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A stabilizer interpretation of double shuffle Lie algebras. (English summary)
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Multiple L-values (MLVs) $L(k_1,\dots,k_m;\zeta_1,\dots,\zeta_m)$ are complex numbers defined by the series

$$L(k_1,\dots,k_m;\zeta_1,\dots,\zeta_m):=\sum_{n_1>\dots>n_m>0}\frac{\zeta_1^{n_1}\cdots\zeta_m^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}}$$

for $m,k_1,\dots,k_m\in\mathbb{Z}_{>0}$ and $\zeta_1,\dots,\zeta_m$ in the group μ_N of Nth roots of unity in $\mathbb{C}$ with an integer $N\ge 1$. The extended (regularized) double shuffle relations [G. Racinet, Publ. Math. Inst. Hautes Études Sci. No. 95 (2002), 185–231; MR1953193; K. Ihara, M. Kaneko and D. B. Zagier, Compos. Math. **142** (2006), no. 2, 307–338; MR2218898] are some of the most fascinating ones. The structure of the regularized double shuffle relations is clarified by the following.

Theorem 1 [G. Racinet, op. cit. (Theorem I, §3.2.3); MR1953193]. The scheme $\boldsymbol{k}\mapsto\mathrm{DMR}_0(\boldsymbol{k})$ forms a prounipotent subgroup scheme of MT. For $\boldsymbol{k}$ a $\mathbb{Q}$-commutative algebra and $\lambda\in\boldsymbol{k}$, the set $\mathrm{DMR}_\lambda(\boldsymbol{k})$ is a torsor (principal homogeneous space) over the group $\mathrm{DMR}_0(\boldsymbol{k})$.

The symbols DMR and MT stand for the French "double mélange et régularisation" and "groupe de Magnus tordu". The central object of Racinet's approach was a certain invertible non-commutative formal power series in $\mathbb{C}\langle\langle X\rangle\rangle^\times$, constructed through iterated integrals and whose coefficients were expressed in terms of MLVs. The above theorem claims that the scheme $\mathrm{DMR}_{2\pi\sqrt{-1}}(\mathbb{C})$ corresponding to the double shuffle relations of MLVs is isomorphic to the group $\mathrm{DMR}_0(\mathbb{C})$, which is prounipotent, and therefore isomorphic to its Lie algebra, which is the degree completion of a positively graded Lie algebra $\mathfrak{dmr}_0$.

This paper introduces a graded Lie subalgebra $(\mathfrak{Lib}(X),\langle,\rangle)$ of a graded Lie algebra $\mathfrak{mt}$, whose completion is the Lie algebra of MT, and shows that $\mathfrak{dmr}_0$ coincides, up to addition of an abelian Lie algebra, with the stabilizer of an element in a $(\mathfrak{Lib}(X),\langle,\rangle)$-module (Theorem 3.10), giving an alternative proof of the Lie algebra nature of $\mathfrak{dmr}_0$. The authors construct, for any commutative $\mathbb{Q}$-algebra $\boldsymbol{k}$, a subgroup $(\exp(\widehat{\mathfrak{Lib}}(X)),\circledast)$ of $\mathrm{MT}(\boldsymbol{k})$, and establish that $\mathrm{DMR}_0(\boldsymbol{k})$ coincides up to a central additive group with the stabilizer of an element of an $(\exp(\widehat{\mathfrak{Lib}}(X)),\circledast)$-module. The authors show the following (§5.4).

Theorem 2. The subgroup $\widetilde{\mathrm{DMR}_0}(\boldsymbol{k})$ coincides with the stabilizer of the element Δ_* of the space $\mathrm{Hom}^{\mathrm{cont}}_{\boldsymbol{k}}(\boldsymbol{k}\langle Y\rangle,\boldsymbol{k}\langle Y\rangle^{\widehat{\otimes}2})$ of continuous $\boldsymbol{k}$-linear maps $\boldsymbol{k}\langle Y\rangle\to\boldsymbol{k}\langle Y\rangle^{\widehat{\otimes}2}$, equipped with the action of $(\exp(\widehat{\mathfrak{Lib}}(X)),\circledast)$, pulled back by $\Theta_{\boldsymbol{k}}$ of the natural action of $\mathrm{MT}(\boldsymbol{k})$.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.