} \frac{1}{(|z'|^2 + 1)^{n/2}} \, dz'.$$

By applying Minkowski's inequality for integrals (Theorem 3.16)

$$\left\| \int_0^\infty F(\cdot, y_n) \, dy_n \right\|_{L^p(\mathbf{R}^{n-1})} \leq \int_0^\infty \|F(\cdot, y_n)\|_{L^p(\mathbf{R}^{n-1})} \, dy_n$$

to the function

$$F(x', y_n) = \int_0^\infty \int_{\mathbf{R}^{n-1}} \frac{|f(y', y_n)|}{(|x'-y'|^2 + (x_n+y_n)^2)^{n/2}}\, dy',$$

we obtain that

$$\begin{aligned} &I(x_n)^{1/p} \\ &\leq \int_0^\infty \left(\int_{\mathbf{R}^{n-1}} \left(\int_{\mathbf{R}^{n-1}} \frac{|f(y', y_n)|}{(|x'-y'|^2 + (x_n+y_n)^2)^{n/2}}\, dy' \right)^p dx' \right)^{1/p} dy_n \\ &= \int_0^\infty \|G * f(\cdot, x_n + y_n)\|_{L^p(\mathbf{R}^{n-1})}\, dy_n, \end{aligned}$$

where

$$G(x', x_n + y_n) := \frac{1}{(|x'|^2 + (x_n+y_n)^2)^{n/2}},$$

and

$$(G * f)(x', x_n + y_n) := \int_{\mathbf{R}^{n-1}} \frac{|f(y', y_n)|}{(|x'-y'|^2 + (x_n+y_n)^2)^{n/2}}\, dy'.$$

Moreover, by Young's inequality (Theorem 3.23) it follows that

$$\begin{aligned} &\|G * f(\cdot, x_n + y_n)\|_{L^p(\mathbf{R}^{n-1})} \\ &\leq \|G(\cdot, x_n + y_n)\|_{L^1(\mathbf{R}^{n-1})} \|f(\cdot, y_n)\|_{L^p(\mathbf{R}^{n-1})}. \end{aligned} \tag{14.13}$$

However, we have, for the above positive constant C_n,

$$\begin{aligned} \|G(\cdot, x_n + y_n)\|_{L^1(\mathbf{R}^{n-1})} &= \int_{\mathbf{R}^{n-1}} \frac{1}{(|y'|^2 + (x_n+y_n)^2)^{n/2}} dy' \\ &= \frac{1}{x_n + y_n} \int_{\mathbf{R}^{n-1}} \frac{1}{(|z'|^2 + 1)^{n/2}}\, dz' \\ &= \frac{C_n}{x_n + y_n}. \end{aligned} \tag{14.14}$$

Therefore, the desired inequality (14.12) follows by combining inequality (14.13) and formula (14.14).

Step (3): By inequality (14.12), it follows that

$$\begin{aligned} &\int_0^\infty I(x_n)\, dx_n \\ &\leq C_n^p \int_0^\infty \left(\int_0^\infty \frac{\left(\int_{\mathbf{R}^{n-1}} |f(y', y_n)|^p\, dy' \right)^{1/p}}{x_n + y_n}\, dy_n \right)^p dx_n \end{aligned} \tag{14.15}$$

$$= C_n^p \int_0^\infty \left(\int_0^\infty \frac{\left(\int_{\mathbf{R}^{n-1}} |f(y', \lambda x_n)|^p \, dy'\right)^{1/p}}{1+\lambda} \, d\lambda \right)^p dx_n$$
$$= C_n^p \int_0^\infty \left(\int_0^\infty \frac{\|f(\cdot, \lambda x_n)\|_{L^p(\mathbf{R}^{n-1})}}{1+\lambda} \, d\lambda \right)^p dx_n.$$

However, by applying Minkowski's inequality for integrals (Theorem 3.16)

$$\left\| \int_0^\infty G(\cdot, \lambda) \, d\lambda \right\|_{L^p(0,\infty)} \leq \int_0^\infty \|G(\cdot, \lambda)\|_{L^p(0,\infty)} \, d\lambda$$

to the function

$$G(x_n, \lambda) = \frac{\|f(\cdot, \lambda x_n)\|_{L^p(\mathbf{R}^{n-1})}}{1+\lambda},$$

we obtain that

$$\int_0^\infty \left(\int_0^\infty \frac{\|f(\cdot, \lambda x_n)\|_{L^p(\mathbf{R}^{n-1})}}{1+\lambda} \, d\lambda \right)^p dx_n \tag{14.16}$$
$$\leq \left(\int_0^\infty \left(\int_0^\infty \left(\frac{\|f(\cdot, \lambda x_n)\|_{L^p(\mathbf{R}^{n-1})}}{1+\lambda} \right)^p dx_n \right)^{1/p} d\lambda \right)^p$$
$$= \left(\int_0^\infty \frac{1}{1+\lambda} \left(\int_0^\infty \|f(\cdot, z_n)\|^p_{L^p(\mathbf{R}^{n-1})} \frac{dz_n}{\lambda} \right)^{1/p} d\lambda \right)^p$$
$$= \left(\int_0^\infty \frac{1}{(1+\lambda)\lambda^{1/p}} \, d\lambda \right)^p \left(\int_0^\infty \int_{\mathbf{R}^{n-1}} |f(z', z_n)|^p \, dz' dz_n \right)$$
$$= D_p^p \, \|f\|^p_{L^p(\mathbf{R}^n_+)},$$

where D_p is a positive constant given by the formula

$$D_p := \int_0^\infty \frac{1}{(1+\lambda)\lambda^{1/p}} \, d\lambda = B\left(\frac{1}{p}, 1 - \frac{1}{p} \right).$$

Therefore, we have, by inequalities (14.11), (14.15) and (14.16),

$$\left\| \widetilde{K} f(x) \right\|^p_{L^p(\mathbf{R}^n_+)} \leq \int_0^\infty I(x_n) \, dx_n$$
$$\leq C_n^p \int_0^\infty \left(\int_0^\infty \frac{\|f(\cdot, \lambda x_n)\|_{L^p(\mathbf{R}^{n-1})}}{1+\lambda} \, d\lambda \right)^p dx_n$$
$$\leq C_n^p \, D_p^p \, \|f\|^p_{L^p(\mathbf{R}^n_+)} \quad \text{for all } f \in L^p(\mathbf{R}^n_+).$$

This proves the desired inequality (14.10) with

$$c(n,p) := C_n\, D_p.$$

The proof of Theorem 14.4 is complete. □

14.2.2 Proof of Theorem 14.2

This subsection is devoted to the proof of Theorem 14.2. By a *density argument*, it suffices to prove the theorem for all $f \in C_0^\infty(\mathbf{R}_+^n)$. The proof is divided into three steps.

Step 1: First, it should be noticed that the function

$$z \longmapsto |z|^n k(x,z)$$

belongs to $C^\infty(\mathbf{R}^n \setminus \{0\})$ for almost all $x \in \mathbf{R}_+^n$, and it is positively homogeneous of degree zero and satisfies the condition

$$\int_{\Sigma_{n-1}} k(x,z)\, d\sigma_z = 0.$$

For each $m = 1, 2, \ldots$ and $k = 1, 2, \ldots, d(m)$, we let

$$a_{km}(x) := \int_{\Sigma_{n-1}} k(x,z)\, Y_{km}(z)\, dz,$$

then, by the completeness of the spherical harmonics $\{Y_{km}\}$ in $L^2(\Sigma_{n-1})$ it follows that

$$\begin{aligned} &|T(x)-y|^n k(x, T(x)-y) \\ &= \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} a_{km}(x)\, Y_{km}(T(x)-y), \quad x \in \mathbf{R}_+^n,\ y \in \mathbf{R}^n \setminus \{0\}. \end{aligned} \tag{14.17}$$

Moreover, we have, by assertion (4.76) with $r := n$ and assertion (4.75) with $\alpha := 0$ of Theorem 4.31,

$$\|a_{km}\|_{L^\infty(\mathbf{R}_+^n)} \le \frac{c_1(n)}{m^{2n}} M, \tag{14.18a}$$

$$\|Y_{km}\|_{L^\infty(\Sigma_{n-1})} \le c_2(n)\, m^{(n-2)/2}, \tag{14.18b}$$

and, by assertion (4.74) of Theorem 4.31,

$$d(m) \le c_3(n)\, m^{n-2}. \tag{14.18c}$$

Step 2: By using the expansion (14.17), we obtain from inequality (14.18a) that

$$\begin{aligned}
&\left\|\widetilde{K}f\right\|_{L^p(\mathbf{R}^n_+)} \qquad (14.19)\\
&= \left\|\int_{\mathbf{R}^n_+} k(\cdot, T(\cdot)-y) f(y)\, dy\right\|_{L^p(\mathbf{R}^n_+)}\\
&= \left\|\sum_{m=1}^{\infty}\sum_{k=1}^{d(m)} a_{km}(\cdot)\int_{\mathbf{R}^n_+} \frac{Y_{km}(T(\cdot)-y)}{|T(\cdot)-y|^n} f(y)\, dy\right\|_{L^p(\mathbf{R}^n_+)}\\
&\le \sum_{m=1}^{\infty}\sum_{k=1}^{d(m)} \|a_{km}\|_{L^\infty(\mathbf{R}^n_+)} \left\|\int_{\mathbf{R}^n_+} \frac{Y_{km}(T(\cdot)-y)}{|T(\cdot)-y|^n} f(y)\, dy\right\|_{L^p(\mathbf{R}^n_+)}\\
&\le c_1(n)\, M \sum_{m=1}^{\infty}\sum_{k=1}^{d(m)} \frac{1}{m^{2n}} \left\|\int_{\mathbf{R}^n_+} \frac{Y_{km}(T(\cdot)-y)}{|T(\cdot)-y|^n} f(y)\, dy\right\|_{L^p(\mathbf{R}^n_+)}.
\end{aligned}$$

However, by inequality (14.18b) it follows from an application of Lemma 14.3 and Theorem 14.4 that

$$\begin{aligned}
&\left\|\int_{\mathbf{R}^n_+} \frac{Y_{km}(T(\cdot)-y)}{|T(\cdot)-y|^n} f(y)\, dy\right\|_{L^p(\mathbf{R}^n_+)} \qquad (14.20)\\
&\le c(n,p)\, c_2(n)\, m^{(n-2)/2}\, \|f\|_{L^p(\mathbf{R}^n_+)}.
\end{aligned}$$

Indeed, we have, by Lemma 14.3,

$$\begin{aligned}
&\left(\int_{\mathbf{R}^n_+}\left|\int_{\mathbf{R}^n_+} \frac{|Y_{km}(T(x)-y)|}{|T(x)-y|^n} f(y)\, dy\right|^p dx\right)^{1/p}\\
&\le \left(\int_{\mathbf{R}^n_+}\left|\int_{\mathbf{R}^n_+} \frac{\|Y_{km}\|_{L^\infty(\Sigma_{n-1})}}{|T(x)-y|^n} |f(y)|\, dy\right|^p dx\right)^{1/p}\\
&\le c_2(n)\, m^{(n-2)/2} \left(\int_{\mathbf{R}^n_+}\left|\int_{\mathbf{R}^n_+} \frac{|f(y)|}{|\widetilde{x}-y|^n}\, dy\right|^p dx\right)^{1/p}\\
&= c_2(n)\, m^{(n-2)/2} \left\|\widetilde{K}(|f|)\right\|_{L^p(\mathbf{R}^n_+)}\\
&\le c(n,p)\, c_2(n)\, m^{(n-2)/2}\, \|f\|_{L^p(\mathbf{R}^n_+)}.
\end{aligned}$$

Step 3: Finally, the desired inequality (14.6) follows by combining inequalities (14.19), (14.20) and (14.18c):

$$\begin{aligned}
&\left\|\widetilde{K}f\right\|_{L^p(\mathbf{R}^n_+)} \\
&\le c_1(n)\,c(n,p)\,c_2(n)\,M\sum_{m=1}^{\infty}\sum_{k=1}^{d(m)}\frac{1}{m^{2n}}m^{(n-2)/2}\,\|f\|_{L^p(\mathbf{R}^n_+)} \\
&\le c_1(n)\,c_2(n)\,c_3(n)\,c(n,p)M\left(\sum_{m=1}^{\infty}\frac{1}{m^{2n}}m^{(n-2)/2}m^{n-2}\right)\|f\|_{L^p(\mathbf{R}^n_+)} \\
&= c_1(n)\,c_2(n)\,c_3(n)\,c(n,p)\,M\left(\sum_{m=1}^{\infty}\frac{1}{m^{n/2+3}}\right)\|f\|_{L^p(\mathbf{R}^n_+)} \\
&\qquad \text{for all } f\in L^p(\mathbf{R}^n_+).
\end{aligned}$$

The proof of Theorem 14.2 is now complete. □

14.3 L^p Boundedness of Boundary Commutators

The next theorem concerns the L^p boundedness of boundary commutators appearing in boundary representation formula (14.2) (cf. [13, Theorem 3.4]):

Theorem 14.5. *Assume that $k(x,z)$ is a real-valued function defined on $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ which satisfies conditions (i) and (ii). If $\varphi \in L^\infty(\mathbf{R}^n)$ and $f \in L^p(\mathbf{R}^n_+)$ for $1 < p < \infty$, we define the boundary commutator $\widetilde{C}[\varphi, K]$ by the formula*

$$\begin{aligned}
\widetilde{C}\,[\varphi,K]\,f(x) &:= \int_{\mathbf{R}^n_+} k(x,T(x)-y)\,[\varphi(x)-\varphi(y)]\,f(y)\,dy \\
&= \varphi(x)\,Kf(x) - K(\varphi f)(x).
\end{aligned}$$

Then there exists a positive constant $c_5 = c_5(n,p,\lambda,M)$ such that

$$\left\|\widetilde{C}\,[\varphi,K]\,f\right\|_{L^p(\mathbf{R}^n_+)} \le c_5\,\|\varphi\|_*\,\|f\|_{L^p(\mathbf{R}^n_+)} \quad \textit{for all } f\in L^p(\mathbf{R}^n_+). \tag{14.21}$$

The proof of Theorem 14.5 will be given in Subsection 14.3.2, due to its length.

Before the proof of Theorem 14.5, we need one more result concerning constant kernel boundary commutators. However, for lack of regularity of $T(x)$, we cannot apply Theorem 11.4. This difficulty may be overcome by taking into consideration the *adjoint operators* $\widetilde{K}^*$ and $\widetilde{C}^*[\varphi, K]$, just as in Bramanti–Cerutti [13].

More precisely, we can prove the following $L^{p'}$ boundedness theorem (cf. [13, Theorem 3.5]):

Theorem 14.6. *If $k(x)$ is a Calderón–Zygmund kernel, then we let*

$$\widetilde{K}^* g(x) := \int_{\mathbf{R}^n_+} k(T(y) - x) g(y)\, dy.$$

If $\varphi \in L^\infty(\mathbf{R}^n)$ and $g \in L^{p'}(\mathbf{R}^n_+)$ for $p' = p/(p-1)$, we define the boundary commutator $\widetilde{C}^*[\varphi, K]$ by the formula

$$\begin{aligned}\widetilde{C}^*\,[\varphi, K]\, g(x) &:= \int_{\mathbf{R}^n_+} k(T(y) - x)\,[\varphi(x) - \varphi(y)]\, g(y)\, dy \\ &= \varphi(x)\, \widetilde{K}^* g(x) - \widetilde{K}^*(\varphi g)(x).\end{aligned}$$

Then there exists a positive constant $c_6 = c_6(n, p', \lambda, M)$ such that

$$\left\| \widetilde{C}^*\,[\varphi, K]\, g \right\|_{L^{p'}(\mathbf{R}^n_+)} \le c_6\, \|\varphi\|_*\, \|g\|_{L^{p'}(\mathbf{R}^n_+)} \quad \text{for all } g \in L^{p'}(\mathbf{R}^n_+). \tag{14.22}$$

Proof. The proof is divided into three steps.

Step 1: First, by Lemma 14.3 it follows that

$$|k(T(y) - x)| \le \frac{C}{|T(y) - x|^n} \le \frac{C'}{|x - \widetilde{y}|^n}, \quad x,\, y \in \mathbf{R}^n_+.$$

However, we have the formula

$$|x - \widetilde{y}|^2 = |x' - y'|^2 + (x_n + y_n)^2 = |\widetilde{x} - y|^2.$$

Hence it follows that

$$|k(T(y) - x)| \le \frac{C'}{|\widetilde{x} - y|^n}, \quad x,\, y \in \mathbf{R}^n_+.$$

Therefore, by applying Theorem 14.4 to the operator $\widetilde{K}^*$ we obtain that $\widetilde{K}^*$ is bounded on $L^{p'}(\mathbf{R}^n)$; more precisely, there exists a constant $c^*(n, p') > 0$ such that

$$\left\| \widetilde{K}^* g \right\|_{L^{p'}(\mathbf{R}^n_+)} \le c^*(n, p')\, \|g\|_{L^{p'}(\mathbf{R}^n_+)} \quad \text{for all } g \in L^{p'}(\mathbf{R}^n_+). \tag{14.23}$$

Step 2: Now let Q be a cube with sides parallel to the coordinate axes contained in $\mathbf{R}^n_+$. If $f \in L^1_{\text{loc}}(\mathbf{R}^n)$, then we define the Hardy–Littlewood maximal function (see Section 4.4)

$$M_+ f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

and the sharp function (see Section 4.6)

$$f^\sharp_+(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\, dy,$$

where the supremum is taken over all cubes containing x and f_Q is the integral average of f over Q

$$f_Q := \frac{1}{|Q|} \int_Q f(z)\, dz.$$

Here it should be emphasized that we have, for all $f \in L^p(\mathbf{R}^n_+)$ with $1 < p < \infty$ (Theorem 4.8 and Remark 4.1),

$$\|M_+ f\|_{L^p(\mathbf{R}^n_+)} \leq c\, \|f\|_{L^p(\mathbf{R}^n_+)}\,,$$

and that we have, for all $f \in L^p(\mathbf{R}^n_+)$ with $1 < p < \infty$ (Corollary 4.15 and Remark 4.3),

$$\|f\|_{L^p(\mathbf{R}^n_+)} \leq c \left\| f^\sharp_+ \right\|_{L^p(\mathbf{R}^n_+)}.$$

Now we consider the operator

$$Sg(x) := \widetilde{C}^* \left[\varphi, \widetilde{K}^*\right] g(x) = \int_{\mathbf{R}^n_+} k(T(y) - x)\, [\varphi(x) - \varphi(y)]\, g(y)\, dy,$$

and, for each $r \in (1, p')$, we shall prove a pointwise estimate

$$\begin{aligned} &(Sg)^\sharp_+(x) \qquad\qquad (14.24)\\ &\leq c(n,r)\, \|\varphi\|_* \left(\left(M_+(|\widetilde{K}^* g|^r)(x) \right)^{1/r} + \left(M_+(|g|^r)(x) \right)^{1/r} \right)\\ &\qquad \text{for all } x \in \mathbf{R}^n_+, \end{aligned}$$

with a constant $c(n,r) > 0$.

Step 3: Assuming estimate (14.24) for the moment, we shall prove Theorem 14.6. The proof of estimate (14.24) will be given in the next Subsection 14.3.1, due to its length.

By applying Corollary 4.15 with $p := p'$ and Remark 4.3 and estimate (14.24), we obtain that

$$\|Sg\|_{L^{p'}(\mathbf{R}^n_+)} \leq c_{p'} \left\| (Sg)^\sharp_+ \right\|_{L^{p'}(\mathbf{R}^n_+)} \qquad (14.25)$$

$$\leq c_{p'}\, c(n,r)\, \|\varphi\|_{*}$$
$$\times \left\{ \left(\int_{\mathbf{R}^n_+} (M_+(|\widetilde{K}^* g|^r)(x))^{p'/r}\, dx \right)^{1/p'} \right.$$
$$\left. \times \left\{ \left(\int_{\mathbf{R}^n_+} + \left(\int_{\mathbf{R}^n_+} (M_+(|g|^r)(x))^{p'/r}\ dx \right)^{1/p'} \right) \right\} \right. .$$

Step 3-1: However, it should be noticed that

$$\left(\int_{\mathbf{R}^n_+} (M_+(|g|^r)(x))^{p'/r}\ dx \right)^{1/p'} = \left(\|M_+(|g|^r)\|_{L^{p'/r}(\mathbf{R}^n)} \right)^{1/r},$$

so that, by Theorem 4.8 with $p := p'/r$ and Remark 4.1,

$$\|M_+(|g|^r)\|_{L^{p'/r}(\mathbf{R}^n_+)} \leq C(p',r)\, \||g|^r\|_{L^{p'/r}(\mathbf{R}^n_+)} = C(p',r)\, \|g\|^r_{L^{p'}(\mathbf{R}^n_+)}.$$

Hence we have the estimate

$$\left(\int_{\mathbf{R}^n_+} (M_+(|g|^r)(x))^{p'/r}\ dx \right)^{1/p'} \leq C(p',r)^{1/r}\, \|g\|_{L^{p'}(\mathbf{R}^n_+)}. \tag{14.26}$$

Step 3-2: Similarly, we have the formula

$$\left(\int_{\mathbf{R}^n_+} (M_+(|\widetilde{K}^* g|^r)(x))^{p'/r}\, dx \right)^{1/p'} = \left(\left\| M_+(|\widetilde{K}^* g|^r) \right\|_{L^{p'/r}(\mathbf{R}^n_+)} \right)^{1/r},$$

and, by Theorem 4.8 with $p := p'/r$ and Remark 4.1,

$$\begin{aligned} \left\| M_+(|\widetilde{K}^* g|^r) \right\|_{L^{p'/r}(\mathbf{R}^n_+)} &\leq C(p',r) \left\| |\widetilde{K}^* g|^r \right\|_{L^{p/r}(\mathbf{R}^n_+)} \\ &= C(p',r) \left\| \widetilde{K}^* g \right\|^r_{L^{p'}(\mathbf{R}^n_+)}. \end{aligned}$$

Hence it follows from an application of Theorem 14.4 that

$$\begin{aligned} &\int_{\mathbf{R}^n_+} \left(M_+(|\widetilde{K}^* g|^r)(x))^{p'/r} dx \right)^{1/p'} \\ &\leq C(p',r)^{1/r} \left\| \widetilde{K}^* g \right\|_{L^{p'}(\mathbf{R}^n_+)} \\ &\leq c(n,p') C(p',r)^{1/r} c^*(n,p')\, \|g\|_{L^{p'}(\mathbf{R}^n_+)}. \end{aligned} \tag{14.27}$$

Therefore, the desired estimate (14.22) follows by combining estimates (14.25), (14.26) and (14.27), with

$$c_6(n,p') = c_{p'}\, c(n,r)\, (c(n,p') c^*(n,p') + 1)\, C(p',r)^{1/r}, \quad 1 < r < p'.$$

The proof of Theorem 14.6 is now complete, apart from the proof of estimate (14.24). $\square$

14.3.1 End of Proof of Theorem 14.6

This subsection is devoted to the proof of estimate (14.24). If Q is a cube contained in $\mathbf{R}^n_+$, then we denote by δ_Q its side length and by x_Q its center, respectively. For each $j \in \mathbf{N}$, we denote by 2^jQ the cube centered at x_Q with side length $2^j\delta_Q$. For any cube Q containing x, we write $Sg(x) := \widetilde{C}^*g(x)$ in the form

$$\begin{aligned}
Sg(x) &= \int_{\mathbf{R}^n_+} k(T(y)-x)\,(\varphi(x)-\varphi_Q)\,g(y)\,dy \\
&\quad + \int_{\mathbf{R}^n_+} k(T(y)-x)\,(\varphi_Q-\varphi(y))\,g(y)\,dy \\
&= (\varphi(x)-\varphi_Q)\int_{\mathbf{R}^n_+} k(T(y)-x)g(y)\,dy \\
&\quad + \int_{\mathbf{R}^n_+\cap(2Q)} k(T(y)-x)\,(\varphi_Q-\varphi(y))\,g(y)\,dy \\
&\quad + \int_{\mathbf{R}^n_+\setminus(2Q)} k(T(y)-x)\,(\varphi_Q-\varphi(y))\,g(y)\,dy \\
&:= A(x)+B(x)+C(x).
\end{aligned}$$

The proof of estimate (14.24) is divided into four steps.

Step I: The estimate of the term $A(x)$. We prove that, for $1 < r < p' = p/(p-1)$,

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |A(x)-A_Q|\,dx \qquad (14.28)\\
&\le c_1(n,r)\,\|\varphi\|_*\left(M_+\left(\left|\widetilde{K}^*g\right|^r\right)(y)\right)^{1/r} \quad \text{for all } y \in Q.
\end{aligned}$$

First, we have the inequality

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |A(x)-A_Q|\,dx &\le \frac{1}{|Q|}\int_Q |A(x)|\,dx + |A_Q| \qquad (14.29)\\
&= \frac{1}{|Q|}\int_Q |A(x)|\,dx + \frac{1}{|Q|}\left|\int_Q A(y)\,dy\right| \\
&\le \frac{2}{|Q|}\int_Q |A(x)|\,dx.
\end{aligned}$$

If r is a number such that $1 < r < p' = p/(p-1)$, then, by Hölder's inequality (Theorem 3.14) it follows that, for $r' = r/(r-1)$,

$$\begin{aligned}
&\frac{2}{|Q|}\int_Q |A(x)|\,dx \qquad (14.30)\\
&= \frac{2}{|Q|}\int_Q \left|(\varphi(x)-\varphi_Q)\int_{\mathbf{R}^n_+} k(T(y)-x)g(y)\,dy\right| dx\\
&\le \frac{2}{|Q|}\int_Q |\varphi(x)-\varphi_Q| \left|\widetilde{K}^* g(x)\right| dx\\
&\le \frac{2}{|Q|}\left(\int_Q |\varphi(x)-\varphi_Q|^{r'}\,dx\right)^{1/r'} \left(\int_Q \left|\widetilde{K}^* g(x)\right|^r dx\right)^{1/r}.\\
&= 2\left(\frac{1}{|Q|}\int_Q |\varphi(x)-\varphi_Q|^{r'}\,dx\right)^{1/r'} \left(\frac{1}{|Q|}\int_Q \left|\widetilde{K}^* g(x)\right|^r dx\right)^{1/r}.
\end{aligned}$$

However, we have, by Theorem 4.11 with $p := r'$ and Remark 4.2,

$$\frac{1}{|Q|}\int_Q |\varphi(y)-\varphi_Q|^{r'}\,dy \le c_1(r')\,\|\varphi\|_*^{r'}. \qquad (14.31)$$

and, by the definition of the maximal functions,

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left|\widetilde{K}^* g(x)\right|^r dx\right)^{1/r} \qquad (14.32)\\
&\le \left(M_+\left(\left|\widetilde{K}^* g\right|^r\right)(y)\right)^{1/r} \quad \text{for all } y \in Q.
\end{aligned}$$

Therefore, the desired estimate (14.28) follows by combining estimates (14.29), (14.30), (14.31) and (14.32) with

$$c_1(n,r) = 2\,c_1(r')^{1/r'}, \quad r' = \frac{r}{r-1}.$$

Step II: The estimate of the term $B(x)$. Secondly, we prove that, for $1 < r < p' = p/(p-1)$,

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |B(x)-B_Q|\,dx \qquad (14.33)\\
&\le c_2(n,q,r)\,\|\varphi\|_*\,(M_+(|g|^r)(y))^{1/r} \quad \text{for all } y \in Q.
\end{aligned}$$

Here

$$1 < q < r < \frac{p}{p-1}.$$

First, we remark that

$$B(x) = \int_{\mathbf{R}^n_+ \cap (2Q)} k(T(y) - x)\,(\varphi_Q - \varphi(y))\, g(y)\, dy$$
$$= \int_{\mathbf{R}^n_+} k(T(y) - x)\,(\varphi_Q - \varphi(y))\,\chi_{2Q}(y)\, g(y)\, dy$$
$$= -\widetilde{K}^*\left((\varphi - \varphi_Q)\,\chi_{2Q}\, g\right)(x),$$

where χ_{2Q} is the characteristic function of $2Q$:

$$\chi_{2Q}(x) = \begin{cases} 1 & \text{if } x \in 2Q, \\ 0 & \text{if } x \in \mathbf{R}^n \setminus (2Q). \end{cases}$$

Then we have the inequality

$$\frac{1}{|Q|}\int_Q |B(x) - B_Q|\, dx \tag{14.34}$$
$$\leq \frac{1}{|Q|}\int_Q |B(x)|\, dx + |B_Q|$$
$$= \frac{1}{|Q|}\int_Q |B(x)|\, dx + \frac{1}{|Q|}\left|\int_Q B(y)\, dy\right|$$
$$\leq \frac{2}{|Q|}\int_Q |B(x)|\, dx$$
$$= \frac{2}{|Q|}\int_Q \left|\widetilde{K}^*\left((\varphi - \varphi_Q)\,\chi_{2Q}\, g\right)(x)\right| dx.$$

If q is a number such that $1 < q < r$, then, by Hölder's inequality (Theorem 3.14) it follows that, for $q' = q/(q-1)$,

$$\frac{2}{|Q|}\int_Q \left|\widetilde{K}^*\left((\varphi - \varphi_Q)\,\chi_{2Q}\, g\right)(x)\right| dx \tag{14.35}$$
$$\leq 2\,\frac{1}{|Q|}\left(\int_Q \left|\widetilde{K}^*\left((\varphi - \varphi_Q)\,\chi_{2Q}\, g\right)\right|^q dx\right)^{1/q}\left(\int_Q dx\right)^{1/q'}$$
$$= \frac{2}{|Q|^{1/q}}\left(\int_Q \left|\widetilde{K}^*\left((\varphi - \varphi_Q)\,\chi_{2Q}\, g\right)\right|^q dx\right)^{1/q}.$$

However, we have, by inequality (14.23) with $p' := q$ and then by Hölder's inequality for the exponent r/q,

$$\left(\int_Q \left|\widetilde{K}^*\left((\varphi - \varphi_Q)\,\chi_{2Q}\, g\right)\right|^q dx\right)^{1/q} \tag{14.36}$$

$$\le \left(\int_{\mathbf{R}^n_+} \left| \widetilde{K}^* \left((\varphi - \varphi_Q)\, \chi_{2Q}\, g \right) \right|^q \, dx \right)^{1/q}$$
$$\le c^*(n,q)\, \|(\varphi - \varphi_Q)\chi_{2Q}\, g\|_{L^q(\mathbf{R}^n_+)}$$
$$= c^*(n,q) \left(\int_{2Q} |\varphi(x) - \varphi_Q|^q \cdot |g(x)|^q \, dx \right)^{1/q}$$
$$\le c^*(n,q) \left(\int_{2Q} |g(x)|^r \, dx \right)^{1/r} \left(\int_{2Q} |\varphi(x) - \varphi_Q|^{qr/(r-q)} \, dx \right)^{(r-q)/qr}.$$

Therefore, by combining the inequalities (14.34) through (14.36) we obtain that

$$\frac{1}{|Q|} \int_Q |B(x) - B_Q| \, dx \tag{14.37}$$
$$\le \frac{2}{|Q|} \int_Q \left| \widetilde{K}^* \left((\varphi - \varphi_Q)\, \chi_{2Q}\, g \right)(x) \right| \, dx$$
$$\le \frac{2}{|Q|^{1/q}} \left(\int_Q \left| \widetilde{K}^* \left((\varphi - \varphi_Q)\, \chi_{2Q}\, g \right) \right|^q \, dx \right)^{1/q}$$
$$\le \frac{2\, c^*(n,q)}{|Q|^{1/q}} \left(\int_{2Q} |g(x)|^r \, dx \right)^{1/r} \left(\int_{2Q} |\varphi(x) - \varphi_Q|^{qr/(r-q)} \, dx \right)^{(r-q)/qr}.$$

Now we recall the following inequality (see Claim 10.1): There exists a constant $c(n,q,r) > 0$ such that we have, for $1 < q < r$,

$$\int_{2Q} |\varphi(x) - \varphi_Q|^{rq/(r-q)} \, dx \le c(n,q,r)\, \|\varphi\|_*^{rq/(r-q)}\, |2Q| . \tag{14.38}$$

Therefore, it follows from inequalities (14.37) and (14.38) that

$$\frac{1}{|Q|} \int_Q |B(x) - B_Q| \, dx$$
$$\le \frac{2\, c^*(n,q)}{|Q|^{1/q}} \left(\int_{2Q} |g(x)|^r \, dx \right)^{1/r} \left(c(n,q,r)\, \|\varphi\|_*^{rq/(r-q)}\, |2Q| \right)^{(r-q)/qr}$$
$$= 2^{1+(n/q)}\, c^*(n,q)\, c(n,q,r)^{(r-q)/qr}\, \|\varphi\|_* \left(\frac{1}{|2Q|} \int_{2Q} |g(x)|^r \, dx \right)^{1/r}$$
$$\le 2^{1+(n/q)}\, c^*(n,q)\, c(n,q,r)^{(r-q)/qr}\, \|\varphi\|_* \left(M_+ \left(|g|^r \right)(y) \right)^{1/r}$$

for all $y \in Q$.

This proves the desired estimate (14.33) with

$$c_2(n,q,r) := 2^{1+(n/q)}\, c^*(n,q)\, c(n,q,r)^{(r-q)/qr}, \quad 1 < q < r.$$

Step III: The estimate of the term $C(x)$. Thirdly, we prove that, for $1 < r < p' = p/(p-1)$,

$$\frac{1}{|Q|}\int_Q |C(x) - C_Q|\, dx \tag{14.39}$$
$$\leq c_3(n,r)\,\|\varphi\|_*\,(M_+(|g|^r)(y))^{1/r} \quad \text{for all } y \in Q.$$

The proof of estimate (14.39) is divided into four steps.

Step III-1: We begin with the *pointwise Hörmander condition* for adjoint kernels $k\,(x - T(y))$.

Lemma 14.7. *Let $k(x)$ be a Calderón–Zygmund kernel. If Q is a cube with center x_Q contained in $\mathbf{R}^n_+$, then we have, for all $x \in Q$, $y \in \mathbf{R}^n_+ \setminus (2Q)$ (see Figure 14.2),*

$$|k(x - T(y)) - k(x_Q - T(y))| \leq c\,\frac{|x - x_Q|}{|x_Q - \widetilde{y}|^{n+1}}, \tag{14.40}$$

with a positive constant c. Here recall that the map $T(y)$ is defined by the formula

$$T(y) := T(y;y) = \begin{pmatrix} y_1 - 2y_n \frac{a^{1n}(y)}{a^{nn}(y)} \\ y_2 - 2y_n \frac{a^{2n}(y)}{a^{nn}(y)} \\ \vdots \\ -y_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

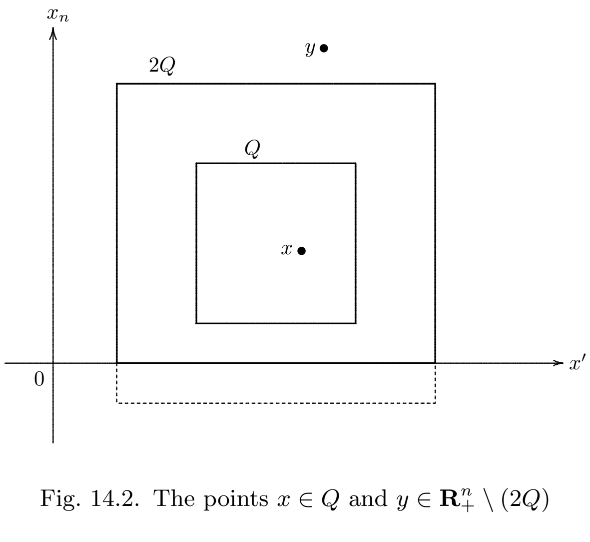

Fig. 14.2. The points $x \in Q$ and $y \in \mathbf{R}^n_+ \setminus (2Q)$

Proof. The proof of Lemma 14.7 is essentially the same as that of Lemma 10.3 if we use Lemma 14.3.

If Q is a cube with center x_Q contained in $\mathbf{R}^n_+$ and if $y \in \mathbf{R}^n_+ \setminus (2Q)$, we let

$$x = (x_1, x_2, \dots, x_n) = (x', x_n) \in Q,$$
$$x_Q = \left(x_Q^1, x_Q^2, \dots, x_Q^n\right) = \left(x'_Q, x_Q^n\right) \in Q,$$

and

$$T(y) = (T_1(y), T_2(y), \dots, T_n(y)) \in \mathbf{R}^n_-,$$
$$\widetilde{y} = (y_1, y_2, \dots, y_{n-1}, -y_n) \in \mathbf{R}^n_-.$$

Then, by the mean value theorem it follows that we have, for $0 < \theta < 1$,

$$\begin{aligned}
&|k(x - T(y)) - k(x_Q - T(y))| \qquad (14.41)\\
&= |k((x_Q - T(y)) + (x - x_Q)) - k(x_Q - T(y))|\\
&\le \left|\frac{\partial k}{\partial x_1}\left(x_Q^1 - T_1(y) + \theta(x_1 - x_Q^1), x_2 - T_2(y), \dots, x_n - T_n(y)\right)\right|\\
&\quad \times \left|x_1 - x_Q^1\right|\\
&\quad + \dots + \dots\\
&\quad + \left|\frac{\partial k}{\partial x_n}\left(x_Q^1 - T_1(y), x_Q^2 - T_2(y), \dots, x_Q^n - T_n(y) + \theta(x_n - x_Q^n)\right)\right|\\
&\quad \times \left|x_n - x_Q^n\right|\\
&\le \left|\frac{\partial k}{\partial x_1}(x_Q^1 - T_1(y) + \theta(x_1 - x_Q^1), x_2 - T_2(y), \dots, x_n - T_n(y))\right|\\
&\quad \times |x - x_Q|\\
&\quad + \dots + \dots\\
&\quad + \left|\frac{\partial k}{\partial x_n}(x_Q^1 - T_1(y), x_Q^2 - T_2(y), \dots, x_Q^n - T_n(y) + \theta(x_n - x_Q^n))\right|\\
&\quad \times |x - x_Q|\,.
\end{aligned}$$

However, we obtain from Lemma 14.3 that

$$\begin{aligned}
c_1\left|x_Q^j - \widetilde{y}_j + \theta(x_j - x_Q^j)\right| &\le \left|x_Q^j - T_j(y) + \theta(x_j - x_Q^j)\right|\\
&\le c_2\left|x_Q^j - \widetilde{y}_j + \theta(x_j - x_Q^j)\right|, \quad 1 \le j \le n,
\end{aligned}$$

and further that, for all $x \in Q$, $y \in \mathbf{R}^n_+ \setminus (2Q)$,

$$\begin{aligned}\left|x_Q^j - \widetilde{y}_j + \theta(x_j - x_Q^j)\right| &\geq \left|x_Q^j - \widetilde{y}_j\right| - \left|x_j - x_Q^j\right| \\ &\geq \frac{1}{2}\left|x_Q^j - \widetilde{y}_j\right|, \quad 1 \leq j \leq n.\end{aligned}$$

In particular, we have, for $1 \leq j \leq n$,

$$\begin{aligned}&\left|\left(x_Q^1 - T_1(y), \ldots, x_Q^j - T_j(y) + \theta(x_j - x_Q^j), \ldots, x_n - T_n(y)\right)\right| \qquad (14.42)\\ &\geq \frac{c_1}{2}\left|x_Q - \widetilde{y}\right|.\end{aligned}$$

Therefore, by using inequality (10.24) we can obtain from inequalities (14.41) and (14.42) that

$$|k(x - T(y)) - k(x_Q - T(y))| \leq \frac{2^{n+1}}{c_1^{n+1}}\, n\, C\, \frac{|x - x_Q|}{|x_Q - \widetilde{y}|^{n+1}}$$
$$\text{for all } x \in Q,\ y \notin 2Q.$$

This proves the desired inequality (14.40), with

$$c := \frac{2^{n+1}}{c_1^{n+1}}\, n\, C = \frac{2^{n+1}}{c_1^{n+1}}\, n \max_{|z|=1} |\nabla k(z)|.$$

The proof of Lemma 14.7 is complete. □

Step III-2: First, it follows that

$$\begin{aligned}&\frac{1}{|Q|}\int_Q |C(x) - C_Q|\, dx \\ &\leq \frac{1}{|Q|}\int_Q |C(x) - C(x_Q)|\, dx + \frac{1}{|Q|}\int_Q |C(x_Q) - C_Q|\, dx.\end{aligned}$$

However, we have the estimate

$$\begin{aligned}|C(x) - C_Q| &= \frac{1}{|Q|}\left|\int_Q (C(x) - C(x_Q))\, dx\right| \\ &\leq \frac{1}{|Q|}\int_Q |C(x) - C(x_Q)|\, dx.\end{aligned}$$

Hence we obtain that

$$\frac{1}{|Q|}\int_Q |C(x) - C_Q|\, dx \leq \frac{2}{|Q|}\int_Q |C(x) - C(x_Q)|\, dx. \qquad (14.43)$$

Moreover, by Lemma 14.7 we can find a constant $c > 0$ such that

$$|k(x - T(y)) - k(x_Q - T(y))| \leq c\, \frac{|x - x_Q|}{|x_Q - \widetilde{y}|^{n+1}}$$

for all $x \in Q$ and $y \notin 2Q$.

Therefore, we have, by Hölder's inequality (Theorem 3.14),

$$\begin{aligned}
&|C(x) - C(x_Q)| \qquad (14.44)\\
&= \left| \int_{\mathbf{R}^n_+ \setminus (2Q)} (k(x - T(y)) - k(x_Q - T(y)))\,(\varphi_Q - \varphi(y))\, g(y)\, dy \right| \\
&\leq \int_{\mathbf{R}^n_+ \setminus (2Q)} |k(x - T(y)) - k(x_Q - T(y))|\, |\varphi(y) - \varphi_Q| |g(y)|\, dy \\
&\leq c \int_{\mathbf{R}^n_+ \setminus (2Q)} \frac{|x - x_Q|}{|x_Q - \widetilde{y}|^{n+1}} |\varphi(x) - \varphi_Q| |g(y)|\, dy \\
&\leq c'\, \delta_Q \int_{\mathbf{R}^n_+ \setminus (2Q)} \frac{|g(y)|}{|x_Q - \widetilde{y}|^{(n+1)/r}} \frac{|\varphi(x) - \varphi_Q|}{|x_Q - \widetilde{y}|^{(n+1)/r'}}\, dy \\
&\leq c'\, \delta_Q \left(\int_{\mathbf{R}^n_+ \setminus (2Q)} \frac{|g(z)|^r}{|x_Q - \widetilde{z}|^{n+1}}\, dz \right)^{1/r} \left(\int_{\mathbf{R}^n_+ \setminus (2Q)} \frac{|\varphi(z) - \varphi_Q|^{r'}}{|x_Q - \widetilde{z}|^{n+1}}\, dz \right)^{1/r'}.
\end{aligned}$$

Here δ_Q is the side length of Q.

Step III-3: Now we prove the following two estimates (14.45) and (14.46):

$$I(x) := \int_{\mathbf{R}^n_+ \setminus (2Q)} \frac{|g(z)|^r}{|x_Q - \widetilde{z}|^{n+1}}\, dz \leq \frac{C_1}{\delta_Q}\, M_+(|g|^r)(y) \quad \text{for all } y \in Q. \tag{14.45}$$

$$II(x) := \int_{\mathbf{R}^n_+ \setminus (2Q)} \frac{|\varphi(x) - \varphi_Q|^{r'}}{|x_Q - \widetilde{z}|^{n+1}}\, dz \leq \frac{C_2}{\delta_Q}\, \|\varphi\|_*^{r'}. \tag{14.46}$$

of Estimate (14.45). Indeed, we have, for all $x_Q^n \geq 0$, $z_n \geq 0$,

$$\begin{aligned}
|x_Q - \widetilde{z}|^2 &= |x'_Q - z'|^2 + (x_Q^n + z_n)^2 \\
&\geq |x'_Q - z'|^2 + (x_Q^n - z_n)^2 = |x_Q - z|^2.
\end{aligned}$$

If Q'_j is the cube in $\mathbf{R}^n_+$ containing $2^j Q \cap \mathbf{R}^n_+$ and having the same measure as $2^j Q$, then we obtain that

$$\begin{aligned}
\int_{\mathbf{R}^n_+ \setminus (2Q)} \frac{|g(z)|^r}{|x_Q - \widetilde{z}|^{n+1}}\, dz &\leq \int_{\mathbf{R}^n_+ \setminus (2Q)} \frac{|g(z)|^r}{|x_Q - z|^{n+1}}\, dz \\
&= \sum_{j=2}^{\infty} \int_{\mathbf{R}^n_+ \cap (2^j Q \setminus (2^{j-1} Q))} \frac{|g(z)|^r}{|x_Q - z|^{n+1}}\, dz \\
&\leq \frac{C}{\delta_Q} \sum_{j=2}^{\infty} \frac{1}{2^j} \left(\frac{1}{|Q'_j|} \int_{Q'_j} |g(z)|^r\, dz \right)
\end{aligned}$$

$$\leq \frac{C}{\delta_Q} \left(\sum_{j=2}^{\infty} \frac{1}{2^j} \right) M_+(|g|^r)(y) \quad \text{for all } y \in Q.$$

This proves the desired estimate (14.45), with

$$C_1 := C \sum_{j=2}^{\infty} \frac{1}{2^j} = \frac{C}{2}.$$

The proof of Estimate (14.45) is complete. □

of Estimate (14.46). Similarly, we have the estimate

$$\begin{aligned} &\int_{\mathbf{R}^n_+ \setminus (2Q)} \frac{|\varphi(z) - \varphi_Q|^{r'}}{|x_Q - \widetilde{z}|^{n+1}} \, dz \qquad (14.47) \\ &\leq \sum_{j=2}^{\infty} \int_{2^j Q \setminus (2^{j-1} Q)} \frac{1}{2^j \delta_Q} \frac{|\varphi(z) - \varphi_Q|^{r'}}{|x_Q - z|^{n+1}} \, dz \\ &\leq \frac{C}{\delta_Q} \sum_{j=2}^{\infty} \left(\frac{1}{|2^j Q|} \int_{2^j Q} |\varphi(z) - \varphi_Q|^{r'} \, dz \right). \end{aligned}$$

However, by Lemma 4.2 with $f := \varphi$ it follows that

$$\begin{aligned} |\varphi(z) - \varphi_Q| &\leq |\varphi(z) - \varphi_{2^j Q}| + |\varphi_{2^j Q} - \varphi_Q| \\ &\leq |\varphi(z) - \varphi_{2^j Q}| + c(n) j \, \|\varphi\|_* , \end{aligned}$$

so that

$$\begin{aligned} &\frac{1}{|2^j Q|} \int_{2^j Q} |\varphi(z) - \varphi_Q|^{r'} \, dz \qquad (14.48) \\ &\leq 2^{r'-1} \left(\frac{1}{|2^j Q|} \int_{2^j Q} |\varphi(z) - \varphi_{2^j Q}|^{r'} \, dz \right) + \frac{1}{|2^j Q|} \int_{2^j Q} \left(c(n) j \, \|\varphi\|_*\right)^{r'} dz \\ &\leq 2^{r'-1} \left(\frac{1}{|2^j Q|} \int_{2^j Q} |\varphi(z) - \varphi_{2^j Q}|^{r'} \, dz \right) + 2^{r'-1} \left(c(n) j \, \|\varphi\|_*\right)^{r'}. \end{aligned}$$

However, we have, by Theorem 4.11 with $p := r'$ and Remark 4.2,

$$\frac{1}{|2^j Q|} \int_{2^j Q} |\varphi(z) - \varphi_{2^j Q}|^{r'} \, dz \leq c_1(r') \, \|\varphi\|_*^{r'}. \qquad (14.49)$$

Hence, it follows from estimates (14.48) and (14.49) that

$$\frac{1}{|2^j Q|} \int_{2^j Q} |\varphi(z) - \varphi_Q|^{r'} \, dz \leq 2^{r'-1} \left(c(n)^{r'} j^{r'} + c_1(r') \right) \|\varphi\|_*^{r'}. \qquad (14.50)$$

Therefore, by combining estimates (14.47) and (14.50) we obtain that

$$\begin{aligned}
&\int_{\mathbf{R}^n_+\setminus(2Q)} \frac{|\varphi(z)-\varphi_Q|^{r'}}{|x_Q-z|^{n+1}}\,dz \\
&= \sum_{j=2}^{\infty} \int_{2^jQ\setminus(2^{j-1}Q)} \frac{|\varphi(z)-\varphi_Q|^{r'}}{|x_Q-z|^{n+1}}\,dz \\
&\le 2^{r'-1}\,C \sum_{j=2}^{\infty} \frac{1}{2^j\delta_Q} \left(c(n)^{r'}\,j^{r'} + c_1(r')\right) \|\varphi\|_*^{r'} \\
&= 2^{r'-1}\frac{C}{\delta_Q}\left[c_1(r')\left(\sum_{j=2}^{\infty}\frac{1}{2^j}\right) + c(n)^{r'}\left(\sum_{j=2}^{\infty}\frac{j^{r'}}{2^j}\right)\right]\|\varphi\|_*^{r'}.
\end{aligned}$$

This proves the desired estimate (14.46), with

$$C_2 := 2^{r'-1}\,C\left[c_1(r')\left(\sum_{j=2}^{\infty}\frac{1}{2^j}\right) + c(n)^{r'}\left(\sum_{j=2}^{\infty}\frac{j^{r'}}{2^j}\right)\right].$$

The proof of Estimate (14.46) is complete. □

Step III-4: Therefore, by combining the estimates (14.43) through (14.46) we obtain that

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |C(x)-C_Q|\,dx \\
&\le \frac{2}{|Q|}\int_Q |C(x)-C(x_Q)|\,dx \\
&\le \frac{2}{|Q|}\,c'\,C_1^{1/r}\,C_2^{1/r'}\,\|\varphi\|_*\,(M_+(|g|^r)(y))^{1/r}\int_Q dx \\
&= 2\,c'\,C_1^{1/r}\,C_2^{1/r'}\,\|\varphi\|_*\,(M_+(|g|^r)(y))^{1/r} \quad \text{for all } y\in Q.
\end{aligned}$$

This proves the desired estimate (14.39), with

$$c_3(n,r) := 2\,c'\,C_1^{1/r}\,C_2^{1/r'}, \quad r' = \frac{r}{r-1}.$$

Step IV: If we let

$$c(n,r) := \max\{c_1(n,r), c_2(n,r), c_3(n,r)\},$$

then, by combining estimates (14.28), (14.33) and (14.39) we obtain that

$$\frac{1}{|Q|}\int_Q |Sg(x)-(Sg)_Q|\,dx$$

$$
\begin{aligned}
&\leq \frac{1}{|Q|}\int_Q |A(x) - A_Q|\, dx + \frac{1}{|Q|}\int_Q |B(x) - B_Q|\, dx \\
&\quad + \frac{1}{|Q|}\int_Q |C(x) - C_Q|\, dx \\
&\leq c(n,r)\, \|\varphi\|_* \left(\left(M_+(|\widetilde{K}^* g|^r)(y)\right)^{1/r} + \left(M_+(|g|^r)(y)\right)^{1/r} \right) \\
&\qquad \text{for all } y \in Q.
\end{aligned}
$$

This proves that

$$
\begin{aligned}
&(Sg)^\sharp_+(y) \\
&\leq c(n,r)\, \|\varphi\|_* \left(\left(M_+(|\widetilde{K}^* g|^r)(y)\right)^{1/r} + \left(M_+(|g|^r)(y)\right)^{1/r} \right) \\
&\qquad \text{for all } y \in \mathbf{R}^n,
\end{aligned}
$$

since Q is arbitrary.

The proof of estimate (14.24) (and hence that of Theorem 14.6) is now complete. □

14.3.2 Proof of Theorem 14.5

This subsection is devoted to the proof of Theorem 14.5. By a *density argument*, it suffices to prove the theorem for all $f \in C_0^\infty(\mathbf{R}^n_+)$. The proof is divided into four steps.

Step 1: First, by passing to adjoint operators we obtain from Theorem 14.6 the following:

Theorem 14.8. *If $k(x)$ is a Calderón–Zygmund kernel, then we let*

$$
\widetilde{K} f(x) := \int_{\mathbf{R}^n_+} k(T(x) - y) f(y)\, dy \quad \text{for all } f \in L^p(\mathbf{R}^n_+).
$$

If $\varphi \in L^\infty(\mathbf{R}^n)$, we define the boundary commutator $\widetilde{C}[\varphi, \widetilde{K}]$ by the formula

$$
\begin{aligned}
\widetilde{C}\left[\varphi, \widetilde{K}\right] f(x) &:= \int_{\mathbf{R}^n_+} k(T(x) - y)\, [\varphi(x) - \varphi(y)]\, f(y)\, dy \\
&= \varphi(x)\, \widetilde{K} f(x) - \widetilde{K}(\varphi f)(x)
\end{aligned}
$$

for all $f \in L^p(\mathbf{R}^n_+)$ with $1 < p < \infty$. Then there exists a positive *constant* $c_7 = c_7(n, p, \lambda, M)$ *such that*

$$\left\| \widetilde{C}\,[\varphi, K]\, f \right\|_{L^p(\mathbf{R}^n_+)} \leq c_7 \, \|\varphi\|_* \, \|f\|_{L^p(\mathbf{R}^n_+)} \quad \text{for all } f \in L^p(\mathbf{R}^n_+). \tag{14.51}$$

Proof. First, we prove the L^p boundedness of the operator $\widetilde{K}$. For any $f \in L^p(\mathbf{R}^n_+)$ and any $g \in L^{p'}(\mathbf{R}^n_+)$, we have, by Theorem 14.6 and Fubini's theorem (Theorem 3.10),

$$\begin{aligned} \left\langle f, \widetilde{K}^* g \right\rangle &= \int_{\mathbf{R}^n_+} f(y) \left(\int_{\mathbf{R}^n_+} k(T(x) - y) g(x)\, dx \right) dy \\ &= \int_{\mathbf{R}^n_+} \left(\int_{\mathbf{R}^n_+} k(T(x) - y) f(y)\, dy \right) g(x)\, dx \\ &= \left\langle \widetilde{K} f, g \right\rangle. \end{aligned}$$

Therefore, by passing to the adjoint operator we obtain from inequality (14.23) that

$$\widetilde{K} f \in L^p(\mathbf{R}^n_+),$$

and further that

$$\begin{aligned} \left\| \widetilde{K} f \right\|_{L^p(\mathbf{R}^n_+)} &= \sup_{\|g\|_{p'} \leq 1} \left| \left\langle \widetilde{K} f, g \right\rangle \right| = \sup_{\|g\|_{p'} \leq 1} \left| \left\langle f, \widetilde{K}^* g \right\rangle \right| \\ &\leq c^*(n, p') \, \|f\|_{L^p(\mathbf{R}^n_+)} . \end{aligned}$$

The operators $\widetilde{K} \colon L^p(\mathbf{R}^n_+) \to L^p(\mathbf{R}^n_+)$ and $\widetilde{K}^* \colon L^{p'}(\mathbf{R}^n_+) \to L^{p'}(\mathbf{R}^n_+)$ can be visualized as follows:

$$\begin{array}{ccc} f \in L^p(\mathbf{R}^n_+) & \xrightarrow{\ \widetilde{K}\ } & L^p(\mathbf{R}^n_+) \ni \widetilde{K} f \\ \updownarrow & & \updownarrow \\ \widetilde{K}^* g \in L^{p'}(\mathbf{R}^n_+) & \xleftarrow{\ \widetilde{K}^*\ } & L^{p'}(\mathbf{R}^n_+) \ni g \end{array}$$

Fig. 14.3. The operators $\widetilde{K}$ and $\widetilde{K}^*$

Secondly, we prove inequality (14.51). For any $f \in L^p(\mathbf{R}^n_+)$ and any $g \in L^{p'}(\mathbf{R}^n_+)$, we have, by Theorem 14.6 and Fubini's theorem (Theorem 3.10),

$$\left\langle f, \widetilde{C}^* \, [\varphi, K] \, g \right\rangle$$

$$
\begin{aligned}
&= \int_{\mathbf{R}_+^n} f(y) \left(\int_{\mathbf{R}_+^n} k(T(x)-y)\,[\varphi(x)-\varphi(y)]\,g(x)\,dx \right) dy \\
&= \int_{\mathbf{R}_+^n} \left(\int_{\mathbf{R}_+^n} k(T(x)-y)\,[\varphi(x)-\varphi(y)]\,f(y)\,dy \right) g(x)\,dx \\
&= \left\langle \widetilde{C}\,[\varphi, K]\,f, g \right\rangle .
\end{aligned}
$$

Therefore, by passing to the adjoint operator we obtain from inequality (14.22) that

$$\widetilde{C}\,[\varphi, K]\,f \in L^p(\mathbf{R}_+^n),$$

and further that

$$
\begin{aligned}
\left\| \widetilde{C}\,[\varphi, K]\,f \right\|_{L^p(\mathbf{R}_+^n)} &= \sup_{\|g\|_{p'} \le 1} \left| \left\langle \widetilde{C}\,[\varphi, K]\,f, g \right\rangle \right| \\
&= \sup_{\|g\|_{p'} \le 1} \left| \left\langle f, \widetilde{C}^*\,[\varphi, K]\,g \right\rangle \right| \\
&\le c_6\,\|\varphi\|_*\,\|f\|_{L^p(\mathbf{R}_+^n)} .
\end{aligned}
$$

The operators

$$\widetilde{C}\,[\varphi, K] : L^p(\mathbf{R}_+^n) \longrightarrow L^p(\mathbf{R}_+^n)$$

and

$$\widetilde{C}^*\,[\varphi, K] : L^{p'}(\mathbf{R}_+^n) \longrightarrow L^{p'}(\mathbf{R}_+^n)$$

can be visualized as follows:

$$
\begin{array}{ccc}
f \in L^p(\mathbf{R}_+^n) & \xrightarrow{\widetilde{C}[\varphi,K]} & L^p(\mathbf{R}_+^n) \ni \widetilde{C}\,[\varphi, K]\,f \\
\updownarrow & & \updownarrow \\
\widetilde{C}^*\,[\varphi, K]\,g \in L^{p'}(\mathbf{R}_+^n) & \xleftarrow{\widetilde{C}^*[\varphi,K]} & L^{p'}(\mathbf{R}_+^n) \ni g
\end{array}
$$

Fig. 14.4. The operators $\widetilde{C}\,[\varphi, K]$ and $\widetilde{C}^*\,[\varphi, K]$

The proof of Theorem 14.8 is complete. □

Step 2: Now let $k(x, z)$ be a real-valued function on $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ that satisfies conditions (i) and (ii). It should be noticed that the function

$$z \longmapsto |z|^n k(x, z)$$

belongs to $C^\infty(\mathbf{R}^n \setminus \{0\})$ for almost all $x \in \mathbf{R}^n_+$, and it is positively homogeneous of degree zero and satisfies the condition

$$\int_{\Sigma_{n-1}} k(x,z)\, d\sigma_z = 0.$$

For each $m = 1, 2, \ldots$ and $k = 1, 2, \ldots, d(m)$, we let

$$a_{km}(x) := \int_{\Sigma_{n-1}} k(x,z) Y_{km}(z)\, dz,$$

then, by the completeness of the spherical harmonics $\{Y_{km}\}$ in $L^2(\Sigma_{n-1})$ it follows that

$$\begin{aligned} &|T(x)-y|^n k(x, T(x)-y) \\ &= \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} a_{km}(x) Y_{km}(T(x)-y), \quad x \in \mathbf{R}^n_+,\ y \in \mathbf{R}^n \setminus \{0\}. \end{aligned} \tag{14.52}$$

Moreover, we have, by Theorem 4.31,

$$\|a_{km}\|_{L^\infty(\mathbf{R}^n_+)} \leq \frac{c_1(n)}{m^{2n}} M, \tag{14.18a}$$

$$\|Y_{km}\|_{L^\infty(\Sigma_{n-1})} \leq c_2(n)\, m^{(n-2)/2}, \tag{14.18b}$$

and

$$d(m) \leq c_3(n)\, m^{n-2}. \tag{14.18c}$$

Step 3: By using the expansion (14.52), we obtain that

$$\begin{aligned} \widetilde{K} f(x) &= \int_{\mathbf{R}^n_+} k(x, T(x)-y) f(y)\, dy \\ &= \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} a_{km}(x) \int_{\mathbf{R}^n_+} \frac{Y_{km}(T(x)-y)}{|T(x)-y|^n} f(y)\, dy. \end{aligned}$$

Hence it follows that

$$\begin{aligned} &\widetilde{C}\, [\varphi, K]\, f(x) \\ &= \int_{\mathbf{R}^n_+} k(x, T(x)-y)\, [\varphi(x)-\varphi(y)]\, f(y)\, dy \\ &= \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} a_{km}(x) \int_{\mathbf{R}^n_+} [\varphi(x)-\varphi(y)]\, \frac{Y_{km}(T(x)-y)}{|T(x)-y|^n} f(y)\, dy, \end{aligned}$$

and so

$$\widetilde{C}\,[\varphi, K]\, f(x) = \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} a_{km}(x) \widetilde{C}_{km}\,[\varphi, K]\, f(x),$$

where

$$\widetilde{C}_{km}\,[\varphi, K]\, f(x) := \int_{\mathbf{R}^n_+} [\varphi(x) - \varphi(y)]\, \frac{Y_{km}(T(x) - y)}{|T(x) - y|^n} f(y)\, dy.$$

Therefore, we have, by inequality (14.18a),

$$\begin{aligned} &\left\| \widetilde{C}\,[\varphi, K]\, f \right\|_{L^p(\mathbf{R}^n_+)} \qquad (14.53) \\ &= \left\| \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} a_{km} \widetilde{C}_{km}\,[\varphi, K]\, f \right\|_{L^p(\mathbf{R}^n_+)} \\ &\le \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} \|a_{km}\|_{L^\infty(\mathbf{R}^n_+)} \left\| \int_{\mathbf{R}^n_+} \widetilde{C}_{km}\,[\varphi, K]\, f \right\|_{L^p(\mathbf{R}^n_+)} \\ &\le c_1(n) M \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} \frac{1}{m^{2n}} \left\| \int_{\mathbf{R}^n_+} \widetilde{C}_{km}\,[\varphi, K]\, f \right\|_{L^p(\mathbf{R}^n_+)}. \end{aligned}$$

However, by inequality (14.18b) it follows from an application of Theorem 14.8 that

$$\left\| \widetilde{C}_{km}\,[\varphi, K]\, f \right\|_{L^p(\mathbf{R}^n_+)} \le c_7 c_2(n)\, m^{(n-2)/2}\, \|\varphi\|_*\, \|f\|_{L^p(\mathbf{R}^n_+)}. \qquad (14.54)$$

Step 4: Finally, the desired inequality (14.21) follows by combining inequalities (14.53), (14.54) and (14.18c):

$$\begin{aligned} &\left\| \widetilde{C}\,[\varphi, K]\, f \right\|_{L^p(\mathbf{R}^n_+)} \\ &\le c_1(n)\, M \sum_{m=1}^{\infty} \sum_{k=1}^{d(m)} \frac{1}{m^{2n}} c_2(n)\, m^{(n-2)/2}\, \|\varphi\|_*\, \|f\|_{L^p(\mathbf{R}^n_+)} \\ &\le c_1(n)\, c_2(n)\, c_3(n)\, c_7 M \left(\sum_{m=1}^{\infty} \frac{1}{m^{2n}} m^{(n-2)/2} m^{n-2} \right) \|\varphi\|_*\, \|f\|_{L^p(\mathbf{R}^n_+)} \\ &= c_1(n)\, c_2(n)\, c_3(n)\, c_7 M \left(\sum_{m=1}^{\infty} \frac{1}{m^{n/2+3}} \right) \|\varphi\|_*\, \|f\|_{L^p(\mathbf{R}^n_+)}. \end{aligned}$$

Now the proof of Theorem 14.5 is complete. □

The next result is a local version of Theorem 14.5 (see [19, Theorem 2.7]):

Corollary 14.9. *Let* $\varphi \in \mathrm{VMO} \cap L^\infty(\mathbf{R}^n)$ *and* η *the* VMO *modulus of* φ. *Then, for each* $\varepsilon > 0$, *there exists a positive constant* $\rho_0 = \rho_0(\varepsilon, \eta)$ *such that, for any* $r \in (0, \rho_0)$, *we have the inequality*

$$\left\| \widetilde{C}\,[\varphi, K]\, f \right\|_{L^p(B_r^+)} \leq c_5\, \varepsilon\, \|f\|_{L^p(B_r^+)} \quad \text{for all } f \in L^p(B_r^+), \tag{14.55}$$

where (see Figure 14.5)

$$B_r^+ := B_r \cap \mathbf{R}^n_+.$$

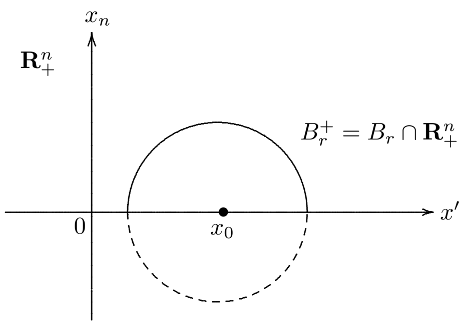

Fig. 14.5. The semi-ball B_r^+ in the half-space $\mathbf{R}^n_+$

Proof. The proof of Corollary 14.9 is divided into three steps.

Step (1): By using Theorem 4.3, for each $\varepsilon > 0$ we can find a bounded, uniformly continuous function $a(x)$ on $\mathbf{R}^n$ such that

$$\|a - \varphi\|_* < \frac{\varepsilon}{2}. \tag{14.56}$$

Let $\omega_a(r)$ be the modulus of uniform continuity of $a(x)$ defined by the formula

$$\omega_a(r) = \sup_{|x-y| \leq r} |a(x) - a(y)|,$$

and choose a constant $\rho_0 = \rho_0(\varepsilon, \eta) > 0$ such that

$$\omega_a(\rho_0) < \frac{\varepsilon}{2}. \tag{14.57}$$

If $B_r = B_r(x_0)$ is a ball of radius r about x_0, we let

$$b(x) := \begin{cases} a(x) & \text{if } x \in B_r(x_0), \\ a\left(x_0 + r\frac{x-x_0}{|x-x_0|}\right) & \text{if } x \in \mathbf{R}^n \setminus B_r(x_0). \end{cases}$$

It should be noticed that the function $b(x)$ is uniformly continuous on $\mathbf{R}^n$ and that the oscillation of $b(x)$ in $\mathbf{R}^n$ equals the oscillation of $a(x)$ in B_r.

Step (2): Now we remark that

$$\begin{aligned} \widetilde{C}\,[\varphi, K]\,f &= \varphi\,(Kf) - K\,(\varphi f) \\ &= (\varphi - a)\,Kf - K\,((\varphi - a)f) + a\,(Kf) - K\,(af) \\ &= \widetilde{C}\,[\varphi - a, K]\,f + \widetilde{C}\,[a, K]\,f, \end{aligned}$$

so that

$$\left\|\widetilde{C}\,[\varphi, K]\,f\right\|_{L^p(B_r^+)} \leq \left\|\widetilde{C}\,[\varphi - a, K]\,f\right\|_{L^p(B_r^+)} + \left\|\widetilde{C}\,[a, K]\,f\right\|_{L^p(B_r^+)}.$$

However, we have, by Theorem 14.5,

$$\left\|\widetilde{C}\,[\varphi - a, K]\,f\right\|_{L^p(B_r^+)} \leq c_5\,\|\varphi - a\|_*\,\|f\|_{L^p(B_r^+)}, \tag{14.58}$$

and also

$$\left\|\widetilde{C}\,[a, K]\,f\right\|_{L^p(B_r^+)} = \left\|\widetilde{C}\,[b, K]\,f\right\|_{L^p(B_r^+)} \leq c_5\,\|b\|_*\,\|f\|_{L^p(B_r^+)}. \tag{14.59}$$

Moreover, it is easy to see that

$$\|b\|_* \leq \omega_b(r) = \omega_a(r). \tag{14.60}$$

Hence, by combining inequalities (14.59) and (14.60) we obtain that

$$\left\|\widetilde{C}\,[a, K]\,f\right\|_{L^p(B_r^+)} \leq c_5\,\omega_a(r)\,\|f\|_{L^p(B_r^+)}. \tag{14.61}$$

Step (3): Therefore, it follows from inequalities (14.58) and (14.61) that we have, for all $0 < r < \rho_0$,

$$\begin{aligned} &\left\|\widetilde{C}\,[\varphi, K]\,f\right\|_{L^p(B_r^+)} \\ &\leq \left\|\widetilde{C}\,[\varphi - a, K]\,f\right\|_{L^p(B_r^+)} + \left\|\widetilde{C}\,[a, K]\,f\right\|_{L^p(B_r^+)} \\ &\leq c_5\,\|\varphi - a\|_*\,\|f\|_{L^p(B_r^+)} + c_5\,\omega_a(r)\,\|f\|_{L^p(B_r^+)} \\ &\leq c_5\,(\|\varphi - a\|_* + \omega_a(\rho_0))\,\|f\|_{L^p(B_r^+)} \quad \text{for all } f \in L^p(B_r^+). \end{aligned}$$

By inequalities (14.56) and (14.57), this inequality proves the desired inequality (14.55).

The proof of Corollary 14.9 is complete. □

Remark 14.1. Roughly speaking, inequality (14.55) may be expressed as follows:

$$\left\| \widetilde{C}\,[\varphi, K]\, f \right\|_{L^p(B_r^+)} \leq c_5\, \eta(r)\, \|f\|_{L^p(B_r^+)} \quad \text{for all } f \in L^p(B_r^+). \tag{14.55$'$}$$

Here

$$\eta(r) = \sup_{\rho \leq r} \frac{1}{|B|} \int_B |\varphi(x) - \varphi_B|\, dx$$

is the VMO modulus of φ (see Section 4.2).

14.3.3 L^p Boundedness of Integral Operators with Positive Kernels

Just as in the proof of Theorem 14.5, we can prove an L^p boundedness of integral operators which have *positive kernels* depending on the difference $|\varphi(x) - \varphi(y)|$. In fact, we can obtain the following ([12, Theorem 0.1]):

Theorem 14.10. *If $\varphi \in$ BMO and $f \in L^p(\mathbf{R}^n_+)$ for $1 < p < \infty$, we define the commutator $\widetilde{C}_\varphi$ by the formula*

$$\widetilde{C}_\varphi f(x) := \int_{\mathbf{R}^n_+} \frac{|\varphi(x) - \varphi(y)|}{|T(x) - y|^n} f(y)\, dy$$
$$= \varphi(x)$$

Then there exists a positive constant $c = c(n,p)$ such that

$$\left\| \widetilde{C}_\varphi f \right\|_{L^p(\mathbf{R}^n_+)} \leq c\, \|\varphi\|_*\, \|f\|_{L^p(\mathbf{R}^n_+)} \quad \textit{for all } f \in L^p(\mathbf{R}^n_+).$$

Remark 14.2. It should be emphasized that the results in Section 14.3 remain valid if the function $|T(x) - y|$ is replaced by an equivalent function.

14.4 Local Boundary Estimates

First, we recall (see Section 14.1) that

$$C_{\gamma_0} = \{u|_{B_r^+} : u \in C_0^\infty(B_r),\ u = 0 \text{ on } B_r \cap \{x_n = 0\}\},$$

and that $W^{2,p}_{\gamma_0}(B_r^+)$ is the closure of C_{γ_0} in the Sobolev space $W^{2,p}(B_r^+)$. In particular, it should be emphasized that $u \in W^{2,p}_0(B_r)$ belongs to $W^{2,p}_{\gamma_0}(B_r^+)$ if and only if it vanishes on $B_r \cap \{x_n = 0\}$ (see [2, Theorem 5.37]).

Now we can prove local boundary *a priori* estimates for the second derivatives of solutions of the homogeneous Dirichlet problem, similar to Lemma 13.2 (see [19, Theorem 3.3]):

Lemma 14.11. *Let $1 < q < p < \infty$, and let*

$$M := \max_{1 \le i,j \le n} \max_{|\alpha| \le 2n} \left\| \frac{\partial^\alpha}{\partial z^\alpha} \Gamma_{ij}(\cdot,\cdot) \right\|_{L^\infty(B_\sigma^+ \times \Sigma_{n-1})}.$$

Then there exists a constant $\rho_0 = \rho_0(n,p,q,M,\lambda,\eta)$, with $0 < \rho_0 < \sigma$, such that if $u \in W^{2,q}_{\gamma_0}(B_r^+)$ with $\mathcal{L}u \in L^p(B_r^+)$ for $0 < r < \rho_0$, it follows that $u \in W^{2,p}(B_r^+)$. Moreover, we have the boundary a priori *estimate*

$$\left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^p(B_r^+)} \le C \left\| \mathcal{L}u \right\|_{L^p(B_r^+)} \quad \text{for } 1 \le i,j \le n, \tag{14.62}$$

with a positive constant $C = C(n,p,M,\lambda,\eta)$. Here (see Figure 14.5)

$$B_r^+ = B_r \cap \mathbf{R}^n_+ \quad \text{for } 0 < r < \rho_0.$$

In other words, the second-order, elliptic differential operator $\mathcal{L}$ controls all second-order partial derivatives in the L^p-norm for $1 < p < \infty$ in the semi-ball B_r^+ (cf. Lemma 13.2).

Proof. Let $\nu \in [q,p]$. For $1 \le i,j,h,k \le n$, we introduce linear operators S_{ijhk} and $\widetilde{S}_{ijhk}$ on $L^\nu(B_r^+)$, $0 < r \le \sigma$, as follows:

(1) If $1 \le i,j,h,k \le n$, we let

$$S_{ijhk}(f) := \text{v. p.} \int_{B_r^+} \Gamma_{ij}(x, x-y) \left[a^{hk}(x) - a^{hk}(y)\right] f(y)\, dy.$$

(2) If $1 \le i,j \le n-1$ and $1 \le h,k \le n$, we let

$$\widetilde{S}_{ijhk}(f) := \int_{B_r^+} \Gamma_{ij}(x, T(x)-y) \left[a^{hk}(x) - a^{hk}(y)\right] f(y)\, dy.$$

(3) If $1 \le i \le n-1$, $j = n$ and $1 \le h,k \le n$, we let

$$\widetilde{S}_{inhk}(f) := \int_{B_r^+} \left(\sum_{j=1}^n \Gamma_{ij}(x, T(x)-y) A_j(x) \right) \left[a^{hk}(x) - a^{hk}(y)\right] f(y)\, dy.$$

(4) If $i = n$, $j = n$ and $1 \le h, k \le n$, we let

$$\widetilde{S}_{nnhk}(f) := \int_{B_r^+} \left(\sum_{i,j=1}^{n} \Gamma_{ij}(x, T(x) - y) A_i(x) A_j(x) \right) \times \left[a^{hk}(x) - a^{hk}(y) \right] f(y)\, dy.$$

Here we recall that the map $T(x)$ is defined by the formula

$$T(x) := T(x;x) = \begin{pmatrix} x_1 - 2x_n \frac{a^{1n}(x)}{a^{nn}(x)} \\ x_2 - 2x_n \frac{a^{2n}(x)}{a^{nn}(x)} \\ \vdots \\ -x_n \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

and that the vector $A(x)$ is defined by the formula

$$A(x) = \begin{pmatrix} A_1(x) \\ A_2(x) \\ \vdots \\ A_n(x) \end{pmatrix} := \frac{\partial T}{\partial x_n}(x;z)\Big|_{z=x} = \begin{pmatrix} -2\frac{a^{1n}(x)}{a^{nn}(x)} \\ -2\frac{a^{2n}(x)}{a^{nn}(x)} \\ \vdots \\ -1 \end{pmatrix}.$$

By using Corollaries 11.5 and 14.9, we can find a constant $\rho_0 > 0$ such that

$$\sum_{i,j,h,k=1}^{n} \left\| S_{ijhk} + \widetilde{S}_{ijhk} \right\| < \frac{1}{2}, \quad 0 < \rho < \rho_0. \tag{14.63}$$

Here the norm of the operators $S_{ijhk} + \widetilde{S}_{ijhk}$ is the norm in the space of bounded, linear operators on $L^\nu(B_r^+)$ for $0 < r < \rho_0$ and $q \le \nu \le p$.

Let $u \in W^{2,p}_{\gamma_0}(B_r^+)$ such that $\mathcal{L}u \in L^p(B_r^+)$, $0 < r < \rho_0$, and let

$$h_{ij}(x) = \text{v. p.} \int_{B_r^+} \Gamma_{ij}(x, x-y) \mathcal{L}u(y)\, dy + c_{ij}(x)\ \mathcal{L}u(x) - \widetilde{I}_{ij}(x) \quad \text{for } 1 \le i,j \le n.$$

Here the terms $\widetilde{I}_{ij}(x)$ are defined respectively as follows:

$$\widetilde{I}_{ij}(x) := \int_{B_r^+} \Gamma_{ij}(x, T(x) - y) \mathcal{L}u(y)\, dy \quad \text{for } 1 \le i,j \le n-1;$$

$$\widetilde{I}_{in}(x) := \widetilde{I}_{ni}(x) = \int_{B_r^+} \left(\sum_{\ell=1}^{n} \Gamma_{ij}(x, T(x) - y) A_\ell(y) \right) \mathcal{L}u(y)\, dy$$

for $1 \leq i \leq n-1$,

$$\widetilde{I}_{nn}(x) := \int_{B_r^+} \left(\sum_{\ell,m=1}^{n} \Gamma_{ij}(x, T(x)-y) A_\ell(x) A_m(x) \right) \mathcal{L}u(y)\, dy.$$

Then, since $\mathcal{L}u \in L^p(B_r^+)$, it follows from an application of Theorem 14.2 that

$$h_{ij} \in L^p(B_r^+).$$

Therefore, if we introduce a mapping

$$T\colon \left(L^p(B_r^+)\right)^{n^2} \longrightarrow \left(L^p(B_r^+)\right)^{n^2}$$

by the formula

$$\begin{aligned} T\mathbf{w} &= \left(\sum_{h,k=1}^{n} \left(S_{ijhk} + \widetilde{S}_{ijhk} \right) (w_{hk}) + h_{ij}(x) \right)_{1 \leq i,j \leq n}, \\ \mathbf{w} &= (w_{ij})_{1 \leq i,j \leq n}, \end{aligned}$$

then we find that T is a *contraction mapping*. Indeed, we have, by condition (14.63),

$$\begin{aligned} \left\| T\mathbf{w}^{(\mathbf{1})} - T\mathbf{w}^{(\mathbf{2})} \right\| &= \sum_{i,j=1}^{n} \left\| \sum_{h,k=1}^{n} \left(S_{ijhk} + \widetilde{S}_{ijhk} \right) \left(w_{hk}^{(1)} - w_{hk}^{(2)} \right) \right\|_{L^r(B_r^+)} \\ &\leq \sum_{i,j,h,k=1}^{n} \left\| S_{ijhk} + \widetilde{S}_{ijhk} \right\| \left\| w_{hk}^{(1)} - w_{hk}^{(2)} \right\|_{L^r(B_r^+)} \\ &\leq \sum_{h,k=1}^{n} \left(\sum_{i,j=1}^{n} \left\| S_{ijhk} + \widetilde{S}_{ijhk} \right\| \right) \left\| w_{hk}^{(1)} - w_{hk}^{(2)} \right\|_{L^r(B_r^+)} \\ &\leq \frac{1}{2} \sum_{h,k=1}^{n} \left\| w_{hk}^{(1)} - w_{hk}^{(2)} \right\|_{L^r(B_r^+)} \\ &= \frac{1}{2} \left\| \mathbf{w}^{(\mathbf{1})} - \mathbf{w}^{(\mathbf{2})} \right\|. \end{aligned}$$

Hence it follows an application of the *contraction mapping principle* (Theorem 2.1) that the mapping T has a unique fixed point

$$\begin{aligned} \mathbf{w} &= (w_{ij}) \in \left(L^{q_1}(B_r^+)\right)^{n^2}, \\ T\mathbf{w} &= \mathbf{w}. \end{aligned}$$

On the other hand, boundary representation formula (14.2) implies that the Hessian matrix

$$\nabla^2 u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right) \in \left(L^q(B_r^+) \right)^{n^2}$$

is also a fixed point of T. However, since we have the inclusion

$$\left(L^p(B_r^+) \right)^{n^2} \subset \left(L^q(B_r^+) \right)^{n^2},$$

by the uniqueness of fixed points of T, it follows that

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = w_{ij} \in L^p(B_r^+) \quad \text{for } 1 \leq i, j \leq n.$$

Therefore, we obtain that

$$u \in W^{2,p}(B_r^+).$$

Furthermore, by Theorem 14.1 we can write the second derivatives

$$\frac{\partial^2 u}{\partial x_i \partial x_j} \quad \text{for } 1 \leq i, j \leq n$$

in the form

$$\begin{aligned} &\frac{\partial^2 u}{\partial x_i \partial x_j}(x) \\ &= \sum_{h,k=1}^{n} C\left[a^{hk}, K_{ij}\right] \left(\frac{\partial^2 u}{\partial x_h \partial x_k} \right)(x) \\ &\quad + K_{ij}(\mathcal{L}u)(x) + \left(\int_{|t|=1} \Gamma_i(x,t) t_j \, d\sigma_t \right) \mathcal{L}u(x) - I_{ij}(x). \end{aligned}$$

Here the singular integral operators K_{ij} and boundary commutators $C[\varphi, K_{ij}]$ are defined respectively as follows:

$$\begin{aligned} K_{ij} f(x) &:= \text{v. p.} \int_{B_r^+} \Gamma_{ij}(x, x-y) f(y) \, dy, \\ C\left[\varphi, K_{ij}\right] f(x) &:= \text{v. p.} \int_{B_r^+} \Gamma_{ij}(x, x-y) \left[\varphi(x) - \varphi(y)\right] f(y) \, dy, \end{aligned}$$

and the terms $I_{ij}(x)$ are defined respectively as follows:

$$I_{ij}(x) := \sum_{h,k=1}^{n} \widetilde{C}\left[a^{hk}, K_{ij}\right] \left(\frac{\partial^2 u}{\partial x_h \partial x_k} \right)(x) + \widetilde{K}_{ij}(\mathcal{L}u)(x),$$
$$\text{for } 1 \leq i, j \leq n-1;$$

$$I_{in}(x) = I_{in}(x) := \sum_{j=1}^{n} A_j(x)$$
$$\times \left(\sum_{h,k=1}^{n} \widetilde{C}\left[a^{hk}, K_{ij}\right] \left(\frac{\partial^2 u}{\partial x_h \partial x_k}\right)(x) + \widetilde{K}_{ij}(\mathcal{L}u)(x) \right),$$
$$\text{for } 1 \leq i \leq n-1;$$
$$I_{nn}(x) := \sum_{i,j=1}^{n} A_i(x) A_j(x)$$
$$\times \left(\sum_{h,k=1}^{n} \widetilde{C}\left[a^{hk}, K_{ij}\right] \left(\frac{\partial^2 u}{\partial x_h \partial x_k}\right)(x) + \widetilde{K}_{ij}(\mathcal{L}u)(x) \right),$$

and the singular integral operators $\widetilde{K}_{ij}$ and boundary commutators $\widetilde{C}[\varphi, K_{ij}]$ are defined respectively as follows:

$$\widetilde{K}_{ij} f(x) := \int_{B_r^+} \Gamma_{ij}(x, T(x) - y) f(y)\, dy,$$
$$\widetilde{C}\left[\varphi, K_{ij}\right] f(x) := \int_{B_r^+} \Gamma_{ij}(x, T(x) - y) \left[\varphi(x) - \varphi(y)\right] f(y)\, dy.$$

Therefore, the desired boundary *a priori* estimate (14.62) follows from an application of Theorems 11.3, 14.2, 14.5 and Corollary 14.9 with

$$k(x, z) := \Gamma_{ij}(x, z), \quad \varphi(x) := a^{hk}(x).$$

The proof of Lemma 14.11 is complete. □

14.5 Proof of Theorem 12.2

The purpose of this section is to prove Theorem 12.2. The proof of Theorem 12.2 is divided into two steps.

Step 1: Since the boundary $\partial\Omega$ is of class $C^{1,1}$, for each point $x_0 \in \partial\Omega$ we can find an open neighborhood $U(x_0)$ and a $C^{1,1}$-diffeomorphism $G = (G_1, G_2, \ldots, G_n)$ which maps $U(x_0) \cap \Omega$ onto B_r^+ (see Figure 14.6). Then it is easy to see that the coefficients $a^{ij}(x)$ of $\mathcal{L}$ are transformed into the functions

$$b^{ij}(y) = \sum_{h,k=1}^{n} a^{hk}(x) \frac{\partial G_i}{\partial x_h}(x) \frac{\partial G_j}{\partial x_k}(x), \quad x = G^{-1}(y), \quad y \in B_r^+,$$

where $G^{-1}\colon B_r^+ \to U(x_0) \cap \Omega$ is the inverse of G. Moreover, without loss of generality, we may assume that the functions $b^{ij}(y)$ are defined on the whole space $\mathbf{R}^n$. Then we can verify that the functions $b^{ij}(y)$ are in VMO and further that their VMO moduli are estimated in terms of the VMO moduli of a^{hk}, the $C^{1,1}$-norm of G, the uniform continuity moduli of the derivatives $\partial G_i/\partial x_h$ (see [1, Proposition 1.3]).

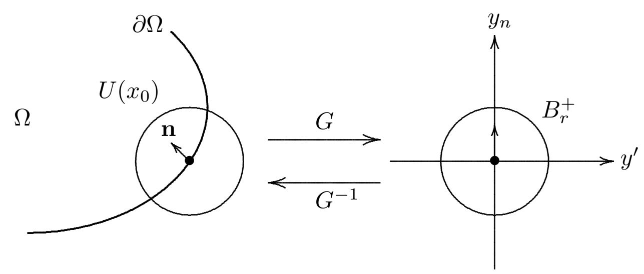

Fig. 14.6. An open neighborhood $U(x_0)$ and the $C^{1,1}$-diffeomorphism G of $U(x_0) \cap \Omega$ onto B_r^+

Step 2: The desired global *a priori* estimate (12.3) follows in a standard way from Theorem 12.1 and Lemma 14.11 by a covering argument, by flattening the boundary $\partial\Omega$ and by using interpolation inequalities (Theorem 13.4).

Step 2-1: Now we choose a finite covering $\{U_j\}_{j=1}^N$ of the boundary $\partial\Omega$ by open subsets of $\mathbf{R}^n$ and $C^{1,1}$-diffeomorphisms G_j of $U_j \cap \Omega$ onto B_r^+ in each of which inequality (14.62) holds true. Furthermore, we choose an open subset U_0 of Ω, bounded away from $\partial\Omega$, such that (see Figure 14.7)

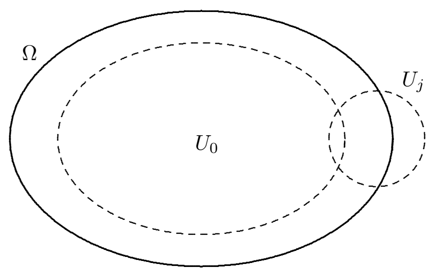

Fig. 14.7. The open covering $\{U_j\}$ of $\partial\Omega$ and the open set U_0 bounded away from $\partial\Omega$

$$\Omega \subset U_0 \cup \left(\bigcup_{j=1}^N U_j \right),$$

and take a partition of unity $\{\alpha_k\}_{k=0}^N$ subordinate to the open covering $\{U_k\}_{k=0}^N$ of Ω.

Step 2-2: Now we assume that a function

$$u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$$

satisfies the equation

$$\mathcal{L}u = f \quad \text{in } \Omega.$$

Then, by applying the interior *a priori* estimate (12.2) to the function $\alpha_0 u$ we obtain that

$$\begin{aligned}
&\|u\|_{W^{2,p}(\Omega)} \qquad (14.64)\\
&\le \sum_{k=0}^{N} \|\alpha_k u\|_{W^{2,p}(\Omega)}\\
&\le \|\alpha_0 u\|_{W^{2,p}(\Omega)}\\
&\quad + \sum_{k=1}^{N} \left(\left\|\nabla^2(\alpha_k u)\right\|_{L^p(\Omega)} + \|\nabla(\alpha_k u)\|_{L^p(\Omega)} + \|\alpha_k u\|_{L^p(\Omega)} \right)\\
&\le C_0 \left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} \right)\\
&\quad + \sum_{k=1}^{N} \left(\left\|\nabla^2(\alpha_k u)\right\|_{L^p(\Omega\cap U_k)} + \|\nabla(\alpha_k u)\|_{L^p(\Omega\cap U_k)} + \|u\|_{L^p(\Omega)} \right).
\end{aligned}$$

(a) In order to estimate the terms $\nabla(\alpha_k u)$, $1 \le k \le N$, we recall the interpolation inequality (Theorem 13.4)

$$\|\nabla v\|_{L^p(\Omega)} \le \varepsilon \; \left\|\nabla^2 v\right\|_{L^p(\Omega)} + \frac{C}{\varepsilon} \; \|v\|_{L^p(\Omega)}, \quad v \in W^{2,p}(\Omega). \quad (13.17)$$

Since we have the formula

$$\nabla(\alpha_k u) = \alpha_k \nabla u + u(\nabla \alpha_k),$$

by applying inequality (13.17) to the function $u \in W^{2,p}(\Omega)$, we can find positive constants C_1 and C_2 such that

$$\begin{aligned}
&\|\nabla(\alpha_k u)\|_{L^p(\Omega\cap U_k)} \qquad (14.65)\\
&\le \|\alpha_k(\nabla u)\|_{L^p(\Omega\cap U_k)} + \|u(\nabla\alpha_k)\|_{L^p(\Omega\cap U_k)}\\
&\le \|\nabla u\|_{L^p(\Omega)} + C_1 \|u\|_{L^p(\Omega)}\\
&\le \varepsilon \; \left\|\nabla^2 u\right\|_{L^p(\Omega)} + \frac{C_2}{\varepsilon} \|u\|_{L^p(\Omega)} + C_1 \|u\|_{L^p(\Omega)} \qquad \text{for } 1 \le k \le N.
\end{aligned}$$

(b) Moreover, by applying inequality (14.62) to the functions $\alpha_k u \in W^{2,p}_{\gamma_0}(\Omega \cap U_k)$, $1 \leq k \leq N$, we obtain that

$$\left\|\nabla^2(\alpha_k u)\right\|_{L^p(\Omega\cap U_k)} \leq C \left\|\mathcal{L}(\alpha_k u)\right\|_{L^p(\Omega\cap U_k)}. \tag{14.66}$$

However, we have the formula

$$\mathcal{L}(\alpha_k u) = \alpha_k f + u\,\mathcal{L}(\alpha_k) + 2\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial \alpha_k}{\partial x_i}\frac{\partial u}{\partial x_j} \tag{14.67}$$

$$\text{for } 1 \leq k \leq N.$$

Hence, it follows from inequality (14.66), formula (14.67) and inequality (13.17) that

$$\begin{aligned}
&\left\|\nabla^2(\alpha_k u)\right\|_{L^p(\Omega\cap U_k)} \\
&\leq C \left\|\mathcal{L}(\alpha_k u)\right\|_{L^p(\Omega\cap U_k)} \\
&\leq C\Bigg(\left\|\alpha_k f\right\|_{L^p(\Omega\cap U_k)} + \left\|u\,\mathcal{L}(\alpha_k)\right\|_{L^p(\Omega\cap U_k)} \\
&\quad + 2\sum_{i,j=1}^{n}\left\|a^{ij}\frac{\partial \alpha_k}{\partial x_i}\frac{\partial u}{\partial x_j}\right\|_{L^p(\Omega\cap U_k)}\Bigg) \\
&\leq C_3\left(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}\right) \\
&\leq C_3\left(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right) + \varepsilon\ \left\|\nabla^2 u\right\|_{L^p(\Omega)} + \frac{C_4}{\varepsilon}\|u\|_{L^p(\Omega)} \\
&\qquad \text{for } 1 \leq k \leq N.
\end{aligned} \tag{14.68}$$

(c) Therefore, by combining inequalities (14.64), (14.65) and (14.68) we obtain that

$$\begin{aligned}
\|u\|_{W^{2,p}(\Omega)} &\leq 2N\varepsilon\ \left\|\nabla^2 u\right\|_{L^p(\Omega)} + C_5\left(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right) \\
&\quad + \frac{C_6}{\varepsilon}\|u\|_{L^p(\Omega)}.
\end{aligned}$$

This proves the desired global *a priori* estimate (12.3), if we take

$$\varepsilon := \frac{1}{4N}.$$

The proof of Theorem 12.2 is now complete. □

14.6 Notes and Comments

This chapter is adapted from Chiarenza–Frasca–Longo [19] and also Bramanti–Cerutti [13].

15

Unique Solvability of the Dirichlet Problem

This chapter is devoted to the study of the *homogeneous* Dirichlet problem for a second-order, uniformly elliptic differential operator with VMO coefficients in the framework of Sobolev spaces of L^p style. We prove an existence and uniqueness theorem for the Dirichlet problem (Theorem 15.1). Our proof is based on some interior and boundary *a priori* estimates for the solutions of problem (15.2) (Theorem 12.1 and Theorem 12.2). Both the interior and boundary *a priori* estimates are consequences of explicit representation formulas (13.1) and (14.1) for the solutions of problem (15.2) (Theorem 13.1 and Theorem 14.1) and also of the L^p-boundedness of Calderón–Zygmund singular integral operators and boundary commutators appearing in those representation formulas (Theorem 14.2 and Theorem 14.5). It should be emphasized that the VMO assumption on the coefficients a^{ij} is of the greatest relevance in the study of singular commutators. The uniqueness result in Theorem 15.1 follows from a variant of the Bakel'man–Aleksandrov maximum principle in the framework of Sobolev spaces due to Bony [9] (Theorem 8.5). Moreover, in order to prove the existence theorem for the homogeneous Dirichlet problem, we make good use of the method of continuity (Theorem 2.14).

Now, let Ω be a bounded domain in Euclidean space $\mathbf{R}^n$ $(n \geq 3)$ with boundary $\partial\Omega$ of class $C^{1,1}$, and we consider a second-order, elliptic differential operator $\mathcal{L}$ with real *discontinuous* coefficients of the form

$$\mathcal{L}u := \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Here the coefficients a^{ij} satisfy the following three conditions (i), (ii) and (iii):

(i) $a^{ij}(x) \in \mathrm{VMO} \cap L^\infty(\mathbf{R}^n)$ for all $1 \leq i, j \leq n$.
(ii) $a^{ij}(x) = a^{ji}(x)$ for all $1 \leq i, j \leq n$ and for almost all $x \in \Omega$.
(iii) There exists a positive constant λ such that

$$\frac{1}{\lambda} |\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2 \tag{15.1}$$

for almost all $x \in \Omega$ and for all $\xi \in \mathbf{R}^n$.

The purpose of this chapter is devoted to the study of the following *homogeneous* Dirichlet problem:

$$\begin{cases} \mathcal{L}u = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f & \text{in } \Omega, \\ \gamma_0 u = 0 & \text{on } \partial\Omega. \end{cases} \tag{15.2}$$

The next existence and uniqueness theorem for problem (15.2) is essentially due to Chiarenza–Frasca–Longo [19, Theorems 4.3 and 4.4] (see also [76, Theorem 1.1]):

Theorem 15.1 (the existence and uniqueness theorem)**.** *Let $1 < p < \infty$. Then, for any given $f \in L^p(\Omega)$ there exists a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ of problem* (15.2)*. Moreover, we have the* a priori *estimate*

$$\|u\|_{W^{2,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)}, \tag{15.3}$$

with a positive constant $C = C(n, p, \lambda, \eta, M, \partial\Omega)$.

The proof of Theorem 15.1 can be visualized in the following diagram:

15.1 VMO Functions and Friedrichs' Mollifiers

The VMO assumption means a kind of continuity in the average sense, not in the pointwise sense. This property guarantees that VMO functions may be approximated by smooth functions. In fact, we can prove the following (cf. condition (iii) of Theorem 4.5):

Proposition 15.2. *Let $f(x) \in \mathrm{VMO}$. For any $\varepsilon > 0$, there exists a smooth function $g_\varepsilon(x) \in \mathrm{VMO}$ such that*

$$\|f - g_\varepsilon\|_* < \varepsilon, \tag{15.4}$$

and further that

$$\eta_\varepsilon(r) \leq \eta(r),$$

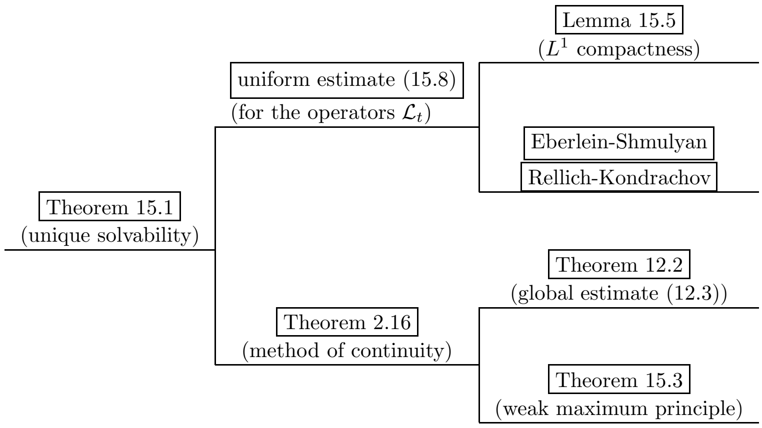

Table 15.1. *A flowchart for the proof of Theorem 15.1*

where $\eta(r)$ and $\eta_\varepsilon(r)$ are the VMO *moduli of f and g_ε, respectively. In other words,* VMO *functions can be approximated by smooth functions.*

Proof. Just as in Subsection 3.7.2, we take a bell-shaped function $\varphi(x)$ on $\mathbf{R}^n$ that satisfies the following four conditions:

$$\varphi(x) \in C_0^\infty(\mathbf{R}^n).$$
$$\varphi(x) \geq 0 \quad \text{on } \mathbf{R}^n.$$
$$\operatorname{supp} \varphi \subset B(0,1) := \{x \in \mathbf{R}^n : |x| \leq 1\}.$$
$$\int_{\mathbf{R}^n} \varphi(x)\, dx = 1.$$

For each $\varepsilon > 0$, we let

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right).$$

Then it is easy to verify the following four conditions:

$$\varphi_\varepsilon(x) \in C_0^\infty(\mathbf{R}^n).$$
$$\varphi_\varepsilon(x) \geq 0 \quad \text{on } \mathbf{R}^n.$$
$$\operatorname{supp} \varphi_\varepsilon \subset B(0,\varepsilon) := \{x \in \mathbf{R}^n : |x| \leq \varepsilon\},$$
$$\int_{\mathbf{R}^n} \varphi_\varepsilon(x)\, dx = 1.$$

We recall that the functions $\{\varphi_\varepsilon\}$ are *Friedrichs' mollifiers.*

If we let

$$f_\varepsilon(x) := (f * \varphi_\varepsilon)(x) = \frac{1}{\varepsilon^n} \int_{\mathbf{R}^n} \varphi\left(\frac{x-y}{\varepsilon}\right) f(y)\, dy$$
$$= \int_{|z|\le 1} \varphi(z)\, f(x - \varepsilon z)\, dz,$$

then we have the following assertions:

(1) $f_\varepsilon(x) \in C^\infty(\mathbf{R}^n)$.

(2) By applying Minkowski's inequality for integrals (Theorem 3.16), we obtain that

$$\|f_\varepsilon - f\|_* = \left\| \int_{|z|\le 1} \varphi(z)\, (f(\cdot - \varepsilon z) - f(\cdot))\, dz \right\|_* \tag{15.5}$$
$$\le \int_{|z|\le 1} \varphi(z)\, \|f(\cdot - \varepsilon z) - f(\cdot)\|_*\, dz.$$

However, by Theorem 4.3, (iii) it follows that there exists a positive constant C, independent of ε, such that

$$\|f(\cdot - \varepsilon z) - f(\cdot)\|_* \le C\, \eta(\varepsilon) \quad \text{for } |z| \le 1. \tag{15.6}$$

By combining inequalities (15.5) and (15.6), we find that

$$\|f_\varepsilon - f\|_* \le \int_{|z|\le 1} \varphi(z)\, \|f(\cdot - \varepsilon z) - f(\cdot)\|_*\, dz$$
$$\le C \int_{|z|\le 1} \varphi(z)\, dz \cdot \eta(\varepsilon)$$
$$= C\, \eta(\varepsilon).$$

This proves the desired inequality (15.4) with $g_\varepsilon := f_\varepsilon$.

(3) Moreover, we have, by Theorem 3.16,

$$\|f_\varepsilon\|_* = \left\| \int_{|z|\le 1} \varphi(z)\, f(\cdot - \varepsilon z)\, dz \right\|_*$$
$$\le \int_{|z|\le 1} \varphi(z)\, \|f(\cdot - \varepsilon z)\|_*\, dz = \int_{|z|\le 1} \varphi(z)\, dz \cdot \|f\|_*$$
$$= \|f\|_*\,.$$

This proves that

$$g_\varepsilon = f_\varepsilon \in \mathrm{VMO} \cap C^\infty(\mathbf{R}^n),$$
$$\eta_\varepsilon(r) \le \eta(r).$$

The proof of Proposition 15.2 is complete. □

15.2 Proof of Theorem 15.1

The proof of Theorem 15.1 is divided into four steps (see [18], [19], [47]).

Step 1: First, by applying Theorem 12.2 we obtain the following global *a priori* estimate:

$$\begin{aligned} &\|u\|_{W^{2,p}(\Omega)} \leq c_1 \left(\|\mathcal{L}u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \right), \\ &u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \end{aligned} \tag{15.7}$$

where

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega) : \gamma_0 u = 0 \text{ on } \partial\Omega \right\}.$$

Here $c_1 > 0$ is a constant depending on the coefficients a^{ij} only through the ellipticity constant λ, the bound on the norms $\|a^{ij}\|_{L^p(\Omega)}$ and the VMO moduli of the a^{ij}.

Step 2: The *uniqueness result* of problem (15.2) follows from an application of the global regularity theorem (Theorem 12.2) and the following weak maximum principle (Theorem 8.5) due to Bony [9]:

Theorem 15.3 (the weak maximum principle)**.** *Assume that a function u in $W^{2,p}(\Omega)$ with $n < p < \infty$ satisfies the condition*

$$\mathcal{L}u(x) \geq 0 \quad \textit{for almost all } x \in \Omega.$$

Then it follows that

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+,$$

where

$$u^+(x) := \max\{u(x), 0\} \quad \textit{for } x \in \overline{\Omega} = \Omega \cup \partial\Omega.$$

Remark 15.1. In fact, Bony proved this maximum principle under the weaker condition that $a^{ij} \in L^\infty(\Omega)$ (see the proof of Theorem 8.5 in Chapter 8).

In light of the global regularity theorem (Theorem 12.2), we may assume that

$$n < p < \infty.$$

Hence we have, by Sobolev's imbedding theorem (see Theorem 7.3),

$$W^{2,p}(\Omega) \subset C^1(\overline{\Omega}),$$

since $2 - n/p > 1$ for $n < p < \infty$. By applying the weak maximum principle (Theorem 8.5) to the functions $\pm u(x)$, we find that

$$\begin{cases} \mathcal{L}u = 0 & \text{almost everywhere in } \Omega, \\ \gamma_0 u = 0 & \text{on } \partial\Omega \end{cases}$$
$$\Longrightarrow \ u = 0 \quad \text{in } \Omega.$$

This proves the uniqueness result of problem (15.2):

Step 3: In order to prove the *existence result* of problem (15.2), we make use of the *method of continuity* (Theorem 2.14).

We shall apply Theorem 2.14 with

$$\begin{aligned} &\mathcal{B} := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \\ &\mathcal{V} := L^p(\Omega), \\ &\mathcal{L}_1 := \mathcal{L} = \sum_{i,j=1}^n a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \\ &\mathcal{L}_0 := \Delta. \end{aligned}$$

Here it should be noticed that

$$\mathcal{B} = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) = \left\{ u \in W^{2,p}(\Omega) : \gamma_0 u = 0 \text{ on } \partial\Omega \right\}$$

is a closed subspace of $W^{2,p}(\Omega)$, since the trace theorem (Theorem 7.6) asserts that the trace map $\gamma_0 : W^{2,p}(\Omega) \to B^{2-1/p,p}(\partial\Omega)$ is continuous.

Substep 3-1: The essential step in our proof is how to show the *a priori* estimate (corresponding to inequality (2.9)) for a family of elliptic differential operators

$$\begin{aligned} \mathcal{L}_t &:= t\mathcal{L} + (1-t)\Delta \\ &= \sum_{i,j=1}^n \left(t\, a^{ij}(x) + (1-t)\delta^{ij} \right) \frac{\partial^2}{\partial x_i \partial x_j}, \quad 0 \leq t \leq 1. \end{aligned}$$

More precisely, we consider, instead of the original homogeneous Dirichlet problem ($t = 1$)

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ \gamma_0 u = 0 & \text{on } \partial\Omega, \end{cases} \tag{15.2}$$

a family of homogeneous Dirichlet problems

$$\begin{cases} \mathcal{L}_t u = f & \text{in } \Omega, \\ \gamma_0 u = 0 & \text{on } \partial\Omega. \end{cases}$$

By the uniqueness result of problem (15.2), we can get rid of the term

$\|u\|_{L^p(\Omega)}$ on the right-hand side of estimate (15.4). Namely, we shall prove that the *a priori* estimate

$$\|u\|_{W^{2,p}(\Omega)} \leq c_2 \|\mathcal{L}_t u\|_{L^p(\Omega)}, \quad u \in \mathfrak{B} \tag{15.8}$$

holds true for the elliptic differential operators $\mathcal{L}_t$, $0 \leq t \leq 1$. Here $c_2 > 0$ is a structure constant depending on the coefficients a^{ij} only through the ellipticity constant λ, the bound on the norms $\|a^{ij}\|_{L^p(\Omega)}$ and the VMO moduli of the a^{ij}.

Now it should be noticed that the coefficients

$$a^{ij}_{(t)}(x) := t\, a^{ij}(x) + (1-t)\,\delta^{ij}, \quad 0 \leq t \leq 1,$$

satisfy the following three conditions (i), (ii) and (iii):

(i) $a^{ij}_{(t)}(x) \in \mathrm{VMO} \cap L^\infty(\mathbf{R}^n)$ for all $1 \leq i, j \leq n$.
Indeed, we have, for all $0 \leq t \leq 1$,

$$\left\|a^{ij}_{(t)}\right\|_{L^\infty(\mathbf{R}^n)} \leq \left\|a^{ij}\right\|_{L^\infty(\mathbf{R}^n)} + 1,$$

and

$$\left\|a^{ij}_{(t)}\right\|_* \leq t \left\|a^{ij}\right\|_* \leq \left\|a^{ij}\right\|_* .$$

(ii) $a^{ij}_{(t)}(x) = a^{ji}_{(t)}(x)$ for almost all $x \in \Omega$.

(iii) We have, for almost all $x \in \Omega$ and for all $\xi \in \mathbf{R}^n$,

$$\frac{1}{\lambda+1}|\xi|^2 \leq \sum_{i,j=1}^n a^{ij}_{(t)}(x)\xi_i\xi_j \leq (\lambda+1)|\xi|^2,$$

where λ is the same constant as in condition (15.1).

Our proof of the *a priori* estimate (15.8) is based on a reduction to absurdity. We assume, to the contrary, that the *a priori* estimate (15.8) does not hold true. Then we can find a sequence of elliptic differential operators

$$\mathcal{L}^{(m)} = \sum_{i,j=1}^n a^{ij}_{(m)}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \quad m = 1, 2, \ldots,$$

and a sequence of functions

$$u^{(m)} \in \mathcal{B} = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega), \quad m = 1, 2, \ldots,$$

such that the coefficients $a^{ij}_{(m)}(x)$ and the functions $\{u^{(m)}\}$ satisfy the following four conditions (i), (ii), (iii) and (iv):

(i) $a^{ij}_{(m)}(x) \in \mathrm{VMO} \cap L^\infty(\mathbf{R}^n)$ for all $1 \leq i, j \leq n$ and

$$\left\| a^{ij}_{(m)} \right\|_{L^\infty(\mathbf{R}^n)} \leq \left\| a^{ij} \right\|_{L^\infty(\mathbf{R}^n)} + 1, \tag{15.9a}$$

$$\eta^{ij}_{(m)}(r) \leq \eta^{ij}(r). \tag{15.9b}$$

Here we recall that

$$\eta^{ij}_{(m)}(r) = \sup_{\rho \leq r} \frac{1}{|B|} \int_B \left| a^{ij}_{(m)}(y) - \left(a^{ij}_{(m)} \right)_B \right| dy,$$

$$\eta^{ij}(r) = \sup_{\rho \leq r} \frac{1}{|B|} \int_B \left| a^{ij}(y) - \left(a^{ij} \right)_B \right| dy,$$

where the supremum is taken over all balls B with radius $\rho \leq r$.

(ii) $a^{ij}_{(m)}(x) = a^{ji}_{(m)}(x)$ for almost all $x \in \Omega$.

(iii) We have, for almost all $x \in \Omega$ and for all $\xi \in \mathbf{R}^n$,

$$\frac{1}{\lambda + 1} |\xi|^2 \leq \sum_{i,j=1}^n a^{ij}_{(m)}(x) \xi_i \xi_j \leq (\lambda + 1) |\xi|^2. \tag{15.10}$$

(iv) $u^{(m)} \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ and

$$\left\| u^{(m)} \right\|_{W^{2,p}(\Omega)} = 1, \tag{15.11a}$$

$$\left\| \mathcal{L}^{(m)} u^{(m)} \right\|_{L^p(\Omega)} \longrightarrow 0. \tag{15.11b}$$

(3-1a): First, it follows that the sequence $\{(a^{ij}_{(m)})_B\}$ is bounded for every $1 \leq i, j \leq n$. Indeed, we have, by inequality (15.9a),

$$\left| \left(a^{ij}_{(m)} \right)_B \right| \leq \left\| a^{ij}_{(m)} \right\|_{L^\infty(\mathbf{R}^n)} \leq \left\| a^{ij} \right\|_{L^\infty(\mathbf{R}^n)} + 1.$$

Moreover, we have the following lemma:

Lemma 15.4. *For any ball B in $\mathbf{R}^n$, every sequence*

$$\left\{ a^{ij}_{(m)}(x) - \left(a^{ij}_{(m)} \right)_B \right\}, \quad 1 \leq i, j \leq n,$$

is compact *in the space $L^1(B)$.*

Proof. The proof of Lemma 15.4 is divided into two steps (1) and (2).

(1) For simplicity, we write

$$a_{(m)}(x) := a^{ij}_{(m)}(x),$$

and let

$$f_{(m)}(x) := a_{(m)}(x) - \left(a_{(m)} \right)_B, \tag{15.12a}$$

$$f^{\varepsilon}_{(m)}(x) := a_{(m)} * \varphi_{\varepsilon}(x) - \left(a_{(m)} * \varphi_{\varepsilon}\right)_B, \tag{15.12b}$$

where

$$a_{(m)} * \varphi_{\varepsilon}(x) = \int_{\mathbf{R}^n} a_{(m)}(x-y)\varphi_{\varepsilon}(y)\,dy$$

is a *mollification* of the coefficients $a_{(m)}$.

In light of the mollification, the VMO functions may be approximated by smooth functions. In fact, we can prove the following two claims 15.1 and 15.2:

Claim 15.1. The sequence $\{f^{\varepsilon}_{(m)}\}$ is uniformly bounded and equicontinuous in B, for each $\varepsilon > 0$. Hence $\{f^{\varepsilon}_{(m)}\}$ has a subsequence that is uniformly convergent in B to a continuous function, for each $\varepsilon > 0$.

Proof. We have only to prove the first statement. Indeed, the second statement follows from an application of the Ascoli–Arzelà theorem (Corollary 2.69).

(a) The *uniform boundedness* of $\{f^{\varepsilon}_{(m)}\}$: First, we have, for all $x \in B$,

$$\begin{aligned}|a_{(m)} * \varphi_{\varepsilon}(x)| &\leq \int_{\mathbf{R}^n} |a_{(m)}(x-y)|\,\varphi_{\varepsilon}(y)\,dy \leq \left\|a_{(m)}\right\|_{L^{\infty}(\mathbf{R}^n)} \\ &\leq \|a\|_{L^{\infty}(\mathbf{R}^n)} + 1,\end{aligned}$$

and also

$$\begin{aligned}\left|\left(a_{(m)} * \varphi_{\varepsilon}\right)_B\right| &\leq \frac{1}{|B|}\int_{\mathbf{R}^n} |a_{(m)}(x-y)|\,\varphi_{\varepsilon}(y)\,dy \leq \left\|a_{(m)}\right\|_{L^{\infty}(\mathbf{R}^n)} \\ &\leq \|a\|_{L^{\infty}(\mathbf{R}^n)} + 1.\end{aligned}$$

Hence it follows that

$$\begin{aligned}|f^{\varepsilon}_{(m)}(x)| &\leq |a_{(m)} * \varphi_{\varepsilon}(x)| + \left|\left(a_{(m)} * \varphi_{\varepsilon}\right)_B\right| \\ &\leq 2\left(\|a\|_{L^{\infty}(\mathbf{R}^n)} + 1\right) \quad \text{for all } x \in B.\end{aligned}$$

(b) The *equicontinuity* of $\{f^{\varepsilon}_{(m)}\}$: By the mean value theorem, it suffices to show that the gradient $\{\nabla f^{\varepsilon}_{(m)}\}$ is uniformly bounded in B, for each $\varepsilon > 0$.

We have, for all $x \in B$,

$$\begin{aligned}\nabla_x f^{\varepsilon}_{(m)}(x) &= \nabla_x\left(\int_{B(0,\varepsilon)} a_{(m)}(x-y)\varphi_{\varepsilon}(y)\,dy - \left(a_{(m)} * \varphi_{\varepsilon}\right)_{B(0,\varepsilon)}\right) \\ &= \nabla_x\left(\int_{B(x,\varepsilon)} \varphi_{\varepsilon}(x-y)a_{(m)}(y)\,dy\right)\end{aligned}$$

$$= \frac{1}{\varepsilon^n} \nabla_x \left(\int_{B(x,\varepsilon)} \varphi \left(\frac{x-y}{\varepsilon} \right) a_{(m)}(y)\, dy \right).$$

However, it should be noticed that

$$\nabla_x \left(\varphi \left(\frac{x-y}{\varepsilon} \right) \right) = -\nabla_y \left(\varphi \left(\frac{x-y}{\varepsilon} \right) \right) = \frac{1}{\varepsilon} (\nabla \varphi) \left(\frac{x-y}{\varepsilon} \right),$$

and further that

$$\int_{B(x,\varepsilon)} \nabla \varphi \left(\frac{x-y}{\varepsilon} \right) dy = 0.$$

Hence we can express the function $\nabla_x f^{\varepsilon}_{(m)}(x)$ as follows:

$$\begin{aligned} \nabla_x f^{\varepsilon}_{(m)}(x) &= \frac{1}{\varepsilon^{n+1}} \int_{B(x,\varepsilon)} \nabla \varphi \left(\frac{x-y}{\varepsilon} \right) a_{(m)}(y)\, dy \\ &= \frac{1}{\varepsilon^{n+1}} \int_{B(x,\varepsilon)} \nabla \varphi \left(\frac{x-y}{\varepsilon} \right) \left(a_{(m)}(y) - \left(a_{(m)} \right)_{B(x,\varepsilon)} \right) dy. \end{aligned}$$

Therefore, we have, for all $x \in B$,

$$\begin{aligned} &\left| \nabla f^{\varepsilon}_{(m)}(x) \right| \qquad (15.13) \\ &\leq \frac{1}{\varepsilon^{n+1}} \int_{B(x,\varepsilon)} \left| a_{(m)}(y) - \left(a_{(m)} \right)_{B(x,\varepsilon)} \right| \cdot \left| \nabla \varphi \left(\frac{x-y}{\varepsilon} \right) \right| dy \\ &\leq \frac{1}{\varepsilon^{n+1}} \left(\sup_{\mathbf{R^n}} |\nabla \varphi| \right) \int_{B(x,\varepsilon)} \left| a_{(m)}(y) - \left(a_{(m)} \right)_{B(x,\varepsilon)} \right| dy \\ &\leq \frac{1}{\varepsilon^{n+1}} \left(\sup_{\mathbf{R^n}} |\nabla \varphi| \right) |B(x,\varepsilon)|\, \eta_{(m)}(\varepsilon) \\ &= \frac{1}{\varepsilon^{n+1}} \left(\sup_{\mathbf{R^n}} |\nabla \varphi| \right) (\varepsilon^n \omega_n)\, \eta_{(m)}(\varepsilon) \\ &\leq \frac{\omega_n}{\varepsilon} \left(\sup_{\mathbf{R^n}} |\nabla \varphi| \right) \eta(\varepsilon), \quad \varepsilon > 0. \end{aligned}$$

Here we recall that

$$\begin{aligned} &\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \quad \text{(the surface area of the unit sphere } \Sigma_{n-1} \text{ in } \mathbf{R}^n), \\ &\eta_{(m)}(r) := \eta^{ij}_{(m)}(r) = \sup_{\rho \leq r} \frac{1}{|B|} \int_B \left| a^{ij}_{(m)}(y) - \left(a^{ij}_{(m)} \right)_B \right| dy, \\ &\eta(r) := \eta^{ij}(r) = \sup_{\rho \leq r} \frac{1}{|B|} \int_B \left| a^{ij}(y) - \left(a^{ij} \right)_B \right| dy. \end{aligned}$$

By inequality (15.13), it follows that $\{\nabla f^{\varepsilon}_{(m)}\}$ is uniformly bounded in B, for each $\varepsilon > 0$.

The proof of Claim 15.1 is complete. □

Claim 15.2. There exists a positive constant C, independent of ε, such that

$$\frac{1}{|B|}\int_B \left|f^{\varepsilon}_{(m)}(x) - f_{(m)}(x)\right| dx \leq C\,\eta(\varepsilon) \quad \text{for } \varepsilon > 0. \tag{15.14}$$

In particular, we have the assertion

$$\left\|f^{\varepsilon}_{(m)} - f_{(m)}\right\|_* \leq C\,\eta(\varepsilon) \quad \text{for } \varepsilon > 0.$$

Proof. First, by Fubini's theorem (Theorem 3.10) it follows that

$$\begin{aligned}
\left(a_{(m)} * \varphi_\varepsilon\right)_B &= \frac{1}{|B|}\int_B a_{(m)} * \varphi_\varepsilon(x)\,dx \qquad (15.15)\\
&= \frac{1}{|B|}\int_B \left(\int_{B(0,\varepsilon)} a_{(m)}(x-y)\varphi_\varepsilon(y)\,dy\right) dx\\
&= \int_{B(0,\varepsilon)} \left(\frac{1}{|B|}\int_B a_{(m)}(x-y)\,dx\right)\varphi_\varepsilon(y)\,dy\\
&= \int_{B(0,\varepsilon)} \left(\frac{1}{|B-y|}\int_{B-y} a_{(m)}(z)\,dz\right)\varphi_\varepsilon(y)\,dy\\
&= \int_{B(0,\varepsilon)} \left(a_{(m)}\right)_{B-y}\varphi_\varepsilon(y)\,dy,
\end{aligned}$$

where $B-y$ is the translation of the ball B by y-units

$$B - y = \{x - y : x \in B\}\,.$$

Hence we have, by formulas (15.12) and (15.15),

$$\begin{aligned}
&\int_B \left|f^{\varepsilon}_{(m)}(x) - f_{(m)}(x)\right| dx \qquad (15.16)\\
&= \int_B \left|a_{(m)} * \varphi_\varepsilon(x) - \left(a_{(m)} * \varphi_\varepsilon\right)_B - \left(a_{(m)}(x) - \left(a_{(m)}\right)_B\right)\right| dx\\
&= \int_B \left|\int_{B(0,\varepsilon)} \left(a_{(m)}(x-y) - a_{(m)}(x)\right)\right.\\
&\qquad \left. - \left(\left(a_{(m)}\right)_{B-y} - \left(a_{(m)}\right)_B\right)\varphi_\varepsilon(y)\,dy\right| dx\\
&\leq \int_B \int_{B(0,\varepsilon)} \left|\left(a_{(m)}(x-y) - a_{(m)}(x)\right)\right.\\
&\qquad \left. - \left(\left(a_{(m)}\right)_{B-y} - \left(a_{(m)}\right)_B\right)\right| \varphi_\varepsilon(y)\,dy\,dx
\end{aligned}$$

$$\leq \int_{B(0,\varepsilon)} \varphi_\varepsilon(y) \left(\int_B \Big| \left(a_{(m)}(x-y) - a_{(m)}(x)\right) - \left(\left(a_{(m)}\right)_{B-y} - \left(a_{(m)}\right)_B \right) \Big| dx \right) dy.$$

However, we have the formula

$$\begin{aligned}\left(a_{(m)}\right)_{B-y} &= \frac{1}{|B-y|} \int_{B-y} a_{(m)}(w)\, dw = \frac{1}{|B|} \int_B a_{(m)}(z-y)\, dz \\ &= a_{(m)}(\cdot - y)_B.\end{aligned}$$

Hence it follows from an application of inequality (4.9) with $f := a_{(m)}$ that

$$\begin{aligned}&\frac{1}{|B|} \int_B \left| (a_{(m)}(x-y) - a_{(m)}(x)) - \left(\left(a_{(m)}\right)_{B-y} - \left(a_{(m)}\right)_B \right) \right| dx \qquad (15.17)\\ &= \frac{1}{|B|} \int_B \left| \left(a_{(m)}(x-y) - a_{(m)}(x)\right) - \left(a_{(m)}(\cdot - y) - a_{(m)}\right)_B \right| dx \\ &\leq \left\| a_{(m)}(\cdot - y) - a_{(m)}(\cdot) \right\|_* \\ &\leq C\, \eta_{(m)}(\varepsilon), \quad |y| < \varepsilon.\end{aligned}$$

Therefore, by combining inequalities (15.16) and (15.17) we obtain that

$$\begin{aligned}\int_B \left| f^\varepsilon_{(m)}(x) - f_{(m)}(x) \right| dx &\leq C\, \eta_{(m)}(\varepsilon) \left(\int_{B(0,\varepsilon)} \varphi_\varepsilon(y)\, dy \right) |B| \\ &= C\, \eta_{(m)}(\varepsilon)\, |B| \\ &\leq C\, |B|\, \eta(\varepsilon).\end{aligned}$$

This proves the desired inequality (15.14).

The proof of Claim 15.2 is complete. □

(2) By combining Claims 15.1 and 15.2, we find from Theorems 2.4 and 2.5 that the sequence $\{f_{(m)}\}$ is compact in the space $L^1(B)$. Indeed, it suffices to apply Example 2.1 with $X := L^1(B)$, $A := \{f_{(m)}\}$ and $A_h := \{f^h_{(m)}\}$.

The proof of Lemma 15.4 is complete. □

(3-1b): By using Lemma 15.4 and the Bolzano–Weierstrass theorem, we can choose a subsequence of the sequence

$$a^{ij}_{(m)}(x) = \left(a^{ij}_{(m)}(x) - \left(a^{ij}_{(m)}\right)_B \right) + \left(a^{ij}_{(m)}\right)_B$$

that converges almost everywhere in B. Therefore, by considering an exhaustive sequence of balls of $\mathbf{R}^n$ we can choose a subsequence of $a^{ij}_{(m)}$, denoted again by $a^{ij}_{(m)}$, that converges to a function α^{ij} almost everywhere in $\mathbf{R}^n$, as $m \to \infty$. Then it is easy to verify the following three assertions (i), (ii) and (iii):

(i) $\alpha^{ij}(x) \in \mathrm{VMO} \cap L^\infty(\mathbf{R}^n)$ for all $1 \leq i, j \leq n$.

Indeed, since we have the inequality

$$\left\| a^{ij}_{(m)} \right\|_{L^\infty(\mathbf{R}^n)} \leq \left\| a^{ij} \right\|_{L^\infty(\mathbf{R}^n)} + 1,$$

it follows that

$$\left\| \alpha^{ij} \right\|_{L^\infty(\mathbf{R}^n)} \leq \left\| a^{ij} \right\|_{L^\infty(\mathbf{R}^n)} + 1.$$

Moreover, since we have, for all balls B with radius $\rho \leq r$,

$$\frac{1}{|B|} \int_B |a^{ij}_{(m)}(x) - \left(a^{ij}_{(m)} \right)_B | \, dx \leq \eta^{ij}_{(m)}(r) \leq \eta^{ij}(r),$$

and since we have, by the Lebesgue dominated convergence theorem (Theorem 3.8),

$$\begin{aligned} (a^{ij}_{(m)})_B &= \frac{1}{|B|} \int_B a^{ij}_{(m)}(y) \, dy \\ &\longrightarrow \frac{1}{|B|} \int_B \alpha^{ij}(y) \, dy = \left(\alpha^{ij} \right)_B \quad \text{as } m \to \infty, \end{aligned}$$

it follows that we have, for all balls B with radius $\rho \leq r$,

$$\begin{aligned} \frac{1}{|B|} \int_B |\alpha^{ij}(x) - \left(\alpha^{ij} \right)_B | \, dx &= \lim_{m \to \infty} \frac{1}{|B|} \int_B |a^{ij}_{(m)}(x) - \left(a^{ij}_{(m)} \right)_B | \, dx \\ &\leq \eta^{ij}(r) \quad \text{for } 1 \leq i, j \leq n. \end{aligned}$$

This proves that the VMO modulus of α^{ij} is dominated by $\eta^{ij}(r)$:

$$\sup_{\rho \leq r} \frac{1}{|B|} \int_B |\alpha^{ij}(x) - \left(\alpha^{ij} \right)_B | \, dx \leq \eta^{ij}(r) \quad \text{for } 1 \leq i, j \leq n.$$

(ii) $\alpha^{ij}(x) = \alpha^{ji}(x)$ for almost all $x \in \Omega$ and $1 \leq i, j \leq n$.

Indeed, we have, for almost all $x \in \Omega$,

$$\alpha^{ij}(x) = \lim_{m \to \infty} a^{ij}_{(m)}(x) = \lim_{m \to \infty} a^{ji}_{(m)}(x) = \alpha^{ji}(x).$$

(iii) We have, for almost all $x \in \Omega$ and for all $\xi \in \mathbf{R}^n$,

$$\frac{1}{\lambda + 1} |\xi|^2 \leq \sum_{i,j=1}^n \alpha^{ij}(x) \xi_i \xi_j \leq (\lambda + 1) |\xi|^2.$$

These inequalities can be obtained by passing to the limit in the inequalities (15.10).

We introduce a new second-order, uniformly elliptic differential operator $\mathcal{A}$ by the formula

$$\mathcal{A} := \sum_{i,j=1}^{n} \alpha^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}.$$

(3-1c): Now it is known (see [2, Theorem 3.6]) that the Sobolev space $W^{2,p}(\Omega)$ is reflexive if $1 < p < \infty$. By applying the Eberlein–Shmulyan theorem (Theorem 2.30) and the Rellich–Kondrachov theorem (Theorem 7.4) to our situation, we can obtain the following two assertions (A) and (B):

(A) Let $\{w_m\}$ be any sequence that is norm bounded in the Sobolev space $W^{2,p}(\Omega)$ for $1 < p < \infty$. Then there exists a subsequence $\{w_{m'}\}$ that converges *weakly* to an element of $W^{2,p}(\Omega)$.
(B) The injection $W^{2,p}(\Omega) \to W^{1,p}(\Omega)$ is *compact* for $1 < p < \infty$ ($j := 1$ and $p = q$ in Theorem 7.4).

Therefore, by using condition (15.11a) we can find a subsequence of the sequence

$$\left\{u^{(m)}\right\} \subset W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$

denoted again by $\{u^{(m)}\}$, that converges *weakly* to some function

$$v \in W^{2,p}(\Omega)$$

and also converges *strongly* to v in $L^p(\Omega)$:

$$u^{(m)} \longrightarrow v \quad \text{weakly in } W^{2,p}(\Omega) \text{ as } m \to \infty,$$
$$u^{(m)} \longrightarrow v \quad \text{in } L^p(\Omega) \text{ as } m \to \infty.$$

We shall prove that $v = 0$, that is,

$$u^{(m)} \longrightarrow 0 \quad \text{weakly in } W^{2,p}(\Omega) \text{ as } m \to \infty, \tag{15.18a}$$
$$u^{(m)} \longrightarrow 0 \quad \text{in } L^p(\Omega) \text{ as } m \to \infty. \tag{15.18b}$$

First, we show that the function $v \in W^{2,p}(\Omega)$ satisfies the boundary condition

$$v = 0 \quad \text{on } \partial\Omega,$$

that is,

$$v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega). \tag{15.19}$$

To do this, it should be noticed that the subspace

$$\mathfrak{B} = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$$

is *strongly closed* in $W^{2,p}(\Omega)$. Indeed, it suffices to note that if $\{u_j\}$ is any sequence in $\mathfrak{B}$ that converges strongly to a function u in $W^{2,p}(\Omega)$, then we have, by the trace theorem (Theorem 7.6),

$$\gamma_0 u = \lim_{j\to\infty} \gamma_0(u_j) = 0 \quad \text{in } B^{2-1/p,p}(\partial\Omega).$$

Hence, we find that the subspace $\mathfrak{B}$ is *weakly closed* in $W^{2,p}(\Omega)$, by applying Mazur's theorem (or Theorem 2.29) with

$$X := W^{2,p}(\Omega), \quad M := \mathfrak{B}.$$

Therefore, we obtain the desired assertion (15.19), since the sequence $\{u^{(m)}\}$ in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ converges weakly to v in $W^{2,p}(\Omega)$.

To prove assertions (15.18), let φ be an arbitrary function in $L^q(\Omega)$ with $q = p/(p-1)$. Then we have, by Hölder's inequality (Theorem 3.14) and condition (15.18a),

$$\begin{aligned}
&\left| \int_\Omega \left(\mathcal{L}^{(m)} u^{(m)} - \mathcal{A}v \right) \varphi \, dx \right| \qquad (15.20)\\
&= \left| \sum_{i,j=1}^n \int_\Omega \left(a^{ij}_{(m)}(x) \frac{\partial^2 u^{(m)}}{\partial x_i \partial x_j} - \alpha^{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} \right) \varphi \, dx \right| \\
&\le \sum_{i,j=1}^n \left| \int_\Omega \left(a^{ij}_{(m)}(x) \frac{\partial^2 u^{(m)}}{\partial x_i \partial x_j} - \alpha^{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} \right) \varphi \, dx \right| \\
&\le \sum_{i,j=1}^n \left| \int_\Omega \left(a^{ij}_{(m)}(x) - \alpha^{ij}(x) \right) \frac{\partial^2 u^{(m)}}{\partial x_i \partial x_j} \, \varphi \, dx \right| \\
&\quad + \sum_{i,j=1}^n \left| \int_\Omega \alpha^{ij}(x) \left(\frac{\partial^2 u^{(m)}}{\partial x_i \partial x_j} - \frac{\partial^2 v}{\partial x_i \partial x_j} \right) \varphi \, dx \right| \\
&\le \sum_{i,j=1}^n \left\| \nabla^2 u^{(m)} \right\|_{L^p(\Omega)} \left\| \left(a^{ij}_{(m)} - \alpha^{ij} \right) \varphi \right\|_{L^q(\Omega)} \\
&\quad + \sum_{i,j=1}^n \left| \int_\Omega \alpha^{ij}(x) \varphi(x) \cdot \left(\frac{\partial^2 u^{(m)}}{\partial x_i \partial x_j} - \frac{\partial^2 v}{\partial x_i \partial x_j} \right) dx \right| \\
&\le \sum_{i,j=1}^n \left\| \left(a^{ij}_{(m)} - \alpha^{ij} \right) \varphi \right\|_{L^q(\Omega)}
\end{aligned}$$

$$+ \sum_{i,j=1}^{n} \left| \int_\Omega \alpha^{ij}(x)\varphi(x) \cdot \left(\frac{\partial^2 u^{(m)}}{\partial x_i \partial x_j} - \frac{\partial^2 v}{\partial x_i \partial x_j} \right) dx \right|.$$

However, by condition (15.9a) it follows from an application of the Lebesgue dominated convergence theorem (Theorem 3.8) that the first term on the last inequality (15.20) tends to zero as $m \to \infty$:

$$\sum_{i,j=1}^{n} \left\| \left(a^{ij}_{(m)} - \alpha^{ij} \right) \varphi \right\|_{L^q(\Omega)} \longrightarrow 0.$$

Moreover, we recall that $\{u^{(m)}\}$ converges weakly to a function $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Since $\alpha^{ij}(x)\varphi(x) \in L^q(\Omega)$, we find that the second term on the last inequality (15.20) tends to zero as $m \to \infty$:

$$\sum_{i,j=1}^{n} \left| \int_\Omega \alpha^{ij}(x)\varphi(x) \cdot \left(\frac{\partial^2 u^{(m)}}{\partial x_i \partial x_j} - \frac{\partial^2 v}{\partial x_i \partial x_j} \right) dx \right| \longrightarrow 0.$$

Hence we have, by inequality (15.20),

$$\int_\Omega \mathcal{L}^{(m)} u^{(m)} \cdot \varphi \, dx \longrightarrow \int_\Omega \mathcal{A} v \cdot \varphi \, dx \quad \text{as } m \to \infty.$$

On the other hand, by Hölder's inequality (Theorem 3.14) and condition (15.11b) it follows that

$$\left| \int_\Omega \mathcal{L}^{(m)} u^{(m)} \cdot \varphi \, dx \right| \leq \left\| \mathcal{L}^{(m)} u^{(m)} \right\|_{L^p(\Omega)} \cdot \|\varphi\|_{L^q(\Omega)} \longrightarrow 0 \quad \text{as } m \to \infty.$$

This proves that

$$\int_\Omega \mathcal{A} v \cdot \varphi \, dx = \lim_{n \to \infty} \int_\Omega \mathcal{L}^{(m)} u^{(m)} \cdot \varphi \, dx = 0 \quad \text{for all } \varphi \in L^q(\Omega).$$

Summing up, we have proved that

$$v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$
$$\mathcal{A} v = 0 \quad \text{almost everywhere in } \Omega.$$

By applying Theorem 15.3 (the weak maximum principle) to the operator $\mathcal{A}$, we obtain that

$$v = 0 \quad \text{in } \Omega.$$

This proves the desired assertions (15.18).

(3-1d): By combining assertions (15.11b) and (15.18b), we have proved that, as $m \to \infty$,

$$\mathcal{L}^{(m)} u^{(m)} \longrightarrow 0 \quad \text{in } L^p(\Omega),$$

$$u^{(m)} \longrightarrow 0 \quad \text{in } L^p(\Omega).$$

Therefore, by applying estimate (15.7) to the operators $\{\mathcal{L}^{(m)}\}$ we obtain that

$$\left\| u^{(m)} \right\|_{W^{2,p}(\Omega)} \leq c_1 \left(\left\| \mathcal{L}^{(m)} u^{(m)} \right\|_{L^p(\Omega)} + \left\| u^{(m)} \right\|_{L^p(\Omega)} \right),$$

so that

$$u^{(m)} \longrightarrow 0 \quad \text{in } W^{2,p}(\Omega) \text{ as } m \to \infty.$$

However, this assertion contradicts condition (15.11a):

$$\left\| u^{(m)} \right\|_{W^{2,p}(\Omega)} = 1.$$

This contradiction proves the desired *a priori* estimate (15.8).

Substep 3-2: By virtue of estimate (15.8), we can apply the *method of continuity* (Theorem 2.14) to obtain that the Dirichlet problem is uniquely solvable for the operator $\mathcal{L}_0$ if and only if it is uniquely solvable for the operator $\mathcal{L}_1$. However, it is known (see [33, Theorem 13.15]) that the homogeneous Dirichlet problem is uniquely solvable for the Laplace operator $\mathcal{L}_0 = \Delta$: More precisely, for any $f \in L^p(\Omega)$ there exists a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ of the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \gamma_0 u = 0 & \text{on } \partial\Omega. \end{cases}$$

In other words, the operator $\mathcal{L}_0 := \Delta$ maps $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ onto $L^p(\Omega)$.

Therefore, it follows from an application of Theorem 2.14 that the operator $\mathcal{L}_1 := \mathcal{L}$ maps $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ onto $L^p(\Omega)$. Namely, for any $f \in L^p(\Omega)$, there exists a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ of problem (15.2).

Finally, the desired *a priori* estimate (15.3) follows from estimate (15.8) with $t := 1$.

Now the proof of Theorem 15.1 is complete. □

15.3 Notes and Comments

The results discussed here are adapted from Chiarenza–Frasca–Longo [19] and Gilbarg–Trudinger [33].