Čoupek, Pavel; **Šťovíček, Jan**
Cotilting sheaves on Noetherian schemes. (English) Zbl 07242444
Math. Z. 296, No. 1-2, 275-312 (2020).

Tilting theory is a machinery composed of well established methods for investigating equivalences between triangulated categories in homological algebra, struggling to answer the question how to characterize the situation that, given two abelian categories $\mathcal{A}$ and $\mathcal{H}$, which may not be equivalent, their derived categories are equivalent, namely,

$$\boldsymbol{D}(\mathcal{H}) \simeq \boldsymbol{D}(\mathcal{A})$$

If $\mathcal{H}$ is a module category, say, $\mathcal{H} = \boldsymbol{Mod} - R$, it is well known that existence of a tilting complex in $\boldsymbol{D}(\mathcal{A})$ whose endomorphism ring is R guarantees such a derived equivalence.

This paper begins with the question how to detect the case that $\mathcal{H}$ has an injective cogenerator W, e.g. if $\mathcal{H}$ is a Grothendieck category, in which the image C of W in $\boldsymbol{D}(\mathcal{A})$ should hopefully be a cotilting complex. There are several definitions in the case where C is required to be an object of $\mathcal{A}$ [*R. Colpi* et al., J. Algebra 191, No. 2, 461–494 (1997; Zbl 0876.16004); J. Algebra 307, No. 2, 841–863 (2007; Zbl 1120.18008); Commun. Algebra 25, No. 10, 3225–3237 (1997; Zbl 0893.16017); *L. Fiorot* et al., Proc. Am. Math. Soc. 145, No. 4, 1505–1514 (2017; Zbl 1359.18006); *C. E. Parra* and *M. Saorín*, J. Pure Appl. Algebra 219, No. 9, 4117–4143 (2015; Zbl 1333.18017); *J. Šťovíček*, Adv. Math. 263, 45–87 (2014; Zbl 1301.18015); "The tilting-cotilting correspondence", Int. Math. Res. Notices rnz116, `doi:10.1093/imrn/rnz116`], and rather recent research deals with the case that C is an actual complex [*P. Nicolás* et al., J. Pure Appl. Algebra 223, No. 6, 2273–2319 (2019; Zbl 1436.18013); *C. Psaroudakis* and *J. Vitória*, Math. Z. 288, No. 3–4, 965–1028 (2018; Zbl 1407.18014)]. One of the main difficulties in manipulating cotilting complexes is that, unlike the ring in its module category, injective cogenerators are usually very far from being finitely generated in any reasonable sense. The authors focus on derived equivalences coming from turning around a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, which is a very general method introduced by *D. Happel* et al. [Tilting in abelian categories and quasitilted algebras. Providence, RI: American Mathematical Society (AMS) (1996; Zbl 0849.16011)].

The principal objective in this paper is, inspired by the recent progress in the study of cotilting sheaves of affine schemes [*M. Hrbek* and *J. Šťovíček*, Forum Math. 32, No. 1, 235–267 (2020; Zbl 07175508); *L. A. Hügel* et al., Trans. Am. Math. Soc. 366, No. 7, 3487–3517 (2014; Zbl 1291.13018)], to comprehend the situation in detail for the particular case that $\mathcal{A}$ is also a Grothendieck category, and preferably even the category QCoh_X of quasi-coherent sheaves on a Noetherian scheme X.

A synopsis of the paper, consisting of six section together with two appendices, goes as follows. §2 establishes basic theory of cotitling objects of injective dimension at most 1, showing that basic aspects of cotilting modules [*R. Colpi* et al., J. Algebra 191, No. 2, 461–494 (1997; Zbl 0876.16004); Commun. Algebra 25, No. 10, 3225–3237 (1997; Zbl 0893.16017)] generaiize to Grothendieck categories rather easily and that the definition here matches perfectly with the one in [*L. Fiorot* et al., Proc. Am. Math. Soc. 145, No. 4, 1505–1514 (2017; Zbl 1359.18006); *P. Nicolás* et al., J. Pure Appl. Algebra 223, No. 6, 2273–2319 (2019; Zbl 1436.18013)]. §3 is concerned with pure-injectivity of cotilting objects, describing the cotilting torsion-free classes in $\mathcal{A}$ (Theorem 3.10), and showing that if is a locally Noetherian Grothendieck category, cotilting objects are parametrized, up to equivalence, by torsion pairs in the category $\mathcal{A}_0$ of Noetherian objects in $\mathcal{A}$.

§4 is concerned with derived equivalences, its exposition being standard, except that a cotilting object in a Grothendieck category is shown to induce a derived equivalence to another Grothendieck category in crucial use of pure-injectivity of cotilting objects.

§5 aims to specialize to categories of quasi-coherent sheaves on Noetherian schemes, being devoted to the description of suitable torsion pairs in the categories of coherent and quasi-coherent sheaves.

§6 aims at classifying those cotilting torsion-free classes in QCoh_X, which are closed under taking injective envelopes or, equivalently, for those X which have an ample family of line bundles, under tensoring with line bundles.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18C10 Theories (e.g., algebraic theories), structure, and semantics
03E55 Large cardinals
03G30 Categorical logic, topoi
18-02 Research exposition (monographs, survey articles) pertaining to category theory

Keywords:

Grothendieck category; cotilting objects; pure-injective objects; Noetherian scheme; classification

Full Text: DOI

References:

[1] Adámek, J., Rosický, J.: Locally Presentable and Accessible Categories, London Mathematical Society Lecture Note Series, vol. 189. Cambridge University Press, Cambridge (1994) · Zbl 0795.18007

[2] Angeleri Hügel, L., Happel, D., Krause, H. (eds.), Handbook of Tilting Theory, London Mathematical Society Lecture Note Series, vol. 332. Cambridge University Press, Cambridge (2007) · Zbl 1106.16300

[3] Angeleri Hügel, L.; Pospíšil, D.; Šťovíček, J.; Trlifaj, J., Tilting, cotilting, and spectra of commutative noetherian rings, Trans. Am. Math. Soc., 366, 3487-3517 (2014) · Zbl 1291.13018

[4] Auslander, M., Reiten, I., Smalø, S.O.: Representation Theory of Artin Algebras, Cambridge Studies in Advanced Mathematics, vol. 36. Cambridge University Press, Cambridge (1997). Corrected reprint of the 1995 original · Zbl 0834.16001

[5] Bashir, RE, Covers and directed colimits, Algebras Represent. Theory, 9, 5, 423-430 (2006) · Zbl 1114.16002

[6] Bazzoni, S., Cotilting modules are pure-injective, Proc. Am. Math. Soc., 131, 12, 3665-3672 (2003) · Zbl 1045.16004

[7] Beĭlinson, AA, Coherent sheaves on \(\mathbb{P}^n\) and problems of linear algebra, Funct. Anal. Appl., 12, 3, 214-216 (1978) · Zbl 0424.14003

[8] Bruns, W., Herzog, J.: Cohen-Macaulay Rings. rev. ed. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (1998)

[9] Bondal, A.; van den Bergh, M., Generators and representability of functors in commutative and noncommutative geometry, Mosc. Math. J., 3, 1, 1-36 (2003) · Zbl 1135.18302

[10] Borelli, M., Some results on ampleness and divisoral schemes, Pac. J. Math., 23, 217-227 (1967) · Zbl 0156.41103

[11] Buan, AB; Krause, H., Cotilting modules over tame hereditary algebras, Pac. J. Math., 211, 1, 41-59 (2003) · Zbl 1070.16014

[12] Colpi, R.: Tilting in Grothendieck categories, Forum Mathematicum, vol. 11. De Gruyter, Berlin, pp. 735-760 (1999) · Zbl 0934.18010

[13] Colpi, R.; Fuller, KR, Tilting objects in abelian categories and quasitilted rings, Trans. Am. Math. Soc., 359, 2, 741-765 (2007) · Zbl 1111.16008

[14] Colpi, R.; Trlifaj, J., Tilting modules and tilting torsion theories, J. Algebra, 178, 2, 614-634 (1995) · Zbl 0849.16033

[15] Colpi, R.; D'Este, G.; Tonolo, A., Quasi-tilting modules and counter equivalences, J. Algebra, 191, 2, 461-494 (1997) · Zbl 0876.16004

[16] Colpi, R.; Gregorio, E.; Mantese, F., On the heart of a faithful torsion theory, J. Algebra, 307, 2, 841-863 (2007) · Zbl 1120.18008

[17] Colpi, R.; Tonolo, A.; Trlifaj, J., Partial cotilting modules and the lattices induced by them, Commun. Algebra, 25, 10, 3225-3237 (1997) · Zbl 0893.16017

[18] Crawley-Boevey, W., Locally finitely presented additive categories, Commun. Algebra, 22, 5, 1641-1674 (1994) · Zbl 0798.18006

[19] Dickson, SE, A torsion theory for Abelian categories, Trans. Am. Math. Soc., 121, 1, 223-235 (1966) · Zbl 0138.01801

[20] Eisenbud, D., Commutative Algebra: With a View Toward Algebraic Geometry (1995), New York: Springer, New York · Zbl 0819.13001

[21] Enochs, EE; Estrada, S., Relative homological algebra in the category of quasi-coherent sheaves, Adv. Math., 194, 2, 284-295 (2005) · Zbl 1090.16003

[22] Fiorot, L.; Mattiello, F.; Saorín, M., Derived equivalences induced by nonclassical tilting objects, Proc. Am. Math. Soc., 145, 4, 1505-1514 (2017) · Zbl 1359.18006

[23] Gabriel, P., Des catégories abéliennes, Bull. Soc. Math. Fr., 90, 323-448 (1962) · Zbl 0201.35602

[24] Göbel, R.; Trlifaj, J., Approximations and Endomorphism Algebras of Modules (2012), Berlin: De Gruyter, Berlin ·

Zbl 1292.16001

[25] Görtz, U.; Wedhorn, T., Algebraic Geometry I (2010), Wiesbaden: Vieweg + Teubner, Wiesbaden

[26] Happel, D., On the derived category of a finite-dimensional algebra, Comment. Math. Helv., 62, 339-389 (1987) · Zbl 0626.16008

[27] Happel, D., Reiten, I., Smalø, S.O.: Tilting in Abelian Categories and Quasitilted Algebras, Mem. Amer. Math. Soc. vol. 120, no. 575, viii+ 88 (1996) · Zbl 0849.16011

[28] Hartshorne, R., Residues and Duality (1966), Berlin: Springer, Berlin

[29] Hartshorne, R., Algebraic Geometry (1977), New York: Springer, New York

[30] Hrbek, M., Šťovíček, J.: Tilting classes over commutative rings. Forum. Math. (2019). 10.1515/forum-2017-0219 · Zbl 07175508

[31] Hunter, RH, Balanced subgroups of abelian groups, Trans. Am. Math. Soc., 215, 81-98 (1976) · Zbl 0321.20035

[32] Illusie, L., Existence de résolutions globales, 160-221 (1971), Berlin: Springer, Berlin · Zbl 0241.14002

[33] Jensen, CU; Lenzing, H., Model-Theoretic Algebra with Particular Emphasis on Fields, Rings, Modules, Algebra, Logic and Applications (1989), New York: Gordon and Breach Science Publishers, New York · Zbl 0728.03026

[34] Krause, H., The stable derived category of a noetherian scheme, Compos. Math., 141, 423-430 (2005)

[35] Lenzing, H.: Homological Transfer from Finitely Presented to Infinite Modules, Abelian Group Theory (Honolulu, Hawaii, 1983), Lecture Notes in Mathematics, vol. 1006. Springer, Berlin, pp. 734-761 (1983) · Zbl 0529.16016

[36] Murfet, D.: The Mock Homotopy Category of Projectives and Grothendieck Duality, Ph.D. thesis, Aust. National U. (2007)

[37] Nicolás, P.; Saorín, M.; Zvonareva, A., Silting theory in triangulated categories with coproducts, J. Pure Appl. Algebra, 223, 6, 2273-2319 (2019) · Zbl 1436.18013

[38] Noohi, B., Explicit HRS-tilting, J. Noncommut. Geom., 3, 2, 223-259 (2009) · Zbl 1172.18003

[39] Odabaşı, S., Locally torsion-free quasi-coherent sheaves, J. Pure Appl. Algebra, 218, 9, 1760-1770 (2014) · Zbl 1287.13010

[40] Parra, CE; Saorín, M., Direct limits in the heart of a t-structure: the case of a torsion pair, J. Pure Appl. Algebra, 219, 9, 4117-4143 (2015) · Zbl 1333.18017

[41] Positselski, L., Šťovíček, J.: The tilting-cotilting correspondence. Int. Math. Res. Not. IMRN (2019). 10.1093/imrn/rnz116 · Zbl 07178875

[42] Prest, M., Purity, Spectra and Localisation, Encyclopedia of Mathematics and its Applications (2009), Cambridge: Cambridge University Press, Cambridge

[43] Psaroudakis, C.; Vitória, J., Realisation functors in tilting theory, Math. Z., 288, 3-4, 965-1028 (2018) · Zbl 1407.18014

[44] Rickard, J., Morita theory for derived categories, J. Lond. Math. Soc., s2-39, 3, 436-456 (1989) · Zbl 0642.16034

[45] Slávik, A., Šťovíček, J.: On flat generators and Matlis duality for quasicoherent sheaves (arXiv preprint). arXiv:1902.05740 (2019)

[46] Stenström, B., Rings of Quotients (1975), New York: Springer, New York · Zbl 0296.16001

[47] Šťovíček, J., All n-cotilting modules are pure-injective, Proc. Am. Math. Soc., 134, 7, 1891-1897 (2006) · Zbl 1095.16002

[48] Šťovíček, J., Derived equivalences induced by big cotilting modules, Adv. Math., 263, 45-87 (2014) · Zbl 1301.18015

[49] Šťovíček, J.; Kerner, O.; Trlifaj, J., Tilting via torsion pairs and almost hereditary noetherian rings, J. Pure Appl. Algebra, 215, 9, 2072-2085 (2011) · Zbl 1269.16010

[50] Šťovíček, J.; Trlifaj, J.; Herbera, D., Cotilting modules over commutative noetherian rings, J. Pure Appl. Algebra, 218, 1696-1711 (2014) · Zbl 1297.13013

[51] The Stacks Project Authors, Stacks Project (2019). http://stacks.math.columbia.edu

[52] Thomason, RW, The classification of triangulated subcategories, Compos. Math., 105, 1, 1-27 (1997) · Zbl 0873.18003

[53] Thomason, R.W., Trobaugh, T.: Higher Algebraic K-Theory of Schemes and of Derived Categories, The Grothendieck Festschrift, Progress in Mathematics, vol. 88, Birkhäuser, Boston, pp. 247-435 (1990)

[54] Wakamatsu, T., On modules with trivial self-extensions, J. Algebra, 114, 1, 106-114 (1988) · Zbl 0646.16025